Lesson 20: Four Interesting Transformations of Functions

Classwork

Opening Exercise

Fill in the blanks of the table with the appropriate heading or descriptive information.

<table>
<tr><th>Graph of $y = f(x)$</th><th colspan="3">Vertical</th><th colspan="3">Horizontal</th></tr>
<tr><td rowspan="2">Translate</td><td rowspan="2">$y = f(x) + k$</td><td>$k > 0$</td><td>Translate up by $|k|$ units</td><td rowspan="2"></td><td>$k > 0$</td><td>Translate right by $|k|$ units</td></tr>
<tr><td></td><td>Translate down by $|k|$ units</td><td>$k < 0$</td><td></td></tr>
<tr><td rowspan="4">Scale by scale factor k</td><td rowspan="4"></td><td>$k > 1$</td><td></td><td rowspan="4">$y = f\left(\frac{1}{k}x\right)$</td><td></td><td>Horizontal stretch by a factor of $|k|$</td></tr>
<tr><td>$0 < k < 1$</td><td>Vertical shrink by a factor of $|k|$</td><td>$0 < k < 1$</td><td></td></tr>
<tr><td></td><td>Vertical shrink by a factor of $|k|$ and reflection over x-axis</td><td>$-1 < k < 0$</td><td>Horizontal shrink by a factor of $|k|$ and reflection across y-axis</td></tr>
<tr><td>$k < -1$</td><td></td><td>$k < -1$</td><td>Horizontal stretch by a factor of $|k|$ and reflection over y-axis</td></tr>
</table>

Exploratory Challenge 1

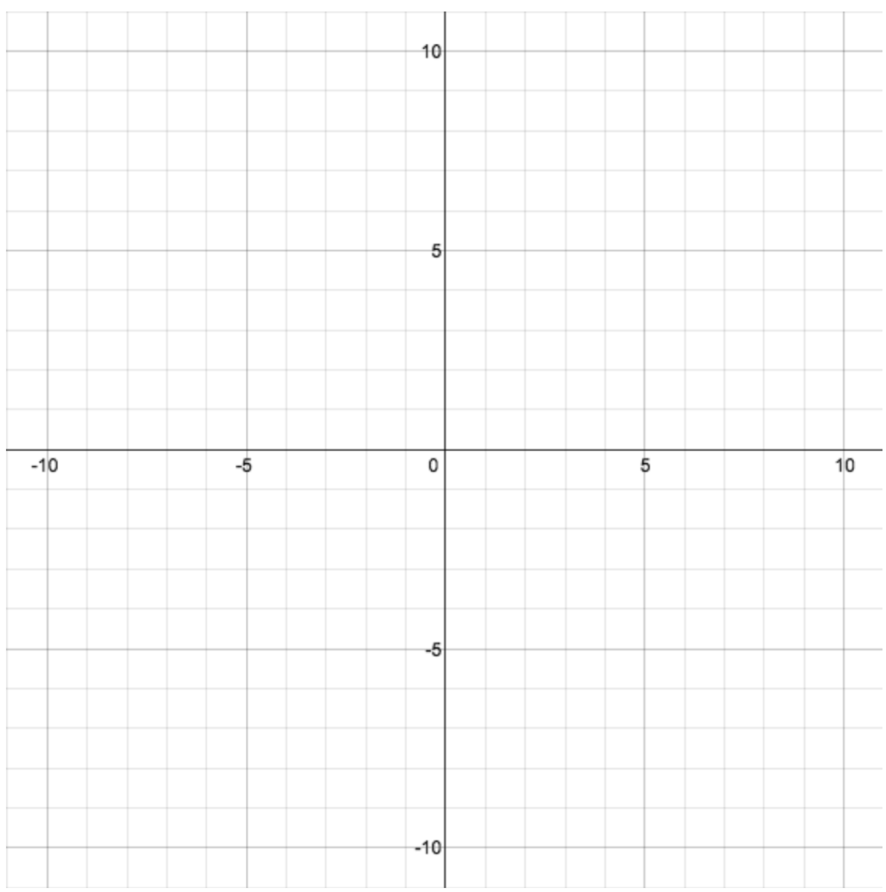
A transformation of the absolute value function $f(x) = |x - 3|$ is rewritten here as a piecewise function. Describe in words how to graph this piecewise function.

