



## Table of Contents<sup>1</sup>

# Polynomial, Rational, and Radical Relationships

<b>Module Overview</b> .....	3
<b>Topic A: Polynomials—From Base Ten to Base X (A-SSE.A.2, A-APR.C.4)</b> .....	12
Lesson 1: Successive Differences in Polynomials .....	14
Lesson 2: The Multiplication of Polynomials .....	25
Lesson 3: The Division of Polynomials .....	37
Lesson 4: Comparing Methods—Long Division, Again? .....	48
Lesson 5: Putting It All Together .....	56
Lesson 6: Dividing by $x - a$ and by $x + a$ .....	65
Lesson 7: Mental Math .....	75
Lesson 8: The Power of Algebra—Finding Primes .....	86
Lesson 9: Radicals and Conjugates .....	98
Lesson 10: The Power of Algebra—Finding Pythagorean Triples .....	108
Lesson 11: The Special Role of Zero in Factoring .....	117
<b>Topic B: Factoring—Its Use and Its Obstacles (N-Q.A.2, A-SSE.A.2, A-APR.B.2, A-APR.B.3, A-APR.D.6, F-IF.C.7c)</b> .....	128
Lesson 12: Overcoming Obstacles in Factoring .....	130
Lesson 13: Mastering Factoring .....	141
Lesson 14: Graphing Factored Polynomials .....	149
Lesson 15: Structure in Graphs of Polynomial Functions .....	164
Lessons 16–17: Modeling with Polynomials—An Introduction .....	178
Lesson 18: Overcoming a Second Obstacle in Factoring—What If There Is a Remainder? .....	191
Lesson 19: The Remainder Theorem .....	200

<sup>1</sup> Each lesson is ONE day, and ONE day is considered a 45-minute period.

Lessons 20–21: Modeling Riverbeds with Polynomials ..... 211

**Mid-Module Assessment and Rubric** ..... 227

*Topics A through B (assessment 1 day, return, remediation, or further applications 1 day)*

Topic C: Solving and Applying Equations—Polynomial, Rational, and Radical (**A-APR.D.6, A-REI.A.1, A-REI.A.2, A-REI.B.4b, A-REI.C.6, A-REI.C.7, G-GPE.A.2**) ..... 237

Lesson 22: Equivalent Rational Expressions ..... 239

Lesson 23: Comparing Rational Expressions ..... 249

Lesson 24: Multiplying and Dividing Rational Expressions ..... 261

Lesson 25: Adding and Subtracting Rational Expressions ..... 272

Lesson 26: Solving Rational Equations ..... 283

Lesson 27: Word Problems Leading to Rational Equations ..... 293

Lesson 28: A Focus on Square Roots ..... 304

Lesson 29: Solving Radical Equations ..... 313

Lesson 30: Linear Systems in Three Variables ..... 320

Lesson 31: Systems of Equations ..... 329

Lesson 32: Graphing Systems of Equations ..... 341

Lesson 33: The Definition of a Parabola ..... 352

Lesson 34: Are All Parabolas Congruent? ..... 369

Lesson 35: Are All Parabolas Similar? ..... 388

Topic D: A Surprise from Geometry—Complex Numbers Overcome All Obstacles (**N-CN.A.1, N-CN.A.2, N-CN.C.7, A-REI.A.2, A-REI.B.4b, A-REI.C.7**) ..... 403

Lesson 36: Overcoming a Third Obstacle to Factoring—What If There Are No Real Number Solutions? ..... 405

Lesson 37: A Surprising Boost from Geometry ..... 418

Lesson 38: Complex Numbers as Solutions to Equations ..... 431

Lesson 39: Factoring Extended to the Complex Realm ..... 445

Lesson 40: Obstacles Resolved—A Surprising Result ..... 455

**End-of-Module Assessment and Rubric** ..... 465

*Topics A through D (assessment 1 day, return 1 day, remediation or further applications 1 day)*

## Algebra II • Module 1

# Polynomial, Rational, and Radical Relationships

## OVERVIEW

In this module, students draw on their foundation of the analogies between polynomial arithmetic and base-ten computation, focusing on properties of operations, particularly the distributive property (**A-SSE.B.2**, **A-APR.A.1**). Students connect multiplication of polynomials with multiplication of multi-digit integers, and division of polynomials with long division of integers (**A-APR.A.1**, **A-APR.D.6**). Students identify zeros of polynomials, including complex zeros of quadratic polynomials, and make connections between zeros of polynomials and solutions of polynomial equations (**A-APR.B.3**). Students explore the role of factoring, as both an aid to the algebra and to the graphing of polynomials (**A-SSE.2**, **A-APR.B.2**, **A-APR.B.3**, **F-IF.C.7c**). Students continue to build upon the reasoning process of solving equations as they solve polynomial, rational, and radical equations, as well as linear and non-linear systems of equations (**A-REI.A.1**, **A-REI.A.2**, **A-REI.C.6**, **A-REI.C.7**). The module culminates with the fundamental theorem of algebra as the ultimate result in factoring. Students pursue connections to applications in prime numbers in encryption theory, Pythagorean triples, and modeling problems.

An additional theme of this module is that the arithmetic of rational expressions is governed by the same rules as the arithmetic of rational numbers. Students use appropriate tools to analyze the key features of a graph or table of a polynomial function and relate those features back to the two quantities that the function is modeling in the problem (**F-IF.C.7c**).

## Focus Standards

### Reason quantitatively and use units to solve problems.

**N-Q.A.2<sup>2</sup>** Define appropriate quantities for the purpose of descriptive modeling.\*

### Perform arithmetic operations with complex numbers.

**N-CN.A.1** Know there is a complex number  $i$  such that  $i^2 = -1$ , and every complex number has the form  $a + bi$  with  $a$  and  $b$  real.

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<sup>2</sup> This standard will be assessed in Algebra II by ensuring that some modeling tasks (involving Algebra II content or securely held content from previous grades and courses) require the student to create a quantity of interest in the situation being described (i.e., this is not provided in the task). For example, in a situation involving periodic phenomena, the student might autonomously decide that amplitude is a key variable in a situation and then choose to work with peak amplitude.

- N-CN.A.2** Use the relation  $i^2 = -1$  and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

### Use complex numbers in polynomial identities and equations.

- N-CN.C.7** Solve quadratic equations with real coefficients that have complex solutions.

### Interpret the structure of expressions.

- A-SSE.A.2<sup>3</sup>** Use the structure of an expression to identify ways to rewrite it. *For example, see  $x^4 - y^4$  as  $(x^2)^2 - (y^2)^2$ , thus recognizing it as a difference of squares that can be factored as  $(x^2 - y^2)(x^2 + y^2)$ .*

### Understand the relationship between zeros and factors of polynomials.

- A-APR.B.2<sup>4</sup>** Know and apply the Remainder Theorem: For a polynomial  $p(x)$  and a number  $a$ , the remainder on division by  $x - a$  is  $p(a)$ , so  $p(a) = 0$  if and only if  $(x - a)$  is a factor of  $p(x)$ .
- A-APR.B.3<sup>5</sup>** Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.

### Use polynomial identities to solve problems.

- A-APR.C.4** Prove<sup>6</sup> polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity  $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$  can be used to generate Pythagorean triples.*

### Rewrite rational expressions.

- A-APR.D.6<sup>7</sup>** Rewrite simple rational expressions in different forms; write  $a(x)/b(x)$  in the form  $q(x) + r(x)/b(x)$ , where  $a(x)$ ,  $b(x)$ ,  $q(x)$ , and  $r(x)$  are polynomials with the degree of  $r(x)$  less than the degree of  $b(x)$ , using inspection, long division, or, for the more complicated examples, a computer algebra system.

### Understand solving equations as a process of reasoning and explain the reasoning.

- A-REI.A.1<sup>8</sup>** Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

<sup>3</sup> In Algebra II, tasks are limited to polynomial, rational, or exponential expressions. Examples: see  $x^4 - y^4$  as  $(x^2)^2 - (y^2)^2$ , thus recognizing it as a difference of squares that can be factored as  $(x^2 - y^2)(x^2 + y^2)$ . In the equation  $x^2 + 2x + 1 + y^2 = 9$ , see an opportunity to rewrite the first three terms as  $(x+1)^2$ , thus recognizing the equation of a circle with radius 3 and center  $(-1, 0)$ . See  $(x^2 + 4)/(x^2 + 3)$  as  $((x^2+3) + 1)/(x^2+3)$ , thus recognizing an opportunity to write it as  $1 + 1/(x^2 + 3)$ .

<sup>4</sup> Include problems that involve interpreting the Remainder Theorem from graphs and in problems that require long division.

<sup>5</sup> In Algebra II, tasks include quadratic, cubic, and quadratic polynomials and polynomials for which factors are not provided. For example, find the zeros of  $(x^2 - 1)(x^2 + 1)$ .

<sup>6</sup> Prove and apply (in preparation for Regents Exams).

<sup>7</sup> Include rewriting rational expressions that are in the form of a complex fraction.

<sup>8</sup> In Algebra II, tasks are limited to simple rational or radical equations.

- A-REI.A.2** Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.

### Solve equations and inequalities in one variable.

- A-REI.B.4<sup>9</sup>** Solve quadratic equations in one variable.
- b. Solve quadratic equations by inspection (e.g., for  $x^2 = 49$ ), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as  $a \pm bi$  for real numbers  $a$  and  $b$ .

### Solve systems of equations.

- A-REI.C.6<sup>10</sup>** Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
- A-REI.C.7** Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. *For example, find the points of intersection between the line  $y = -3x$  and the circle  $x^2 + y^2 = 3$ .*

### Analyze functions using different representations.

- F-IF.C.7** Graph functions expressed symbolically and show key features of the graph (by hand in simple cases and using technology for more complicated cases).<sup>\*</sup>
- c. Graph polynomial functions, identifying zeros when suitable factorizations are available and showing end behavior.

### Translate between the geometric description and the equation for a conic section.

- G-GPE.A.2** Derive the equation of a parabola given a focus and directrix.

## Extension Standards

The (+) standards below are provided as an extension to Module 1 of the Algebra II course to provide coherence to the curriculum. They are used to introduce themes and concepts that will be fully covered in the Precalculus course. *Note: None of the (+) standards below will be assessed on the Regents Exam or PARCC Assessments until Precalculus.*

### Use complex numbers in polynomial identities and equations.

- N-CN.C.8** (+) Extend polynomial identities to the complex numbers. *For example, rewrite  $x^2 + 4$  as  $(x + 2i)(x - 2i)$ .*

<sup>9</sup> In Algebra II, in the case of equations having roots with nonzero imaginary parts, students write the solutions as  $a \pm bi$ , where  $a$  and  $b$  are real numbers.

<sup>10</sup> In Algebra II, tasks are limited to  $3 \times 3$  systems.

**N-CN.C.9** (+) Know the Fundamental Theorem of Algebra; show that it is true for quadratic polynomials.

### Rewrite rational expressions.

**A-APR.C.7** (+) Understand that rational expressions form a system analogous to the rational numbers, closed under addition, subtraction, multiplication, and division by a nonzero rational expression; add, subtract, multiply, and divide rational expressions.

## Foundational Standards

### Use properties of rational and irrational numbers.

**N-RN.B.3** Explain why the sum or product of two rational numbers is rational; that the sum of a rational number and an irrational number is irrational; and that the product of a nonzero rational number and an irrational number is irrational.

### Reason quantitatively and use units to solve problems.

**N-Q.A.1** Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.\*

### Interpret the structure of expressions.

**A-SSE.A.1** Interpret expressions that represent a quantity in terms of its context.\*

- Interpret parts of an expression, such as terms, factors, and coefficients.
- Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret  $P(1 + r)^n$  as the product of  $P$  and a factor not depending on  $P$ .*

### Write expressions in equivalent forms to solve problems.

**A-SSE.B.3** Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.\*

- Factor a quadratic expression to reveal the zeros of the function it defines.

### Perform arithmetic operations on polynomials.

**A-APR.A.1** Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; add, subtract, and multiply polynomials.

**Create equations that describe numbers or relationships.**

- A-CED.A.1** Create equations and inequalities in one variable and use them to solve problems. *Include equations arising from linear and quadratic functions, and simple rational and exponential functions.*\*
- A-CED.A.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.\*
- A-CED.A.3** Represent constraints by equations or inequalities and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. *For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.*\*
- A-CED.A.4** Rearrange formulas to highlight a quantity of interest, using the same reasoning used in solving equations. *For example, rearrange Ohm's law  $V = IR$  to highlight resistance  $R$ .*\*

**Solve equations and inequalities in one variable.**

- A-REI.B.3** Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
- A-REI.B.4** Solve quadratic equations in one variable.
- a. Use the method of completing the square to transform any quadratic equation in  $x$  into an equation of the form  $(x - p)^2 = q$  that has the same solutions. Derive the quadratic formula from this form.

**Solve systems of equations.**

- A-REI.C.5** Prove that, given a system of two equations in two variables, replacing one equation by the sum of that equation and a multiple of the other produces a system with the same solutions.

**Represent and solve equations and inequalities graphically.**

- A-REI.D.10** Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).
- A-REI.D.11** Explain why the  $x$ -coordinates of the points where the graphs of the equations  $y = f(x)$  and  $y = g(x)$  intersect are the solutions of the equation  $f(x) = g(x)$ ; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where  $f(x)$  and/or  $g(x)$  are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.\*

**Translate between the geometric description and the equation for a conic section.**

- G-GPE.A.1** Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.

## Focus Standards for Mathematical Practice

- MP.1** **Make sense of problems and persevere in solving them.** Students discover the value of equating factored terms of a polynomial to zero as a means of solving equations involving polynomials. Students solve rational equations and simple radical equations, while considering the possibility of extraneous solutions and verifying each solution before drawing conclusions about the problem. Students solve systems of linear equations and linear and quadratic pairs in two variables. Further, students come to understand that the complex number system provides solutions to the equation  $x^2 + 1 = 0$  and higher-degree equations.
- MP.2** **Reason abstractly and quantitatively.** Students apply polynomial identities to detect prime numbers and discover Pythagorean triples. Students also learn to make sense of remainders in polynomial long division problems.
- MP.4** **Model with mathematics.** Students use primes to model encryption. Students transition between verbal, numerical, algebraic, and graphical thinking in analyzing applied polynomial problems. Students model a cross-section of a riverbed with a polynomial, estimate fluid flow with their algebraic model, and fit polynomials to data. Students model the locus of points at equal distance between a point (focus) and a line (directrix) discovering the parabola.
- MP.7** **Look for and make use of structure.** Students connect long division of polynomials with the long-division algorithm of arithmetic and perform polynomial division in an abstract setting to derive the standard polynomial identities. Students recognize structure in the graphs of polynomials in factored form and develop refined techniques for graphing. Students discern the structure of rational expressions by comparing to analogous arithmetic problems. Students perform geometric operations on parabolas to discover congruence and similarity.
- MP.8** **Look for and express regularity in repeated reasoning.** Students understand that polynomials form a system analogous to the integers. Students apply polynomial identities to detect prime numbers and discover Pythagorean triples. Students recognize factors of expressions and develop factoring techniques. Further, students understand that all quadratics can be written as a product of linear factors in the complex realm.

## Terminology

### New or Recently Introduced Terms

- **A Square Root of a Number** (A *square root of a number*  $x$  is a number whose square is  $x$ . In symbols, a square root of  $x$  is a number  $a$  such that  $a^2 = x$ . Negative numbers do not have any square roots, zero has exactly one square root, and positive numbers have two square roots.)
- **The Square Root of a Number** (Every positive real number  $x$  has a unique positive square root called the *square root* or *principle square root* of  $x$ ; it is denoted  $\sqrt{x}$ . The square root of zero is zero.)
- **Pythagorean Triple** (A *Pythagorean triple* is a triplet of positive integers  $(a, b, c)$  such that  $a^2 + b^2 = c^2$ . The triplet  $(3, 4, 5)$  is a Pythagorean triple but  $(1, 1, \sqrt{2})$  is not, even though the numbers are side lengths of an isosceles right triangle.)

- **End Behavior** (Let  $f$  be a function whose domain and range are subsets of the real numbers. The end behavior of a function  $f$  is a description of what happens to the values of the function
  - as  $x$  approaches positive infinity, and
  - as  $x$  approaches negative infinity.)
- **Even Function** (Let  $f$  be a function whose domain and range is a subset of the real numbers. The function  $f$  is called *even* if the equation,  $f(x) = f(-x)$ , is true for every number  $x$  in the domain. Even-degree polynomial functions are sometimes even functions, such as  $f(x) = x^{10}$ , and sometimes not, such as  $g(x) = x^2 - x$ .)
- **Odd Function** (Let  $f$  be a function whose domain and range is a subset of the real numbers. The function  $f$  is called *odd* if the equation,  $f(-x) = -f(x)$ , is true for every number  $x$  in the domain. Odd-degree polynomial functions are sometimes odd functions, such as  $f(x) = x^{11}$ , and sometimes not, such as  $h(x) = x^3 - x^2$ .)
- **Rational Expression** (A *rational expression* is either a numerical expression or a variable symbol, or the result of placing two previously generated rational expressions into the blanks of the addition operator ( $\_ + \_$ ), the subtraction operator ( $\_ - \_$ ), the multiplication operator ( $\_ \times \_$ ), or the division operator ( $\_ \div \_$ ).
- **Parabola** (A *parabola* with *directrix line*  $L$  and *focus point*  $F$  is the set of all points in the plane that are equidistant from the point  $F$  and line  $L$ .)
- **Axis of Symmetry** (The *axis of symmetry of a parabola* given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus.)
- **Vertex of a Parabola** (The *vertex of a parabola* is the point where the axis of symmetry intersects the parabola.)
- **Dilation at the Origin** (A dilation at the origin  $D_k$  is a horizontal scaling by  $k > 0$  followed by a vertical scaling by the same factor  $k$ . In other words, this dilation of the graph of  $y = f(x)$  is the graph of the equation  $y = kf\left(\frac{1}{k}x\right)$ . A dilation at the origin is a special type of a dilation.)

## Familiar Terms and Symbols<sup>11</sup>

- Sequence
- Arithmetic Sequence
- Numerical Symbol
- Variable Symbol
- Algebraic Expression
- Numerical Expression
- Polynomial Expression
- Monomial
- Degree of a Monomial
- Binomial

<sup>11</sup> These are terms and symbols students have seen previously.

- Trinomial
- Coefficient of a Monomial
- Terms of a Polynomial
- Like Terms of a Polynomial
- Standard Form of a Polynomial in One Variable
- Degree of a Polynomial in One Variable
- Equivalent Polynomial Expressions
- Polynomial Identity
- Function
- Polynomial Function
- Degree of a Polynomial Function
- Constant Function
- Linear Function
- Quadratic Function
- Discriminant of a Quadratic Function
- Cubic Function
- Zeros or Roots of a Function
- Increasing/Decreasing
- Relative Maximum
- Relative Minimum
- Graph of  $f$
- Graph of  $y = f(x)$

## Suggested Tools and Representations

- Graphing Calculator
- Wolfram Alpha Software
- Geometer's Sketchpad Software

## Assessment Summary

Assessment Type	Administered	Format	Standards Addressed
Mid-Module Assessment Task	After Topic B	Constructed response with rubric	N-Q.A.2, A-SSE.A.2, A-APR.B.2, A-APR.B.3, A-APR.C.4, A-REI.A.1, A-REI.B.4b, F-IF.C.7c
End-of-Module Assessment Task	After Topic D	Constructed response with rubric	N-Q.A.2, A.SSE.A.2, A-APR.B.2, A-APR.B.3, A-APR.C.4, A-APR.D.6, A-REI.A.1, A-REI.A.2, A-REI.B.4b, A-REI.C.6, A-REI.C.7, F-IF.C.7c, G-GPE.A.2



Topic A:

# Polynomials—From Base Ten to Base X

## A-SSE.A.2, A-APR.C.4

<b>Focus Standard:</b>	A-SSE.A.2	Use the structure of an expression to identify ways to rewrite it. <i>For example, see <math>x^4 - y^4</math> as <math>(x^2)^2 - (y^2)^2</math>, thus recognizing it as a difference of squares that can be factored as <math>(x^2 - y^2)(x^2 + y^2)</math>.</i>
	A-APR.C.4	Prove polynomial identities and use them to describe numerical relationships. <i>For example, the polynomial identity <math>(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2</math> can be used to generate Pythagorean triples.</i>
<b>Instructional Days:</b>	11	
<b>Lesson 1:</b>	Successive Differences in Polynomials (P) <sup>1</sup>	
<b>Lesson 2:</b>	The Multiplication of Polynomials (P)	
<b>Lesson 3:</b>	The Division of Polynomials (E)	
<b>Lesson 4:</b>	Comparing Methods—Long Division, Again? (P)	
<b>Lesson 5:</b>	Putting It All Together (P)	
<b>Lesson 6:</b>	Dividing by $x - a$ and by $x + a$ (P)	
<b>Lesson 7:</b>	Mental Math (P)	
<b>Lesson 8:</b>	The Power of Algebra—Finding Primes (P)	
<b>Lesson 9:</b>	Radicals and Conjugates (P)	
<b>Lesson 10:</b>	The Power of Algebra—Finding Pythagorean Triples (P)	
<b>Lesson 11:</b>	The Special Role of Zero in Factoring (S)	

In Topic A, students draw on their foundation of the analogies between polynomial arithmetic and base ten computation, focusing on properties of operations, particularly the distributive property. In Lesson 1, students write polynomial expressions for sequences by examining successive differences. They are engaged in a lively lesson that emphasizes thinking and reasoning about numbers and patterns and equations. In Lesson 2, they use a variation of the area model referred to as the tabular method to represent polynomial multiplication and connect that method back to application of the distributive property.

In Lesson 3, students continue using the tabular method and analogies to the system of integers to explore division of polynomials as a missing factor problem. In this lesson, students also take time to reflect on and

<sup>1</sup> Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

arrive at generalizations for questions such as how to predict the degree of the resulting sum when adding two polynomials. In Lesson 4, students are ready to ask and answer whether long division can work with polynomials too and how it compares with the tabular method of finding the missing factor. Lesson 5 gives students additional practice on all operations with polynomials and offers an opportunity to examine the structure of expressions such as recognizing that  $\frac{n(n+1)(2n+1)}{6}$  is a 3<sup>rd</sup> degree polynomial expression with leading coefficient  $\frac{1}{3}$  without having to expand it out.

In Lesson 6, students extend their facility with dividing polynomials by exploring a more generic case; rather than dividing by a factor such as  $(x + 3)$ , they divide by the factor  $(x + a)$  or  $(x - a)$ . This gives them the opportunity to discover the structure of special products such as  $(x - a)(x^2 + ax + a^2)$  in Lesson 7 and go on to use those products in Lessons 8–10 to employ the power of algebra over the calculator. In Lesson 8, they find they can use special products to uncover mental math strategies and answer questions such as whether or not  $2^{100} - 1$  is prime. In Lesson 9, they consider how these properties apply to expressions that contain square roots. Then, in Lesson 10, they use special products to find Pythagorean triples.

The topic culminates with Lesson 11 and the recognition of the benefits of factoring and the special role of zero as a means for solving polynomial equations.



## Lesson 1: Successive Differences in Polynomials

### Student Outcomes

- Students write explicit polynomial expressions for sequences by investigating successive differences of those sequences.

### Lesson Notes

This first lesson of the year tells students that this course is about thinking and reasoning with mathematics. It reintroduces the study of polynomials in a surprising new way involving sequences. This gives you a chance to evaluate how much your students recall from Grade 9. The lesson starts with discussions of expressions, polynomials, sequences, and equations. In this lesson, students continue the theme that began in Grade 6 of evaluating and building expressions. Explore ways to test your students' recall of the vocabulary terms listed at the end of this lesson.

Throughout this lesson, listen carefully to your students' discussions. Their reactions will inform you how to best approach the rest of the module. The homework set to this lesson should also give you insight into how much they remember from previous grades and how well they can read instructions. In particular, if they have trouble with evaluating or simplifying expressions or solving equations, then you might want to revisit Lessons 6–9 in Grade 9, Module 1, and Lesson 2 in Grade 9, Module 4. If they are having trouble solving equations, use Lessons 10–12, 15–16, and 19 in Grade 9, Module 1 to give them extra practice.

Finally, the use of the term *constant* may need a bit of extra discussion. It is used throughout this PK–12 curriculum in two ways: either as a constant number (e.g., the  $a$  in  $ax^2 + bx + c$  is a number chosen once-and-for-all at the beginning of a problem) or as a constant rate (e.g., a copier that reproduces at a constant rate of 40 copies/minute). We offer both uses in this lesson.

### Classwork

#### Opening Exercise (7 minutes)

This exercise provides an opportunity to think about and generalize the main concept of today's lesson: that the second differences of a quadratic polynomial are constant. This generalizes to the  $n^{\text{th}}$  differences of a degree  $n$  polynomial. Your goal is to help students investigate, discuss, and generalize the second and higher differences in this exercise.

Present the exercise to your students and ask them (in groups of two) to study the table and explain to their partner how to calculate each line in the table. If they get stuck, help them find entry points into this question, possibly by drawing segments connecting the successive differences on their papers (e.g., connect 5.76 and 11.56 to 5.8 and ask, "How are these three numbers related?"). This initial problem of the school year is designed to encourage students to persevere and look for and express regularity in repeated reasoning.

Teachers may also use the Opening Exercise to informally assess students' pattern-finding abilities and fluency with rational numbers.

#### Scaffolding:

Before presenting the problem below, you might start by displaying the first two rows of the table on the board and asking students to investigate the relationship between them, including making a conjecture about the nature of the relationship.

MP.1  
&  
MP.8

Opening Exercise

John noticed patterns in the arrangement of numbers in the table below.

<u>Number</u>	2.4	3.4	4.4	5.4	6.4
<u>Square</u>	5.76	11.56	19.36	29.16	40.96
<u>First Differences</u>		5.8	7.8	9.8	11.8
<u>Second Differences</u>			2	2	2

Assuming that the pattern would continue, he used it to find the value of  $7.4^2$ . Explain how he used the pattern to find  $7.4^2$ , and then use the pattern to find  $8.4^2$ .

To find  $7.4^2$ , John assumed the next term in the first differences would have to be 13.8 since 13.8 is 2 more than 11.8. Therefore, the next term in the square numbers would have to be  $40.96 + 13.8$ , which is 54.76. Checking with a calculator, we also find  $7.4^2 = 54.76$ .

To find  $8.4^2$ , we follow the same process: The next term in the first differences would have to be 15.8, so the next term in the square numbers would be  $54.76 + 15.8$ , which is 70.56. Check:  $8.4^2 = 70.56$ .

How would you label each row of numbers in the table?

*Number, Square, First Differences, Second Differences*

Scaffolding:

For students working below grade level, consider using positive integers  $\{1, 2, 3, \dots\}$  and corresponding squares  $\{1, 4, 9, \dots\}$  instead of using  $\{2.4, 3.4, 4.4, \dots\}$ .

MP.8

Discuss with students the relationship between each row and the row above it and how to label the rows based upon that relationship. Feel free to have this discussion before or after they find  $7.4^2$  and  $8.4^2$ . They are likely to come up with labels such as “subtract” or “difference” for the third and fourth row. However, guide them to call the third and fourth rows “First Differences” and “Second Differences,” respectively.

Discussion (3 minutes)

The pattern illustrated in the Opening Exercise is a particular case of a general phenomenon about polynomials. In Algebra I, Module 3, we saw how to recognize linear functions and exponential functions by recognizing similar growth patterns; that is, linear functions grow by a “constant difference over successive intervals of equal length,” and exponential functions grow by a “constant factor over successive intervals of equal length.” In this lesson, we generalize the linear growth pattern to polynomials of second degree (quadratic expressions) and third degree (cubic expressions).

Discussion

Let the sequence  $\{a_0, a_1, a_2, a_3, \dots\}$  be generated by evaluating a polynomial expression at the values 0, 1, 2, 3,.... The numbers found by evaluating  $a_1 - a_0, a_2 - a_1, a_3 - a_2, \dots$  form a new sequence which we will call the *first differences* of the polynomial. The differences between successive terms of the first differences sequence are called the *second differences* and so on.

It is a good idea to use an actual sequence of numbers such as the square numbers  $\{1, 4, 9, 16, \dots\}$  to help explain the meaning of the terms “first differences” and “second differences.”

**Example 1 (4 minutes)**

Although you may be tempted to work through Example 1 using numbers instead of  $a$  and  $b$ , using symbols  $a$  and  $b$  actually makes the *structure* of the first differences sequence obvious, whereas numbers could hide that structure. Also, working with constant coefficients gives the generalization all at once.

