



Lesson 14: Linear and Exponential Models—Comparing Growth Rates

Student Outcomes

- Students compare linear and exponential models by focusing on how the models change over intervals of equal length. Students observe from tables that a function that grows exponentially will eventually exceed a function that grows linearly.

Classwork

Example 1 (12 minutes)

Example 1

Linear Functions

- a. Sketch points $P_1 = (0, 4)$ and $P_2 = (4, 12)$. Are there values of m and b such that the graph of the linear function described by $f(x) = mx + b$ contains P_1 and P_2 ? If so, find those values. If not, explain why they do not exist.

The graph of the linear function contains $(0, 4)$ when the y -intercept, b , is equal to 4. So all the linear functions whose graphs contain $(0, 4)$ are represented by

$$f(x) = mx + 4, \text{ where } m \text{ is a real number.}$$

For the graph of f to also contain $(4, 12)$,

$$\begin{aligned} 12 &= m(4) + 4 \\ m &= 2 \end{aligned}$$

The linear function whose graph passes through $(0, 4)$ and $(4, 12)$ is

$$f(x) = 2x + 4.$$

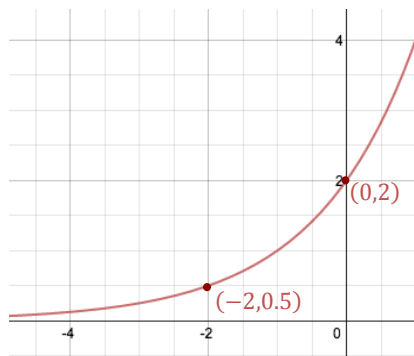
- b. Sketch $P_1 = (0, 4)$ and $P_2 = (0, -2)$. Are there values of m and b so that the graph of a linear function described by $f(x) = mx + b$ contains P_1 and P_2 ? If so, find those values. If not, explain why they do not exist.

For a function f , for each value of x , there can only be one $f(x)$ -value. Therefore, there is no linear function $f(x) = mx + b$ that contains P_1 and P_2 since each point has two different $f(x)$ values for the same x -value.

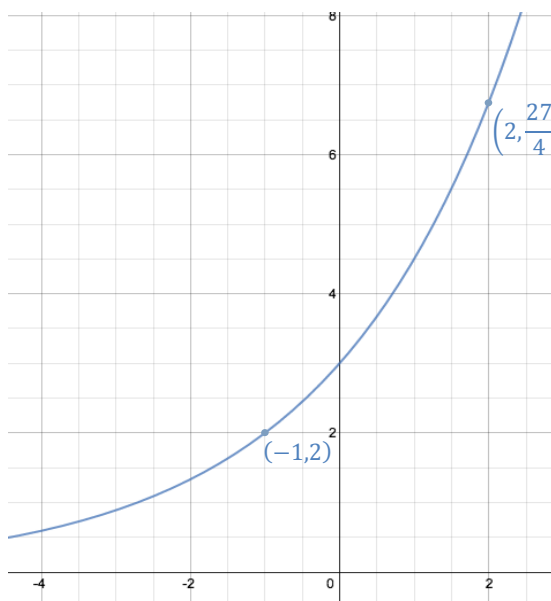
Exponential Functions

Graphs (c) and (d) are both graphs of an exponential function of the form $g(x) = ab^x$. Rewrite the function $g(x)$ using the values for a and b that are required for the graph shown to be a graph of g .

c. $g(x) = 2(2)^x$



d. $g(x) = 3\left(\frac{3}{2}\right)^x$



Discussion (5 minutes)

Discuss the rates of change in Example 1.

- Consider Example 1 part (a). Restrict the linear function $f(x) = 2x + 4$ to the positive integers and consider the sequence $f(0), f(1), f(2), \dots$, i.e., the sequence 4, 6, 8, Ask students what they observe about this sequence, looking for the answer, "Each term is always the sum of the previous term and 2."

Suggest they prove that statement by showing $f(n + 1) = f(n) + 2$: For n a positive integer, $f(n + 1) = 2(n + 1) + 4 = 2n + 2 + 4 = (2n + 4) + 2$. Since $f(n) = 2n + 4$, we see that $f(n + 1) = f(n) + 2$.

In particular, the recursion formula $f(n + 1) = f(n) + 2$ shows that the linear function grows additively by 2 over intervals of length 1 (i.e., the length of the intervals between consecutive integers). This fact holds for any interval of length 1 (the equation $f(x + 1) = f(x) + 2$ holds for any real number x , not just integers).

Similarly, for any linear function of the form $f(x) = mx + b$, that linear function grows additively by m over any interval of length 1. In fact, it grows additively by mx over any interval of length l (if time permits, have students prove this by calculating $f(x + l) - f(x)$).

