Lesson 15: Piecewise Functions

Classwork

Opening Exercise

For each real number a, the *absolute value of* a is the distance between 0 and a on the number line and is denoted $|a|$.

1. Solve each one variable equation.

   a. $|x| = 6$

   b. $|x - 5| = 4$

   c. $2|x + 3| = -10$

2. Determine at least five solutions for each two-variable equation. Make sure some of the solutions include negative values for either x or y.

   a. $y = |x|$

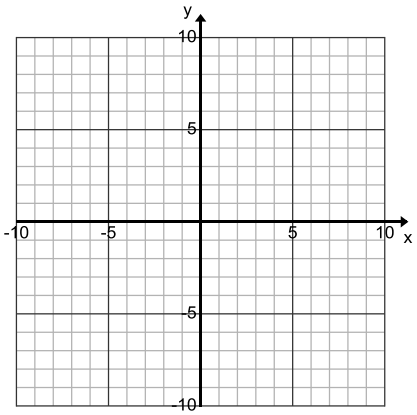
   b. $y = |x - 5|$

   c. $x = |y|$

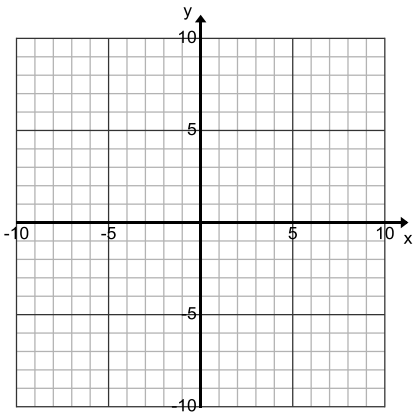
Exploratory Challenge 1

For parts (a)–(c) create graphs of the solution set of each two-variable equation from Opening Exercise 2.

a.



b.



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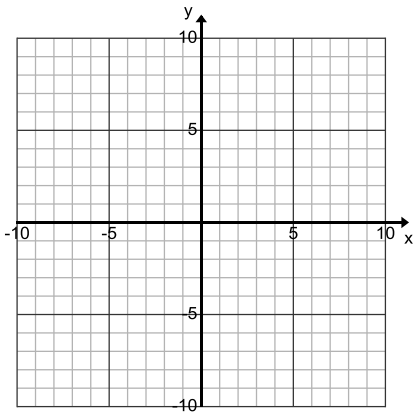
c.



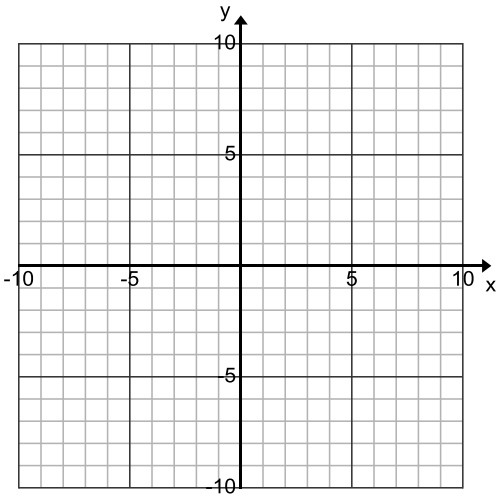
d. Write a brief summary comparing and contrasting the three solution sets and their graphs.

For parts (e)–(j) consider the function $f(x) = |x|$ where x can be any real number.

e. Explain the meaning of the function f in your own words.

f. State the domain and range of this function.

g. Create a graph of the function f. You might start by listing several ordered pairs that represent the corresponding domain and range elements.