Note: Consider using Example 1 to informally assess students' fluency with algebraic manipulations.

**Example 1**

What is the sequence of first differences for the linear polynomial given by  $ax + b$ , where  $a$  and  $b$  are constant coefficients?

*The terms of the first differences sequence are found by subtracting consecutive terms in the sequence generated by the polynomial expression  $ax + b$ , namely,  $\{b, a + b, 2a + b, 3a + b, 4a + b, \dots\}$ .*

*1<sup>st</sup> term:  $(a + b) - b = a$ ,*

*2<sup>nd</sup> term:  $(2a + b) - (a + b) = a$ ,*

*3<sup>rd</sup> term:  $(3a + b) - (2a + b) = a$ ,*

*4<sup>th</sup> term:  $(4a + b) - (3a + b) = a$ .*

*The first differences sequence is  $\{a, a, a, a, \dots\}$ . For first degree polynomial expressions, the first differences are constant and equal to  $a$ .*

What is the sequence of second differences for  $ax + b$ ?

*Since  $a - a = 0$ , the second differences are all 0. Thus the sequence of second differences is  $\{0, 0, 0, 0, \dots\}$ .*

**Scaffolding:**

Try starting the example by first asking students to generate sequences of first differences for  $2x + 3$ ,  $3x - 1$ , and  $4x + 2$ . For example, the sequence generated by  $2x + 3$  is  $\{3, 5, 7, 9, \dots\}$  and its sequence of first differences is  $\{2, 2, 2, 2, \dots\}$ .

You can then use these three sequences as a source of examples from which to make and verify conjectures.

- How is this calculation similar to the arithmetic sequences you studied in Algebra I, Module 3?
  - *The constant derived from the first differences of a linear polynomial is the same constant addend used to define the arithmetic sequence generated by the polynomial. That is, the  $a$  in  $A(n) = an + b$  for  $n \geq 0$ . Written recursively this is  $A(0) = b$  and  $A(n + 1) = A(n) + a$  for  $n \geq 0$ .*

For Examples 2 and 3, let students work in groups of two to fill in the blanks of the tables (3 min. max for each table). Walk around the room, checking student work for understanding. Afterward, discuss the paragraphs below each table as a whole class.

Example 2 (5 minutes)

**Example 2**  
Find the first, second, and third differences of the polynomial  $ax^2 + bx + c$  by filling in the blanks in the following table.

$x$	$ax^2 + bx + c$	First Differences	Second Differences	Third Differences
0	$c$	$a + b$		
1	$a + b + c$	$3a + b$	$2a$	$0$
2	$4a + 2b + c$	$5a + b$	$2a$	$0$
3	$9a + 3b + c$	$7a + b$	$2a$	$0$
4	$16a + 4b + c$	$9a + b$	$2a$	
5	$25a + 5b + c$			

MP.7 & MP.8

The table shows that the second differences of the polynomial  $ax^2 + bx + c$  all have the constant value  $2a$ . The second differences hold for any sequence of values of  $x$  where the values in the sequence differ by 1, as the Opening Exercise shows. For example, if we studied the second differences for  $x$ -values  $\pi, \pi + 1, \pi + 2, \pi + 3, \dots$ , we would find that the second differences would also be  $2a$ . In your homework, you will show that this fact is indeed true by finding the second differences for the values  $n + 0, n + 1, n + 2, n + 3, n + 4$ .

Ask students to describe what they notice in the sequences of first, second, and third differences. Have them make a conjecture about the third and fourth differences of a sequence generated by a third degree polynomial.

Students are likely to say that the third differences have the constant value  $3a$  (which is incorrect). Have them work through the next example to help them discover what the third differences really are. This is a good example of why we need to follow up conjecture based on observation with proof.

Example 3 (7 minutes)

**Example 3**  
Find the second, third, and fourth differences of the polynomial  $ax^3 + bx^2 + cx + d$  by filling in the blanks in the following table.

$x$	$ax^3 + bx^2 + cx + d$	First Differences	Second Differences	Third Differences	Fourth Differences
0	$d$	$a + b + c$			
1	$a + b + c + d$	$7a + 3b + c$	$6a + 2b$	$6a$	
2	$8a + 4b + 2c + d$	$19a + 5b + c$	$12a + 2b$	$6a$	$0$
3	$27a + 9b + 3c + d$	$37a + 7b + c$	$18a + 2b$	$6a$	$0$
4	$64a + 16b + 4c + d$	$61a + 9b + c$	$24a + 2b$		
5	$125a + 25b + 5c + d$				

The third differences of  $ax^3 + bx^2 + cx + d$  all have the constant value  $6a$ . Also, if a different sequence of values for  $x$  that differed by 1 was used instead, the third differences would still have the value  $6a$ .

- Ask students to make a conjecture about the fourth differences of a sequence generated by degree 4 polynomial. Students who were paying attention to their (likely wrong) conjecture of the third differences before doing this example may guess that the fourth differences are constant and equal to  $(1 \cdot 2 \cdot 3 \cdot 4)a$ , which is  $24a$ . This pattern continues: the  $n^{\text{th}}$  differences of any sequence generated by an  $n^{\text{th}}$  degree polynomial with leading coefficient  $a$  will be constant and have the value  $a \cdot (n!)$ .
- Ask students to make a conjecture about the  $(n + 1)^{\text{st}}$  differences of a degree  $n$  polynomial, for example, the 5<sup>th</sup> differences of a fourth degree polynomial.

Students are now ready to tackle the main goal of this lesson—using differences to recognize polynomial relationships and build polynomial expressions.

**Example 4 (7 minutes)**

When collecting bivariate data on an event or experiment, the data does not announce, “I satisfy a quadratic relationship,” or “I satisfy an exponential relationship.” We need ways to recognize these relationships in order to model them with functions. In Algebra I, Module 3, students studied the conditions upon which they could conclude that the data satisfied a linear or exponential relationship. Either the first differences were constant or “first factors” were constant. By checking that the second or third differences of the data are constant, students now have a way to recognize a quadratic or cubic relationship and can write an equation to describe that relationship (A-CED.A.3, F-BF.A.1a).

Give students an opportunity to attempt this problem in groups of two. Walk around the room helping them find the leading coefficient.

**Example 4**

What type of relationship does the set of ordered pairs  $(x, y)$  satisfy? How do you know? Fill in the blanks in the table below to help you decide. (The first differences have already been computed for you.)

$x$	$y$	First Differences	Second Differences	Third Differences
0	2	-1		
1	1	5	<u>6</u>	<u>6</u>
2	6	17	<u>12</u>	<u>6</u>
3	23	35	<u>18</u>	<u>6</u>
4	58	59	<u>24</u>	
5	117			

*Since the third differences are constant, the pairs could represent a cubic relationship between  $x$  and  $y$ .*

Find the equation of the form  $y = ax^3 + bx^2 + cx + d$  that all ordered pairs  $(x, y)$  above satisfy. Give evidence that your equation is correct.

*Since third differences of a cubic polynomial are equal to  $6a$ , using the table above, we get  $6a = 6$ , so that  $a = 1$ . Also, since  $(0, 2)$  satisfies the equation, we see that  $d = 2$ . Thus, we need only find  $b$  and  $c$ . Substituting  $(1, 1)$  and  $(2, 6)$  into the equation, we get*

$$\begin{aligned} 1 &= 1 + b + c + 2 \\ 6 &= 8 + 4b + 2c + 2. \end{aligned}$$

*Subtracting two times the first equation from the second, we get  $4 = 6 + 2b - 2$ , so that  $b = 0$ . Substituting 0 in for  $b$  in the first equation gives  $c = -2$ . Thus, the equation is  $y = x^3 - 2x + 2$ .*

- After finding the equation, have students check that the pairs  $(3, 23)$  and  $(4, 58)$  satisfy the equation.

Help students to persevere in finding the coefficients (MP.1). They will most likely try to plug three ordered pairs into the equation, which gives a  $3 \times 3$  system of linear equations in  $a$ ,  $b$ , and  $c$  after they find that  $d = 2$ . Using the fact that the third differences of a cubic polynomial are  $6a$  will greatly simplify the problem. (It implies  $a = 1$  immediately, which reduces the system to the easy  $2 \times 2$  system above.) Walk around the room as they work, and ask questions that lead them to realize that they can use the third differences fact if they get too stuck. Alternatively, find a student who used the fact, and then have the class discuss and understand his or her approach.

### Closing (7 minutes)

- What are some of the key ideas that we learned today?
  - Sequences whose second differences are constant satisfy a quadratic relationship.
  - Sequences whose third differences are constant satisfy a cubic relationship.

The following terms were introduced and taught in Module 1 of Algebra I. The terms are listed here for completeness and reference.

#### Relevant Vocabulary

**Numerical Symbol:** A *numerical symbol* is a symbol that represents a specific number. Examples: 1, 2, 3, 4,  $\pi$ ,  $-3.2$ .

**Variable Symbol:** A *variable symbol* is a symbol that is a placeholder for a number from a specified set of numbers. The set of numbers is called the *domain of the variable*. Examples:  $x$ ,  $y$ ,  $z$ .

**Algebraic Expression:** An *algebraic expression* is either

- a numerical symbol or a variable symbol or
- the result of placing previously generated algebraic expressions into the two blanks of one of the four operators  $(\_\_)+(\_\_)$ ,  $(\_\_)-(\_\_)$ ,  $(\_\_)\times(\_\_)$ ,  $(\_\_)\div(\_\_)$  or into the base blank of an exponentiation with an exponent that is a rational number.

Following the definition above,  $((x) \times (x)) \times (x) + ((3) \times (x))$  is an algebraic expression, but it is generally written more simply as  $x^3 + 3x$ .

**Numerical Expression:** A *numerical expression* is an algebraic expression that contains only numerical symbols (no variable symbols) that evaluates to a single number. Example: The numerical expression  $\frac{(3 \cdot 2)^2}{12}$  evaluates to 3.

**Monomial:** A *monomial* is an algebraic expression generated using only the multiplication operator  $(\_\_)\times(\_\_)$ . The expressions  $x^3$  and  $3x$  are both monomials.

**Binomial:** A *binomial* is the sum of two monomials. The expression  $x^3 + 3x$  is a binomial.

**Polynomial Expression:** A *polynomial expression* is a monomial or sum of two or more monomials.

**Sequence:** A *sequence* can be thought of as an ordered list of elements. The elements of the list are called the *terms of the sequence*.

**Arithmetic Sequence:** A sequence is called *arithmetic* if there is a real number  $d$  such that each term in the sequence is the sum of the previous term and  $d$ .

### Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 1: Successive Differences in Polynomials

### Exit Ticket

1. What type of relationship is indicated by the following set of ordered pairs? Explain how you know.

$x$	$y$
0	0
1	2
2	10
3	24
4	44

2. Find an equation that all ordered pairs above satisfy.

Exit Ticket Sample Solutions

1. What type of relationship is indicated by the following set of ordered pairs? Explain how you know.

$x$	$y$	First Differences	Second Differences
0	0		
1	2	2	
2	10	8	6
3	24	14	6
4	44	20	6

Since the second differences are constant, there is a quadratic relationship between  $x$  and  $y$ .

2. Find an equation that all ordered pairs above satisfy.

Since  $(0, 0)$  satisfies an equation of the form  $y = ax^2 + bx + c$ , we have that  $c = 0$ . Using the points  $(1, 2)$  and  $(2, 10)$ , we have

$$\begin{aligned} 2 &= a + b \\ 10 &= 4a + 2b \end{aligned}$$

Subtracting twice the first equation from the second gives  $6 = 2a$ , which means  $a = 3$ . Substituting 3 into the first equation gives  $b = -1$ . Thus,  $y = 3x^2 - x$  is the equation.

OR

Since the pairs satisfy a quadratic relationship, the second differences must be equal to  $2a$ . Therefore,  $6 = 2a$ , so  $a = 3$ . Since  $(0, 0)$  satisfies the equation,  $c = 0$ . Using the point  $(1, 2)$ , we have that  $2 = 3 + b + 0$ , so  $b = -1$ . Thus,  $y = 3x^2 - x$  is the equation that is satisfied by these points.

Problem Set Sample Solutions

1. Create a table to find the second differences for the polynomial  $36 - 16t^2$  for integer values of  $t$  from 0 to 5.

$t$	$36 - 16t^2$	First Differences	Second Differences
0	36		
1	20	-16	
2	-28	-48	-32
3	-108	-80	-32
4	-220	-112	-32
5	-364	-144	-32

2. Create a table to find the third differences for the polynomial  $s^3 - s^2 + s$  for integer values of  $s$  from  $-3$  to  $3$ .

$s$	$s^3 - s^2 + s$	First Differences	Second Differences	Third Differences
-3	-39			
-2	-14	25		
-1	-3	11	-14	
0	0	3	-8	6
1	1	1	-2	6
2	6	5	4	6
3	21	15	10	6

3. Create a table of values for the polynomial  $x^2$ , using  $n, n + 1, n + 2, n + 3, n + 4$  as values of  $x$ . Show that the second differences are all equal to 2.

$x$	$x^2$	First Differences	Second Differences
$n$	$n^2$		
$n + 1$	$n^2 + 2n + 1$	$2n + 1$	
$n + 2$	$n^2 + 4n + 4$	$2n + 3$	2
$n + 3$	$n^2 + 6n + 9$	$2n + 5$	2
$n + 4$	$n^2 + 8n + 16$	$2n + 7$	2

4. Show that the set of ordered pairs  $(x, y)$  in the table below satisfies a quadratic relationship. (Hint: Find second differences.) Find the equation of the form  $y = ax^2 + bx + c$  that all of the ordered pairs satisfy.

$x$	0	1	2	3	4	5
$y$	5	4	-1	-10	-23	-40

Students show that second differences are constant and equal to  $-4$ . The equation:  $y = -2x^2 + x + 5$ .

5. Show that the set of ordered pairs  $(x, y)$  in the table below satisfies a cubic relationship. (Hint: Find third differences.) Find the equation of the form  $y = ax^3 + bx^2 + cx + d$  that all of the ordered pairs satisfy.

$x$	0	1	2	3	4	5
$y$	20	4	0	20	76	180

Students show that third differences are constant and equal to 12. The equation:  $y = 2x^3 - 18x + 20$ .

6. The distance  $d$  ft. required to stop a car traveling at  $10v$  mph under dry asphalt conditions is given by the following table.

$v$	0	1	2	3	4	5
$d$	0	5	19.5	43.5	77	120

- a. What type of relationship is indicated by the set of ordered pairs?

*Students show that second differences are constant and equal to 9.5. Therefore, the relationship is quadratic.*

- b. Assuming that the relationship continues to hold, find the distance required to stop the car when the speed reaches 60 mph, when  $v = 6$ .

*172.5 ft.*

- c. (Challenge) Find an equation that describes the relationship between the speed of the car  $v$  and its stopping distance  $d$ .

*$d = 4.75v^2 + 0.25v$  (Note: Students do not need to find the equation to answer part (b).)*

7. Use the polynomial expressions  $5x^2 + x + 1$  and  $2x + 3$  to answer the questions below.

- a. Create a table of second differences for the polynomial  $5x^2 + x + 1$  for the integer values of  $x$  from 0 to 5.

$x$	$5x^2 + x + 1$	First Differences	Second Differences
0	1		
1	7	6	10
2	23	16	10
3	49	26	10
4	85	36	10
5	131	46	

- b. Justin claims that for  $n \geq 2$ , the  $n^{\text{th}}$  differences of the sum of a degree  $n$  polynomial and a linear polynomial are the same as the  $n^{\text{th}}$  differences of just the degree  $n$  polynomial. Find the second differences for the sum  $(5x^2 + x + 1) + (2x + 3)$  of a degree 2 and a degree 1 polynomial and use the calculation to explain why Justin might be correct in general.

*Students compute that the second differences are constant and equal to 10, just as in part (a). Justin is correct because the differences of the sum are the sum of the differences. Since the second (and all other higher) differences of the degree 1 polynomial are constant and equal to zero, only the  $n^{\text{th}}$  differences of the degree  $n$  polynomial contribute to the  $n^{\text{th}}$  difference of the sum.*

- c. Jason thinks he can generalize Justin's claim to the product of two polynomials. He claims that for  $n \geq 2$ , the  $(n + 1)^{\text{th}}$  differences of the product of a degree  $n$  polynomial and a linear polynomial are the same as the  $n^{\text{th}}$  differences of the degree  $n$  polynomial. Use what you know about second and third differences (from Examples 2 and 3) and the polynomial  $(5x^2 + x + 1)(2x + 3)$  to show that Jason's generalization is incorrect.

*The second differences of a quadratic polynomial are  $2a$ , so the second differences of  $5x^2 + x + 1$  are always 10. Since  $(5x^2 + x + 1)(2x + 3) = 10x^3 + 17x^2 + 5x + 3$  and third differences are equal to  $6a$ , we have that the third differences of  $(5x^2 + x + 1)(2x + 3)$  are always 60, which is not 10.*



## Lesson 2: The Multiplication of Polynomials

### Student Outcomes

- Students develop the distributive property for application to polynomial multiplication. Students connect multiplication of polynomials with multiplication of multi-digit integers.

### Lesson Notes

This lesson begins to address standards **A-SSE.A.2** and **A-APR.C.4** directly, and provides opportunities for students to practice MP.7 and MP.8. The work is scaffolded to allow students to discern patterns in repeated calculations, leading to some general polynomial identities that will be explored further in the remaining lessons of this module.

As in the last lesson, if students struggle with this lesson, they may need to review concepts covered in previous grades, such as:

- The connection between area properties and the distributive property: Grade 7, Module 6, Lesson 21.
- Introduction to the table method of multiplying polynomials: Algebra I, Module 1, Lesson 9.
- Multiplying polynomials (in the context of quadratics): Algebra I, Module 4, Lessons 1 and 2.

Since division is the inverse operation of multiplication, it is important to make sure that your students understand how to multiply polynomials before moving on to division of polynomials in Lesson 3 of this module. In Lesson 3, division is explored using the “reverse tabular method,” so it is important for students to work through the table diagrams in this lesson to prepare them for the upcoming work.

We continue to draw a sharp distinction in this curriculum between justification and proof, such as justifying the identity  $(a + b)^2 = a^2 + 2ab + b^2$  using area properties and proving the identity using the distributive property. The key point is that the area of a figure is *always* a nonnegative quantity, and so cannot be used to prove an algebraic identity where the letters can stand for negative numbers (there is no such thing as a geometric figure with negative area). This is one of many reasons that manipulatives such as Algebra Tiles need to be handled with extreme care: depictions of “negative area” actually teach incorrect mathematics. (A correct way to model expressions involving the subtraction of two positive quantities using an area model is depicted in the last problem of the Problem Set.)

The tabular diagram described in this lesson is purposely designed to “look like” an area model without actually being an area model. It is a convenient way to keep track of the use of the distributive property, which is a basic property of the number system and is assumed to be true for all real numbers—regardless of whether they are positive or negative, fractional or irrational.

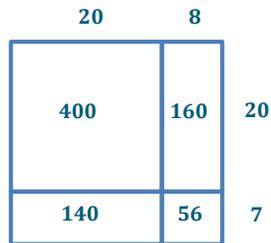
### Classwork

#### Opening Exercise (5 minutes)

The Opening Exercise is a simple use of an area model to justify why the distributive property works when multiplying  $28 \times 27$ . When drawing the area model, remember that it really matters that the length of the side of the big square is about  $2\frac{1}{2}$  times the length of the top side of the upper right rectangle (20 units versus 8 units) in the picture below and similarly for the lengths going down the side of the large rectangle. It should be an accurate representation of the area of a rectangular region that measures 28 units by 27 units.

Opening Exercise

Show that  $28 \times 27 = (20 + 8)(20 + 7)$  using an area model. What do the numbers you placed inside the four rectangular regions you drew represent?



$$\begin{aligned}
 28 \times 27 &= (20 + 8)(20 + 7) \\
 &= 400 + 140 + 160 + 56 \\
 &= 756
 \end{aligned}$$

Scaffolding:

For students above grade level, you might ask them to prove that  $(a + b)(c + d) = ac + bc + ad + bd$ , where  $a, b, c$ , and  $d$  are all positive real numbers.

The numbers placed into the blanks represent the number of unit squares (or square units) in each sub-rectangle.

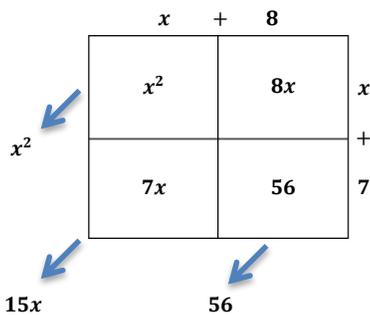
Example 1 (9 minutes)

MP.7

Explain that the goal today is to generalize the Opening Exercise to multiplying polynomials. Start by asking students how the expression  $(x + 8)(x + 7)$  is similar to the expression  $28 \times 27$ . Then suggest that students replace 20 with  $x$  in the area model. Since  $x$  in  $(x + 8)(x + 7)$  can stand for a negative number, but lengths and areas are always positive, we cannot use an area model to represent the polynomial expression  $(x + 8)(x + 7)$  without also saying that  $x > 0$ . So it is not correct to say that the area model above (with 20 replaced by  $x$ ) represents the polynomial expression  $(x + 8)(x + 7)$  for all values of  $x$ . The tabular method below is meant to remind students of the area model as a visual representation, but it is not an area model.

Example 1

Use tabular method to multiply  $(x + 8)(x + 7)$  and combine like terms.



$$(x + 8)(x + 7) = x^2 + 15x + 56$$

- Explain how the result  $x^2 + 15x + 56$  is related to 756 determined in the Opening Exercise.
  - If  $x$  is replaced with 20 in  $x^2 + 15x + 56$ , then the calculation becomes the same as the one shown in the Opening Exercise:  $(20)^2 + 15(20) + 56 = 400 + 300 + 56 = 756$ .

- How can we multiply these binomials without using a table?
  - *Think of  $x + 8$  as a single number and distribute over  $x + 7$ :*

$$\boxed{x + 8}(x + 7) = \boxed{x + 8} \cdot x + \boxed{x + 8} \cdot 7$$

Next, distribute the  $x$  over  $x + 8$  and the  $7$  over  $x + 8$ . Combining like terms shows that  $(x + 8)(x + 7) = x^2 + 15x + 56$ .

$$(x + 8)(x + 7) = (x + 8) \cdot x + (x + 8) \cdot 7 = x^2 + 8x + 7x + 56$$

- What property did we repeatedly use to multiply the binomials?
  - *The distributive property.*
- The table in the calculation above looks like the area model in the Opening Exercise. What are the similarities? Differences?
  - *The expressions placed in each table entry correspond to the expressions placed in each rectangle of the area model. The sum of the table entries represents the product, just as the sum of the areas of the sub-rectangles is the total area of the large rectangle.*
  - *One difference is that we might have  $x < 0$  so that  $7x$  and  $8x$  are negative, which does not make sense in an area model.*
- How would you have to change the table so that it represents an area model?
  - *First, all numbers and variables would have to represent positive lengths. So, in the example above, we would have to assume that  $x > 0$ . Second, the lengths should be commensurate with each other; that is, the side length for the rectangle represented by  $7$  should be slightly shorter than the side length represented by  $8$ .*
- How is the tabular method similar to the distributive property?
  - *The sum of the table entries is equal to the result of repeatedly applying the distributive property to  $(x + 8)(x + 7)$ . The tabular method graphically organizes the results of using the distributive property.*
- Does the table work even when the binomials do not represent lengths? Why?
  - *Yes. Because the table is an easy way to summarize calculations done with the distributive property—a property that works for all polynomial expressions.*

**Scaffolding:**  
If you feel students need to work another problem, ask students to use an area model to find  $16 \times 19$ , and then use the tabular method to find  $(x + 6)(x + 9)$ .

MP.7

**Exercises 1–2 (6 minutes)**

Allow students to work in groups or pairs on these exercises. While Exercise 1 is analogous to the previous example, in Exercise 2, students may need time to think about how to handle the zero coefficient of  $x$  in  $x^2 - 2$ . Allow them to struggle and discuss possible solutions.

Exercises 1–2

1. Use the tabular method to multiply  $(x^2 + 3x + 1)(x^2 - 5x + 2)$  and combine like terms.

Sample student work:

	$x^2$	$3x$	$1$	
$x^2$	$x^4$	$3x^3$	$x^2$	$x^2$
$-5x$	$-5x^3$	$-15x^2$	$-5x$	$-5x$
$2$	$2x^2$	$6x$	$2$	$2$

Diagonals shown:  $x^4$ ,  $-2x^3$ ,  $-12x^2$ ,  $x$ ,  $2$

$$(x^2 + 3x + 1)(x^2 - 5x + 2) = x^4 - 2x^3 - 12x^2 + x + 2$$

2. Use the tabular method to multiply  $(x^2 + 3x + 1)(x^2 - 2)$  and combine like terms.

Sample student work:

	$x^2$	$3x$	$1$	
$x^2$	$x^4$	$3x^3$	$x^2$	$x^2$
$0x$	$0x^3$	$0x^2$	$0x$	$0x$
$-2$	$-2x^2$	$-6x$	$-2$	$-2$

Diagonals shown:  $x^4$ ,  $3x^3$ ,  $-x^2$ ,  $-6x$ ,  $-2$

$$(x^2 + 3x + 1)(x^2 - 2) = x^4 + 3x^3 - x^2 - 6x - 2$$

Another solution method would be to omit the row for  $0x$  in the table and to manually add all table entries instead of adding along the diagonals:

$$\begin{aligned} (x^2 + 3x + 1)(x^2 - 2) &= x^4 + 3x^3 + x^2 - 2x^2 - 6x - 2 \\ &= x^4 + 3x^3 - x^2 - 6x - 2 \end{aligned}$$

Scaffolding:

For further scaffolding, you might ask students to see the pattern using numerical expressions, such as:

$$(2 - 1)(2^1 + 1)$$

$$(2 - 1)(2^2 + 2 + 1)$$

$$(2 - 1)(2^3 + 2^2 + 2 + 1).$$

Can they describe in words or symbols the meaning of these quantities?

Example 2 (6 minutes)

Prior to Example 2, consider asking students to find the products of each of these expressions.

$$(x - 1)(x + 1)$$

$$(x - 1)(x^2 + x + 1)$$

$$(x - 1)(x^3 + x^2 + x + 1)$$

Students may work on this in mixed-ability groups and come to generalize the pattern.

**Example 2**

Multiply the polynomials  $(x - 1)(x^4 + x^3 + x^2 + x + 1)$  using a table. Generalize the pattern that emerges by writing down an identity for  $(x - 1)(x^n + x^{n-1} + \dots + x^2 + x + 1)$  for  $n$  a positive integer.

	$x$	$-1$	
$x^5$ ←	$x^5$	$-x^4$	$x^4$
$0x^4$ ←	$x^4$	$-x^3$	$x^3$
$0x^3$ ←	$x^3$	$-x^2$	$x^2$
$0x^2$ ←	$x^2$	$-x$	$x$
$0x$ ←	$x$	$-1$	$1$
	$-1$		

$$(x - 1)(x^4 + x^3 + x^2 + x + 1) = x^5 - 1$$

The pattern suggests  $(x - 1)(x^n + x^{n-1} + \dots + x^2 + x + 1) = x^{n+1} - 1$ .

MP.8

- What quadratic identity from Algebra I does the identity above generalize?
  - This generalizes  $(x - 1)(x + 1) = x^2 - 1$ , or more generally, the difference of squares formula  $(x - y)(x + y) = x^2 - y^2$  with  $y = 1$ . We will explore this last identity in more detail in Exercises 2 and 3.

**Exercises 3–4 (10 minutes)**

MP.1

Before moving on to Exercise 3, it may be helpful to scaffold the problem by asking students to multiply  $(x - y)(x + y)$  and  $(x - y)(x^2 + xy + y^2)$ . Ask students to make conjectures about the form of the answer to Exercise 3.

**Exercises 3–4**

3. Multiply  $(x - y)(x^3 + x^2y + xy^2 + y^3)$  using the distributive property and combine like terms. How is this calculation similar to Example 2?

*Distribute the expression  $x^3 + x^2y + xy^2 + y^3$  through  $x - y$  to get*

$$(x - y)(x^3 + x^2y + xy^2 + y^3) = x^4 + x^3y + x^2y^2 + xy^3 - x^3y - x^2y^2 - xy^3 - y^4 = x^4 - y^4.$$

*Substitute 1 in for  $y$  to get the identity for  $n = 3$  in Example 2.*

*This calculation is similar to Example 2 because it has the same structure. Substituting 1 for  $y$  results in the same expression as Example 2.*

Exercise 3 shows why the mnemonic FOIL is not very helpful—and in this case does not make sense. By now, students should have had enough practice multiplying to no longer require such mnemonics to help them. They understand that the multiplications they are doing are really “repeated use of the distributive property,” an idea which started when they learned the multiplication algorithm in 4<sup>th</sup> grade. However, you may still need to summarize the process with a mnemonic. If this is the case, we recommend “Each-With-Each,” or EWE, which is short for the process of multiplying each term of one polynomial with each term of the a second polynomial and combining like terms.