- Now consider Example 1, part (c). Restrict the exponential function $g(x) = 2(2)^x$ to the positive integers, and consider the sequence $g(0), g(1), g(2), g(3), \dots$, i.e., the sequence 2, 4, 8, 16, Ask students what they observe about this sequence, looking for the answer, "Each term is always the product of the previous term and 2."

Suggest they prove that statement by showing $g(n + 1) = g(n) \cdot 2$: for n a positive integer, $g(n + 1) = 2(2)^{n+1} = 2(2)^n(2)^1 = [2(2)^n] \cdot 2$. Since $g(n) = 2(2)^n$, we see that $g(n + 1) = g(n) \cdot 2$.

In particular, the recursion formula $g(n + 1) = g(n) \cdot 2$ shows that the exponential function grows multiplicatively by 2 over intervals of length 1 (i.e., the length of the intervals between consecutive integers). This fact holds for any interval of length 1 (the equation $g(x + 1) = g(x) \cdot 2$ holds for any real number x , not just integers).

Similarly, for any exponential function of the form $f(x) = ab^x$, that exponential function grows multiplicatively by b over any interval of length 1.

Scaffolding:

For students performing above grade level: Ask students to make and test a conjecture about how an exponential function grows multiplicatively over any interval of length l .

Answer:

$$g(x + l) = g(x) \cdot b^l$$

- Moral: Linear functions grow additively while exponential functions grow multiplicatively.

Example 2 (15 minutes)

Example 2

A lab researcher records the growth of the population of a yeast colony and finds that the population doubles every hour.

- a. Complete the researcher's table of data:

Hours into study	0	1	2	3	4
Yeast colony population (thousands)	5	10	20	40	80

- b. What is the exponential function that models the growth of the colony's population?

$$P(t) = 5(2)^t$$

- c. Several hours into the study, the researcher looks at the data and wishes there were more frequent measurements. Knowing that the colony doubles every hour, how can the researcher determine the population in half-hour increments? Explain.

Let x represent the factor by which the population grows in half an hour. Since the population grows by the same factor in the next half hour, also x , the population will grow by x^2 in 1 hour. However, the colony's population doubles every hour:

$$\begin{aligned} x^2 &= 2 \\ x &= \sqrt{2} \end{aligned}$$

The researcher should multiply the population by $\sqrt{2}$ every half hour.

- d. Complete the new table that includes half-hour increments.

Hours into study	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3
Yeast colony population (thousands)	5	7.071	10	14.142	20	28.284	40

- e. How would the calculation for the data change for time increments of 20 minutes? Explain.

Now let x represent the factor by which the population grows in 20 minutes. Since the population grows by the same factor in the next 20 minutes and the 20 minutes after that, the population will grow by x^3 in 1 hour. Since the colony's population doubles every hour:

$$\begin{aligned} x^3 &= 2 \\ x &= \sqrt[3]{2} \end{aligned}$$

The researcher should multiply the population by $\sqrt[3]{2}$ every 20 minutes.

- f. Complete the new table that includes 20-minute increments.

Hours into study	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	$\frac{5}{3}$	2
Yeast colony population (thousands)	5	6.3	7.937	10	12.599	15.874	20

- g. The researcher's lab assistant studies the data recorded and makes the following claim:

Since the population doubles in 1 hour, then half of that growth happens in the first half hour and the other half of that growth happens in the second half hour. We should be able to find the population at $t = \frac{1}{2}$ by taking the average of the populations at $t = 0$ and $t = 1$.

Is the assistant's reasoning correct? Compare this strategy to your work in parts (c) and (e).

The assistant's reasoning is not correct. By the assistant's reasoning, the population growth at the first half-hour mark would be $t(\frac{1}{2}) = 7.5$ because then the population would have grown by 2.5 thousand cells from 5 thousand cells in the first half hour and another 2.5 thousand cells to 10 thousand cells in the second half hour. This is linear growth, or the same amount of population growth in each half hour. However, the percent growth by the assistant's reasoning is 50% from $t = 0$ to $t = 1$ and 33% from $t = 0$ to $t = 1$. For the population to double in 1 hour, there must be constant percent growth in each half hour.

You might have a student who says that the assistant is correct if he is using the geometric mean. Recall that the geometric mean of two numbers is the square root of their product. In this case, it is $\sqrt{5 \times 10} \approx 7.071$, which is the same value as the half-hour mark in part (d). Explore with your students the connection between the answer in part (c) and the definition of geometric mean. Does the geometric mean give the same value as other half-hour marks in the table to part (d)?