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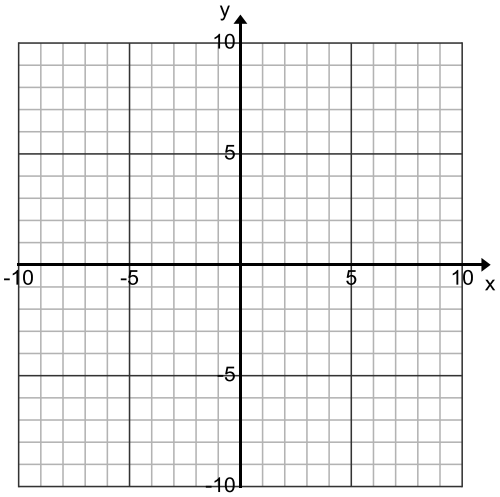
h. How does the graph of the absolute value function compare to the graph of $y = |x|$?

i. Define a function whose graph would be identical to the graph of $y = |x - 5|$?

j. Could you define a function whose graph would be identical to the graph of $x = |y|$? Explain your reasoning.

k. Let $f_1(x) = -x$ for $x < 0$ and let $f_2(x) = x$ for ≥ 0 . Graph the functions f_1 and f_2 on the same Cartesian plane. How does the graph of these two functions compare to the graph in part (g)?



Definition:

The *absolute value function* f is defined by setting $f(x) = |x|$ for all real numbers. Another way to write f is as a piecewise linear function:

$$f(x) = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

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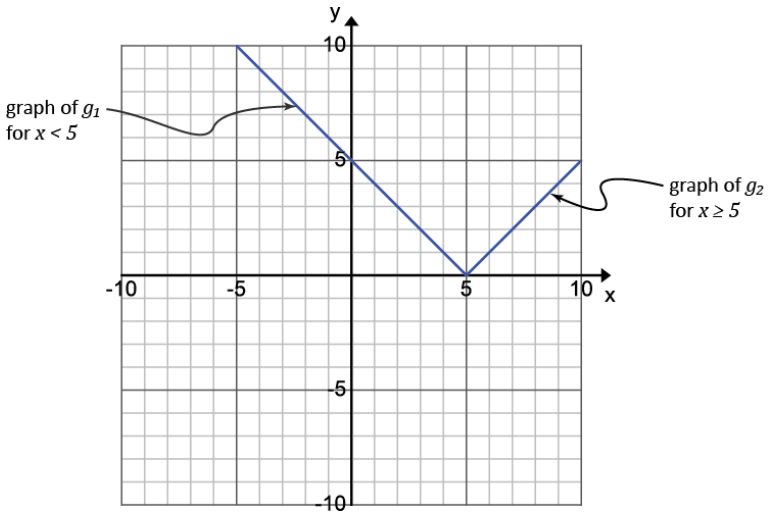
Example 1

Let $g(x) = |x - 5|$. The graph of g is the same as the graph of the equation $y = |x - 5|$ you drew in Exploratory Challenge 1, part (b). Use the redrawn graph below to re-write the function g as a piecewise function.



Label the graph of the linear function with negative slope by g_1 and the graph of the linear function with positive slope by g_2 as in the picture above.

Function g_1: Slope of g_1 is -1 (why?), and the y-intercept is 5; therefore, $g_1(x) = -x + 5$.

Function g_2: Slope of g_2 is 1 (why?), and the y-intercept is -5 (why?); therefore, $g_2(x) = x - 5$.

Writing g as a piecewise function is just a matter of collecting all of the different "pieces" and the intervals upon which they are defined:

$$g(x) = \begin{cases} -x + 5 & x < 5 \\ x - 5 & x \geq 5 \end{cases}$$

Exploratory Challenge 2

The *floor* of a real number x, denoted by $\lfloor x \rfloor$, is the largest integer not greater than x. The *ceiling* of a real number x, denoted by $\lceil x \rceil$, is the smallest integer not less than x. The *sawtooth* number of a positive number is the "fractional part" of the number that is to the right of its floor on the number line. In general, for a real number x, the sawtooth number of x is the value of the expression $x - \lfloor x \rfloor$. Each of these expressions can be thought of as functions with domain the set of real numbers.

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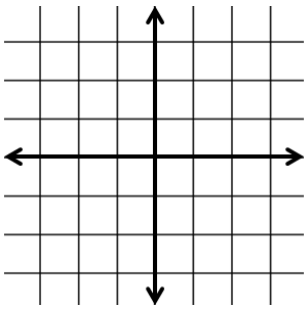
a. Complete the following table to help you understand how these functions assign elements of the domain to elements of the range. The first and second rows have been done for you.