To introduce Exercise 4, you might start with a group activity to help illuminate the generalization. For example, students could work in groups again to investigate the pattern found in expanding these expressions.

$$(x^2 + y^2)(x^2 - y^2)$$

$$(x^3 + y^3)(x^3 - y^3)$$

$$(x^4 + y^4)(x^4 - y^4)$$

$$(x^5 + y^5)(x^5 - y^5)$$

**Scaffolding:**

For further scaffolding, you might ask students to see the pattern using numerical expressions, such as:

$$(3^2 + 2^2)(3^2 - 2^2)$$

$$(3^3 + 2^3)(3^3 - 2^3)$$

$$(3^4 + 2^4)(3^4 - 2^4).$$

How do  $13 \cdot 5$  and  $3^4 - 2^4$  relate to the first line? How do  $35 \cdot 19$  and  $3^6 - 2^6$  relate to the second line? Etc.

4. Multiply  $(x^2 - y^2)(x^2 + y^2)$  using the distributive property and combine like terms. Generalize the pattern that emerges to write down an identity for  $(x^n - y^n)(x^n + y^n)$  for positive integers  $n$ .

$$(x^2 - y^2)(x^2 + y^2) = (x^2 - y^2) \cdot x^2 + (x^2 - y^2) \cdot y^2 = x^4 - x^2y^2 + x^2y^2 - y^4 = x^4 - y^4.$$

**Generalization:**  $(x^n - y^n)(x^n + y^n) = x^{2n} - y^{2n}$ .

**Sample student work:**

- We will use the generalized identity  $x^{2n} - y^{2n} = (x^n - y^n)(x^n + y^n)$  several times in this module. For example, it will help us recognize that  $2^{130} - 1$  is not a prime number because it can be written as  $(2^{65} - 1)(2^{65} + 1)$ . Some of the problems in the Problem Set rely on this type of thinking.

**Closing (4 minutes)**

Ask students to share two important ideas from the day’s lesson with their neighbor. You can also use this opportunity to informally assess their understanding. Then summarize:

- Multiplying two polynomials is performed by repeatedly applying the distributive property and combining like terms.
- There are several useful identities:
  - $(a + b)(c + d) = ac + ad + bc + bd$  (an example of “each-with-each”)
  - $(a + b)^2 = a^2 + 2ab + b^2$
  - $(x^n - y^n)(x^n + y^n) = x^{2n} - y^{2n}$ , including  $(x - y)(x + y) = x^2 - y^2$  and  $(x^2 - y^2)(x^2 + y^2) = x^4 - y^4$
  - $(x - 1)(x^n + x^{n-1} + \dots + x^2 + x + 1) = x^{n+1} - 1$
- (Optional) Consider a quick whiteboard activity in which students build fluency with applying these identities.

The vocabulary used in this lesson was introduced and taught in Algebra I. The definitions included in this lesson are for reference. To support students, consider creating a poster with these vocabulary words for the classroom wall.

#### Relevant Vocabulary

**Equivalent Polynomial Expressions:** Two polynomial expressions in one variable are *equivalent* if, whenever a number is substituted into all instances of the variable symbol in both expressions, the numerical expressions created are equal.

**Polynomial Identity:** A *polynomial identity* is a statement that two polynomial expressions are equivalent. For example,  $(x + 3)^2 = x^2 + 6x + 9$  for any real number  $x$  is a polynomial identity.

**Coefficient of a Monomial:** The *coefficient of a monomial* is the value of the numerical expression found by substituting the number 1 into all the variable symbols in the monomial. The coefficient of  $3x^2$  is 3, and the coefficient of the monomial  $(3xyz) \cdot 4$  is 12.

**Terms of a Polynomial:** When a polynomial is expressed as a monomial or a sum of monomials, each monomial in the sum is called a *term* of the polynomial.

**Like Terms of a Polynomial:** Two terms of a polynomial that have the same variable symbols each raised to the same power are called *like terms*.

**Standard Form of a Polynomial in One Variable:** A polynomial expression with one variable symbol,  $x$ , is in *standard form* if it is expressed as

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where  $n$  is a non-negative integer, and  $a_0, a_1, a_2, \dots, a_n$  are constant coefficients with  $a_n \neq 0$ .

A polynomial expression in  $x$  that is in standard form is often just called a *polynomial in  $x$*  or a *polynomial*.

The *degree of the polynomial in standard form* is the highest degree of the terms in the polynomial, namely  $n$ . The term  $a_n x^n$  is called the *leading term* and  $a_n$  (thought of as a specific number) is called the *leading coefficient*. The *constant term* is the value of the numerical expression found by substituting 0 into all the variable symbols of the polynomial, namely  $a_0$ .

#### Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 2: The Multiplication of Polynomials

### Exit Ticket

Multiply  $(x - 1)(x^3 + 4x^2 + 4x - 1)$  and combine like terms. Explain how you reached your answer.

Exit Ticket Sample Solutions

Multiply  $(x - 1)(x^3 + 4x^2 + 4x - 1)$  and combine like terms. Explain how you reached your answer.

*Tabular method:*

	$x$	$-1$	
$x^4$ ←	$x^4$	$-x^3$	$x^3$
$3x^3$ ←	$4x^3$	$-4x^2$	$4x^2$
$0x^2$ ←	$4x^2$	$-4x$	$4x$
$-5x$ ←	$-x$	$1$	$-1$
	$1$ ←		

$(x - 1)(x^3 + 4x^2 + 4x - 1) = x^4 + 3x^3 - 5x + 1$

*Using the distributive property (Each-With-Each):*

$(x - 1)(x^3 + 4x^2 + 4x - 1) = x^4 + 4x^3 + 4x^2 - x - x^3 - 4x^2 - 4x + 1 = x^4 + 3x^3 - 5x + 1.$

Problem Set Sample Solutions

1. Complete the following statements by filling in the blanks.

a.  $(a + b)(c + d + e) = ac + ad + ae + \underline{\hspace{1cm}} + \underline{\hspace{1cm}} + \underline{\hspace{1cm}}$   $bc, \quad bd, \quad be$

b.  $(r - s)^2 = (\underline{\hspace{1cm}})^2 - (\underline{\hspace{1cm}})rs + s^2$   $r, \quad 2$

c.  $(2x + 3y)^2 = (2x)^2 + 2(2x)(3y) + (\underline{\hspace{1cm}})^2$   $3y$

d.  $(w - 1)(1 + w + w^2) = \underline{\hspace{1cm}} - 1$   $w^3$

e.  $a^2 - 16 = (a + \underline{\hspace{1cm}})(a - \underline{\hspace{1cm}})$   $4, \quad 4$

f.  $(2x + 5y)(2x - 5y) = \underline{\hspace{1cm}} - \underline{\hspace{1cm}}$   $4x^2, \quad 25y^2$

g.  $(2^{21} - 1)(2^{21} + 1) = \underline{\hspace{1cm}} - 1$   $2^{42}$

h.  $[(x - y) - 3][(x - y) + 3] = (\underline{\hspace{1cm}})^2 - 9$   $x - y$

2. Use the tabular method to multiply and combine like terms.

a.  $(x^2 - 4x + 4)(x + 3)$

Sample student work:

	$x^2$	$-4x$	$4$	
$x^3$	$x^3$	$-4x^2$	$4x$	$x$
$-x^2$	$3x^2$	$-12x$	$12$	$3$
	$-8x$	$12$		

$$(x^2 - 4x + 4)(x + 3) = x^3 - x^2 - 8x + 12$$

b.  $(11 - 15x - 7x^2)(25 - 16x^2)$

Sample student work:

	$-7x^2$	$-15x$	$11$	
$112x^4$	$112x^4$	$240x^3$	$-176x^2$	$-16x^2$
$240x^3$	$0 \cdot x^3$	$0 \cdot x^2$	$0 \cdot x$	$0 \cdot x$
$-351x^2$	$-175x^2$	$-375x$	$275$	$25$
	$-375x$	$275$		

$$(11 - 15x - 7x^2)(25 - 16x^2) = 112x^4 + 240x^3 - 351x^2 - 375x + 275$$

c.  $(3m^3 + m^2 - 2m - 5)(m^2 - 5m - 6)$

Sample student work:

	$3m^3$	$m^2$	$-2m$	$-5$	
$3m^5$	$3m^5$	$m^4$	$-2m^3$	$-5m^2$	$m^2$
$-14m^4$	$-15m^4$	$-5m^3$	$10m^2$	$25m$	$-5m$
$-25m^3$	$-18m^3$	$-6m^2$	$12m$	$30$	$-6$
	$-25m^3$	$-m^2$	$37m$	$30$	

$$(3m^3 + m^2 - 2m - 5)(m^2 - 5m - 6) = 3m^5 - 14m^4 - 25m^3 - m^2 + 37m + 30$$

d.  $(x^2 - 3x + 9)(x^2 + 3x + 9)$

Sample student work:

	$x^2$	$-3x$	$9$	
$x^4$	$x^4$	$-3x^3$	$9x^2$	$x^2$
$0x^3$	$3x^3$	$-9x^2$	$27x$	$3x$
$9x^2$	$9x^2$	$-27x$	$81$	$9$
	$9x^2$	$0x$	$81$	

$$(x^2 - 3x + 9)(x^2 + 3x + 9) = x^4 + 9x^2 + 81$$

3. Multiply and combine like terms to write as the sum or difference of monomials.

a.  $2a(5 + 4a)$   
 $8a^2 + 10a$

b.  $x^3(x + 6) + 9$   
 $x^4 + 6x^3 + 9$

c.  $\frac{1}{8}(96z + 24z^2)$   
 $12z + 3z^2$

d.  $2^{23}(2^{84} - 2^{81})$   
 $2^{107} - 2^{104}$

e.  $(x - 4)(x + 5)$   
 $x^2 + x - 20$

f.  $(10w - 1)(10w + 1)$   
 $100w^2 - 1$

g.  $(3z^2 - 8)(3z^2 + 8)$   
 $9z^4 - 64$

h.  $(-5w - 3)w^2$   
 $-5w^3 - 3w^2$

i.  $8y^{1000}(y^{12200} + 0.125y)$   
 $8y^{13200} + y^{1001}$

j.  $(2r + 1)(2r^2 + 1)$   
 $4r^3 + 2r^2 + 2r + 1$

k.  $(t - 1)(t + 1)(t^2 + 1)$   
 $t^4 - 1$

l.  $(w - 1)(w^5 + w^4 + w^3 + w^2 + w + 1)$   
 $w^6 - 1$

m.  $(x + 2)(x + 2)(x + 2)$   
 $x^3 + 6x^2 + 12x + 8$

n.  $n(n + 1)(n + 2)$   
 $n^3 + 3n^2 + 2n$

o.  $n(n + 1)(n + 2)(n + 3)$   
 $n^4 + 6n^3 + 11n^2 + 6n$

p.  $n(n + 1)(n + 2)(n + 3)(n + 4)$   
 $n^5 + 10n^4 + 35n^3 + 50n^2 + 24n$

q.  $(x + 1)(x^3 - x^2 + x - 1)$   
 $x^4 - 1$

r.  $(x + 1)(x^5 - x^4 + x^3 - x^2 + x - 1)$   
 $x^6 - 1$

s.  $(x + 1)(x^7 - x^6 + x^5 - x^4 + x^3 - x^2 + x - 1)$   
 $x^8 - 1$

t.  $(m^3 - 2m + 1)(m^2 - m + 2)$   
 $m^5 - m^4 + 3m^2 - 5m + 2$

4. Polynomial expressions can be thought of as a generalization of place value.

a. Multiply  $214 \times 112$  using the standard paper-and-pencil algorithm.

$$\begin{array}{r} 214 \\ \times 112 \\ \hline 428 \\ 214 \phantom{0} \\ + 214 \phantom{00} \\ \hline 23,968 \end{array}$$

b. Multiply  $(2x^2 + x + 4)(x^2 + x + 2)$  using the tabular method and combine like terms.

	$2x^2$	$x$	$4$	
$2x^4$	$2x^4$	$x^3$	$4x^2$	$x^2$
$3x^3$	$2x^3$	$x^2$	$4x$	$x$
$9x^2$	$4x^2$	$2x$	$8$	$2$
	$6x$	$8$		

$$(2x^2 + x + 4)(x^2 + x + 2) = 2x^4 + 3x^3 + 9x^2 + 6x + 8$$

c. Put  $x = 10$  into your answer from part (b).

23,968

d. Is the answer to part (c) equal to the answer from part (a)? Compare the digits you computed in the algorithm to the coefficients of the entries you computed in the table. How do the place-value units of the digits compare to the powers of the variables in the entries?

*Yes. The digits computed in the algorithm are the same as the coefficients computed in the table entries. The zero-degree term in the table corresponds to the ones unit, the first-degree terms in the table correspond to the tens unit, the second-degree terms in the table correspond to the hundreds unit, and so on.*

5. Jeremy says  $(x - 9)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$  must equal  $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$  because when  $x = 10$ , multiplying by  $x - 9$  is the same as multiplying by 1.

a. Multiply  $(x - 9)(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$ .

$$x^8 - 8x^7 - 8x^6 - 8x^5 - 8x^4 - 8x^3 - 8x^2 - 8x - 9$$

b. Put  $x = 10$  into your answer.

$$100,000,000 - 80,000,000 - 8,000,000 - 800,000 - 80,000 - 8,000 - 800 - 80 - 9$$

$$100,000,000 - 88,888,889 = 11,111,111$$

c. Is the answer to part (b) the same as the value of  $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$  when  $x = 10$ ?

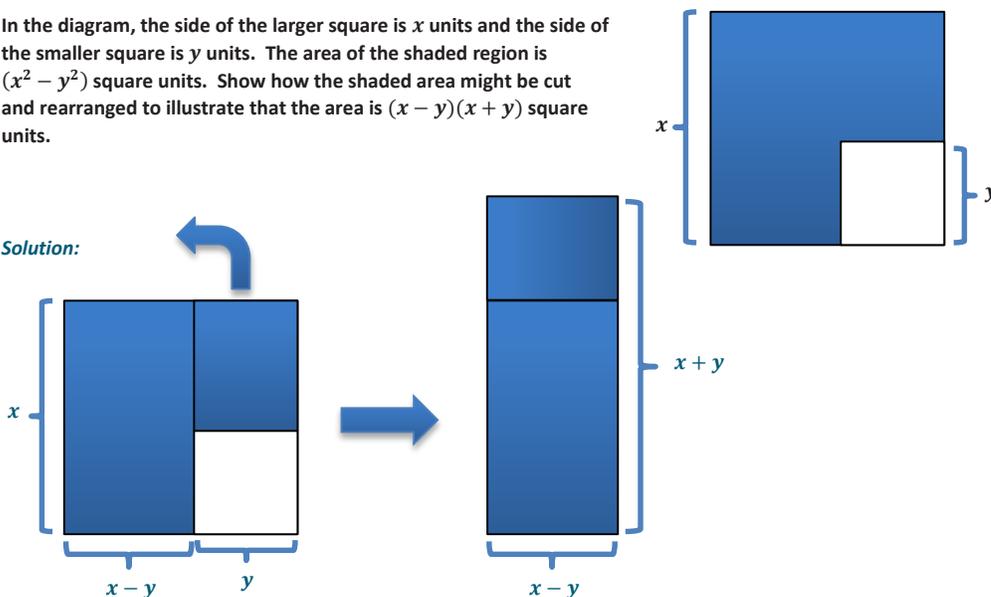
*Yes.*

d. Was Jeremy right?

*No, just because it is true when  $x$  is 10, does not make it true for all real  $x$ . The two expressions are not algebraically equivalent.*

6. In the diagram, the side of the larger square is  $x$  units and the side of the smaller square is  $y$  units. The area of the shaded region is  $(x^2 - y^2)$  square units. Show how the shaded area might be cut and rearranged to illustrate that the area is  $(x - y)(x + y)$  square units.

*Solution:*





## Lesson 3: The Division of Polynomials

### Student Outcomes

- Students develop a division algorithm for polynomials by recognizing that division is the inverse operation of multiplication.

### Lesson Notes

This lesson begins to address standards **A-SSE.A.2** and **A-APR.C.4** and provides many opportunities for students to practice MP.7 and 8. Students will explore reversing the tabular method they learned in Lesson 2 as a means to divide two polynomials. They will develop a procedure for polynomial division using a table. The problems are scaffolded to lead students to discover this method in small groups, allowing them to discern patterns in repeated calculations, reinforcing the work from Lessons 1 and 2. Discussion questions and the lesson closure will be key to guiding struggling students. In this lesson it is critical to emphasize the relationships between the table entries and the position of the dividend and divisor and how the diagonals in the table sum to the terms of the dividend. In the next lesson, students will connect this method to the traditional long division algorithm of polynomial division while reinforcing the relationship between polynomial division and integer arithmetic. All of the problems in this lesson will divide without a remainder; division with remainders will be addressed in later lessons in this module. General polynomial identities that will be used heavily and explored further in the remaining lessons of this module are also touched on in this lesson.

Since division is the inverse operation of multiplication, it is important to make sure that your students understand how to multiply polynomials before moving on to division of polynomials found in this lesson. If your students are struggling with the content of this lesson, they may need to review problems from Lesson 2 in this module and the following lessons from previous grades:

- Studying the connection between area properties and the distributive property: Grade 7, Module 6, Lesson 21.
- Introduction to the tabular method of multiplying polynomials: Algebra I, Module 1, Lesson 9.

In the early lessons on division (Lessons 3–7) the issue of potential zeros in the denominator has not been emphasized, but it becomes prominent in Lesson 21 when rational expressions are introduced. When introducing polynomial division in this lesson, you may or may not choose to emphasize that we need to exclude certain values of  $x$  that cause division by zero.

### Classwork

#### Opening Exercise (3 minutes)

This exercise will provide students with an opportunity to practice the tabular method of multiplication of polynomials. This problem will be continued in the Discussion and Exploratory Challenge that follow.

#### Scaffolding:

For students who are unfamiliar with the vocabulary in this lesson (divisor, quotient, dividend, factor, product), take time to pre-teach these terms. Use a vocabulary journal or notebook to help them learn and use proper terminology throughout the lesson.

A visual representation like the one below may be of assistance.

$$(Factor)(Factor) = Product$$

$$\frac{Dividend}{Divisor} = Quotient$$

**Opening Exercise**

- a. Multiply these polynomials using the tabular method.

$$(2x + 5)(x^2 + 5x + 1)$$

*The product is  $2x^3 + 15x^2 + 27x + 5$*

- b. How can you use the expression in part (a) to quickly multiply  $25 \times 151$ ?

*If you let  $x = 10$ , then the product is*

$$2(10)^3 + 15(10)^2 + 27(10) + 5 = 2,000 + 1,500 + 270 + 5 = 3,775.$$

**Discussion (5 minutes)**

Lead a discussion that connects multiplication and division. Display the following problem on the board, and ask how it could be transformed into a division problem.

$$25 \times 151 = 3,775$$

- How can a multiplication problem be re-written as a division problem?
  - *One of the factors is the divisor and the other is the quotient. The product is the dividend.*

$$\frac{3775}{25} = 151 \quad \text{OR} \quad \frac{3775}{151} = 25$$

- How can we re-write the Opening Exercise as a division problem?

$$(2x + 5)(x^2 + 5x + 1) = 2x^3 + 15x^2 + 27x + 5$$

- *It would be re-written the same way. One of the factors is the divisor and the other is the quotient. The product is the dividend.*

$$\frac{2x^3 + 15x^2 + 27x + 5}{2x + 5} = x^2 + 5x + 1 \quad \text{OR} \quad \frac{2x^3 + 15x^2 + 27x + 5}{x^2 + 5x + 1} = 2x + 5$$

- Let  $x = 10$ . Substitute that value into each polynomial, and compare the results of multiplying and dividing the polynomials with the arithmetic problem. How do polynomial multiplication and division compare to multiplication and division of integers?
  - *When  $x = 10$ , the polynomial problems result in the same values as the number problems. This reinforces the fact that arithmetic operations with polynomials are similar to arithmetic operations with integers.*

**Exploratory Challenge (27 minutes): Reverse the Tabular Method to Divide Polynomials**

Give students time to discuss how they would fill in the rows and columns in the table below. Let them struggle to make sense of the problem and look for patterns. If they are proficient with the tabular method of multiplication, they should quickly discover a method for populating the table and then verifying that the quotient will be in the top row. Read through this entire section before attempting this with your students so you clearly understand how the process works. Start this challenge by asking students to consider how they might reverse the tabular method for multiplying to solve a polynomial division problem. Encourage them to discuss where to position the polynomials now that they have the “answer” (the product) and one of the factors. In the last lessons on multiplication, the factors positioned along the top



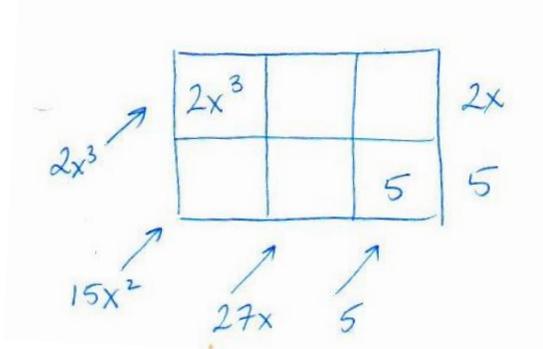
After students have had a few minutes to discuss their ideas in groups, lead a short discussion (as needed) if they appear to be stuck. You may use these questions to prompt groups as you circulate around the room as well.

- The quotient will be the polynomial that would go along the top of the table. Remember we know it should be  $x^2 + 5x + 1$ . How can you get started to confirm this using the dividend and the divisor?
  - *We need to think about how the divisor terms fill back into the table, and then we need to use the  $2x$  and the  $5$  terms to determine the other factor for each column at the top of the table.*
- Are there any cells in the table that we can fill in based on the information we have? What must be in the top left cell? Why?
  - *The  $2x^3$  term will be the top left entry and the  $5$  will be the bottom right entry. Because the diagonals add to produce the terms of the product, we know that the top left and bottom right entries must be the first and last terms of the dividend since those terms do not involve combining any like terms when we compute the product.*
- What must the first term of the missing polynomial in the top row be?
  - *It would have to be  $x^2$  since that term and  $2x$  must multiply to be  $2x^3$ .*
- What goes in the rest of the cells in the first column? How can you continue this pattern?
  - *Multiply  $x^2$  and  $5$  to get  $5x^2$ . Then the  $2^{nd}$  column in the first row would need to be  $10x^2$  because  $15x^2 = 10x^2 + 5x^2$ . Since the  $2^{nd}$  column of the first row is now known, we can figure out the remaining term of the quotient since that term times  $2x$  would have to equal  $10x^2$ .*
- Compare your work on this problem with the Opening Exercise. How could you verify that  $x^2 + 5x + 1$  really is the quotient? Explain?
  - *In the Opening Exercise, we multiplied the quotient and divisor and got the dividend. So  $(2x + 5)(x^2 + 5x + 1)$  must equal  $2x^3 + 15x^2 + 27x + 5$ .*

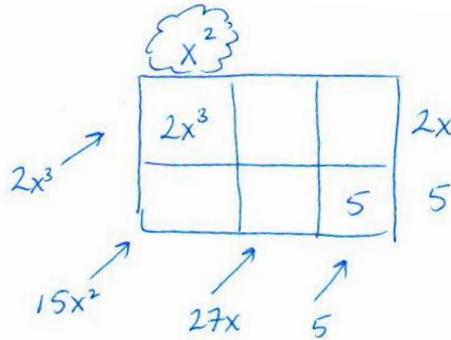
2. Describe the process you used to determine your answer to Exercise 1.  
*Student descriptions will vary but should be similar to the detailed directions provided on the following page.*

Check each group’s work before moving on to the next exercise. Have a couple of groups present their solution methods to the Exercises 1 and 2 on the board, especially if other groups seem to be struggling. Then, if needed, go over the steps outlined below in detail to clarify this process for all students. Provide additional scaffolding if needed.

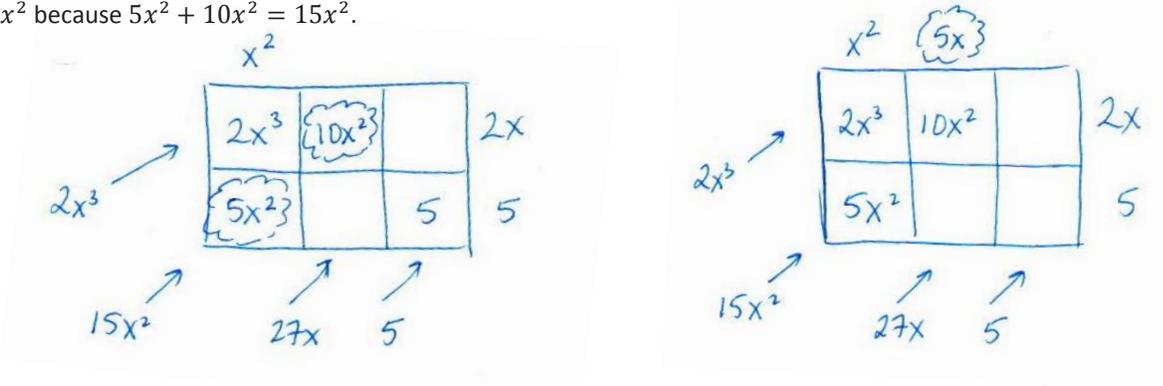
Step 1—Draw a table with factors along the right sides and bottom corner and fill in known entries along the top left and bottom right.



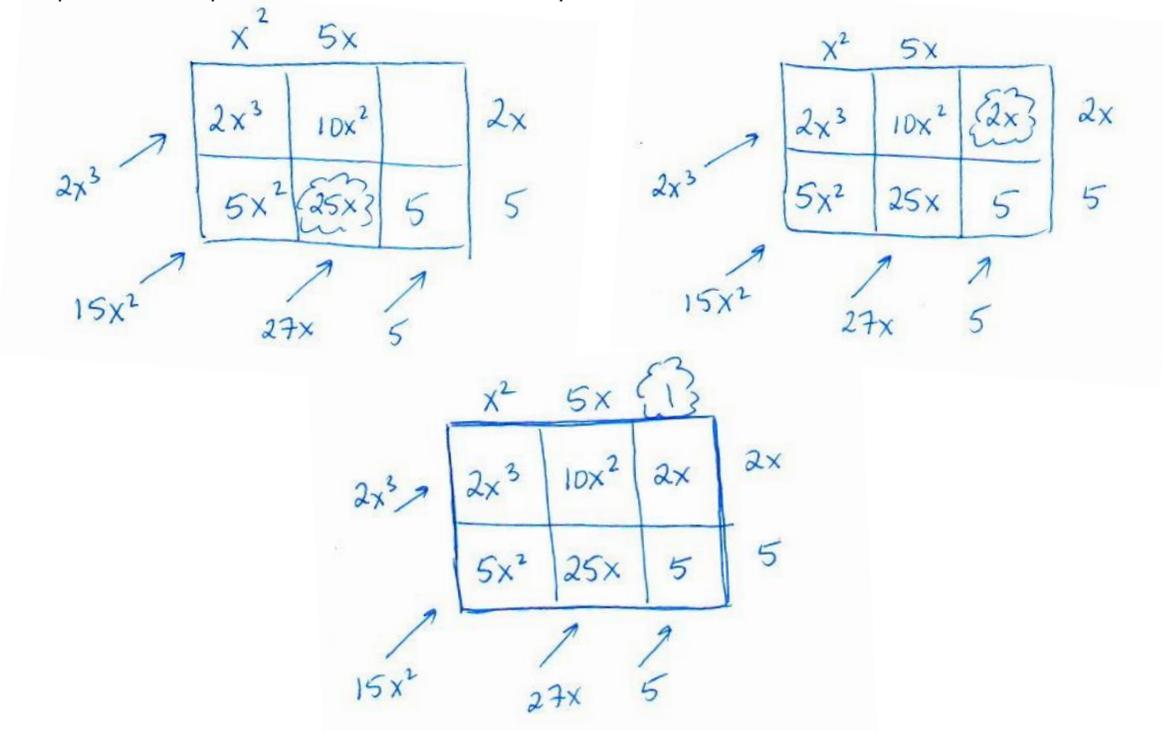
Step 2—Use the top left entry to find the second degree term along the top. It has to be  $x^2$  because  $2x \cdot x^2 = 2x^3$ .