Example 3 (8 minutes)

Example 3

A California Population Projection Engineer in 1920 was tasked with finding a model that predicts the state's population growth. He modeled the population growth as a function of time, t years since 1900. Census data shows that the population in 1900, in thousands, was 1,490. In 1920, the population of the state of California was 3,554 thousand. He decided to explore both a linear and an exponential model.

- a. Use the data provided to determine the equation of the linear function that models the population growth from 1900–1920.

$$f(t) = 103.2t + 1,490$$

- b. Use the data provided and your calculator to determine the equation of the exponential function that models the population growth.

$$g(t) = 1,490(1.044)^t$$

- c. Use the two functions to predict the population for the following years:

	Projected Population Based on Linear Function, $f(t)$ (thousands)	Projected Population Based on Exponential Function, $g(t)$ (thousands)	Census Population Data and Intercensal Estimates for California (thousands)
1935	5,102	6,725	6,175
1960	7,682	19,734	15,717
2010	12,842	169,919	37,253

- d. Which function is a better model for the population growth of California in 1935 and in 1960?

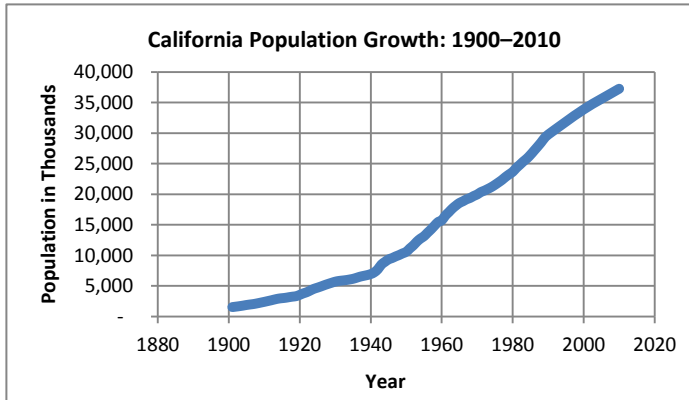
The exponential model.

Scaffolding:

Encourage students to use graphing calculators to determine the linear and exponential regression functions. The steps for finding the exponential regression are listed below.

- e. Does either model closely predict the population for 2010? What phenomenon explains the real population value?

Neither model closely predicts the population. After a population boom from 1900–1960, the population growth slowed down. The following graph shows census and intercensal estimates for California's population between 1900 and 2010.



Closing (2 minutes)

- What is the difference between the way a linear function increases and the way an exponential function increases?
 - *A linear function grows additively and an exponential function grows multiplicatively*
- Is it fair to say that an exponential function always grows faster than a linear?
 - *Not necessarily. At the beginning a linear function may be increasing faster than an exponential.*
- What can we say about an increasing exponential function when compared with an increasing linear function?
 - *An increasing exponential function will always eventually exceed any increasing linear function.*

Lesson Summary

- Given a linear function of the form $L(x) = mx + k$ and an exponential function of the form $E(x) = ab^x$ for x a real number and constants m , k , a , and b , consider the sequence given by $L(n)$ and the sequence given by $E(n)$, where $n = 1, 2, 3, 4, \dots$. Both of these sequences can be written recursively:

$$L(n+1) = L(n) + m \text{ and } L(0) = k, \text{ and} \\ E(n+1) = E(n) \cdot b \text{ and } E(0) = a.$$

The first sequence shows that a linear function grows additively by the same summand m over equal length intervals (i.e., the intervals between consecutive integers). The second sequence shows that an exponential function grows multiplicatively by the same factor b over equal length intervals (i.e., the intervals between consecutive integers).

- An increasing exponential function will eventually exceed any linear function. That is, if $f(x) = ab^x$ is an exponential function with $a > 0$ and $b > 1$, and $g(x) = mx + k$ is a linear function, then there is a real number M such that for all $x > M$, then $f(x) > g(x)$. Sometimes this is not apparent in a graph displayed on a graphing calculator; that is because the graphing window does not show enough of the graphs for us to see the sharp rise of the exponential function in contrast with the linear function.

Exit Ticket (3 minutes)

Name _____

Date _____

Lesson 14: Linear and Exponential Models—Comparing Growth Rates

Exit Ticket

A big company settles its new headquarters in a small city. The city council plans road construction based on traffic increasing at a linear rate, but based on the company's massive expansion, traffic is really increasing exponentially.

What will be the repercussions of the city council's current plans? Include what you know about linear and exponential growth in your discussion.

Exit Ticket Sample Solutions

A big company settles its new headquarters in a small city. The city council plans road construction based on traffic increasing at a linear rate, but based on the company's massive expansion, traffic is really increasing exponentially.

What will be the repercussions of the city council's current plans? Include what you know about linear and exponential growth in your discussion.