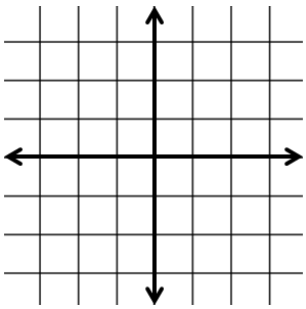
x	$floor(x) = \lfloor x \rfloor$	$ceiling(x) = \lceil x \rceil$	$sawtooth(x) = x - \lfloor x \rfloor$
4.8	4	5	0.8
-1.3	-2	-1	0.7
2.2			
6			
-3			
$-\frac{2}{3}$			
π			

b. Create a graph of each function.

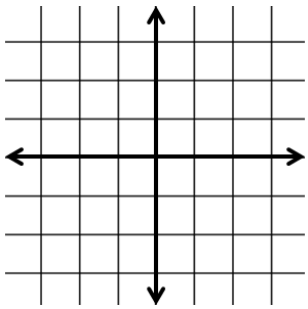
$\boldsymbol{floor(x) = \lfloor x \rfloor}$



$\boldsymbol{ceiling(x) = \lceil x \rceil}$



$\boldsymbol{sawtooth(x) = x - \lfloor x \rfloor}$



c. For the floor, ceiling, and sawtooth functions, what would be the range values for all real numbers x on the interval $[0,1)$? The interval $(1,2]$? The interval $[-2,-1)$? The interval $[1.5,2.5]$?

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Relevant Vocabulary

Piecewise-Linear Function: Given a number of non-overlapping intervals on the real number line, a *(real) piecewise-linear function* is a function from the union of the intervals to the set of real numbers such that the function is defined by (possibly different) linear functions on each interval.

Absolute Value Function: The absolute value of a number x, denoted by $|x|$, is the distance between 0 and x on the number line. The *absolute value function* is the piecewise-linear function such that for each real number x, the value of the function is $|x|$.

We often name the absolute value function by saying, "Let $f(x) = |x|$ for all real numbers x."

Floor Function: The *floor* of a real number x, denoted by $\lfloor x \rfloor$, is the largest integer not greater than x. The *floor function* is the piecewise-linear function such that for each real number x, the value of the function is $\lfloor x \rfloor$.

We often name the floor function by saying, "Let $f(x) = \lfloor x \rfloor$ for all real numbers x."

Ceiling Function: The *ceiling* of a real number x, denoted by $\lceil x \rceil$, is the smallest integer not less than x. The *ceiling function* is the piecewise-linear function such that for each real number x, the value of the function is $\lceil x \rceil$.

We often name the ceiling function by saying, "Let $f(x) = \lceil x \rceil$ for all real numbers x."

Sawtooth Function: The *sawtooth function* is the piecewise-linear function such that for each real number x, the value of the function is given by the expression $x - \lfloor x \rfloor$.

The sawtooth function assigns to each positive number the part of the number (the non-integer part) that is to the right of the floor of the number on the number line. That is, if we let $f(x) = x - \lfloor x \rfloor$ for all real numbers x, then

$f\left(\frac{1}{3}\right) = \frac{1}{3}$, $f\left(1\frac{1}{3}\right) = \frac{1}{3}$, $f(1000.02) = 0.02$, $f(-0.3) = 0.7$, etc.

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Problem Set

1. Explain why the sawtooth function, $sawtooth(x) = x - \lfloor x \rfloor$ for all real numbers x, takes only the "fractional part" of a number when the number is positive.

2. Let $g(x) = \lceil x \rceil - \lfloor x \rfloor$ where x can be any real number. In otherwords, g is the difference between the ceiling and floor functions. Express g as a piecewise function.

3. The Heaviside function is defined using the formula below.

$$H(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

   Graph this function and state its domain and range.

4. The following piecewise function is an example of a step function.

$$S(x) = \begin{cases} 3 & -5 \le x < -2 \\ 1 & -2 \le x < 3 \\ 2 & 3 \le x \le 5 \end{cases}$$

   a. Graph this function and state the domain and range.
   b. Why is this type of function called a step function?