Step 3—Use  $x^2$  and 5 to find the bottom left entry,  $5x^2$ , and then use that entry to find the top middle term. It has to be  $10x^2$  because  $5x^2 + 10x^2 = 15x^2$ .



Step 4—Repeat these steps until to fill in the bottom entry in the 2<sup>nd</sup> and last columns.



Students do not already know the quotient in Exercise 3, making this more challenging than the previous exercises.

3. Reverse the tabular method of multiplication to find the quotient:  $\frac{2x^2+x-10}{x-2}$ .

After these two exercises, lead a short discussion to help students learn how to reverse their thinking about the tabular method of multiplication to divide polynomials.

- We will call this approach to dividing polynomials the *reverse tabular method*. Let's make a conjecture about the number of rows and columns you will need to perform the division. How can you predict how many rows you will need in your table? How can you predict how many columns you will need in your table?
  - *The number of rows will be one more than the degree of the divisor. The number of columns will be one more than the difference between the degree of the dividend and the degree of the divisor.*

4. Test your conjectures. Create your own table and use the *reverse tabular method* to find the quotient.

$$\frac{x^4 + 4x^3 + 3x^2 + 4x + 2}{x^2 + 1}$$

*The quotient is  $x^2 + 4x + 2$ .*

MP.3

If necessary, remind groups to include a 0 coefficient term place holder when missing a needed term.

5. Test your conjectures. Use the *reverse tabular method* to find the quotient.

$$\frac{3x^5 - 2x^4 + 6x^3 - 4x^2 - 24x + 16}{x^2 + 4}$$

The quotient is  $3x^3 - 2x^2 - 6x + 4$ .

6. What is the quotient of  $\frac{x^5-1}{x-1}$ ? Of  $\frac{x^6-1}{x-1}$ ?

The quotients are  $x^4 + x^3 + x^2 + x + 1$  and  $x^5 + x^4 + x^3 + x^2 + x + 1$ , respectively.

*Scaffolding:*

If students are struggling to extend this pattern in Exercise 6, begin with simpler problems such as:

$$\frac{(x^2 - 1)}{(x - 1)} = x + 1$$

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1$$

$$\frac{x^4 - 1}{x - 1} = x^3 + x^2 + x + 1.$$

They should solve these simpler problems using the tabular method. Perhaps have different groups each work a different problem, and then record the results on the board. Conduct a class discussion so they can generalize the pattern.

After students complete Exercise 6, see if they can extend the patterns to make predictions about similar problems.

MP.7 & MP.8

- What would be the result of  $\frac{x^8-1}{x-1}$ ?
  - $x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
- What would be the result of  $\frac{x^n-1}{x-1}$ ?
  - $x^{n-1} + x^{n-2} + \dots + x + 1$
- What would be the result of  $\frac{x^5+1}{x+1}$ ?
  - $x^4 - x^3 + x^2 - x + 1$

Wrap up this lesson by having groups present their solutions to Exercises 3–5 and discuss the strategies they used to solve the problems. It is fine if two groups present the same problem since they may have had slightly different approaches to completing the table. Students may notice that it is easier to fill in the table by columns. Some may have started working from the lower right corner where the constant is located. Students may realize that they do not need to fill in every cell to finish the problem but may wish to do so to check their work. If time permits, have them verify their conjectures to the follow-up questions for Exercise 6.

**Closing (5 minutes)**

These questions help to reinforce the relationship between multiplication and division and some of the patterns that emerge in using the reverse tabular method to divide polynomials with no remainder. Have students respond to the questions below (either in writing or with a partner) to provide a summary and formative assessment information. The questions that ask about the degree and leading coefficient help to reinforce the A.APR standards addressed in this lesson.

- What strategies were helpful when you set up and solved these problems? What patterns did you notice as you solved these problems?
  - *The number of rows will be one more than the degree of the divisor. The number of columns will be one more than the difference between the degree of the dividend and the degree of the divisor. For example, when dividing  $2x^4 - x^3 + 4x^2 - 1$  by  $x^3 + 2x + 1$ , we will have four rows since the divisor has degree 3. Then we will need  $4 - 3 + 1 = 2$  columns for the quotient.*

- *You need to use a place holder for missing terms. For example,  $x^2 + 1 = x^2 + 0x + 1$ .*
- *The diagonals must add to the correct terms, and the cells in the table must be the product of the factors along the sides of the table.*
- What happens to the degree of the product when you multiply two polynomials?
  - *When you multiply polynomials, the degree of the product will be the sum of the degrees of each factor.*
- What happens to the degree of the quotient when you divide two polynomials?
  - *When you divide polynomials, the degree of the quotient will be the difference of the degrees of the dividend and divisor.*
- What happens to leading coefficients when you multiply or divide polynomials?
  - *The leading coefficient of the result is the product or quotient of the leading coefficients of the original polynomials.*

**Exit Ticket (5 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 3: The Division of Polynomials

### Exit Ticket

Find the quotient. Justify your answer.

$$\frac{x^5 + 2x^4 - 7x^2 - 19x + 15}{x^2 + 2x + 5}$$

Exit Ticket Sample Solutions

Find the quotient. Justify your answer.

$$\frac{x^5 + 2x^4 - 7x^2 - 19x + 15}{x^2 + 2x + 5}$$

The quotient is  $x^3 - 5x + 3$ .

	$x^3$	$0x^2$	$-5x$	$3$	
	$x^5$	$0x^4$	$-5x^3$	$3x^2$	$x^2$
$x^5$	$2x^4$	$0x^3$	$-10x^2$	$6x$	$2x$
$2x^4$	$5x^3$	$0x^2$	$-25x$	$15$	$5$
$0x^3$		$-7x^2$	$-19x$	$15$	

Problem Set Sample Solutions

Use the reverse tabular method to solve these division problems.

- $(2x^3 + x^2 - 16x + 15) \div (2x - 3)$   $x^2 + 2x - 5$
- $(3x^5 + 12x^4 + 11x^3 + 2x^2 - 4x - 2) \div (3x^2 - 1)$   $x^3 + 4x^2 + 4x + 2$
- $\frac{x^3 - 4x^2 + 7x - 28}{x^2 + 7}$   $x - 4$
- $\frac{x^4 - 2x^3 - 29x - 12}{x^3 + 2x^2 + 8x + 3}$   $x - 4$
- $\frac{6x^5 + 4x^4 - 6x^3 + 14x^2 - 8}{6x + 4}$   $x^4 - x^2 + 3x - 2$
- $(x^3 - 8) \div (x - 2)$   $x^2 + 2x + 4$
- $\frac{x^3 + 2x^2 + 2x + 1}{x + 1}$   $x^2 + x + 1$
- $\frac{x^4 + 2x^3 + 2x^2 + 2x + 1}{x + 1}$   $x^3 + x^2 + x + 1$

9. Use the results of Problems 7 and 8 to predict the quotient of  $\frac{x^5+2x^4+2x^3+2x^2+2x+1}{x+1}$ . Explain your prediction. Then check your prediction using the reverse tabular method.
- $x^4 + x^3 + x^2 + x + 1$ . In Problems 7 and 8, the result was a polynomial of degree one less than the dividend where all the coefficients were 1. The dividend in this problem had the same structure except it was degree 5 and it was also divided by  $x + 1$ .
10. Use the results of Exercise 5 in the Exploratory Challenge and Problems 7–9 above to predict the quotient of  $\frac{x^4-2x^3+2x^2-2x+1}{x-1}$ . Explain your prediction. Then check your prediction using the reverse tabular method.
- $x^3 - x^2 + x - 1$
11. Make and test a conjecture about the quotient of  $\frac{x^6+x^5+2x^4+2x^3+2x^2+x+1}{x^2+1}$ . Explain your reasoning.
- $x^4 + x^3 + x^2 + x + 1$ . Since we are missing the  $x$  term, there will not be two  $x^5$  or two  $x$  terms. Otherwise it will follow the same pattern as Problems 7–9.
12. Given the following quotients:
- $$\frac{4x^2 + 8x + 3}{2x + 1} \quad \text{and} \quad \frac{483}{21}$$
- a. How are these expressions related?
- If we let  $x = 10$ , then the values of the two expressions are equivalent.*
- b. Find each quotient.
- $2x + 3$  and  $23$*
- c. Explain the connection between the quotients.
- If we let  $x = 10$ , then the values of the two quotients are equivalent.*



## Lesson 4: Comparing Methods—Long Division, Again?

### Student Outcomes

- Students connect long division of polynomials with the long division algorithm of arithmetic and use this algorithm to rewrite rational expressions that divide without a remainder.

### Lesson Notes

This lesson reinforces the analogous relationship between arithmetic of numbers and the arithmetic of polynomials (A-APR.6, A-APR.7). These standards address working with rational expressions and focus on using a long division algorithm to rewrite simple rational expressions. In addition, it provides another method for students to fluently calculate the quotient of two polynomials after the Opening Exercises.

### Classwork

#### Opening

Have students work individually on the Opening Exercises to confirm their understanding of the previous lesson’s outcomes. Circulate around the room to observe their progress, or have students check their work with a partner after a few minutes. Today’s lesson will transition to another method for dividing polynomials.

#### Opening Exercises (5 minutes)

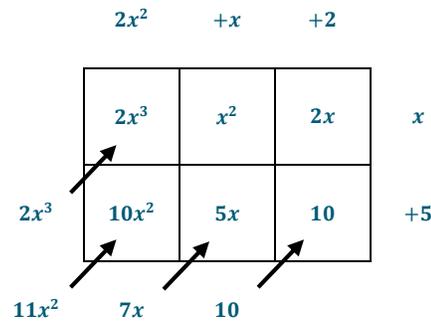
**Opening Exercises**

1. Use the reverse tabular method to determine the quotient  $\frac{2x^3+11x^2+7x+10}{x+5}$ .

	$2x^2$	$+ x$	$+ 2$	
	$2x^3$	$x^2$	$2x$	$x$
$2x^3$	$10x^2$	$5x$	$10$	$+ 5$
	$+ 11x^2$	$+ 7x$	$+ 10$	

2. Use your work from Exercise 1 to write the polynomial  $2x^3 + 11x^2 + 7x + 10$  in factored form, and then multiply the factors to check your work above.

$$(x + 5)(2x^2 + x + 2)$$



The product is  $2x^3 + 11x^2 + 7x + 10$ .

Division and multiplication of polynomials is very similar to those operations with real numbers. In these problems, if you let  $x = 10$ , the result would match an arithmetic problem. Yesterday, we divided two polynomials using the reverse tabular method. Today, we want to see how polynomial division is related to the long division algorithm learned in elementary school.

### Discussion (5 minutes)

We have seen how division of polynomials relates to multiplication and that both of these operations are similar to the arithmetic operations you learned in elementary school.

- Can we relate division of polynomials to the long division algorithm?
  - We would need to use the fact that the terms of a polynomial expression represent place value when  $x = 10$ .

Prompt students to consider the long division algorithm they learned in elementary school, and ask them to apply it to evaluate  $1573 \div 13$ . Have a student model the algorithm on the board as well. The solution to this problem is included in the example below.

### Example 1 (5 minutes): The Long Division Algorithm for Polynomial Division

When you solve the problem in Example 1, be sure to record the polynomial division problem next to the arithmetic problem already on the board. Guide students through this example to demonstrate the parallels between the long division algorithm for numbers and this method. You can emphasize that the long division algorithm they learned in elementary school is a special case of polynomial long division. They should record the steps on their handouts or in their notebooks. Have students check their work by solving this problem using the reverse tabular method. Use the questions below as you work the example.

MP.7

See the sample problem written out after Example 1, and use the questions that follow as discussion points while you model this algorithm.

- What expression multiplied by  $x$  will result in  $x^3$ ?
  - $x^2$
- When you do long division, you multiply the first digit of the quotient by the divisor and then subtract the result. It works the same with polynomial division. How do we represent multiplication and subtraction of polynomials?
  - *You apply the distributive property to multiply, and to subtract you add the opposite.*
- Then, we repeat the process to determine the next term in the quotient. What do we need to bring down to complete the process?
  - *You should bring down the next term.*

**Example 1**

If  $x = 10$ , then the division  $1573 \div 13$  can be represented using polynomial division.

$$x+3 \overline{)x^3 + 5x^2 + 7x + 3}$$

*The quotient is  $x^2 + 2x + 1$ .*

*Scaffolding:*

- For further scaffolding consider starting with a simpler problem such as  $126 \div 18$ . Have students compare this problem to the polynomial division problem  $(x^2 + 2x + 6) \div (x + 8)$  by explaining the structural similarities. Have students consider this when you place them side by side on the board. This will show students that if we let  $x = 10$ , the polynomial division problem is analogous to the integer division problem.
- For advanced learners, you could challenge them to create two examples, a numerical one and a polynomial one, that illustrate the structural similarities. Note, however, that not every problem will work nicely. For example,  $800 \div 32 = 25$ , but  $8x^2 \div (3x + 2) \neq 2x + 5$  because there are many polynomials in  $x$  that evaluate to 25 when  $x = 10$ .

The completed board work for this example should look something like this:

**Example 2 (5 minutes): The Long Division Algorithm for Polynomial Division**

We can divide any two numbers as long as the divisor is not equal to 0. Similarly, we can divide any two polynomials as long as the divisor is not equal to 0. Note: The number 0 is also a polynomial. Because we are now dealing with a general case of polynomials and not simply numbers, we can solve problems where the coefficients of the terms are any real numbers. It would difficult, but not impossible, if the coefficients of the terms of our polynomials were irrational. In the next example, model again how this process works. Be sure to point out that students must use a 0 coefficient place holder for the missing  $x$  term.

**Example 2**

Use the long division algorithm for polynomials to evaluate

$$\frac{2x^3 - 4x^2 + 2}{2x - 2}$$

The quotient is  $x^2 - x - 1$ .

Before beginning the next exercises, take the time to reinforce the idea that polynomial division is analogous to whole number division by posing a reflection question. Students can discuss this with a partner or respond in writing.

- Why are we able to do long division with polynomials?
  - *Polynomials form a system analogous to the integers. The same operations that hold for integers hold for polynomials.*

**Exercises 1–8 (15 minutes)**

These problems start simple and become more complicated. Monitor student progress as they work. Have students work these problems independently or in pairs and use this as an opportunity to informally assess their understanding. After students have completed the exercises, post the solutions on the board but not the work. Have students with errors team up with a partner and trade papers. Ask students to find the mistakes in their partner's work. You may choose an incorrect solution to display on the board and then lead a class discussion to point out where students are likely to make errors and how to prevent them. Students will typically make careless errors in multiplying or subtracting terms. Other errors can occur if they forget to include the zero coefficient place holder terms when needed. If students appear to be running short on time, have them check every other result using the reverse tabular method. Alternately, students could check their work using multiplication.

**Exercises 1–8**

Use the long division algorithm to determine the quotient. For each problem, check your work by using the reverse tabular method.

1.  $\frac{x^2 + 6x + 9}{x + 3}$

$x + 3$

2.  $(7x^3 - 8x^2 - 13x + 2) \div (7x - 1)$

$x^2 - x - 2$

3.  $(x^3 - 27) \div (x - 3)$

$x^2 + 3x + 9$

4.  $\frac{2x^4 + 14x^3 + x^2 - 21x - 6}{2x^2 - 3}$

$x^2 + 7x + 2$

5.  $\frac{5x^4 - 6x^2 + 1}{x^2 - 1}$

$$5x^2 - 1$$

6.  $\frac{x^6 + 4x^4 - 4x - 1}{x^3 - 1}$

$$x^3 + 4x + 1$$

7.  $(2x^7 + x^5 - 4x^3 + 14x^2 - 2x + 7) \div (2x^2 + 1)$

$$x^5 - 2x + 7$$

8.  $\frac{x^6 - 64}{x + 2}$

$$x^5 - 2x^4 + 4x^3 - 8x^2 + 16x - 32$$

### Closing (5 minutes)

Ask students to summarize the important parts of this lesson either in writing, to a partner, or as a class. Use this opportunity to informally assess their understanding prior to starting the Exit Ticket. Important elements are included in the Lesson Summary box below. The questions that follow are recommended to guide the discussions with sample student responses included in italics. Depending on the structure of your closure activity, the sample responses would be similar to student-written, partner, or whole-class summaries.

- Which method do you prefer, long division or the reverse tabular method?
  - *Student responses will vary. The reverse tabular method may appeal to visual learners. The long division algorithm works well as long as you avoid careless mistakes.*
- Is one method “easier” than another?
  - *This will depend on student preferences, but some will like the connection to prior methods for dividing and multiplying. Perhaps when many terms are missing (as in Exercise 8), the reverse tabular method can go more quickly than long division.*
- What advice would you give to a friend that is just learning how to do these problems quickly and accurately?
  - *Be careful when multiplying terms and working with negative terms.*

#### Lesson Summary

The long division algorithm to divide polynomials is analogous to the long division algorithm for integers. The long division algorithm to divide polynomials produces the same results as the reverse tabular method.

### Exit Ticket (5 minutes)



Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 4: Comparing Methods—Long Division, Again?

### Exit Ticket

Write a note to a friend explaining how to use long division to find the quotient.

$$\frac{2x^2 - 3x - 5}{x + 1}$$

## Exit Ticket Sample Solutions

Write a note to a friend explaining how to use long division to find the quotient.

$$\frac{2x^2 - 3x - 5}{x + 1}$$

Set up the divisor outside the division sign and the dividend underneath it. Then ask yourself what number multiplied by  $x$  is  $2x^2$ . Then multiply that number by  $x + 1$ , and record the results underneath  $2x^2 - 3x$ . Subtract these terms and bring down the  $-5$ . Then repeat the process.

## Problem Set Sample Solutions

Use the long division algorithm to determine the quotient.

1. 
$$\frac{2x^3 - 13x^2 - x + 3}{2x + 1}$$

$$x^2 - 7x + 3$$

2. 
$$\frac{3x^3 + 4x^2 + 7x + 22}{x + 2}$$

$$3x^2 - 2x + 11$$

3. 
$$\frac{x^4 + 6x^3 - 7x^2 - 24x + 12}{x^2 - 4}$$

$$x^2 + 6x - 3$$

4.  $(12x^4 + 2x^3 + x - 3) \div (2x^2 + 1)$

$$6x^2 + x - 3$$

5.  $(2x^3 + 2x^2 + 2x) \div (x^2 + x + 1)$

$$2x$$

6. Use long division to find the polynomial,  $p$ , that satisfies the equation below.

$$2x^4 - 3x^2 - 2 = (2x^2 + 1)(p(x))$$

$$x^2 - 2$$

7. Given  $q(x) = 3x^3 - 4x^2 + 5x + k$ .

a. Determine the value of  $k$  so that  $3x - 7$  is a factor of the polynomial  $q$ .

$$k = -28$$

b. What is the quotient when you divide the polynomial  $q$  by  $3x - 7$ ?

$$x^2 + x + 4$$

8. In parts (a)–(b) and (d)–(e), use long division to evaluate each quotient. Then, answer the remaining questions.

a.  $\frac{x^2-9}{x+3}$   
 $x - 3$

b.  $\frac{x^4-81}{x+3}$   
 $x^3 - 3x^2 + 9x - 27$

c. Is  $x + 3$  a factor of  $x^3 - 27$ ? Explain your answer using the long division algorithm.

*No. The remainder is not 0 when you perform long division.*

d.  $\frac{x^3+27}{x+3}$   
 $x^2 - 3x + 9$

e.  $\frac{x^5+243}{x+3}$   
 $x^4 - 3x^3 + 9x^2 - 27x + 81$

f. Is  $x + 3$  a factor of  $x^2 + 9$ ? Explain your answer using the long division algorithm.

*No. The remainder is not 0 when you perform long division.*

g. For which positive integers  $n$  is  $x + 3$  a factor of  $x^n + 3^n$ ? Explain your reasoning.

*Only if  $n$  is an odd number. By extending the patterns in parts (a)–(c) and (e), we can generalize that  $x + 3$  divides evenly into  $x^n + 3^n$  for odd powers of  $n$  only.*

h. If  $n$  is a positive integer, is  $x + 3$  a factor of  $x^n - 3^n$ ? Explain your reasoning.

*Only for even numbers  $n$ . By extending the patterns in parts (a)–(c), we can generalize that  $x + 3$  will always divide evenly into the dividend.*



## Lesson 5: Putting It All Together

### Student Outcomes

- Students perform arithmetic operations on polynomials and write them in standard form.
- Students understand the structure of polynomial expressions by quickly determining the first and last terms if the polynomial were to be written in standard form.

### Lesson Notes

In this lesson, students work with all four polynomial operations. The first part of the lesson is a relay exercise designed to build fluency, and the second part of the lesson includes combining two or more operations to write a polynomial in standard form. You will need to prepare a set of notecards as described below for Exercises 1–15.

The Algebra Progressions name three different forms for a quadratic expression: standard, factored, and vertex. Examples of these forms are shown below.

- Standard Form:  $x^2 + 4x - 5$
- Factored Form:  $(x + 5)(x - 1)$
- Vertex Form:  $(x + 2)^2 - 9$

We can define a standard and factored form of a degree  $n$  polynomial expression as well. The final lessons in this module introduce the Fundamental Theorem of Algebra which states that a degree  $n$  polynomial with real coefficients can be written as the product of  $n$  linear factors. Precise definitions of the following terms are provided for teacher reference at the end of this lesson: monomial, polynomial expression, coefficient of a monomial, degree of a monomial, terms of a polynomial, standard form of a polynomial in one variable, and degree of a polynomial in one variable. This lesson concludes by challenging students to quickly determine the first and last term of a polynomial expression if it were to be written in standard form. Students who have this capacity can quickly analyze the end behavior of the graph of a polynomial function or compute key features such as a  $y$ -intercept without having to fully re-write the expression in standard form. This allows for quick analysis when applying properties of polynomials in future mathematics courses such as Grade 12 mathematics or Calculus.

#### *Scaffolding:*

Before beginning these exercises it may be necessary to review the definition and/or post an example of a polynomial in standard form. You can also remind students of vertex and factored form of quadratic expressions using the examples in the Lesson Notes if needed.

For example,

- Standard Form:  $x^2 + 4x - 5$
- Factored Form:  $(x + 5)(x - 1)$
- Vertex Form:  $(x + 2)^2 - 9$

### Classwork

#### Exercises 1–15 (20 minutes): Polynomial Pass

Prior to the lesson, write each Exercise 1 to 15 on an index card, one exercise per card. On the back of each card, write the solution to the previous exercise. For example, on the back of the card for Exercise 2, write the answer to Exercise 1. On the back of the index card for Exercise 1, write the answer to Exercise 15. Students should be seated in a circle if possible. If more than 15 different cards are needed, create additional exercises that require addition, subtraction, multiplication, or division without remainder of linear, quadratic, or cubic polynomials.

To complete the exercise, have students work in pairs. Pass out cards to pairs in numerical order so that when the cards are passed, each pair gets a new exercise that contains the answer to the one they just worked on the back of the card. (Alternatively, have students work individually and make two sets of cards so that cards 16–30 are duplicates of 1–15.) Allow one minute for students to work each problem on a separate sheet of paper, and then have them pass the card to the next pair and receive a card from the previous pair. They should check their answers and begin the next exercise. After all the rounds are completed, have students complete the graphic organizer on their student handouts to model each operation.

**Exercises 1–15: Polynomial Pass**

Perform the indicated operation to write each polynomial in standard form.

- |  |                                       |
|--|---------------------------------------|
| 1. $(x^2 - 3)(x^2 + 3x - 1)$                         | $x^4 + 3x^3 - 4x^2 - 9x + 3$          |
| 2. $(5x^2 - 3x - 7) - (x^2 + 2x - 5)$                | $4x^2 - 5x - 2$                       |
| 3. $(x^3 - 8) \div (x - 2)$                          | $x^2 + 2x + 4$                        |
| 4. $(x + 1)(x - 2)(x + 3)$                           | $x^3 + 2x^2 - 5x - 6$                 |
| 5. $(x + 1) - (x - 2) - (x + 3)$                     | $-x$                                  |
| 6. $(x + 2)(2x^2 - 5x + 7)$                          | $2x^3 - x^2 - 3x + 14$                |
| 7. $\frac{x^3 - 2x^2 - 65x + 18}{x - 9}$             | $x^2 + 7x - 2$                        |
| 8. $(x^2 - 3x + 2) - (2 - x + 2x^2)$                 | $-x^2 - 2x$                           |
| 9. $(x^2 - 3x + 2)(2 - x + 2x^2)$                    | $2x^4 - 7x^3 + 9x^2 - 8x + 4$         |
| 10. $\frac{x^3 - x^2 - 5x - 3}{x - 3}$               | $x^2 + 2x + 1$                        |
| 11. $(x^2 + 7x - 12)(x^2 - 9x + 1)$                  | $x^4 - 2x^3 - 74x^2 + 115x - 12$      |
| 12. $(2x^3 - 6x^2 - 7x - 2) + (x^3 + x^2 + 6x - 12)$ | $3x^3 - 5x^2 - x - 14$                |
| 13. $(x^3 - 8)(x^2 - 4x + 4)$                        | $x^5 - 4x^4 + 4x^3 - 8x^2 + 32x - 32$ |
| 14. $(x^3 - 2x^2 - 5x + 6) \div (x + 2)$             | $x^2 - 4x + 3$                        |
| 15. $(x^3 + 2x^2 - 3x - 1) + (4 - x - x^3)$          | $2x^2 - 4x + 3$                       |

**Exercise 16 (5 minutes)**

Use this exercise to help students see the structure of the expressions and to create a graphic organizer of how to perform the four polynomial operations for future reference. They should complete this exercise either with a partner or in groups of four.

Exercises 16–22	
16. Review Exercises 1–15 and then select one exercise for each category and record the steps in the operation below as an example. Be sure to show all your work.	
Addition Exercise	Multiplication Exercise
Subtraction Exercise	Division Exercise

**Exercises 17–20 (5 minutes)**

Before starting the next exercises, lead a short discussion transitioning into the problems below that combine polynomial operations. Say (or paraphrase) the following:

- In the previous exercises you applied one operation to two or three polynomials, but many times we work with expressions that contain more than one operation. In the next exercises, more than one operation is indicated. How do you determine which operation to perform first?
  - *The parentheses and the use of the fraction bar for division tell us which operations to perform first. The order of operations also can be applied to polynomials. For example, you have to multiply before you can add or subtract.*

After this discussion, have students work with the same partner (or group) that they worked with on Exercise 16. Have different groups present their work to the class. Students could use whiteboards or chart paper to present their solutions to the class.

<p>For Exercises 17–20, re-write each polynomial in standard form by applying the operations in the appropriate order.</p> <p>17. <math>\frac{(x^2+5x+20)+(x^2+6x-6)}{x+2}</math></p> <p><math>2x + 7</math></p> <p>18. <math>(x^2 - 4)(x + 3) - (x^2 + 2x - 5)</math></p> <p><math>x^3 + 2x^2 - 6x - 7</math></p>
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$$19. \frac{(x-3)^3}{x^2-6x+9}$$

$$x - 3$$

$$20. (x+7)(2x-3) - (x^3 - 2x^2 + x - 2) \div (x-2)$$

$$x^2 + 11x - 22$$

**Exercise 21 (4 minutes)**

This exercise along with Exercise 22 helps students understand that they can learn quite a bit about the nature of a polynomial expression without performing all the operations required to write it in standard form.

- Sometimes we do not need to perform the entire operation to understand the structure of an expression. Can you think of a situation where we might only need to know the first term or the last term of a polynomial expression?
  - *If we wanted to understand the shape of the graph of a polynomial, like  $p(x) = x^2 + 2x + 3$ , we would only need to know the coefficient and degree of the first term to get a general idea of its behavior.*
  - *The constant term of a polynomial expression indicates the  $y$ -intercept of the corresponding graph.*

21. What would be the first and last terms of the polynomial if it was re-written in standard form? Answer these quickly without performing all of the indicated operations.

a.  $(2x^3 - x^2 - 9x + 7) + (11x^2 - 6x^3 + 2x - 9)$

*First term:  $-4x^3$ , Last term:  $-2$*

b.  $(x-3)(2x+3)(x-1)$

*First term:  $2x^3$ , Last term:  $9$*

c.  $(2x-3)(3x+5) - (x+1)(2x^2-6x+3)$

*First term:  $-2x^3$ , Last term:  $-18$*

d.  $(x+5)(3x-1) - (x-4)^2$

*First term:  $2x^2$ , Last term:  $-21$*

After students complete this exercise, lead a short discussion.