$$f(x) = \begin{cases} -x + 3, & x < 3 \\ x - 3, & x \geq 3 \end{cases}$$

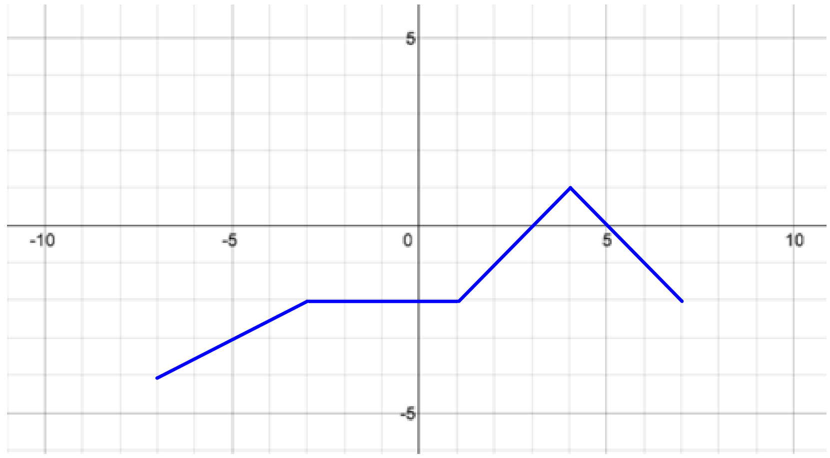
Exercises 1–2

1. Describe how to graph the following piecewise function. Then graph $y = f(x)$ below.

$$f(x) = \begin{cases} -3x - 3, & x \leq -2 \\ 0.5x + 4, & -2 < x < 2 \\ -2x + 9, & x \geq 2 \end{cases}$$



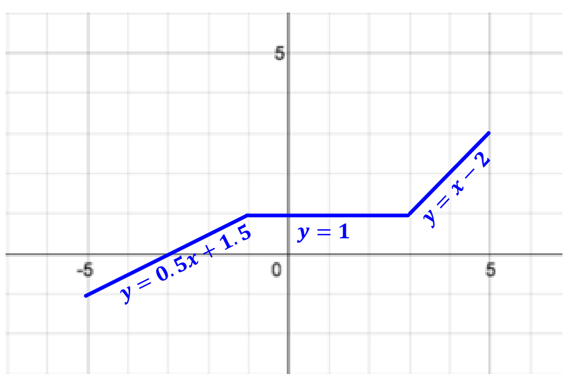
2. Using the graph of f below, write a formula for f as a piecewise function.



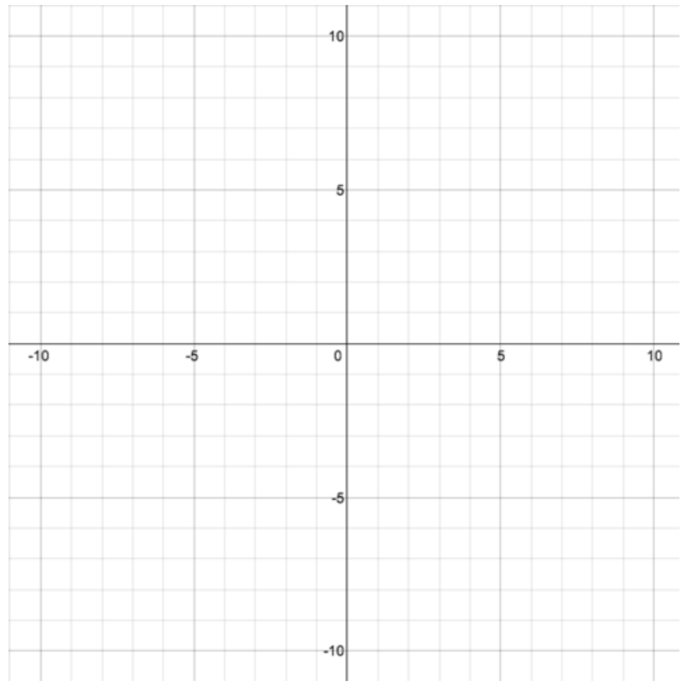
Exploratory Challenge 2

The graph $y = f(x)$ of a piecewise function f is shown. The domain of f is $-5 \le x \le 5$, and the range is $-1 \le y \le 3$.

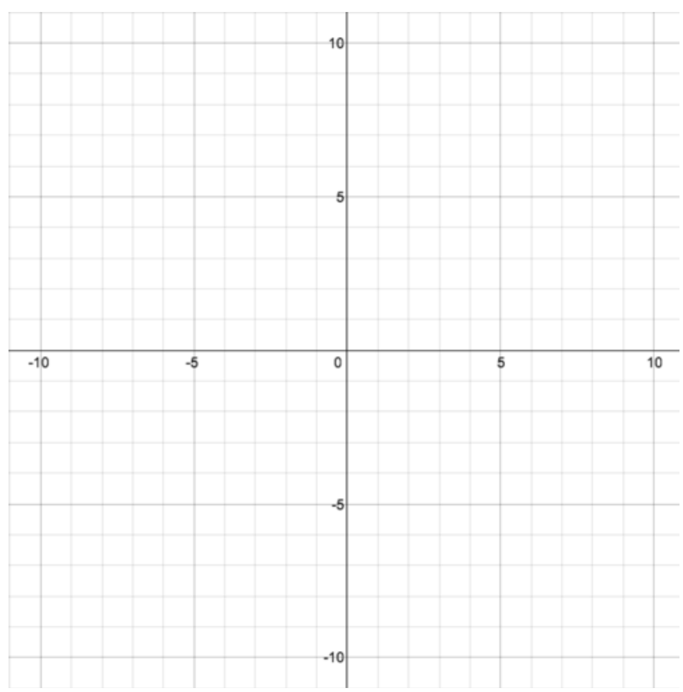
a. Mark and identify four strategic points helpful in sketching the graph of $y = f(x)$.