The city will not have the roads or capacity to handle the kind of traffic that will exist. Even if the linear growth of traffic initially outruns the exponential growth, eventually the exponential growth will catch up and exceed it. This means people will sit in traffic longer and the city will be generally congested for longer than if it had been planned for much heavier traffic flow.

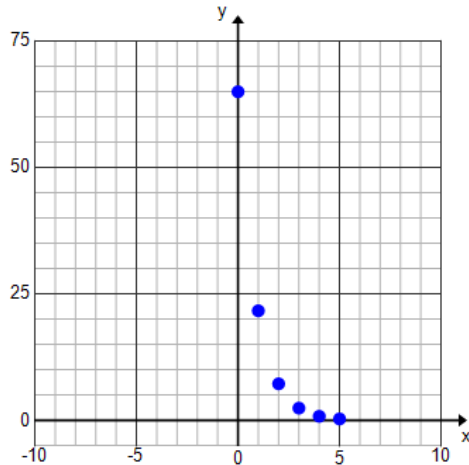
Problem Set Sample Solutions

1. When a ball bounces up and down, the maximum height it reaches decreases with each bounce in a predictable way. Suppose for a particular type of squash ball dropped on a squash court, the maximum height, $h(x)$, after x number of bounces can be represented by $h(x) = 65 \left(\frac{1}{3}\right)^x$.

- a. How many times higher is the height after the first bounce compared to the height after the third bounce?

9 times higher.

- b. Graph the points $(x, h(x))$ for x -values of 0, 1, 2, 3, 4, and 5.



2. Australia experienced a major pest problem in the early 20th century. The pest? Rabbits. In 1859, 24 rabbits were released by Thomas Austin at Barwon Park. In 1926, there were an estimated 10 billion rabbits in Australia. Needless to say, the Australian government spent a tremendous amount of time and money to get the rabbit problem under control. (To find more on this topic, visit Australia's Department of Environment and Primary Industries website under Agriculture.)

- a. Based only on the information above, write an exponential function that would model Australia's rabbit population growth.

$$R(t) = 24(1.3448)^t$$

- b. The model you created from the data in the problem is obviously a huge simplification from the actual function of the number of rabbits in any given year from 1859 to 1926. Name at least one complicating factor (about rabbits) that might make the graph of your function look quite different than the graph of the actual function.

Example: A drought could have wiped out a huge percentage of rabbits in a single year, showing a dip in the graph of the actual function.

3. After graduating from college, Jane has two job offers to consider. Job A is compensated at \$100,000 a year but with no hope of ever having an increase in pay. Jane knows a few of her peers are getting that kind of an offer right out of college. Job B is for a social media start-up, which guarantees a mere \$10,000 a year. The founder is sure the concept of the company will be the next big thing in social networking and promises a pay increase of 25% at the beginning of each new year.

- a. Which job will have a greater annual salary at the beginning of the fifth year? By approximately how much?

Job A, \$100,000 vs. \$24,400, a difference of about \$75,600.

- b. Which job will have a greater annual salary at the beginning of the tenth year? By approximately how much?

Job A, \$100,000 vs. \$74,500, a difference of about \$25,500.

- c. Which job will have a greater annual salary at the beginning of the twentieth year? By approximately how much?

Job B, \$694,000 vs. \$100,000, a difference of about \$594,000.

- d. If you were in Jane's shoes, which job would you take?

Answers will vary. Encourage students to voice reasons for each position. Note: The accumulated total after n years for Job B is $B(n) = (40,000)(1.25^n - 1)$, and the accumulated total for Job A is $A(n) = (100,000)n$. After 16.9 years, Job B begins to have a bigger total financial payoff than Job A.

4. The population of a town in 2007 was 15,000 people. The town has gotten its fresh water supply from a nearby lake and river system with the capacity to provide water for up to 30,000 people. Due to its proximity to a big city and a freeway, the town's population has begun to grow more quickly than in the past. The table below shows the population counts for each year from 2007–2012.

- a. Write a function of x that closely matches these data points for x -values of 0, 1, 2, 3, 4, and 5.

Year	Years Past 2007	Population of the Town
2007	0	15,000
2008	1	15,600
2009	2	16,224
2010	3	16,873
2011	4	17,548
2012	5	18,250

The value of the ratio of population in one year to the population in the previous year appears to be the same for any two consecutive years in the table. The value of the ratio is 1.04, so a function that would model this data for x -values of 0, 1, 2, ..., 5, is $f(x) = 15,000(1.04)^x$.

- b. Assume the function is a good model for the population growth from 2012–2032. At what year during the time frame 2012–2032 will the water supply be inadequate for the population?

If this model continues to hold true, the population will be larger than 30,000 when x is 18, which corresponds to the year 2025.