- How did you determine your answer quickly?
  - *I knew that the first term would be the one with the highest degree, so I focused on that operation. I knew that the last term would be the constant term unless it turned out to equal 0.*
- Did any of the solutions surprise you?
  - *In parts (a) and (c), I had to pay attention to the order of the terms and which operation would produce the largest degree term.*

MP.7

Exercise 22 (4 minutes)

22. What would the first and last terms of the polynomial be if it was re-written in standard form?

a.	$(n + 1)(n + 2)(n + 3) \dots (n + 9)(n + 10)$	First term: $n^{10}$ , Last term: $10!$
b.	$(x - 2)^{10}$	First term: $x^{10}$ , Last term: $(-2)^{10}$
c.	$\frac{(x-2)^{10}}{(x-2)}$	First term: $x^9$ , Last term: $(-2)^9$
d.	$\frac{n(n+1)(2n+1)}{6}$	First term: $\frac{1}{3}n^3$ , Last term: $\frac{n}{6}$

**Scaffolding:**  
For advanced learners or early finishers, you can increase the complexity of these exercises by posing the following problem (or one similar to it):

- Generate three different polynomial expressions NOT already expressed in standard form that would have a first term of  $-3x^3$  and a last term of  $\frac{1}{2}$  if the polynomial expression was written in standard form.

Closing (2 minutes)

Consider having students record their answers to these questions in writing or by sharing their thoughts with a partner.

- How is polynomial arithmetic similar to integer arithmetic?
  - The four operations produce a new polynomial. The four operations can be combined by following the order of operations conventions.
- How can you quickly determine the first and last terms of a polynomial without performing all of the operations needed to rewrite it in standard form?
  - Analyze the problem to identify the highest degree terms and perform the indicated operation on only those terms. Do the same with the lowest degree terms to determine the last term.

Precise definitions of terms related to polynomials are presented here. These definitions were first introduced in Algebra I, Modules 1 and 4. These definitions are for teacher reference and can be shared with students at your discretion. Following these definitions are discussion questions for closing this lesson.

Relevant Vocabulary

**Polynomial Expression:** A polynomial expression is either a numerical expression or a variable symbol or the result of placing two previously generated polynomial expressions into the blanks of the addition operator ( $\_ + \_$ ) or the multiplication operator ( $\_ \times \_$ ).

The definition of polynomial expression includes subtraction ( $a - b = a + (-1 \cdot b)$ ), exponentiation by a non-negative integer ( $x^3 = (x \cdot x) \cdot x$ ), and dividing by a non-zero number (multiplying by the multiplicative inverse). Because subtraction, exponentiation, and division still apply, we will continue to use the regular notation for these operations. In other words, we will continue to write polynomials simply as  $\frac{(x^3-3x)}{2}$  instead of

$$\frac{1}{2} \cdot \left( ((x \cdot x) \cdot x) + (-1 \cdot (3 \cdot x)) \right)$$

Note, however, that the definition excludes dividing by  $x$  or dividing by any polynomial in  $x$ .

All polynomial expressions are algebraic expressions.

**Monomial:** A *monomial* is a polynomial expression generated using only the multiplication operator ( $\_ \times \_$ ).

Monomials are products whose factors are numerical expressions or variable symbols.

**Coefficient of a Monomial:** The *coefficient of a monomial* is the value of the numerical expression found by substituting the number 1 into all the variable symbols in the monomial.

Sometimes the coefficient is considered as a constant (like the constant  $a$  in  $ax^2$ ) instead of an actual number. In those cases, when a monomial is expressed as a product of a constant and variables, then the constant is called a *constant coefficient*.

**Degree of a Monomial:** The *degree of a non-zero monomial* is the sum of the exponents of the variable symbols that appear in the monomial.

For example, the degree of  $7x^2y^4$  is 6, the degree of  $8x$  is 1, the degree of 8 is 0, and the degree of  $9x^2y^3x^{10}$  is 15.

**Terms of a Polynomial:** When a polynomial is expressed as a monomial or a sum of monomials, each monomial in the sum is called a *term of the polynomial*.

A monomial can now be described as a “polynomial with only one term.” But please realize that the word “term” means many, seemingly disparate things in mathematics (e.g.,  $\sin x$  is often called a term of the expression  $\sin x + \cos x$ ), whereas monomial is a specific object. This is why we have chosen to define monomial first and then use monomial to define a *term of a polynomial*.

**Standard Form of a Polynomial in One Variable:** A polynomial expression with one variable symbol  $x$  is in *standard form* if it is expressed as,

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where  $n$  is a non-negative integer, and  $a_0, a_1, a_2, \dots, a_n$  are constant coefficients with  $a_n \neq 0$ .

A polynomial expression in  $x$  that is in standard form is often called a *polynomial in  $x$* .

The *degree of the polynomial in standard form* is the highest degree of the terms in the polynomial, namely  $n$ . The term  $a_n x^n$  is called the *leading term* and  $a_n$  is called the *leading coefficient*. The *constant term* is the value of the numerical expression found by substituting 0 into all the variable symbols of the polynomial, namely  $a_0$ .

In general, one has to be careful about determining the degree of polynomials that are not in standard form: For example, the degree of the polynomial,

$$(x + 1)^2 - (x - 1)^2,$$

is not 2 since its standard form is  $4x$ .

**Degree of a Polynomial in One Variable:** The *degree of a polynomial expression in one variable* is the degree of the polynomial in standard form that is equivalent to it.

### Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 5: Putting It All Together

### Exit Ticket

Jenny thinks that the expression below is equal to  $x^2 - 4$ . If you agree, show that she is correct. If you disagree, show that she is wrong by re-writing this expression as a polynomial in standard form.

$$\frac{(x - 2)^3}{x - 2}$$

## Exit Ticket Sample Solutions

Jenny thinks that the expression below is equal to  $x^2 - 4$ . If you agree, show that she is correct. If you disagree, show that she is wrong by re-writing this expression as a polynomial in standard form.

$$\frac{(x-2)^3}{x-2}$$

Multiple approaches are possible to justify why Jenny is incorrect. One possible solution is shown below.

Jenny is incorrect. To perform this operation, you can first divide by  $x - 2$  then expand the quotient.

$$\begin{aligned}\frac{(x-2)^3}{x-2} &= (x-2)^2 \\ &= x^2 - 4x + 4\end{aligned}$$

MP.3

## Problem Set Sample Solutions

For Problems 1–7, rewrite each expression as a polynomial in standard form.

1.  $(3x - 4)^3$

$$27x^3 - 108x^2 + 144x - 64$$

2.  $(2x^2 - x^3 - 9x + 1) - (x^3 + 7x - 3x^2 + 1)$

$$-2x^3 + 5x^2 - 16x$$

3.  $(x^2 - 5x + 2)(x - 3)$

$$x^3 - 8x^2 + 17x - 6$$

4.  $\frac{x^4 - x^3 - 6x^2 - 9x + 27}{x - 3}$

$$x^3 + 2x^2 - 9$$

5.  $(x + 3)(x - 3) - (x + 4)(x - 4)$

$$7$$

6.  $(x + 3)^2 - (x + 4)^2$

$$-2x - 7$$

7.  $\frac{x^2 - 5x + 6}{x - 3} + \frac{x^3 - 1}{x - 1}$

$$x^2 + 2x - 1$$

For Problems 8–9: Quick, what would be the first and last terms of the polynomial if it was written in standard form?

8.  $2(x^2 - 5x + 4) - (x + 3)(x + 2)$

*The first and last term are  $x^2$  and 2.*

9.  $\frac{(x-2)^5}{x-2}$

*The first and last terms are  $x^4$  and 16.*

10. The profit a business earns by selling  $x$  items is given by the polynomial function

$$p(x) = x(160 - x) - (100x + 500).$$

What is the last term in the standard form of this polynomial? What does it mean in this situation?

*The last term is  $-500$ , so that  $p(0) = -500$ . This means that if no items are sold the company would lose \$500.*

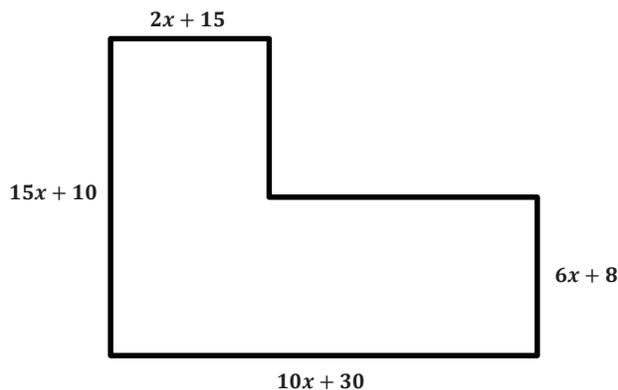
11. Explain why these two quotients are different. Compute each one. What do they have in common? Why?

$$\frac{(x-2)^4}{x-2} \text{ and } \frac{x^4 - 16}{x-2}$$

*The quotients are  $x^3 - 6x^2 + 12x - 8$  and  $x^3 + 2x^2 + 4x + 8$ .*

*They are different because the dividends are not equivalent expressions. The quotients have the first and last terms in common because division is going to reduce the degree by the difference of the degrees of the numerator and denominator, and their leading coefficients were both one. When multiplying, the last term of a polynomial in standard form is the product of the lowest degree terms in each factor. Therefore, when dividing, the last term of the quotient will be the quotient of the last term of the dividend and divisor.*

12. What are the area and perimeter of the figure? Assume a right angle at each vertex.



*The missing horizontal side length is  $8x + 15$ . The missing vertical side length is  $9x + 2$ . I determined these lengths by subtracting the vertical lengths and by subtracting the horizontal lengths. The perimeter is  $50x + 80$ . I got this by adding the lengths of all of the sides together. The area can be found by splitting the shape either horizontally or vertically into two rectangles. If split vertically, the areas of the rectangles are  $(15x + 10)(2x + 15)$  and  $(6x + 8)(8x + 15)$ . The total area of the figure is the sum of these two products,  $78x^2 + 399x + 270$ .*



## Lesson 6: Dividing by $x - a$ and by $x + a$

### Student Outcomes

- Students work with polynomials with constant coefficients to prove polynomial identities.

### Lesson Notes

Students extend their understanding of polynomial division to abstract situations that involve division by  $x - a$  and by  $x + a$ . Through this work they derive the fundamental identities for the difference of two squares and the sum and difference of two cubes. Further, they connect this work back to divisibility of integers. This lesson bridges the first five lessons of Topic A of this module with the upcoming lessons in Topic B in which students extend work with division and polynomial identities to factoring. In those lessons, they will learn the usefulness of the factored form of a polynomial expression to solve polynomial equations and analyze the zeros of the graph of a polynomial function. This lesson addresses aspects of several standards (most notably **A.SSE.A.2** and **A.APR.C.4**) in a way that also emphasizes MP.7 and MP.8. Students recognize and then generalize patterns, and use them to fluently rewrite polynomial expressions and perform polynomial operations.

### Classwork

#### Opening (1 minute)

Students may choose to solve the problems using either the reverse tabular method or long division. Encourage and model both approaches throughout this lesson. Have students work with a partner on these problems and then randomly select pairs to present each of the problems on the board. You can begin by paraphrasing the following statement as students start the Opening Exercise:

- Today we want to observe patterns when we divide certain types of polynomials and make some generalizations to help us quickly compute quotients without having to do the work involved with the reverse tabular method or the long division algorithm.

#### Opening Exercise (4 minutes)

##### Opening Exercise

Find the following quotients, and write the quotient in standard form.

a.  $\frac{x^2 - 9}{x - 3}$   
 $x + 3$

#### Scaffolding:

Provide support for struggling learners by asking them to recall the two methods for dividing polynomials: the reverse tabular method and the long division algorithm. Provide a table with the dividend and divisor in the proper location or a long division problem with the missing zero coefficient terms already in place.

$$x - 3 \overline{) x^2 + 0x - 9}$$

$$x - 3 \overline{) x^3 + 0x^2 + 0x - 27}$$

Or using the tabular method as shown.

	$x$	$+3$	
$x^2$	$x^2$	$3x$	$x$
	$-3x$	$-9$	$-3$
$0x$	$-9$		

Ask advanced learners to generate a similar sequence of problems that have a quotient equal to  $x - 4$  and to then explain how they determined their expressions.

$$\text{b. } \frac{x^3 - 27}{x - 3}$$

$$x^2 + 3x + 9$$

$$\text{c. } \frac{x^4 - 81}{x - 3}$$

$$x^3 + 3x^2 + 9x + 27$$

**Discussion (5 minutes)**

Have students come to the board to present their solutions. After students check and correct their work, discuss the patterns that they notice in these problems.

- What patterns do you notice in the Opening Exercise?
  - The expression  $x - 3$  divides without a remainder into all three dividends, which means it is a factor of each dividend.
  - The dividends are differences of powers of  $x$  and powers of 3. For example,  $x^3 - 27 = x^3 - 3^3$ .
  - The degree of the quotient is 1 less than the degree of the dividend. The terms of the quotient are products of powers of  $x$  and powers of 3. The exponents on  $x$  decrease by one and the exponents on 3 increase by one. Each term is positive.
- Use the patterns you observed in the Opening Exercise to determine the quotient of  $\frac{x^5 - 243}{x - 3}$ . Explain your reasoning.
  - Since  $243 = 3^5$ , we should be able to apply the same pattern, and the quotient should be  $x^4 + 3x^3 + 9x^2 + 27x + 81$ .
- Test your conjecture by using long division or the reverse tabular method to compute the quotient.
  - The result is the same.

**Exercise 1 (5 minutes)**

Have students work in groups of two or three to complete these problems. Have them make and test conjectures about the quotient that results in each problem. Have the groups divide up the work so at least two students are working on each problem. Then have them share their results in their small groups.

**Exercise 1**

1. Use patterns to predict each quotient. Explain how you arrived at your prediction, and then test it by applying the reverse tabular method or long division.

$$\text{a. } \frac{x^2 - 144}{x - 12}$$

$x + 12$ . I arrived at this conclusion by noting that  $144 = 12^2$ , so I could apply the patterns in the previous problems to obtain the result.

b.  $\frac{x^3-8}{x-2}$   
 $x^2 + 2x + 4$ . *The dividend is the difference of two perfect cubes,  $x$  and  $2^3 = 8$ . Based on the patterns in the Opening Exercise, the quotient will be a quadratic polynomial with coefficients that are ascending powers of 2 starting with  $2^0$ .*

c.  $\frac{x^3-125}{x-5}$   
 $x^2 + 5x + 25$

d.  $\frac{x^6-1}{x-1}$   
 $x^5 + x^4 + x^3 + x^2 + 1$

Once this exercise concludes and students have presented their work to the class, they should be ready to generalize a pattern for the quotient of  $\frac{x^n-a^n}{x-a}$ . The next example establishes this identity for the case where  $n = 2$ .

**Example 1 (4 minutes)**

We can use the reverse tabular method to compute quotients like the ones in the Opening Exercise and Exercise 1 for any constant  $a$ . In this way, we can verify that the patterns you noticed work for any value of  $a$ . In this example and Exercises 1 and 2, students work with specific values for the exponents. An interesting extension for advanced students would be to show that  $\frac{x^n-a^n}{x-a} = x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1}x^0$  using the reverse tabular method.

**Example 1**

What is the quotient of  $\frac{x^2-a^2}{x-a}$ ? Use the reverse tabular method or long division.

	$x$	$+a$	
	$x^2$	$ax$	$x$
$x^2$	$-ax$	$-a^2$	$-a$
$0x$	$-a^2$		

*The result of dividing  $x^2 - a^2$  by  $x - a$  is  $x + a$ .*

*Scaffolding:*

Provide additional support here by considering specific values of  $a$  for each part. For example, ask students to work with

$$\frac{x^3 - 8}{x - 2}$$

$$\frac{x^3 - 27}{x - 3}$$

$$\frac{x^3 - 64}{x - 4}$$

and then ask them to solve the problem using the tabular method for the variables  $x$  and  $a$ .

## Exercise 2 (7 minutes)

## Exercises 2–4

2. Work with your group to find the following quotients.

a.  $\frac{x^3 - a^3}{x - a}$   
 $x^2 + ax + a^2$

b.  $\frac{x^4 - a^4}{x - a}$   
 $x^3 + ax^2 + a^2x + a^3$

Before moving on, discuss these results as a whole class. You may need to model a solution to the third question below if students are still struggling with connecting division back to multiplication.

MP.7

- What patterns do you notice in the quotient?
  - *The terms are always added and each term is a product of a power of  $x$  and a power of  $a$ . As the powers of  $x$  decrease by 1 for each consecutive term, the powers of  $a$  increase by 1.*
- How do these patterns compare to the ones you observed in the opening exercises?
  - *They support the patterns we discovered earlier. This work shows that we can quickly compute the quotient for any problem that fits the pattern.*
- How can you rewrite these division problems as multiplication problems?
  - *The dividend is equal to the product of the quotient and the divisor. For example,  $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$ . The other problems would be  $x^2 - a^2 = (x - a)(x + a)$  and  $x^4 - a^4 = (x - a)(x^3 + ax^2 + a^2x + a^3)$ .*

## Exercise 3 (6 minutes)

The focus shifts to division by  $x + a$ . The expression  $x + a$  divides into the difference of two squares  $x^2 - a^2$  without a remainder. Some students may be surprised that it does not divide without a remainder into the difference of two cubes. However,  $x + a$  does divide without a remainder into the sum of two cubes but not into the sum of two squares. Prior to this point, students have not worked with polynomial division problems that result in a remainder. As you discuss these results as a class, you can draw a parallel to division of integers. Polynomial division with remainders will be addressed in later lessons in this same module. At this point, lead students to conclude that some of these quotients produce identities that may be helpful for quickly dividing polynomials and some do not.

Ask students to think about how these problems would be different if we were dividing by  $x + a$ . Have students discuss their ideas with a partner before starting this exercise. Students will most likely assume that there are similar patterns for dividing the sums of squares and cubes, but they may be surprised by their results to parts (b) and (c). For groups that finish early, have them guess and check the results of dividing  $x^4 + a^4$  and  $x^4 - a^4$  by  $x + a$ . You can provide additional concrete examples with numerical values of  $a$  such as  $a = 2, 3, 4, \dots$  if needed to reinforce this concept.

The focus of this part of the lesson is to derive the three identities provided in the Lesson Summary and for students to realize that  $x + a$  and  $x - a$  do not divide into the sum of squares  $x^2 + a^2$  without a remainder.

3. Predict without performing division whether or not the divisor will divide into the dividend without a remainder for the following problems. If so, find the quotient. Then check your answer.

a.  $\frac{x^2 - a^2}{x + a}$

*The quotient is  $x - a$ . This makes sense because we already showed that the result when dividing by  $x - a$  is  $x + a$ .*

b.  $\frac{x^3 - a^3}{x + a}$

*This problem does not divide without a remainder; therefore,  $x + a$  is not a factor of  $x^3 - a^3$ .*

c.  $\frac{x^2 + a^2}{x + a}$

*This problem does not divide without a remainder; therefore,  $x + a$  is not a factor of  $x^2 + a^2$ .*

d.  $\frac{x^3 + a^3}{x + a}$

*The quotient is  $x^2 - ax + a^2$ . This result is similar to our work in Exercise 2 except the middle term is  $-ax$  instead of  $ax$ .*

**Exercise 4 (5 minutes)**

Students consider the special case when  $a = 1$  for different values of  $n$ . They should be able to quickly generalize a pattern. In part (b), you can introduce the use of the ellipsis (...) to indicate the missing powers of  $x$  when displaying the general result since you cannot write all of the terms. In this exercise, we ask students to look for patterns in the quotient  $\frac{x^n - 1}{x - 1}$  for integer exponents  $n$  greater than 1.

4. Find the quotient  $\frac{x^n - 1}{x - 1}$  for  $n = 2, 3, 4$  and  $8$ .

*For  $n = 2$ , the quotient is  $x + 1$ .*

*For  $n = 3$ , the quotient is  $x^2 + x + 1$ .*

*For  $n = 4$ , the quotient is  $x^3 + x^2 + x + 1$ .*

*For  $n = 8$ , the quotient is  $x^7 + x^6 + x^5 + \dots + x + 1$ .*

a. What patterns do you notice?

*The degree of the quotient is 1 less than the degree of the dividend. The degree of each term is 1 less than the degree of the previous term. The last term is 1. The number of terms will be equal to the degree of the dividend.*

b. Use your work in this problem to write an expression equivalent to  $\frac{x^n - 1}{x - 1}$  for any integer  $n > 1$ .

$x^{n-1} + x^{n-2} + x^{n-3} + \dots + x^1 + 1$

MP.8

**Closing (5 minutes)**

The summary details the identities derived in this lesson. Ask students to summarize the important results of this lesson either in writing, to a partner, or as a class. Take the opportunity to informally assess their understanding of this lesson before moving on to the Exit Ticket. Note that we have not formally derived the last identity, merely used inductive reasoning to generalize the pattern based on the work done in Exercise 4.

**Lesson Summary**

Based on the work in this lesson, we can conclude the following statements are true for all real values of  $x$  and  $a$ :

$$\begin{aligned}x^2 - a^2 &= (x - a)(x + a) \\x^3 - a^3 &= (x - a)(x^2 + ax + a^2) \\x^3 + a^3 &= (x + a)(x^2 - ax + a^2),\end{aligned}$$

and it seems that the following statement is also an identity for all real values of  $x$  and  $a$ :

$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + x^{n-3} + \dots + x^1 + 1), \text{ for integers } n > 1.$$

**Exit Ticket (3 minutes)**

In this Exit Ticket, students actually apply the identities they worked with to determine quotients. This Exit Ticket will allow you to test their fluency in working with these new relationships. The Lesson Summary is reinforced by having them rewrite each quotient as a product.

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 6: Dividing by $x - a$ and by $x + a$

### Exit Ticket

Compute each quotient using the identities you discovered in this lesson. Then, write the problem as a product. An example is shown.

QUOTIENT

$$\frac{x^2-9}{x+3} = x - 3$$

PRODUCT

$$x^2 - 9 = (x + 3)(x - 3)$$

1.  $\frac{x^4-16}{x-2}$

2.  $\frac{x^3+1000}{x+10}$

3.  $\frac{x^5-1}{x-1}$

Exit Ticket Sample Solutions

Compute each quotient using the identities you discovered in this lesson. Then, write the problem as a product. An example is shown.

	QUOTIENT	PRODUCT
	$\frac{x^2-9}{x+3} = x - 3$	$x^2 - 9 = (x + 3)(x - 3)$
1.	$\frac{x^4-16}{x-2}$	
	<b>QUOTIENT:</b> $\frac{x^4-16}{x-2} = x^3 + 2x^2 + 4x + 8$	<b>PRODUCT:</b> $x^4 - 16 = (x - 2)(x^3 + 2x^2 + 4x + 8)$
2.	$\frac{x^3+1000}{x+10}$	
	<b>QUOTIENT:</b> $\frac{x^3+1000}{x+10} = x^2 - 10x + 100$	<b>PRODUCT:</b> $x^3 + 1000 = (x + 10)(x^2 - 10x + 100)$
3.	$\frac{x^5-1}{x-1}$	
	<b>QUOTIENT:</b> $\frac{x^5-1}{x-1} = x^4 + x^3 + x^2 + x + 1$	<b>PRODUCT:</b> $x^5 - 1 = (x - 1)(x^4 + x^3 + x^2 + x + 1)$

Problem Set Sample Solutions

1. Compute each quotient.

a.  $\frac{x^2-625}{x-25}$   
 $x + 25$

b.  $\frac{x^3+1}{x+1}$   
 $x^2 - x + 1$

c.  $\frac{x^3-\frac{1}{8}}{x-\frac{1}{2}}$   
 $x^2 + \frac{1}{2}x + \frac{1}{4}$

d.  $\frac{x^2-0.01}{x-0.1}$   
 $x + 0.1$

2. In the next exercises, you can use the same identities you applied in the previous problem. Fill in the blanks in the problems below to help you get started. Check your work by using the reverse tabular method or long division to make sure you are applying the identities correctly.

a.  $\frac{16x^2-121}{4x-11} = \frac{(\quad)^2-(\quad)^2}{4x-11} = (\quad) + 11$

$4x, 11, 4x$

b.  $\frac{25x^2-49}{5x+7} = \frac{(\quad)^2-(\quad)^2}{5x+7} = (\quad) - (\quad) = \underline{\hspace{2cm}}$

$5x, 7, 5x, 7, 5x - 7$

c.  $\frac{8x^3-27}{2x-3} = \frac{(\quad)^3-(\quad)^3}{2x-3} = (\quad)^2 + (\quad)(\quad) + (\quad)^2 = \underline{\hspace{2cm}}$

$2x, 3, 2x, 2x, 3, 3, 4x^2 + 6x + 9$

3. Show how the patterns and relationships learned in this lesson could be applied to solve the following arithmetic problems by filling in the blanks.

a.  $\frac{625-81}{16} = \frac{(\quad)^2-(9)^2}{25-(\quad)} = (\quad) + (\quad) = 34$

$\frac{25^2 - 9^2}{25 - 9} = 25 + 9 = 34$

b.  $\frac{1000-27}{7} = \frac{(\quad)^3-(\quad)^3}{(\quad)-3} = (\quad)^2 + (10)(\quad) + (\quad)^2 = \underline{\hspace{2cm}}$

$\frac{10^3 - 3^3}{10 - 3} = 10^2 + 10(3) + 3^2 = 139$

c.  $\frac{100-9}{7} = \frac{(\quad)^2-(\quad)^2}{(\quad)-3} = \underline{\hspace{2cm}}$

$\frac{10^2 - 3^2}{10 - 3} = 10 + 3 = 13$

d.  $\frac{1000+64}{14} = \frac{(\quad)^3+(\quad)^3}{(\quad)+(\quad)} = (\quad)^2 - (\quad)(\quad) + (\quad)^2 = \underline{\hspace{2cm}}$

$\frac{10^3 + 4^3}{10 + 4} = 10^2 - 10(4) + 4^2 = 76$

4. Apply the identities from this lesson to compute each quotient. Check your work using the reverse tabular method or long division.

a.  $\frac{16x^2-9}{4x+3}$

$4x - 3$

b.  $\frac{81x^2-25}{18x-10}$

$\frac{9}{2}x + \frac{5}{2}$

c.  $\frac{27x^3-8}{3x-2}$

$$9x^2 + 6x + 4$$

5. Extend the patterns and relationships you learned in this lesson to compute the following quotients. Explain your reasoning, and then check your answer by using long division or the tabular method.

a.  $\frac{8+x^3}{2+x}$

*The quotient is  $4 + 2x + x^2$ . This problem just has the variable and constant terms re-written using the commutative property, so it is the same as computing  $(x^3 + 8) \div (x + 2)$ .*

b.  $\frac{x^4-y^4}{x-y}$

*The quotient is  $x^3 + x^2y + xy^2 + y^3$ . This problem is similar to Opening Exercise part (c), except that instead of 81 and 3 in the dividend and quotient, we have a power of  $y$ . You can also extend the patterns for*

*$\frac{x^3-a^3}{x-a} = x^2 + ax^2 + a^2$  using the variable  $y$  instead of the variable  $a$ .*

c.  $\frac{27x^3+8y^3}{3x+2y}$

*The quotient is  $9x^2 - 6xy + 4y^2$ . In this example,  $3x$  is in the  $x$  position and  $2y$  is in the  $a$  position. Then, the divisor fits the pattern of  $x^3 + a^3$ .*

d.  $\frac{x^7-y^7}{x-y}$

*The quotient is  $x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6$ . In this problem, replace 1 with  $y$  and extend the powers of  $y$  pattern using the identities in the Lesson Summary.*



## Lesson 7: Mental Math

### Student Outcomes

- Students perform arithmetic by using polynomial identities to describe numerical relationships.

### Lesson Notes

Students continue exploring the usefulness of polynomial identities to perform arithmetic calculations. This work reinforces the essential understanding of A.APR and A.SEE standards. The lesson concludes by discussing prime and composite numbers and using polynomial identities to check whether a number is prime or composite. This lesson ties into the work in the next lesson which further investigates prime numbers.

The tree diagram analysis, touched upon later in the lesson, offers a connection to some of the probability work that will occur later in this course.

### Classwork

#### Opening (1 minute)

Students will perform arithmetic that they might not have thought possible without the assistance of a calculator or computer. To motivate this lesson, mention a multiplication problem of the form  $(a - b)(a + b)$  that is difficult to calculate without pencil and paper, such as  $87 \cdot 93$ . Perhaps even time students to see how long it takes them to do this calculation without a calculator.