b. Sketch the graph of $y = 2f(x)$ and state the domain and range of the transformed function. How can you use part (a) to help sketch the graph of $y = 2f(x)$?



c. A horizontal scaling with scale factor $\frac{1}{2}$ of the graph of $y = f(x)$ is the graph of $y = f(2x)$. Sketch the graph of $y = f(2x)$ and state the domain and range. How can you use the points identified in part (a) to help sketch $y = f(2x)$?



Exercises 3–4

3. How does the range of f in Exploratory Challenge 2 compare to the range of a transformed function g, where $g(x) = kf(x)$, when $k > 1$?

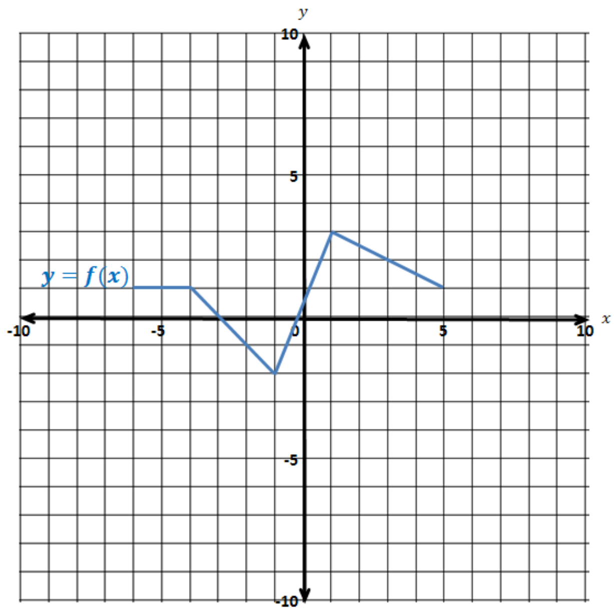
4. How does the domain of f in Exploratory Challenge 2 compare to the domain of a transformed function g, where $g(x) = f\left(\frac{1}{k}x\right)$, when $0 < k < 1$? (Hint: How does a graph shrink when it is horizontally scaled by a factor k?)

Problem Set

1. Suppose the graph of f is given. Write an equation for each of the following graphs after the graph of f has been transformed as described. Note that the transformations are not cumulative.
   a. Translate 5 units upward.
   b. Translate 3 units downward.
   c. Translate 2 units right.
   d. Translate 4 units left.
   e. Reflect about the x-axis.
   f. Reflect about the y-axis.
   g. Stretch vertically by a factor of 2.
   h. Shrink vertically by a factor of $\frac{1}{3}$.
   i. Shrink horizontally by a factor of $\frac{1}{3}$.
   j. Stretch horizontally by a factor of 2.

2. Explain how the graphs of the equations below are related to the graph of $y = f(x)$.
   a. $y = 5f(x)$
   b. $y = f(x - 4)$
   c. $y = -2f(x)$
   d. $y = f(3x)$
   e. $y = 2f(x) - 5$

3. The graph of the equation $y = f(x)$ is provided below. For each of the following transformations of the graph, write a formula (in terms of f) for the function that is represented by the transformation of the graph of $y = f(x)$. Then draw the transformed graph of the function on the same set of axes as the graph of $y = f(x)$.



   a. A translation 3 units left and 2 units up.

   b. A vertical stretch by a scale factor of 3.

   c. A horizontal shrink by a scale factor of $\frac{1}{2}$.

4. Reexamine your work on Exploratory Challenge 2 and Exercises 3 and 4 from this lesson. Parts (b) and (c) of Exploratory Challenge 2 asked how the equations $y = 2f(x)$ and $y = f(2x)$ could be graphed with the help of the strategic points found in (a). In this problem, we investigate whether it is possible to determine the graphs of $y = 2f(x)$ and $y = f(2x)$ by working with the piecewise linear function f directly.

   a. Write the function f in Exploratory Challenge 2 as a piecewise linear function.

   b. Let $g(x) = 2f(x)$. Use the graph you sketched in Exploratory Challenge 2, part (b) of $y = 2f(x)$ to write the formula for the function g as a piecewise-linear function.

   c. Let $h(x) = f(2x)$. Use the graph you sketched in Exploratory Challenge 2, part (c) of $y = f(2x)$ to write the formula for the function h as a piecewise-linear function.

   d. Compare the piecewise linear functions g and h to the piecewise linear function f. Did the expressions defining each piece change? If so, how? Did the domains of each piece change? If so how?