Lesson 26: Recursive Challenge Problem—The Double and Add 5 Game

The *double and add 5* game is *loosely* related to the Collatz conjecture—an *unsolved* conjecture in mathematics named after Lothar Collatz, who first proposed the problem in 1937. The conjecture includes a recurrence relation, *triple and add 1,* as part of the problem statement. A worthwhile activity for you and your class is to read about the conjecture online.

Classwork

Example 1

Fill in the *doubling and adding* 5 below:

Number	Double and add 5
1	$1 \cdot 2 + 5 = 7$
7	______________
______	______________
______	______________
______	______________

Exercise 1

Complete the tables below for the given starting number.

Number	Double and add 5
2	______________
______	______________
______	______________

Number	Double and add 5
3	______________
______	______________
______	______________

Exercise 2

Given a starting number, double it and add 5 to get the result of round 1. Double the result of Round 1 and add 5, and so on. The goal of the game is to find the smallest starting whole number that produces a result of 100 or greater in three rounds or fewer.

Exercise 3

Using a generic initial value, a_0, and the recurrence relation, $a_{i+1} = 2a_i + 5$, for $i \geq 0$, find a formula for a_1, a_2, a_3, a_4 in terms of a_0.

Vocabulary

<u>Sequence</u>: A *sequence* can be thought of as an ordered list of elements. The elements of the list are called the *terms of the sequence.*

For example, (P, O, O, L) is a sequence that is different than (L, O, O, P). Usually the terms are *indexed* (and therefore ordered) by a subscript starting at either 0 or 1: $a_1, a_2, a_3, a_4, \ldots$. The "…" symbol indicates that the pattern described is regular, that is, the next term is a_5, and the next is a_6, and so on. In the first example, $a_1 = P$ is the first term, $a_2 = O$ is the second term, and so on. Both finite and infinite sequences exist everywhere in mathematics. For example, the infinite decimal expansion of $\frac{1}{3} = 0.333333333\ldots$ can be represented as the sequence, $(0.3, 0.33, 0.333, 0.3333, \ldots)$.

<u>Recursive Sequence</u>: An example of a *recursive sequence* is a sequence that is defined by (1) specifying the values of one or more initial terms and (2) having the property that the remaining terms satisfy a recurrence relation that describes the value of a term based upon an algebraic expression in numbers, previous terms, or the index of the term.

The sequence generated by initial term, $a_1 = 3$, and recurrence relation, $a_n = 3a_{n-1}$, is the sequence $(3, 9, 27, 81, 243, \ldots)$. Another example, given by the initial terms, $a_0 = 1$, $a_1 = 1$, and recurrence relation, $a_n = a_{n-1} + a_{n-2}$, generates the famed *Fibonacci sequence* $(1, 1, 2, 3, 5, \ldots)$.

Problem Set

1. Write down the first 5 terms of the recursive sequences defined by the initial values and recurrence relations below:
   a. $a_0 = 0$ and $a_{i+1} = a_i + 1$, for $i \geq 0$,
   b. $a_1 = 1$ and $a_{i+1} = a_i + 1$, for $i \geq 1$,
   c. $a_1 = 2$ and $a_{i+1} = a_i + 2$, for $i \geq 1$,
   d. $a_1 = 3$ and $a_{i+1} = a_i + 3$, for $i \geq 1$,
   e. $a_1 = 2$ and $a_{i+1} = 2a_i$, for $i \geq 1$,
   f. $a_1 = 3$ and $a_{i+1} = 3a_i$, for $i \geq 1$,
   g. $a_1 = 4$ and $a_{i+1} = 4a_i$, for $i \geq 1$,
   h. $a_1 = 1$ and $a_{i+1} = (-1)a_i$, for $i \geq 1$,
   i. $a_1 = 64$ and $a_{i+1} = (-\frac{1}{2})a_i$, for $i \geq 1$,

2. Look at the sequences you created in Problems 1(b) through 1(d). How would you define a recursive sequence that generates multiples of 31?

3. Look at the sequences you created in problems 1(e) through 1(g). How would you define a recursive sequence that generates powers of 15?

4. The following recursive sequence was generated starting with an initial value of a_0, and the recurrence relation $a_{i+1} = 3a_i + 1$, for $i \geq 0$. Fill in the blanks of the sequence
   (______, ______, 94, ______, 850, ______).

5. For the recursive sequence generated by initial value, a_0, and recurrence relation, $a_{i+1} = a_i + 2$, for $i \geq 0$, find a formula for a_1, a_2, a_3, a_4 in terms of a_0. Describe in words what this sequence is generating.

6. For the recursive sequence generated by initial value, a_0, and recurrence relation, $a_{i+1} = 3a_i + 1$, for $i \geq 0$, find a formula for a_1, a_2, a_3, a_4 in terms of a_0.