- Today we use the polynomial identities derived in Lesson 6 to perform a variety of calculations quickly using mental arithmetic.

#### Opening Exercise (3 minutes)

Have students complete the following exercises. Ask students to discuss their ideas with a partner, and then have them summarize their thoughts on the lesson handout. These two exercises build upon the concept of division of polynomials developed in previous lessons by addressing both multiplication *and* division.

##### Opening Exercise

- a. How are these two equations related?

$$\frac{x^2-1}{x+1} = x-1 \text{ and } x^2-1 = (x+1)(x-1)$$

*They represent the same relationship between the expressions  $x^2 - 1$ ,  $x - 1$ , and  $x + 1$ . One shows the relationship as division and the other as multiplication.*

- b. Explain the relationship between the polynomial identities  $x^2 - 1 = (x + 1)(x - 1)$  and  $x^2 - a^2 = (x - a)(x + a)$ .

*The expression  $x^2 - 1$  is of the form  $x^2 - a^2$ , with  $a = 1$ . Note that this works with  $a = -1$  as well.*

**Discussion (8 minutes)**

Call on a student to share his or her solutions to the Opening Exercise. Then invite other students to add their thoughts to the discussion. This discussion should show students how to apply the difference of two squares identity to quickly find the product of two numbers. Use the questions below to prompt a discussion.

- Consider  $(x - 1)(x + 1) = x^2 - 1$ . If  $x = 100$ , what number sentence is represented by this identity? Which side of the equation is easier to compute?
  - *This is  $99 \times 101 = 100^2 - 1$ .*
  - *Computing  $100^2 - 1$  is far easier than the original multiplication.*
- Now let's consider the more general  $x^2 - a^2 = (x - a)(x + a)$ . Keep  $x = 100$ , and test some small positive integer values for  $a$ . What multiplication problem does each one represent?
- How does the identity  $(x - a)(x + a) = x^2 - a^2$  make these multiplication problems easier?
  - *Let  $x = 100$  and  $a = 5$ .*

$$(x - a)(x + a) = x^2 - a^2$$

$$95 \cdot 105 = (100 - 5)(100 + 5) = 100^2 - 5^2$$

Therefore,  $95 \cdot 105 = 100^2 - 5^2 = 10,000 - 25 = 9,975$ .

Let  $x = 100$  and  $a = 7$ .

$$(x - a)(x + a) = x^2 - a^2$$

$$93 \cdot 107 = (100 - 7)(100 + 7) = 100^2 - 7^2$$

Therefore,  $93 \cdot 107 = 100^2 - 7^2 = 10,000 - 49 = 9,951$ .

- Do you notice any patterns?
  - *The products in these examples are differences of squares.*
  - *The factors in the product are exactly 'a' above and 'a' below 100.*
- How could you use the difference of two squares identity to multiply  $92 \cdot 108$ ? How did you determine the values of  $x$  and  $a$ ?
  - *You could let  $x = 100$  and  $a = 8$ . We must figure out each number's distance from 100 on the number line.*
- How would you use the difference of two squares identity to multiply  $87 \cdot 93$ ? What values should you select for  $x$  and  $a$ ? How did you determine them?
  - *We cannot use 100, but these two numbers are 3 above and 3 below 90. So we can use*

$$(90 - 3)(90 + 3) = 90^2 - 3^2 = 8,100 - 9 = 8,091.$$
  - *In general,  $x$  is the mean of the factors, and  $a$  is half of the absolute value of the difference between the factors.*

Depending on the level of your students, you may wish to wait until after Exercise 1 to make a generalized statement about how to determine the  $x$  and  $a$  values used to solve these problems. They may need to experiment with some additional problems before they are ready to generalize a pattern.

**Exercise 1 (4 minutes)**

Have students work individually and then check their answers with a partner. Make sure they write out their steps as in the sample solutions. After a few minutes, invite students to share one or two solutions on the board.

**Exercises 1–3**

1. Compute the following products using the identity  $x^2 - a^2 = (x - a)(x + a)$ . Show your steps.

a.  $6 \cdot 8$

$$\begin{aligned} 6 \cdot 8 &= (7 - 1)(7 + 1) \\ &= 7^2 - 1^2 \\ &= 49 - 1 \\ &= 48 \end{aligned}$$

b.  $11 \cdot 19$

$$\begin{aligned} 11 \cdot 19 &= (15 - 4)(15 + 4) \\ &= 15^2 - 4^2 \\ &= 225 - 16 \\ &= 209 \end{aligned}$$

c.  $23 \cdot 17$

$$\begin{aligned} 23 \cdot 17 &= (20 + 3)(20 - 3) \\ &= 20^2 - 3^2 \\ &= 400 - 9 \\ &= 391 \end{aligned}$$

d.  $34 \cdot 26$

$$\begin{aligned} 34 \cdot 26 &= (30 + 4)(30 - 4) \\ &= 30^2 - 4^2 \\ &= 900 - 16 \\ &= 884 \end{aligned}$$

**Discussion (5 minutes)**

At this point make sure your students have a clear way to determine how to write a product as the difference of two squares. Then put these problems on the board.

$56 \cdot 63$

$24 \cdot 76$

$998 \cdot 1,002$

Give them a few minutes to struggle with these problems. While it is possible to use the identity to rewrite each expression, the first two problems do not make for an easy calculation when written as the difference of two squares. The third problem is easy even though the numbers are large.

- Which product is easier to compute using mental math? Explain your reasoning.
  - *The last one is the easiest. In the first one, the numbers have a mean of 59.5 which is not easy to square mentally. The second example would be  $50^2 - 26^2$ , which is not so each to calculate mentally.*
- Can the product of any two positive integers be written as the difference of two squares?
  - *Yes, but not all of them will be rewritten in a form that makes computation easy.*
- If you wanted to impress your friends with your mental math abilities and they gave you these three problems to choose from, which one would you pick and why?

$509 \cdot 493$

$511 \cdot 489$

$545 \cdot 455$

- *This middle one is the easiest since the numbers are 11 above and below the number 500.*

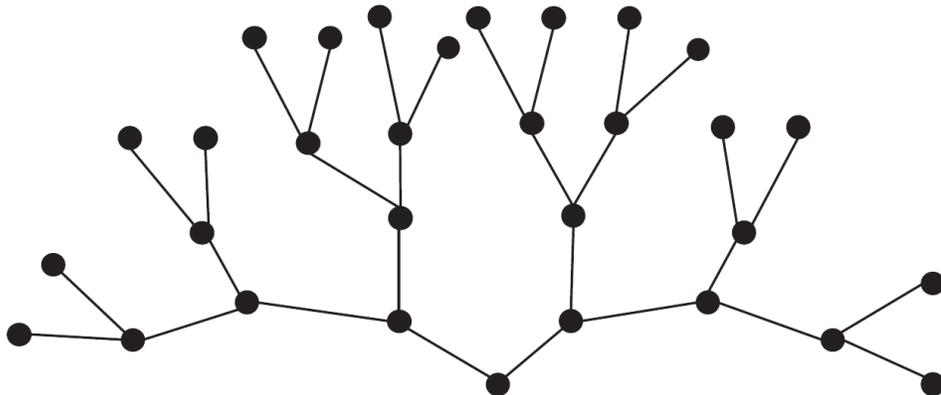
MP.3

**Discussion (10 minutes)**

At this point, we can introduce the power of algebra over the calculator.

- The identity  $x^2 - a^2$  is just the  $n = 2$  case of the identity  $x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-1})$ .
- How might we use this general identity to quickly count mentally?

To see how, let's doodle. A *tree* is a figure made of points called *vertices* and segments called *branches*. My tree splits into two branches at each vertex.

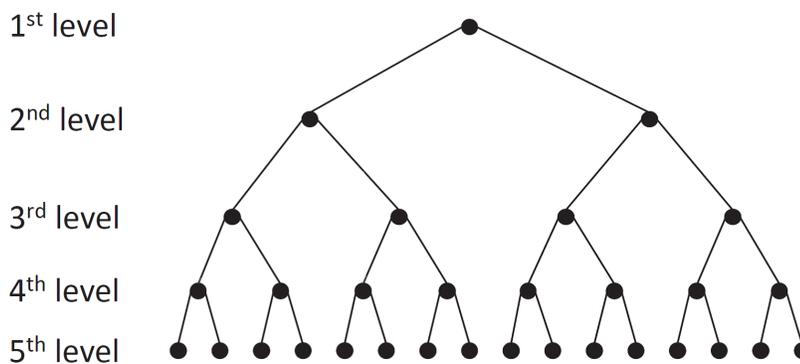


- How many vertices does my tree have?

Allow students to count the vertices for a short while, but don't dwell on the answer.

- It is difficult to count the vertices of this tree, so let's draw it in a more organized way.

Present the following drawing of the tree, with vertices aligned in rows corresponding to their levels.



MP.3

For the following question, give students time to write or share their thinking with a neighbor.

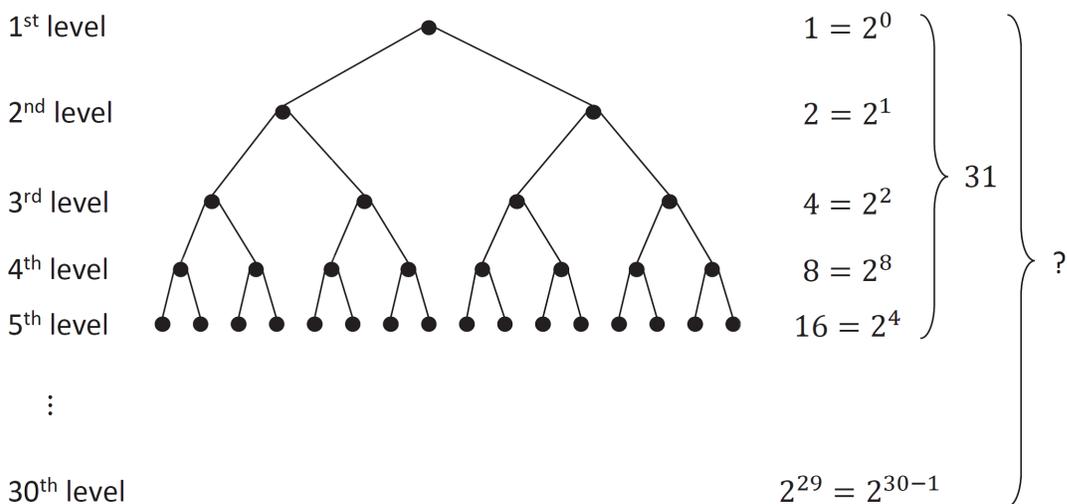
- How many vertices are in each level? Find a formula to describe the number of vertices in level  $n$ .
  - The number of vertices in each level follows this sequence:  $\{1, 2, 4, 8, 16, \dots\}$ , so in level  $n$  there are  $2^{n-1}$  vertices.
- How many vertices are there in all 5 levels? Explain how you know.
  - The number of vertices in our tree, which has five levels, is  $2^4 + 2^3 + 2^2 + 2 + 1$ . First, we recognize that  $2 - 1 = 1$ , so we can rewrite our expression as  $(2 - 1)(2^4 + 2^3 + 2^2 + 2 + 1)$ . If we let  $x = 2$ , this numerical expression becomes a polynomial expression.

$$\begin{aligned}
 2^4 + 2^3 + 2^2 + 2 + 1 &= (2 - 1)(2^4 + 2^3 + 2^2 + 2 + 1) \\
 &= (x - 1)(x^4 + x^3 + x^2 + x + 1) \\
 &= x^5 - 1 \\
 &= 2^5 - 1 \\
 &= 31
 \end{aligned}$$

- How could you find the total number of vertices in a tree like this one with  $n$  levels? Explain.
  - Repeating what we did with  $n = 5$  in the previous step, we have
 
$$\begin{aligned}
 2^{n-1} + 2^{n-2} + \dots + 2 + 1 &= (2 - 1)(2^{n-1} + 2^{n-2} + \dots + 2 + 1) \\
 &= (x - 1)(x^{n-1} + x^{n-2} + \dots + x + 1) \\
 &= x^n - 1 \\
 &= 2^n - 1.
 \end{aligned}$$

Thus, a tree like this one with  $n$  levels has  $2^n - 1$  vertices.

- Now, suppose I drew a tree with 30 levels:



- How many vertices would a tree with 30 levels have?
  - According to the formula we developed in the last step, the number of vertices is
 
$$2^n - 1 = 2^{30} - 1.$$
- Would you prefer to count all 1,073,741,823 vertices?
  - No.

**Discussion (5 minutes)**

This discussion is designed to setup the general identity for  $x^n - a^n$  to identify some composite numbers in the next lesson.

- Recall that a *prime number* is a positive integer greater than 1 whose only positive integer factors are 1 and itself. A *composite number* can be written as the product of positive integers with at least one factor that is not 1 or itself.
- Suppose that  $a, b$ , and  $n$  are positive integers with  $b > a$ . What does the identity  $x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-1})$  suggest about whether or not the number  $b^n - a^n$  is prime?
  - We see that  $b^n - a^n$  is divisible by  $b - a$  and that
 
$$b^n - a^n = (b - a)(b^{n-1} + ab^{n-2} + \dots + a^{n-1}).$$
  - If  $b - a = 1$ , then we do not know if  $b^n - a^n$  is prime because we do not know if  $b^{n-1} + ab^{n-2} + \dots + a^{n-1}$  is prime. For example,  $15^2 - 14^2 = 225 - 196 = 29$  is prime, but  $17^2 - 16^2 = 289 - 256 = 33$  is composite.
  - But, if  $b - a > 1$  then we know that that  $b^n - a^n$  is not prime.
- Use the identity for  $b^n - a^n$  to determine whether or not 143 is prime. Check your work using a calculator.
  - Let  $b = 12, a = 1$ , and  $n = 2$ . Since  $b - a$  is a factor of  $b^n - a^n$ , and  $b - a = 12 - 1 = 11$ , we know that 11 is a factor of 143, which means that 143 is not prime. The calculator shows  $143 = 11 \cdot 13$ .
- We could have used a calculator to determine that  $11 \cdot 13 = 143$ , so that 143 is not prime. Will a calculator help us determine whether  $2^{100} - 1$  is prime? Try it.
  - The calculator will have difficulty calculating a number this large.
- Can we determine whether or not  $2^{100} - 1$  is prime using identities from this lesson?
  - We can try to apply the following identity.
 
$$x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \dots + 1)$$
  - If we let  $n = 100$ , then this identity does not help us because 1 divides both composites and primes.
 
$$2^{100} - 1 = (2 - 1)(2^{99} + 2^{98} + \dots + 2 + 1)$$
- But, what if we look at this problem a bit differently?
 
$$2^{100} - 1 = (2^4)^{25} - 1 = 16^{25} - 1 = (16 - 1)(16^{24} + 16^{23} + \dots + 16 + 1)$$
  - We can see now that  $2^{100} - 1$  is divisible by 15, so  $2^{100} - 1$  is not prime.
- What can we conclude from this discussion?
  - If we can write a positive integer as the difference of squares of non-consecutive integers, then that integer is composite.

*Scaffolding:*

If students are struggling with the words “prime” and “composite,” try doing a quick T-chart activity in which students classify numbers as prime or composite on either side of the T.

**Exercises 2–3 (4 minutes)**

2. Find two additional factors of  $2^{100} - 1$ .

$2^{100} - 1 = (2^5)^{20} - 1$ $= 32^{20} - 1$ $= (32 - 1)(32^{19} + 32^{18} + \dots + 32 + 1)$	$2^{100} - 1 = (2^2)^{50} - 1$ $= 4^{50} - 1$ $= (4 - 1)(4^{49} + 4^{48} + \dots + 4 + 1)$
---	--

*Thus 31 is a factor and so is 3.*

3. Show that  $8^3 - 1$  is divisible by 7.

$$8^3 - 1 = (8 - 1)(8^2 + 8 + 1)$$

### Closing (2 minutes)

Ask students to write a mental math problem that they can now do easily and to explain why the calculation can be done simply.

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this opportunity to informally assess their understanding of the lesson. The following are some important summary elements:

#### Lesson Summary

Based on the work in this lesson, we can convert differences of squares into products (and vice versa) using

$$x^2 - a^2 = (x - a)(x + a).$$

If  $x$ ,  $a$ , and  $n$  are integers and  $n > 1$ , then numbers of the form  $x^n - a^n$  are not prime because

$$x^n - a^n = (x - a)(x^{n-1} + ax^{n-2} + a^2x^{n-3} + \dots + a^{n-2}x + a^{n-1}).$$

### Exit Ticket (3 minutes)



Exit Ticket Sample Solutions

MP.3

1. Explain how you could use the patterns in this lesson to quickly compute  $(57)(43)$ .  
*Subtract 49 from 2,500. That would be 2,451. You can use the identity  $x^2 - a^2 = (x + a)(x - a)$ . In this case,  $x = 50$  and  $a = 7$ .*
2. Jessica believes that  $10^3 - 1$  is divisible by 9. Support or refute her claim using your work in this lesson.  
*Since we recognize that  $9 = 10 - 1$ , then  $\frac{10^3-1}{9}$  fits the pattern of  $\frac{x^3-a^3}{x-a}$  where  $x = 10$  and  $a = 1$ . Therefore,*

$$\frac{10^3-1}{9} = \frac{10^3-1}{10-1} = 10^2 + 10 + 1,$$
*and Jessica is correct.*

Problem Set Sample Solutions

1. Using an appropriate polynomial identity, quickly compute the following products. Show each step. Be sure to state your values for  $x$  and  $a$ .
  - a.  $41 \cdot 19$ 

$$\begin{aligned} a &= 11, x = 30 \\ (x + a)(x - a) &= x^2 - a^2 \\ (30 + 11)(30 - 11) &= 30^2 - 11^2 \\ &= 900 - 121 \\ &= 779 \end{aligned}$$
  - b.  $993 \cdot 1,007$ 

$$\begin{aligned} a &= 7, x = 1,000 \\ (x - a)(x + a) &= x^2 - a^2 \\ (1,000 - 7)(1,000 + 7) &= 1,000^2 - 7^2 \\ &= 1,000,000 - 49 \\ &= 999,951 \end{aligned}$$
  - c.  $213 \cdot 187$ 

$$\begin{aligned} a &= 13, x = 200 \\ (x - a)(x + a) &= x^2 - a^2 \\ (200 - 13)(200 + 13) &= 200^2 - 13^2 \\ &= 40,000 - 169 \\ &= 39,831 \end{aligned}$$
  - d.  $29 \cdot 51$ 

$$\begin{aligned} a &= 11, x = 40 \\ (x - a)(x + a) &= x^2 - a^2 \\ (40 - 11)(40 + 11) &= 40^2 - 11^2 \\ &= 1,600 - 121 \\ &= 1,479 \end{aligned}$$
  - e.  $125 \cdot 75$ 

$$\begin{aligned} a &= 25, x = 100 \\ (x - a)(x + a) &= x^2 - a^2 \\ (100 - 25)(100 + 25) &= 100^2 - 25^2 \\ &= 10,000 - 625 \\ &= 9,375 \end{aligned}$$
2. Give the general steps you take to determine  $x$  and  $a$  when asked to compute a product such as those in Problem 1.  
*The number  $x$  is the mean ("average" is also acceptable) of the two factors, and  $a$  is the positive difference between  $x$  and either factor.*

3. Why is  $17 \cdot 23$  easier to compute than  $17 \cdot 22$ ?

*The mean of 17 and 22 is 19.5, whereas the mean of 17 and 23 is the integer 20. I know that the square of 20 is 400 and the square of 3 is 9. However, I cannot quickly compute the squares of 19.5 and 2.5.*

4. Rewrite the following differences of squares as a product of two integers.

a.  $81 - 1$   $81 - 1 = 9^2 - 1^2$   
 $= (9 - 1)(9 + 1)$   
 $= 8 \cdot 10$

b.  $400 - 121$   $400 - 121 = 20^2 - 11^2$   
 $= (20 - 11)(20 + 11)$   
 $= 9 \cdot 31$

5. Quickly compute the following differences of squares.

a.  $64^2 - 14^2$   $64^2 - 14^2 = (64 - 14)(64 + 14)$   
 $= 50 \cdot 78$   
 $= 3,900$

b.  $112^2 - 88^2$   $112^2 - 88^2 = (112 - 88)(112 + 88)$   
 $= 24 \cdot 200$   
 $= 4,800$

c.  $785^2 - 215^2$   $785^2 - 215^2 = (785 - 215)(785 + 215)$   
 $= 570 \cdot 1,000$   
 $= 570,000$

6. Is 323 prime? Use the fact that  $18^2 = 324$  and an identity to support your answer.

*No, 323 is not prime because it is equal to  $18^2 - 1$ . Therefore,  $323 = (18 - 1)(18 + 1)$ .*

*Note: This problem can also be solved through factoring.*

7. The number  $2^3 - 1$  is prime and so are  $2^5 - 1$  and  $2^7 - 1$ . Does that mean  $2^9 - 1$  is prime? Explain why or why not.

$$2^9 - 1 = (2^3)^3 - 1$$

$$= (2^3 - 1)((2^3)^2 + 2^3(1) + 1^2)$$

*The factors are 7 and 73. As such,  $2^9 - 1$  is not prime.*

8. Show that 9,999,999,991 is not prime without using a calculator or computer.

*Note that  $9,999,999,991 = 10,000,000,000 - 9$ . Since  $10^{10}$  is the square of  $10^5$ ,  $10,000,000,000$  is the square of 100,000. Since 9 is the square of 3;  $9,999,999,991 = 100,000^2 - 3^2$ ; which is divisible by  $100,000 - 3$  and by  $100,000 + 3$ .*

9. Show that 999,973 is not prime without using a calculator or computer.

*Note that  $999,973 = 1,000,000 - 27$ . Since  $27 = 3^3$  and  $1,000,000 = 100^3$  are both perfect cubes, we have  $999,973 = 100^3 - 3^3$ . Therefore, we know that 999,973 is divisible by  $100 - 3 = 97$ .*

10. Find a value of  $b$  so that the expression  $b^n - 1$  is always divisible by 5 for any positive integer  $n$ . Explain why your value of  $b$  works for any positive integer  $n$ .

*There are many correct answers. If  $b = 6$ , then the expression  $6^n - 1$  will always be divisible by 5 because  $5 = 6 - 1$ . This will work for any value of  $b$  that is one more than a multiple of 5, such as  $b = 101$  or  $b = 11$ .*

11. Find a value of  $b$  so that the expression  $b^n - 1$  is always divisible by 7 for any positive integer  $n$ . Explain why your value of  $b$  works for any positive integer  $n$ .

*There are many correct answers. If  $b = 8$ , then the expression  $8^n - 1$  will always be divisible by 7 because  $7 = 8 - 1$ . This will work for any value of  $b$  that is one more than a multiple of 7, such as  $b = 50$  or  $b = 15$ .*

12. Find a value of  $b$  so that the expression  $b^n - 1$  is divisible by both 7 and 9 for any positive integer  $n$ . Explain why your value of  $b$  works for any positive integer  $n$ .

*There are multiple correct answers, but one simple answer is  $b = 64$ . Since  $64 = 8^2$ ,  $64^n - 1 = (8^2)^n - 1$  has a factor of  $8^2 - 1$ , which factors into  $(8 - 1)(8 + 1) = 7 \cdot 9$ .*



## Lesson 8: The Power of Algebra—Finding Primes

### Student Outcomes

- Students apply polynomial identities to the detection of prime numbers.

### Lesson Notes

This lesson applies the identities students have been working with in previous lessons to finding prime numbers, a rich topic with a strong historical background. Many famous mathematicians have puzzled over prime numbers, and their work is the foundation for mathematics used today in the RSA encryption algorithm that provides for secure internet transmissions. This is an engaging topic for students and is readily accessible to them because of its current use in providing safe and secure electronic communications and transactions. Students will be actively engaging several mathematical practice standards during this lesson, including making sense of problems (MP.1), looking for patterns, and seeing the structure in expressions (MP.7 and MP.8), as they investigate patterns with prime numbers. The lesson includes many opportunities to prove conjectures (MP.3) as students gain experience using algebraic properties to prove statements about integers. Several excellent resources are available for students wishing to learn more about prime numbers, their history, and their application to encryption and decryption. A good starting place for additional exploration about prime numbers is the website The Prime Pages, <http://primes.utm.edu/>.

### Classwork

#### Opening (10 minutes)

To motivate students, show the YouTube video on RSA encryption (<http://www.youtube.com/watch?v=M7kEpw1tn50>) to the class. This video will introduce students to encryption and huge numbers. Encryption algorithms are the basis of all secure internet transactions. Today, many encryption algorithms rely on very large prime numbers or very large composite numbers that are the product of two primes to create an encryption key. Often these numbers are Mersenne Primes—primes of the form  $2^p - 1$ , where  $p$  is itself prime. Interestingly, not all numbers in this form are prime. As of December 2013, only 43 Mersenne Primes have been discovered. Encourage students to research the following terms: Mersenne Primes, Data Encryption, and RSA. The Opening Exercise along with the first examples engage students in the exploration of large primes.

Mathematicians have tried for centuries to find a formula that always yields a prime number but have been unsuccessful in their quest. The search for large prime numbers and a formula that will generate all the prime numbers has provided fertile ground for work in number theory. The mathematician Pierre de Fermat (1601–1665, France) applied the difference of two squares identity to factor very large integers as the product of two prime numbers. Up to this point, we have worked with numbers that can be expressed as the difference of two perfect squares. If a prime number could be written as a difference of perfect squares  $a^2 - b^2$ , then it would have to be of the form  $(a + b)(a - b)$ , where  $a$  and  $b$  are consecutive whole numbers and  $a + b$  is prime. The challenge is that not every pair of consecutive whole numbers yields a prime number when added. For example,  $3 + 4 = 7$  is prime, but  $4 + 5 = 9$  is not. This idea is further addressed in the last exercise and in the Problem Set.

**Opening Exercise (10 minutes): When is  $2^n - 1$  prime and when is it composite?**

Before beginning this exercise, have students predict when an expression in this form will be prime and when it will be composite. Questions like this are ideal places to engage students in constructing viable arguments.

MP.3

- When will this expression be prime and when will it be composite?
  - *Student responses will vary. Some may say always prime or always composite. A response that goes back to the identity  $x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \dots + 1)$  from the previous lesson is showing some good initial thinking.*

Students should work the Opening Exercise in small groups. After about seven minutes of group work, have one student from each group come up and fill in the table values and the supporting work. Then, lead a whole group discussion to debrief this problem.

*Scaffolding:*

For more advanced students, consider posing the question: Can you construct an expression what will always yield a prime number? Do this before starting the Opening Exercise and then ask them to test their expressions.

If students are having a hard time constructing an expression, consider asking the following questions: Can you construct an expression that will always yield an even number? Can you construct an expression that will always yield an odd number?

If  $n$  is an integer, then  $n$  is always an even number and  $2n + 1$  is always an odd number.

MP.2  
MP.7  
&  
MP.8

**Opening Exercise: When is  $2^n - 1$  prime and when is it composite?**

Complete the table to investigate which numbers of the form  $2^n - 1$  are prime and which are composite.

Exponent $n$	Expression $2^n - 1$	Value	Prime or Composite? <i>Justify your answer if composite.</i>
1	$2^1 - 1$	1	Prime
2	$2^2 - 1$	3	Prime
3	$2^3 - 1$	7	Prime
4	$2^4 - 1$	15	Composite ( $3 \cdot 5$ )
5	$2^5 - 1$	31	Prime
6	$2^6 - 1$	63	Composite ( $7 \cdot 9$ )
7	$2^7 - 1$	127	Prime
8	$2^8 - 1$	255	Composite (ends in a 5)
9	$2^9 - 1$	511	Composite ( $7 \cdot 73$ )
10	$2^{10} - 1$	1023	Composite ( $(32 - 1)(32 + 1)$ )
11	$2^{11} - 1$	2047	Composite ( $23 \cdot 89$ )

What patterns do you notice in this table about which expressions are prime and which are composite?

*Answers will vary. Suggested responses are in the discussion questions below.*

Encourage students to use tools strategically as they work with these problems. They should have a calculator available to determine if the larger numbers are composite. When debriefing, point out the fact that students can use the difference of two squares identity to factor these expressions when the exponent is an even number.

Use these questions to lead a short discussion on the results of this Opening Exercise.