5. Let $f(x) = \frac{|x|}{x}$ where x can be any real number except 0.

   a. Why is the number 0 excluded from the domain of f?
   b. What is the range of f?
   c. Create a graph of f.
   d. Express f as a piecewise function.
   e. What is the difference between this function and the Heaviside function?

6. Graph the following piecewise functions for the specified domain.

   a. $f(x) = |x + 3|$ for $-5 \le x \le 3$
   b. $f(x) = |2x|$ for $-3 \le x \le 3$
   c. $f(x) = |2x - 5|$ for $0 \le x \le 5$
   d. $f(x) = |3x + 1|$ for $-2 \le x \le 2$
   e. $f(x) = |x| + x$ for $-4 \le x \le 3$
   f. $f(x) = \begin{cases} x & \text{if } x \le 0 \\ x + 1 & \text{if } x > 0 \end{cases}$
   g. $f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ 3 - x & \text{if } x \ge -1 \end{cases}$

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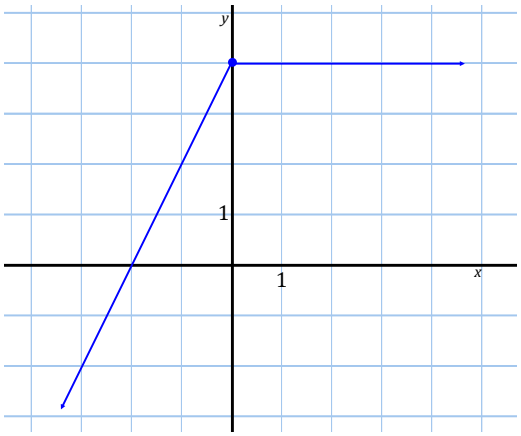
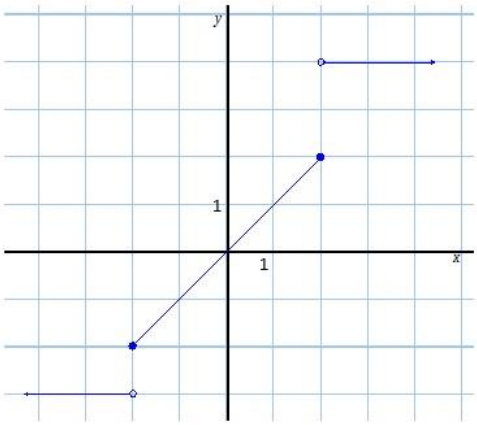
7. Write a piecewise function for each graph below.

a.



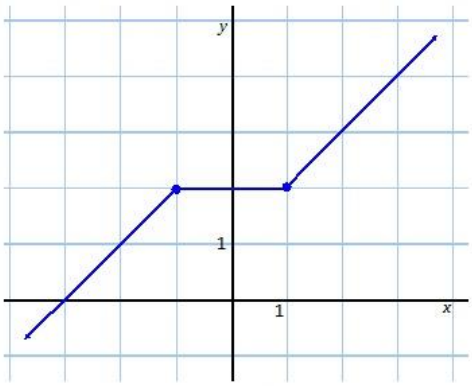
Graph of b

b.



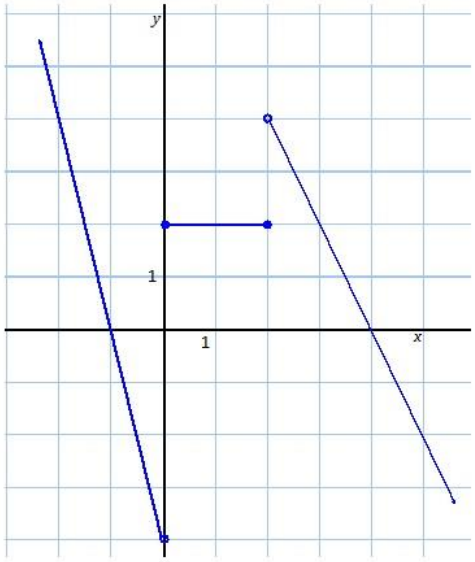
Graph of p

c.



Graph of k

d.



Graph of h

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