- What patterns do you notice about which expressions are composite and which are prime?
  - *When the exponent was an even number greater than 2, the result was composite and can be factored using this identity:  $2^{2n} - 1 = (2^n + 1)(2^n - 1)$ .*
  - *When the exponent is a prime number, the result is sometimes prime and sometimes not prime.  $2^{11} - 1$  was the first number with a prime exponent that was composite.*
  - *When the exponent is a composite odd number, the expression appears to be composite but we have yet to prove that.*

The statements above are examples of the types of patterns students should notice as they complete the Opening Exercise. If your class was not able to prove that the case for even exponents resulted in a composite number, encourage them to consider the identities we learned in the last lesson involving the difference of two squares. See if they can solve the problem with that hint. Of course, that technique does not work when the exponent is odd. Make sure students have articulated the answer to the last problem. To transition to the next section, ask students how they might prove that  $2^{ab} - 1$  is composite when the exponent  $ab$  is an odd composite number.

**Example 1 (5 minutes): Proving a Conjecture**

This example and the next exercise prove patterns students noticed in the table in the Opening Exercise. Some scaffolding is provided, but feel free to adjust as needed for your students. Give students who need less support the conjecture on the board for Example 1 (without the additional scaffolding on the student pages); others may need more assistance to get started.

**Example 1: Proving a Conjecture**

**Conjecture:** If  $m$  is a positive odd composite number, then  $2^m - 1$  is a composite number.

**Start with an identity:**  $x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \dots + x^1 + 1)$

In this case,  $x = 2$ , so the identity above becomes:

$$2^m - 1 = (2 - 1)(2^{m-1} + 2^{m-2} + \dots + 2^1 + 1)$$

$$= (2^{m-1} + 2^{m-2} + \dots + 2^1 + 1),$$

and it is not clear whether or not  $2^m - 1$  is composite.

**Rewrite the expression:** Let  $m = ab$  be a positive odd composite number. Then  $a$  and  $b$  must also be odd, or else the product  $ab$  would be even. The smallest such number  $m$  is 9, so we have  $a \geq 3$  and  $b \geq 3$ .

Then we have

$$2^m - 1 = (2^a)^b - 1$$

$$= (2^a - 1) \underbrace{((2^a)^{b-1} + (2^a)^{b-2} + \dots + (2^a)^1 + 1)}_{\text{Some number larger than 1}}$$

Since  $a \geq 3$ , we have  $2^a \geq 8$ ; thus,  $2^a - 1 \geq 7$ . Since the other factor is also larger than 1,  $2^m - 1$  is composite and we have proven our conjecture.

**Exercises 1–3 (4 minutes)**

In these exercises, students confirm the conjecture proven in Example 1. Emphasize that it does not really matter what the 2<sup>nd</sup> factor is once you know the first one. There is more than one way to solve each of these problems depending on how students decide to factor the exponent on the 2. Students should work in small groups on these exercises. Encourage them to use a calculator to determine the prime factors of 537 in Exercise 3. Have different groups present their results.

**Exercises 1–3**

For Exercises 1–3, find a factor of each expression using the method discussed in Example 1.

1.  $2^{15} - 1$

$$(2^5)^3 - 1 = (2^5 - 1)((2^5)^2 + 2^5 + 1) = (31)(\underbrace{32^2 + 32 + 1}_{\text{Some number larger than 1}})$$

*Thus, 31 is a factor of  $2^{15} - 1$ .*

2.  $2^{99} - 1$

$$(2^3)^{33} = (2^3 - 1)(\underbrace{8^{32} + 8^{31} + \dots + 8 + 1}_{\text{Some number larger than 1}})$$

*Thus, 7 is a factor of  $2^{99} - 1$ .*

3.  $2^{537} - 1$  (Hint: 537 is the product of two prime numbers that are both less than 50.)

Using a calculator we see that  $537 = 17 \cdot 31$ , so

$$(2^{17})^{31} - 1 = (2^{17} - 1)(\underbrace{(2^{17})^{30} + \dots + 2^{17} + 1}_{\text{Some number larger than 1}}).$$

*Thus,  $2^{17} - 1$  is a factor of  $2^{537} - 1$ .*

**Discussion (4 minutes)**

*Cryptography* is the science of making codes, and *cryptanalysis* is the science of breaking codes. The rise of internet commerce has created a demand for encoding methods that are hard for unintended observers to decipher. One encryption method, known as RSA encryption, uses very large numbers with hundreds of digits that are the product of two primes; the product of the prime factors is called the *key*. The key itself is made public so anyone can encode using this system, but in order to break the code, you would have to know how to factor the key, and that is what is so difficult.

- You had a hint in Exercise 3 that made it easier for you to factor a very large number, but what if you do not have any hints?
  - *It would be almost impossible to factor the number because you would have to check all the prime numbers up to the square root of the exponent to find the factors.*

If you know the key, then decoding is not particularly difficult. Programmers select a number that is almost impossible to factor without significant time and computing power and use this as the key to encode data and communications. The last exercise illuminates the logic behind modern encryption algorithms.

**Exercise 4 (6 minutes): How quickly can a computer factor a very large number?**

When we want to determine if a number  $m$  is prime, we can just try dividing by every prime number  $p$  that is less than  $\sqrt{m}$ . This will take the longest time if  $m$  happens to be a perfect square. In this exercise, students consider the speed it takes a certain computer algorithm to factor a square of a large prime number; this is the case where it should take the algorithm the longest to find the factorization. Actual algorithms used to factor large numbers are quicker than this, but it still takes a really long time and works to the advantage of people who want to encrypt information electronically using these large numbers. Introduce the problem and review the table as a whole class; then, have students answer the question in small groups. Have students working in groups apply the given function to estimate the time it would take to factor a 32-digit number. Make sure they convert their answer to years. Because we are using an exponential function, the factorization time grown very rapidly. Take the time after Exercise 4 to remind students that exponential functions increase very rapidly over intervals of equal length, ideas that were introduced in Algebra I and will be revisited in Module 3 of Algebra II.

**Exercise 4: How quickly can a computer factor a very large number?**

4. How long would it take a computer to factor some squares of very large prime numbers?

The time in seconds required to factor an  $n$ -digit number of the form  $p^2$ , where  $p$  is a large prime, can roughly be approximated by  $f(n) = 3.4 \times 10^{(n-13)/2}$ . Some values of this function are listed in the table below.

$p$	$p^2$	Number of Digits	Time needed to factor the number (sec)
10,007	100,140,049	9	0.034
100,003	10,000,600,009	11	0.34
1,000,003	1,000,006,000,009	13	3.4
10,000,019	100,000,380,000,361	15	34
100,000,007	10,000,001,400,000,049	17	340
1000,000,007	1,000,000,014,000,000,049	19	3,400

Use the function given above to determine how long it would take this computer to factor a number that contains 32 digits.

Using the given function,  $f(32) = 1.08 \times 10^{10}$  seconds =  $1.80 \times 10^8$  minutes =  $3 \times 10^6$  hours = 125,000 days, which is about 342.5 years.

After allowing groups to take a few minutes to evaluate the function and convert their answer to years, connect this exercise to the context of this situation by summarizing the following points.

- Using a very fast personal computer with a straightforward algorithm, it would take about 342 years to factor a 32-digit number, making any secret message encoded with that number obsolete before it could be cracked with that computer.
- However, we have extremely fast computers (much faster than one personal computer) and very efficient algorithms designed for those computers for factoring numbers. These computers can factor a number thousands of times faster than the computer used above, but they are still not fast enough to factor huge composite numbers in a reasonable amount of time.
- In 2009, computer scientists were able to factor a 232-digit number in two years by distributing the work over hundreds of fast computers running at the same time. That means any message encoded using that 232-digit number would take two years to decipher, by which time the message would no longer be relevant. Numbers used to encode secret messages typically contain over 300 digits, and extremely important secret messages use numbers that have over 600 digits—a far bigger number than any bank of computers can currently factor in a reasonable amount of time.

**Closing (2 minutes)**

There are better ways of factoring numbers than just checking all of the factors, but even advanced methods take a long time to execute. Products of primes of the magnitude of  $2^{2048}$  are almost impossible to factor in a reasonable amount of time, which is how mathematics is used to guarantee the security of electronic transactions. Give students a few minutes to summarize what they have learned in writing or by discussing it with a partner before starting the Exit Ticket.

- Polynomial identities can help us prove conjectures about numbers and make calculations easier.
- The field of number theory has contributed greatly to the fields of cryptography and cryptanalysis (code-making and code-breaking).

**Exit Ticket (4 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 8: The Power of Algebra—Finding Primes

### Exit Ticket

Express the prime number 31 in the form  $2^p - 1$  where  $p$  is a prime number and as a difference of two perfect squares using the identity  $(a + b)(a - b) = a^2 - b^2$ .

Exit Ticket Sample Solutions

Express the prime number 31 in the form  $2^p - 1$  where  $p$  is a prime number and as a difference of two perfect squares using the identity  $(a + b)(a - b) = a^2 - b^2$ .

$$\begin{aligned} 31 &= (16 - 15)(16 + 15) \\ &= 16^2 - 15^2 \end{aligned}$$

$$31 = 2^5 - 1$$

Problem Set Sample Solutions

1. Factor  $4^{12} - 1$  in two different ways using the identity  $x^n - a^n = (x - a)(x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^n)$  and the difference of squares identity.

$$\begin{aligned} &(4^6 - 1)(4^6 + 1) \\ &(4 - 1)(4^{11} + 4^{10} + \dots + 4 + 1) \end{aligned}$$

2. Factor  $2^{12} + 1$  using the identity  $x^n + a^n = (x + a)(x^n - ax^{n-1} + a^2x^{n-2} - \dots + a^n)$  for odd numbers  $n$ .

$$(2^4)^3 + 1 = (2^4 + 1)((2^4)^2 - 2^4 + 1)$$

3. Is 10,000,000,001 prime? Explain your reasoning.

*No, because it is of the form  $10^{10} + 1$ , which could be written as  $(10^2)^5 + 1 = (10^2 + 1)((10^2)^4 - \dots + 1)$ .*

4. Explain why  $2^n - 1$  is never prime if  $n$  is a composite number.

*If  $n$  is composite, then it can be written in the form  $n = ab$ , where  $a$  and  $b$  are integers larger than 1. Then  $2^n - 1 = 2^{ab} - 1 = (2^a)^b - 1 = (2^a - 1)((2^a)^{b-1} + \dots + 2^a + 1)$ . For  $a > 1$ , this number will be composite because  $2^a - 1$  will be larger than 1.*

5. Fermat numbers are of the form  $2^n + 1$  where  $n$  is positive integer.

- a. Create a table of Fermat numbers for odd values of  $n$  up to 9.

$n$	$2^n + 1$
1	$2^1 + 1 = 3$
3	$2^3 + 1 = 9$
5	$2^5 + 1 = 33$
7	$2^7 + 1 = 129$
9	$2^9 + 1 = 513$

- b. Explain why if  $n$  is odd, the Fermat number  $2^n + 1$  will always be divisible by 3.

*The Fermat number  $2^n + 1$  will factor as  $(2 + 1)(2^{n-1} - 2^{n-2} + \dots + 1)$  using the identity in Exercise 2.*

- c. Complete the table of values for even values of  $n$  up to 12.

$n$	$2^n + 1$
2	$2^2 + 1 = 5$
4	$2^4 + 1 = 17$
6	$2^6 + 1 = 65$
8	$2^8 + 1 = 257$
10	$2^{10} + 1 = 1,025$
12	$2^{12} + 1 = 4,097$

- d. Show that if  $n$  can be written in the form  $2k$  where  $k$  is odd, then  $2^n + 1$  is divisible by 5.

Let  $n = 2k$ , where  $k$  is odd. Then  $2^n + 1 = 2^{2k} + 1 = (2^2)^k + 1 = (2^2 + 1) \underbrace{((2^2)^{k-1} + \dots + 2^2 + 1)}_{\text{number larger than 1}}$ .

Since  $2^2 + 1 = 5$ , we know that 5 is a factor of  $2^n + 1$ . This only holds when  $k$  is an odd number because that is the only case when we can factor expressions of the form  $x^k + 1$ .

- e. Which even numbers are not divisible by an odd number? Make a conjecture about the only Fermat numbers that might be prime.

The powers of 2 are the only positive integers that are not divisible by any odd numbers. This implies that when the exponent  $n$  in  $2^n + 1$  is a power of 2, the Fermat number  $2^n + 1$  might be prime.

6. Complete this table to explore which numbers can be expressed as the difference of two perfect squares.

Number	Difference of Two Squares	Number	Difference of Two Squares
1	$1^2 - 0^2 = 1 - 0 = 1$	11	$6^2 - 5^2 = 36 - 25 = 11$
2	Not possible	12	$4^2 - 2^2 = 16 - 4 = 12$
3	$2^2 - 1^2 = 4 - 1 = 3$	13	$7^2 - 6^2 = 49 - 36 = 13$
4	$2^2 - 0^2 = 4 - 0 = 4$	14	Not possible
5	$3^2 - 2^2 = 9 - 4 = 5$	15	$8^2 - 7^2 = 64 - 49 = 15$
6	Not possible	16	$5^2 - 3^2 = 25 - 9 = 16$
7	$4^2 - 3^2 = 16 - 9 = 7$	17	$9^2 - 8^2 = 81 - 64 = 17$
8	$3^2 - 1^2 = 9 - 1 = 8$	18	Not possible
9	$5^2 - 4^2 = 25 - 16 = 9$	19	$10^2 - 9^2 = 100 - 81 = 19$
10	Not possible	20	$6^2 - 4^2 = 36 - 16 = 20$

- a. For which odd numbers does it appear to be possible to write the number as the difference of two squares?

It appears that we can write any positive odd number as the difference of two squares.

- b. For which even numbers does it appear to be possible to write the number as the difference of two squares?

It appears that we can write any multiple of 4 as the difference of two squares.

- c. Suppose that  $n$  is an odd number that can be expressed as  $n = a^2 - b^2$  for positive integers  $a$  and  $b$ . What do you notice about  $a$  and  $b$ ?

When  $n$  is odd,  $a$  and  $b$  are consecutive whole numbers and  $a + b = n$ .

- d. Suppose that  $n$  is an even number that can be expressed as  $n = a^2 - b^2$  for positive integers  $a$  and  $b$ . What do you notice about  $a$  and  $b$ ?

When  $n$  is an even number that can be written as a difference of squares, then  $n$  is a multiple of 4, and  $a$  and  $b$  are either consecutive even integers or consecutive odd integers. We also have  $a + b = \frac{n}{2}$ .

7. Express the numbers from 21 to 30 as the difference of two squares, if possible.

*This is not possible for 22, 26, and 30. Otherwise we have the following.*

$$\begin{array}{ll} 21 = 11^2 - 10^2 & 27 = 14^2 - 13^2 \\ 23 = 12^2 - 11^2 & 28 = 8^2 - 6^2 \\ 24 = 7^2 - 5^2 & 29 = 15^2 - 14^2 \\ 25 = 13^2 - 12^2 & \end{array}$$

8. Prove this conjecture: Every positive odd number  $m$  can be expressed as the difference of the squares of two consecutive numbers that sum to the original number  $m$ .

a. Let  $m$  be a positive odd number. Then for some integer  $n$ ,  $m = 2n + 1$ . We will look at the consecutive integers  $n$  and  $n + 1$ . Show that  $n + (n + 1) = m$ .

$$\begin{aligned} n + (n + 1) &= n + n + 1 \\ &= 2n + 1 \\ &= m \end{aligned}$$

b. What is the difference of squares of  $n + 1$  and  $n$ ?

$$\begin{aligned} (n + 1)^2 - n^2 &= n^2 + 2n + 1 - n^2 \\ &= 2n + 1 \\ &= m \end{aligned}$$

c. What can you conclude from parts (a) and (b)?

*We can write any positive odd number  $m$  as the difference of squares of two consecutive numbers that sum to  $m$ .*

9. Prove this conjecture: Every positive multiple of 4 can be expressed as the difference of squares of two numbers that differ by 2. Use the table below to organize your work for parts (a)–(c).

a. Write each multiple of 4 in the table as a difference of squares.

$n$	$4n$	Difference of squares $a^2 - b^2$	$a$	$b$
1	4	$2^2 - 0^2$	2	0
2	8	$3^2 - 1^2$	3	1
3	12	$4^2 - 2^2$	4	2
4	16	$5^2 - 3^2$	5	3
5	20	$6^2 - 4^2$	6	4
$n$	$4n$	$(\quad)^2 - (\quad)^2$	$n + 1$	$n - 1$

b. What do you notice about the numbers  $a$  and  $b$  that are squared? How do they relate to the number  $n$ ?

*The values of  $a$  and  $b$  in the differences of two squares differ by 2 every time. They are one larger and one smaller than  $n$ ; that is,  $a = n + 1$  and  $b = n - 1$ .*

- c. Given a positive integer of the form  $4n$ , prove that there are integers  $a$  and  $b$  so that  $4n = a^2 - b^2$  and that  $a - b = 2$ . (Hint: Refer to parts (a) and (b) for the relationship between  $n$  and  $a$  and  $b$ .)

Define  $a = n + 1$  and  $b = n - 1$ . Then we can calculate  $a^2 - b^2$  as follows.

$$\begin{aligned} a^2 - b^2 &= (n + 1)^2 - (n - 1)^2 \\ &= ((n + 1) + (n - 1))((n + 1) - (n - 1)) \\ &= (2n)(2) \\ &= 4n \end{aligned}$$

We can also see that

$$\begin{aligned} a - b &= (n + 1) - (n - 1) \\ &= n + 1 - n + 1 \\ &= 2. \end{aligned}$$

Thus every positive multiple of 4 can be written as a difference of squares of two integers that differ by 2.

10. The steps below prove that the only positive even numbers that can be written as a difference of square integers are the multiples of 4. That is, completing this exercise will prove that it is impossible to write a number of the form  $4n - 2$  as a difference of square integers.

- a. Let  $m$  be a positive even integer that we can write as the difference of square integers  $m = a^2 - b^2$ . Then  $m = (a + b)(a - b)$  for integers  $a$  and  $b$ . How do we know that either  $a$  and  $b$  are both even or  $a$  and  $b$  are both odd?

*If one of  $a$  and  $b$  is even and the other one is odd, then one of  $a^2$  and  $b^2$  is even and the other one is odd. Since the difference of an odd and an even number is odd, this means that  $m = a^2 - b^2$  would be odd. Since we know that  $m$  is even, it must be that either  $a$  and  $b$  are both even or  $a$  and  $b$  are both odd.*

- b. Is  $a + b$  even or odd? What about  $a - b$ ? How do you know?

*Since  $a$  and  $b$  are either both odd or both even, we know that both  $a + b$  and  $a - b$  are even.*

- c. Is 2 a factor of  $a + b$ ? Is 2 a factor of  $a - b$ ? Is 4 a factor of  $(a + b)(a - b)$ ? Explain how you know.

*Because  $a + b$  and  $a - b$  are both even, 2 is a factor of both  $a + b$  and  $a - b$ . Thus,  $2^2 = 4$  is a factor of  $(a + b)(a - b)$ .*

- d. Is 4 a factor of any integer of the form  $4n - 2$ ?

*No. If 4 were a factor of  $4n - 2$ , we could factor it out:  $4n - 2 = 4\left(n - \frac{1}{2}\right)$ . But this means that  $n - \frac{1}{2}$  is an integer, which it clearly is not. This means that 4 is not a factor of any number of the form  $4n - 2$ .*

- e. What can you conclude from your work in parts (a)–(d)?

*If  $m$  is a positive even integer and  $m$  can be written as the difference of two square integers, then  $m$  cannot be of the form  $4n - 2$  for any integer  $n$ . Another way to say this is that the positive integers of the form  $4n - 2$  for some integer  $n$  cannot be written as the difference of two square integers.*

11. Explain why the prime number 17 can only be expressed as the difference of two squares in only one way, but the composite number 24 can be expressed as the difference of two squares in more than one way.

*Since every odd number can be expressed as the difference of two squares,  $a^2 - b^2 = (a + b)(a - b)$ , the number 17 must fit this pattern. Because 17 is prime, there is only one way to factor 17, which is  $17 = 1 \cdot 17$ .*

*Let  $a + b = 17$  and  $a - b = 1$ . The two numbers that satisfy this system of equations are 8 and 9. Thus,*

$$\begin{aligned} 17 &= 1 \cdot 17 \\ &= (9 - 8)(9 + 8) \\ &= 9^2 - 8^2. \end{aligned}$$

*A composite number has more than one factorization, not all of which will lead to writing the number as the difference of squares of two integers. For the number 24, you could use*

$$\begin{aligned} 24 &= 2 \cdot 12 \\ &= (7 - 5)(7 + 5) \\ &= 7^2 - 5^2. \end{aligned}$$

*Or, you could use*

$$\begin{aligned} 24 &= 4 \cdot 6 \\ &= (5 - 1)(5 + 1) \\ &= 5^2 - 1^2. \end{aligned}$$

12. Explain why you cannot use the factors of 3 and 8 to rewrite 24 as the difference of two square integers.

*If  $24 = 3 \cdot 8$ , then  $a - b = 3$  and  $a + b = 8$ . The solution to this system of equations is  $(5.5, 2.5)$ . If we are restricting this problem to the set of whole numbers, then you cannot apply the identity to rewrite 24 as the difference of two perfect squares where  $a$  and  $b$  are whole numbers. It certainly is true that  $24 = (5.5 - 2.5)(5.5 + 2.5) = 5.5^2 - 2.5^2$ , but this is not necessarily an easy way to calculate 24.*

MP.3



## Lesson 9: Radicals and Conjugates

### Student Outcomes

- Students understand that the sum of two square roots (or two cube roots) is not equal to the square root (or cube root) of their sum.
- Students convert expressions to simplest radical form.
- Students understand that the product of conjugate radicals can be viewed as the difference of two squares.

### Lesson Notes

Because this lesson deals with radicals, it might seem out of place amid other lessons on polynomials. A major theme, however, is the parallelism between the product of conjugate radicals and the difference of two squares. There is also parallelism between taking the square root or cube root of a sum and taking the square of a sum; both give rise to an error sometimes called “the freshman’s dream” or “the illusion of linearity”. If students are not careful, they may easily conclude that  $(x + y)^n$  and  $x^n + y^n$  are equivalent expressions for all  $n \geq 0$ , when this is only true for  $n = 1$ . Additionally, this work with radicals will prepare students for later work with radical expressions in Module 3. Throughout this lesson, students employ MP.7, as they see complicated expressions as being composed of simpler ones. Additionally, the Opening Exercise offers further practice in making a conjecture (MP.3).

#### Scaffolding:

If necessary, remind students that  $\sqrt{2}$  and  $-\sqrt[3]{4}$  are irrational numbers. They cannot be written in the form  $\frac{p}{q}$  for integers  $p$  and  $q$ . They are numbers, however, and just like  $\sqrt{4}$  and  $-\sqrt[3]{64}$ , can be found on a number line. On the number line,  $\sqrt{2} = 1.414\dots$  is just to the right of 1.4, and  $-\sqrt[3]{4} = -1.5874\dots$  is just to the left of  $-1.587$ .

### Classwork

#### Opening Exercise (3 minutes)

The Opening Exercise is designed to show students that they need to be cautious when working with radicals. The multiplication and division operations combine with radicals in a predictable way, but the addition and subtraction operations do not.

The square root of a product is the product of the two square roots. For example,

$$\begin{aligned}\sqrt{4 \cdot 9} &= \sqrt{2 \cdot 2 \cdot 3 \cdot 3} \\ &= \sqrt{6 \cdot 6} \\ &= 6 \\ &= 2 \cdot 3 \\ &= \sqrt{4} \cdot \sqrt{9}.\end{aligned}$$

Similarly, the square root of a quotient is the quotient of the two square roots:  $\sqrt{\frac{25}{16}} = \frac{5}{4} = \frac{\sqrt{25}}{\sqrt{16}}$ . And the same holds true for multiplication and division with cube roots, but not for addition or subtraction with square or cube roots.

Begin by posing the following question for students to work on in groups.

**Opening Exercise**

Which of these statements are true for all  $a, b > 0$ ? Explain your conjecture.

i.  $2(a + b) = 2a + 2b$

ii.  $\frac{a + b}{2} = \frac{a}{2} + \frac{b}{2}$

iii.  $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$

**Scaffolding:**

If necessary, circulate to help students get started by suggesting they substitute numerical values for  $a$  and  $b$ . Perfect squares like 1 and 4 are good values to start with when square roots are in the expression.

**Discussion (3 minutes)**

Students should be able to show that the first two equations are true for all  $a$  and  $b$ , and they should be able to find values of  $a$  and  $b$  for which the third equation is not true. (In fact, it is always false, as students will show in the Problem Set.)

- Can you provide cases for which the third equation is not true? (Remind them that a single counterexample is sufficient to make an equation untrue in general.)
  - *Students should give examples such as  $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$ , but  $\sqrt{9 + 16} = \sqrt{25} = 5$ .*

Point out that just as they have learned (in Lesson 2, if not before) that the square of  $(x + y)$  is not equal to the sum of  $x^2$  and  $y^2$  (for  $x, y \neq 0$ ), so it is also true that the square root of  $(x + y)$  is not equal to the sum of the square roots of  $x$  and  $y$  (for  $x, y > 0$ ). Similarly, the cube root of  $(x + y)$  is not equal to the sum of the cube roots of  $x$  and  $y$  (for  $x, y > 0$ ).

**Example 1 (2 minutes)**

Explain to students that an expression is in *simplest radical form* when the radicand (the expression under the radical sign) has no factor that can be raised to a power greater than or equal to the index (either 2 or 3) and there is no radical in the denominator. Present the following example.

**Example 1**

Express  $\sqrt{50} - \sqrt{18} + \sqrt{8}$  in simplest radical form and combine like terms.

$$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$$

$$\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2}$$

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$$

$$\text{Therefore, } \sqrt{50} - \sqrt{18} + \sqrt{8} = 5\sqrt{2} - 3\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}.$$

**Scaffolding:**

If necessary, ask students about expressions that are easier to express in simplest radical form, such as:

$$\sqrt{8}$$

$$\sqrt{6}$$

$$\sqrt{18}$$

$$\sqrt{20}$$

$$\sqrt{20} + \sqrt{8}$$

$$\sqrt{20} - \sqrt{18}.$$

**Exercises 1–5 (8 minutes)**

The following exercises make use of the rules for multiplying and dividing radicals. Express each expression in simplest radical form and combine like terms.

*Scaffolding:*

Circulate to make sure that students understand that they are looking for perfect square factors when the index is 2 and perfect cube factors when the index is 3. They may need reminding that the first few cubes of the counting numbers are 1, 8, 27, 64, and 125.

**Exercises 1–5**

<p>1. <math>\sqrt{\frac{1}{4}} + \sqrt{\frac{9}{4}} - \sqrt{45}</math> <math>2 - 3\sqrt{5}</math></p>	<p>2. <math>\sqrt{2}(\sqrt{3} - \sqrt{2})</math> <math>\sqrt{6} - 2</math></p>
<p>3. <math>\sqrt{\frac{3}{8}}</math> <math>\frac{\sqrt{3}}{\sqrt{8}} = \frac{\sqrt{6}}{\sqrt{16}} = \frac{\sqrt{6}}{4}</math></p>	<p>4. <math>\sqrt[3]{\frac{5}{32}}</math> <math>\frac{\sqrt[3]{5}}{\sqrt[3]{32}} = \frac{\sqrt[3]{10}}{\sqrt[3]{64}} = \frac{\sqrt[3]{10}}{4}</math></p>
<p>5. <math>\sqrt[3]{16x^5}</math> <math>\sqrt[3]{8x^3} \cdot \sqrt[3]{2x^2} = 2x\sqrt[3]{2x^2}</math></p>	

In the example and exercises above, we repeatedly used the following properties of radicals (write the following statements on the board).

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

$$\sqrt[3]{a} \cdot \sqrt[3]{b} = \sqrt[3]{ab}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

$$\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \sqrt[3]{\frac{a}{b}}$$

- When do these identities make sense?
  - *Students should answer that the identities make sense for the square roots whenever  $a \geq 0$  and  $b \geq 0$ , with  $b \neq 0$  when  $b$  is a denominator. They make sense for the cube roots for all  $a$  and  $b$ , with  $b \neq 0$  when  $b$  is a denominator.*

**Example 2 (8 minutes)**

This example is designed to introduce conjugates and their properties.

**Example 2**

**Multiply and combine like terms. Then explain what you notice about the two different results.**

$(\sqrt{3} + \sqrt{2})(\sqrt{3} + \sqrt{2})$

$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$

Solution (with teacher comments and a question):

The first product is  $\sqrt{3} \cdot \sqrt{3} + 2(\sqrt{3} \cdot \sqrt{2}) + \sqrt{2} \cdot \sqrt{2} = 5 + 2\sqrt{6}$ .

The second product is  $\sqrt{3} \cdot \sqrt{3} - (\sqrt{3} \cdot \sqrt{2}) + (\sqrt{3} \cdot \sqrt{2}) - \sqrt{2} \cdot \sqrt{2} = 3 - 2 = 1$ .

The first product is a radical expression; the second is an integer.

The second product has the nice feature that the radicals have been eliminated. In that case, we give the two factors a special name: two binomials of the form  $\sqrt{a} + \sqrt{b}$  and  $\sqrt{a} - \sqrt{b}$  are called *conjugate radicals*:

$\sqrt{a} + \sqrt{b}$  is the conjugate of  $\sqrt{a} - \sqrt{b}$ , and

$\sqrt{a} - \sqrt{b}$  is the conjugate of  $\sqrt{a} + \sqrt{b}$ .

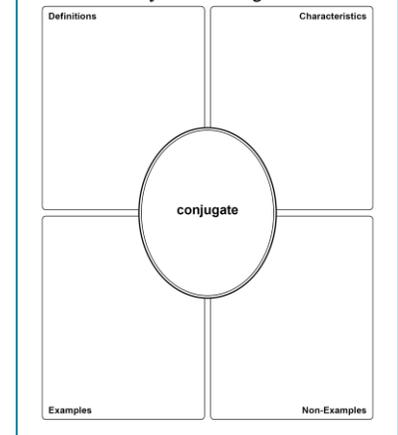
More generally, for any expression in two terms, at least one of which contains a radical, its conjugate is an expression consisting of the same two terms but with the opposite sign separating the terms. For example, the conjugate of  $2 - \sqrt{3}$  is  $2 + \sqrt{3}$ , and the conjugate of  $\sqrt[3]{5} + \sqrt{3}$  is  $\sqrt[3]{5} - \sqrt{3}$ .

- What polynomial identity is suggested by the product of two conjugates?
  - *Students should answer that it looks like the difference of two squares. The product of two conjugates has the form of the difference of squares:  $(x + y)(x - y) = x^2 - y^2$ .*

The following exercise focuses on the use of conjugates.

*Scaffolding:*

Students may have trouble with the word *conjugate*. If so, have them fill out the following diagram.



**Exercise 6 (5 minutes)**

**Exercise 6**

6. Find the product of the conjugate radicals.

$(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3})$	$5 - 3 = 2$
$(7 + \sqrt{2})(7 - \sqrt{2})$	$49 - 2 = 47$
$(\sqrt{5} + 2)(\sqrt{5} - 2)$	$5 - 4 = 1$

- In each case in Exercise 6, is the result the difference of two squares?
  - *Yes. For example, if we think of 5 as  $(\sqrt{5})^2$  and 3 as  $(\sqrt{3})^2$ , then  $5 - 3 = (\sqrt{5})^2 - (\sqrt{3})^2$ .*

**Example 3 (6 minutes)**

This example is designed to show how division by a radical can be reduced to division by an integer by multiplication by the conjugate radical in the numerator and denominator.

**Example 3**

Write  $\frac{\sqrt{3}}{5-2\sqrt{3}}$  in simplest radical form.

$$\frac{\sqrt{3}}{5-2\sqrt{3}} = \frac{\sqrt{3}}{5-2\sqrt{3}} \cdot \frac{5+2\sqrt{3}}{5+2\sqrt{3}} = \frac{\sqrt{3}(5+2\sqrt{3})}{25-12} = \frac{5\sqrt{3}+6}{13}$$

The process for simplifying an expression with a radical in the denominator has two steps:

1. Multiply the numerator and denominator of the fraction by the *conjugate* of the denominator.
2. Simplify the resulting expression.

**Closing (5 minutes)**

- Why is this lesson part of a module on polynomials?
  - *Polynomials can generalize the structure of our place value system. For example, if we set  $x = 10$ , then the polynomial  $3x^2 + 2x + 4$  evaluates to the number 324, where the coefficients of the polynomial correspond to the digits in the number. Polynomials can also generalize the structure of radical expressions. Because radicals with different radicands or different indices do not combine, the irrational number  $\sqrt{a} + \sqrt{b}$  can be generalized by a polynomial  $x + y$ .*
  - *For example, to simplify  $\frac{3}{\sqrt{2}+\sqrt{3}}$ , we treat the denominator like a binomial.*

Substitute  $\sqrt{2} = x$  and  $\sqrt{3} = y$ :

$$\text{then } \frac{3}{\sqrt{2}+\sqrt{3}} = \frac{3}{x+y} = \frac{3}{x+y} \cdot \frac{x-y}{x-y} = \frac{3(x-y)}{x^2-y^2}.$$

Since  $x = \sqrt{2}$  and  $y = \sqrt{3}$ ,  $x^2 - y^2$  is an integer. In this case,  $x^2 - y^2 = -1$ .

$$\frac{3}{\sqrt{2} + \sqrt{3}} \cdot \frac{\sqrt{2} - \sqrt{3}}{\sqrt{2} - \sqrt{3}} = \frac{3(\sqrt{2} - \sqrt{3})}{2 - 3} = -3(\sqrt{2} - \sqrt{3})$$

- Radical expressions with the same index and same radicand combine in the same way as like terms in a polynomial when performing addition and subtraction.

For example,  $\sqrt[3]{3} + \sqrt{2} + 5\sqrt[3]{3} - \sqrt{7} + \sqrt{3} + \sqrt{7} + 3\sqrt{2} = 6\sqrt[3]{3} + 4\sqrt{2} + \sqrt{3}$ .

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.

#### Lesson Summary

- For real numbers  $a \geq 0$  and  $b \geq 0$ , where  $b \neq 0$  when  $b$  is a denominator,

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{and} \quad \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

- For real numbers  $a \geq 0$  and  $b \geq 0$ , where  $b \neq 0$  when  $b$  is a denominator,

$$\sqrt[3]{ab} = \sqrt[3]{a} \cdot \sqrt[3]{b} \quad \text{and} \quad \sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$$

- Two binomials of the form  $\sqrt{a} + \sqrt{b}$  and  $\sqrt{a} - \sqrt{b}$  are called conjugate radicals:

$\sqrt{a} + \sqrt{b}$  is the conjugate of  $\sqrt{a} - \sqrt{b}$ , and

$\sqrt{a} - \sqrt{b}$  is the conjugate of  $\sqrt{a} + \sqrt{b}$ .

For example, the conjugate of  $2 - \sqrt{3}$  is  $2 + \sqrt{3}$ .

- To express a numeric expression with a denominator of the form  $\sqrt{a} + \sqrt{b}$  in simplest radical form, multiply the numerator and denominator by the conjugate  $\sqrt{a} - \sqrt{b}$  and combine like terms.

#### Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 9: Radicals and Conjugates

### Exit Ticket

- Rewrite each of the following radicals as a rational number or in simplest radical form.
  - $\sqrt{49}$
  - $\sqrt[3]{40}$
  - $\sqrt{242}$
- Find the conjugate of each of the following radical expressions.
  - $\sqrt{5} + \sqrt{11}$
  - $9 - \sqrt{11}$
  - $\sqrt[3]{3} + 1.5$
- Rewrite each of the following expressions as a rational number or in simplest radical form.
  - $\sqrt{3}(\sqrt{3} - 1)$
  - $(5 + \sqrt{3})^2$
  - $(10 + \sqrt{11})(10 - \sqrt{11})$

## Exit Ticket Sample Solution

1. Rewrite each of the following radicals as a rational number or in simplest radical form.	
a. $\sqrt{49}$	7
b. $\sqrt[3]{40}$	$2\sqrt[3]{5}$
c. $\sqrt{242}$	$11\sqrt{2}$
2. Find the conjugate of each of the following radical expressions.	
a. $\sqrt{5} + \sqrt{11}$	$\sqrt{5} - \sqrt{11}$
b. $9 - \sqrt{11}$	$9 + \sqrt{11}$
c. $\sqrt[3]{3} + 1.5$	$\sqrt[3]{3} - 1.5$
3. Rewrite each of the following expressions as a rational number or in simplest radical form.	
a. $\sqrt{3}(\sqrt{3} - 1)$	$3 - \sqrt{3}$
b. $(5 + \sqrt{3})^2$	$28 + 10\sqrt{3}$
c. $(10 + \sqrt{11})(10 - \sqrt{11})$	89

## Problem Set Sample Solutions

Problem 10 is different from the others and may require some discussion and explanation before students work on it. You may need to explain that the converse of an *if-then* theorem is obtained by interchanging the clauses introduced by *if* and *then* and that the converse of such a theorem is not necessarily a theorem. The converse of the Pythagorean Theorem will be important for the development of a formula leading to Pythagorean triples in Lesson 10.

1. Express each of the following as a rational number or in simplest radical form. Assume that the symbols $a$ , $b$ , and $x$ represent positive numbers.	
a. $\sqrt{36}$	6
b. $\sqrt{72}$	$6\sqrt{2}$
c. $\sqrt{18}$	$3\sqrt{2}$
d. $\sqrt{9x^3}$	$3x\sqrt{x}$
e. $\sqrt{27x^2}$	$3x\sqrt{3}$

f.	$\sqrt[3]{16}$	$2\sqrt[3]{2}$
g.	$\sqrt[3]{24a}$	$2\sqrt[3]{3a}$
h.	$\sqrt{9a^2 + 9b^2}$	$3\sqrt{a^2 + b^2}$
2.	Express each of the following in simplest radical form, combining terms where possible.	
a.	$\sqrt{25} + \sqrt{45} - \sqrt{20}$	$5 + \sqrt{5}$
b.	$3\sqrt{3} - \sqrt{\frac{3}{4}} + \sqrt{\frac{1}{3}}$	$\frac{17\sqrt{3}}{6}$
c.	$\sqrt[3]{54} - \sqrt[3]{8} + 7\sqrt[3]{\frac{1}{4}}$	$\frac{13\sqrt[3]{2}}{2} - 2$
d.	$\sqrt[3]{\frac{5}{8}} + \sqrt[3]{40} - \sqrt[3]{\frac{8}{9}}$	$\frac{5\sqrt[3]{5}}{2} - \frac{2\sqrt[3]{3}}{3}$
3.	Evaluate $\sqrt{x^2 - y^2}$ when $x = 33$ and $y = 15$ .	$12\sqrt{6}$
4.	Evaluate $\sqrt{x^2 + y^2}$ when $x = 20$ and $y = 10$ .	$10\sqrt{5}$
5.	Express each of the following as a rational expression or in simplest radical form. Assume that the symbols $x$ and $y$ represent positive numbers.	
a.	$\sqrt{3}(\sqrt{7} - \sqrt{3})$	$\sqrt{21} - 3$
b.	$(3 + \sqrt{2})^2$	$11 + 6\sqrt{2}$
c.	$(2 + \sqrt{3})(2 - \sqrt{3})$	1
d.	$(2 + 2\sqrt{5})(2 - 2\sqrt{5})$	-16
e.	$(\sqrt{7} - 3)(\sqrt{7} + 3)$	-2
f.	$(3\sqrt{2} + \sqrt{7})(3\sqrt{2} - \sqrt{7})$	11
g.	$(x - \sqrt{3})(x + \sqrt{3})$	$x^2 - 3$
h.	$(2x\sqrt{2} + y)(2x\sqrt{2} - y)$	$8x^2 - y^2$

6. Simplify each of the following quotients as far as possible.

a.  $(\sqrt{21} - \sqrt{3}) \div \sqrt{3}$   $\sqrt{7} - 1$

b.  $(\sqrt{5} + 4) \div (\sqrt{5} + 1)$   $\frac{1}{4}(1 + 3\sqrt{5})$

c.  $(3 - \sqrt{2}) \div (3\sqrt{2} - 5)$   $-\frac{1}{7}(9 + 4\sqrt{2})$

d.  $(2\sqrt{5} - \sqrt{3}) \div (3\sqrt{5} - 4\sqrt{2})$   $\frac{1}{13}(30 - 3\sqrt{15} + 8\sqrt{10} - 4\sqrt{6})$

7. If  $x = 2 + \sqrt{3}$ , show that  $x + \frac{1}{x}$  has a rational value.

$$x + \frac{1}{x} = 4$$

8. Evaluate  $5x^2 - 10x$  when the value of  $x$  is  $\frac{2-\sqrt{5}}{2}$ .

$$\frac{5}{4}$$

9. Write the factors of  $a^4 - b^4$ . Use the result to obtain the factored form of  $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$ .

**Factors:**  $(a^2 + b^2)(a + b)(a - b)$

**Simplified form:**  $40\sqrt{6}$

10. The converse of the Pythagorean Theorem is also a theorem: If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

Use the converse of the Pythagorean Theorem to show that for  $A, B, C > 0$ , if  $A + B = C$ , then  $\sqrt{A} + \sqrt{B} > \sqrt{C}$ , so that  $\sqrt{A} + \sqrt{B} > \sqrt{A + B}$ .

**Solution A:** Since  $A, B, C > 0$ , we can interpret these quantities as the areas of three squares whose sides have lengths  $\sqrt{A}$ ,  $\sqrt{B}$ , and  $\sqrt{C}$ . Because  $A + B = C$ , then by the converse of the Pythagorean Theorem,  $\sqrt{A}$ ,  $\sqrt{B}$ , and  $\sqrt{C}$  are the lengths of the legs and hypotenuse of a right triangle. In a triangle, the sum of any two sides is greater than the third side. Therefore,  $\sqrt{A} + \sqrt{B} > \sqrt{C}$ , so  $\sqrt{A} + \sqrt{B} > \sqrt{A + B}$ .

**Solution B:** Since  $A, B, C > 0$ , we can interpret these quantities as the areas of three squares whose sides have lengths  $\sqrt{A}$ ,  $\sqrt{B}$ , and  $\sqrt{C}$ . Because  $A + B = C$ , then by the converse of the Pythagorean Theorem,  $\sqrt{A} = a$ ,  $\sqrt{B} = b$ , and  $c = \sqrt{C}$  are the lengths of the legs and hypotenuse of a right triangle, so  $a, b, c > 0$ . Therefore,  $2ab > 0$ . Adding equal positive quantities to each side of that inequality, we get  $a^2 + b^2 + 2ab > c^2$ , which we can rewrite as  $(a + b)^2 > c^2$ . Taking the positive square root of each side, we get  $a + b > c$ , or equivalently,  $\sqrt{A} + \sqrt{B} > \sqrt{C}$ . We then have  $\sqrt{A} + \sqrt{B} > \sqrt{A + B}$ .



## Lesson 10: The Power of Algebra—Finding Pythagorean Triples

### Student Outcomes

- Students explore the difference of two squares identity  $x^2 - y^2 = (x - y)(x + y)$  in the context of finding Pythagorean triples.

### Lesson Notes

This lesson addresses standards **A-SSE.A.2**, **A-APR.C.4**, and MP.7 directly. In particular, this lesson investigates the example suggested by **A-APR.C.4**: Show how “the polynomial identity  $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$  can be used to generate Pythagorean triples.” This polynomial identity is proven in this lesson using the difference of two squares identity by

$$(x^2 + y^2)^2 - (x^2 - y^2)^2 = ((x^2 + y^2) - (x^2 - y^2))((x^2 + y^2) + (x^2 - y^2)) = (2y^2)(2x^2) = (2xy)^2.$$

However, students are first asked to prove the identity on their own in the case when  $y = 1$ . Very few (or likely none) of your students will use the difference of two squares identity, so you will have an opportunity to surprise them with the quick solution presented here.

The lesson starts with a quick review of the most important theorem in all of geometry and arguably in all of mathematics: the Pythagorean Theorem. Students have already studied the Pythagorean Theorem in Grade 8 and high school Geometry, have proven the theorem in numerous ways, and have used it in a wide variety of situations. Students will be asked to prove it in yet a different way in the Problem Set to this lesson. The Pythagorean Theorem will play an important role in both this module and the next.

### Classwork

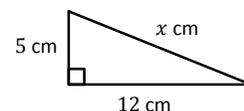
#### Opening Exercise (10 minutes)

This exercise is meant to help students recall facts about the Pythagorean Theorem. Because it is not the main point of this lesson, feel free to move through this exercise quickly. After they have worked the problem, summarize with a statement of the Pythagorean Theorem and its converse, and then move on.

Have students work in groups of two on this problem. Suggest immediately that they draw a diagram to represent the problem.

#### Scaffolding:

You might start by showing a simple example of the Pythagorean Theorem.



$$5^2 + 12^2 = x^2 \Rightarrow x = 13$$

**Opening Exercise**

Sam and Jill decide to explore a city. Both begin their walk from the same starting point.

- Sam walks 1 block north, 1 block east, 3 blocks north, and 3 blocks west.
- Jill walks 4 blocks south, 1 block west, 1 block north, and 4 blocks east.

If all city blocks are the same length, who is the farthest distance from the starting point?

Sam's distance:  $\sqrt{20}$  city block lengths  
 Jill's distance:  $\sqrt{18}$  city block lengths  
 Sam was farthest away from the starting point.

Students may have a question about what the problem is asking: Does distance mean, “Who walked the farthest?”, or “Who is the farthest (as the crow flies) from the starting point?” This question boils down to the difference between the definitions of path length versus distance. While Sam’s path length is 8 city blocks and Jill’s is 10 city blocks, the question asks for the distance between the starting point and their final destinations. To calculate distance, students will need to use the Pythagorean Theorem.

The problem is designed so that answers cannot be guessed easily from precisely drawn pictures.

Another (valid) issue that a student may bring up is whether the streets are considered to have width or not. Discuss this possibility with your class (again, it is a valid point). Suggest that for the purposes of this problem, we will assume that the streets have no width (or, as some may point out, Sam and Jill could walk down the center of the streets—but this is not advisable).

See if you can get your students to conclude that  $\sqrt{18} < \sqrt{20}$  simply because  $18 < 20$ .

- Why must the side length of a square with area 18 square units be smaller than the side length of a square with area 20 square units?
- Can you state the Pythagorean Theorem?
  - If a right triangle has legs of length  $a$  and  $b$  units and hypotenuse of length  $c$  units, then  $a^2 + b^2 = c^2$ .
- What is the converse of the Pythagorean Theorem? Can you state it as an “if-then” statement?
  - If the lengths  $a, b, c$  of the sides of a triangle are related by  $a^2 + b^2 = c^2$ , then the angle opposite the side of length  $c$  is a right angle.
- We will need the converse of the Pythagorean Theorem for this lesson.

MP.6

**Example 1 (15 minutes)**

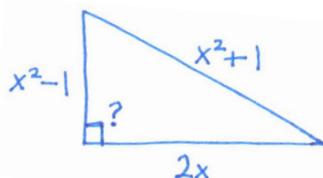
In this example, students explore a specific case of the general method of generating Pythagorean triples, that is, triples of positive integers  $a, b, c$  that satisfy  $a^2 + b^2 = c^2$ . The general form that students will explore in the Problem Set is  $x^2 - y^2, 2xy, x^2 + y^2$  for  $x > y$ .

*Scaffolding:*

To make this example more concrete and accessible, generate (or ask students to generate) a set of triples of this form and verify that they are Pythagorean triples. For example,  $(3,4,5)$  when  $x = 2$ . Furthermore, you might challenge them to find a triple of this form that is *not* a Pythagorean triple.

**Example 1**

Prove that if  $x > 1$ , then a triangle with side lengths  $x^2 - 1, 2x$ , and  $x^2 + 1$  is a right triangle.



For  $x > 1$ , is this triangle right?

**Note:** By the converse to the Pythagorean Theorem, if  $a^2 + b^2 = c^2$  then a triangle with side lengths  $a, b, c$  is a right triangle with right angle opposite the side of length  $c$ . We are given that the triangle exists with these side lengths, so we do not need to explicitly verify that the lengths are positive. Therefore, we need only check that for any  $x > 1$ , we have  $(x^2 - 1)^2 + (2x)^2 = (x^2 + 1)^2$ .

**Proof:** We are given a triangle with side lengths  $2x, x^2 - 1$ , and  $x^2 + 1$  for some real number  $x > 1$ . We need to show that the three lengths  $2x, x^2 - 1$ , and  $x^2 + 1$  form a Pythagorean triple. We will first show that  $(2x)^2$  is equivalent to  $(x^2 + 1)^2 - (x^2 - 1)^2$ .

$$\begin{aligned} (x^2 + 1)^2 - (x^2 - 1)^2 &= ((x^2 + 1) + (x^2 - 1))((x^2 + 1) - (x^2 - 1)) \\ &= (2x^2)(2) \\ &= 4x^2 \\ &= (2x)^2 \end{aligned}$$

Since  $(2x)^2 = (x^2 + 1)^2 - (x^2 - 1)^2$ , we have shown that  $(2x)^2 + (x^2 - 1)^2 = (x^2 + 1)^2$ , and thus the numbers  $x^2 - 1, 2x$ , and  $x^2 + 1$  form a Pythagorean triple. Then by the converse of the Pythagorean Theorem, a triangle with sides of length  $2x, x^2 - 1$ , and  $x^2 + 1$  for some  $x > 1$  is a right triangle.

MP.3

MP.7

Proving that  $(x^2 - 1)^2 + (2x)^2 = (x^2 + 1)^2$  can be done in different ways. You might ask students to try their own method first, and then show the method above. Very few students will use the identity  $a^2 - b^2 = (a - b)(a + b)$ . Most will use  $(x^2 - 1)^2 + 4x^2 = x^4 - 2x^2 + 1 + 4x^2 = x^4 + 2x^2 + 1 = (x^2 + 1)^2$ . This is an excellent exercise as well, since it gets students to wrestle with squares of quadratic polynomials and requires factoring. After they have tried it on their own, they will be surprised by the use of the difference of squares identity.

- A Pythagorean triple is a triple of positive integers  $(a, b, c)$  such that  $a^2 + b^2 = c^2$ . So, while  $(3, 4, 5)$  is a Pythagorean triple, the triple  $(1, 1, \sqrt{2})$  is not, even though  $1, 1$ , and  $\sqrt{2}$  are side lengths of a  $45^\circ$ - $45^\circ$ - $90^\circ$  triangle and  $1^2 + 1^2 = (\sqrt{2})^2$ . While the triangle from Example 1 can have non-integer side lengths, notice that a Pythagorean triple must be comprised of positive integers by definition.
- Note that any multiple of a Pythagorean triple is also a Pythagorean triple: if  $(a, b, c)$  is a Pythagorean triple, then so is  $(na, nb, nc)$  for any positive integer  $n$  (discuss why). Thus,  $(6, 8, 10)$ ,  $(9, 12, 15)$ ,  $(12, 16, 20)$ ,  $(15, 20, 25)$  are all Pythagorean triples because they are multiples of  $(3, 4, 5)$ .
- Also note that if  $(a, b, c)$  is a Pythagorean triple, then  $(b, a, c)$  is also a Pythagorean triple. To reduce redundancy, we often write the smaller number of  $a$  and  $b$  first. Although  $(3, 4, 5)$  and  $(4, 3, 5)$  are both Pythagorean triples, they represent the same triple and we refer to it as  $(3, 4, 5)$ .

- One way to generate Pythagorean triples is to use the expressions from Example 1:  $(x^2 - 1, 2x, x^2 + 1)$ .

Have students try a few as mental math exercises:  $x = 2$  gives  $(4, 3, 5)$ ,  $x = 3$  gives  $(8, 6, 10)$ ,  $x = 4$  gives  $(15, 8, 17)$ , and so on.

One of the Problem Set questions asks students to generalize triples from  $(x^2 - 1, 2x, x^2 + 1)$  to show that triples generated by  $(x^2 - y^2, 2xy, x^2 + y^2)$  also form Pythagorean triples for  $x > y > 0$ . The next example will help students see the general pattern.

**Example 2 (12 minutes)**

This example shows a clever way for students to remember that  $x^2 - 1, 2x,$  and  $x^2 + 1$  can be used to find Pythagorean triples.

**Example 2**

Next we describe an easy way to find Pythagorean triples using the expressions from Example 1. Look at the multiplication table below for  $\{1, 2, \dots, 9\}$ . Notice that the square numbers  $\{1, 4, 9, \dots, 81\}$  lie on the diagonal of this table.

- a. What value of  $x$  is used to generate the Pythagorean triple  $(15, 8, 17)$  by the formula  $(x^2 - 1, 2x, x^2 + 1)$ ? How do the numbers  $(1, 4, 4, 16)$  at the corners of the shaded square in the table relate to the values 15, 8, and 17?

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Using the value 4 for  $x$  gives the triple  $(15, 8, 17)$ . We see that  $1 = 1^2$  and  $16 = 4^2$ , and then we can take  $16 - 1 = 15$ , and  $16 + 1 = 17$ . We also have  $4 + 4 = 8$ .

- b. Now you try one. Form a square on the multiplication table below whose left-top corner is the 1 (as in the example above) and whose bottom-right corner is a square number. Use the sums or differences of the numbers at the vertices of your square to form a Pythagorean triple. Check that the triple you generate is a Pythagorean triple.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Answers will vary. Ask students to report their answers. For example, a student whose square has the bottom-right number 36 will generate  $36 - 1 = 35, 6 + 6 = 12,$  and  $36 + 1 = 37$ . Have students check that  $(12, 35, 37)$  is indeed a Pythagorean triple:  $12^2 + 35^2 = 1369,$  and  $36^2 = 1369$ .

Let's generalize this square to any square in the multiplication table where two opposite vertices of the square are square numbers.

- c. How can you use the sums or differences of the numbers at the vertices of the shaded square to get a triple (16, 30, 34)? Is it a Pythagorean triple?

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Following what we did above, take  $25 - 9 = 16$ ,  $15 + 15 = 30$ , and  $25 + 9 = 34$  to get the triple (16, 30, 34). Yes, it is a Pythagorean triple:  $16^2 + 30^2 = 900 + 256 = 1156 = 34^2$ .

- d. Using  $x$  instead of 5 and  $y$  instead of 3 in your calculations in part (c), write down a formula for generating Pythagorean triples in terms of  $x$  and  $y$ .

The calculation  $25 - 9$  generalizes to  $x^2 - y^2$  as the length of one leg. The length of other leg can be found by  $15 + 15 = 2(3 \cdot 5)$ , which generalizes to  $2xy$ . The length of the hypotenuse,  $25 + 9$ , generalizes to  $x^2 + y^2$ . It seems that Pythagorean triples can be generated by triples  $(x^2 - y^2, 2xy, x^2 + y^2)$  where  $x > y > 0$ .

In the Problem Set, students will prove that if  $x$  and  $y$  are positive integers with  $x > y$ , then  $(x^2 - y^2, 2xy, x^2 + y^2)$  is a Pythagorean triple, mimicking the proof of Example 1.

Closing (3 minutes)

- Pythagorean triples are triplets of positive integers  $(a, b, c)$  that satisfy the relationship  $a^2 + b^2 = c^2$ . Such a triple is called a Pythagorean triple because a right triangle with legs of length  $a$  and  $b$  will have a hypotenuse of length  $c$  by the Pythagorean Theorem.
- To generate a Pythagorean triple, take any two positive integers  $x$  and  $y$  with  $x > y$ , and compute  $(x^2 - y^2, 2xy, x^2 + y^2)$ .

Relevant Facts and Vocabulary

**Pythagorean Theorem:** If a right triangle has legs of length  $a$  and  $b$  units and hypotenuse of length  $c$  units, then  $a^2 + b^2 = c^2$ .

**Converse to the Pythagorean Theorem:** If the lengths  $a, b, c$  of the sides of a triangle are related by  $a^2 + b^2 = c^2$ , then the angle opposite the side of length  $c$  is a right angle.

**Pythagorean Triple:** A *Pythagorean triple* is a triplet of positive integers  $(a, b, c)$  such that  $a^2 + b^2 = c^2$ . The triplet  $(3, 4, 5)$  is a Pythagorean triple but  $(1, 1, \sqrt{2})$  is not, even though the numbers are side lengths of an isosceles right triangle.

Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 10: The Power of Algebra—Finding Pythagorean Triples

### Exit Ticket

Generate six Pythagorean triples using any method discussed during class. Explain each method you use.

## Exit Ticket Sample Solutions

Generate six Pythagorean triples using any method discussed during class. Explain each method you use.

*Answers will vary. One example should use either  $(x^2 - 1, 2x, x^2 + 1)$  or  $(x^2 - y^2, 2xy, x^2 + y^2)$ , but after that students can use the fact that a multiple of a Pythagorean triple is again a Pythagorean triple.*

## Problem Set Sample Solutions

1. Rewrite each expression as a sum or difference of terms.

a.  $(x - 3)(x + 3)$   
 $x^2 - 9$

b.  $(x^2 - 3)(x^2 + 3)$   
 $x^4 - 9$

c.  $(x^{15} + 3)(x^{15} - 3)$   
 $x^{30} - 9$

d.  $(x - 3)(x^2 + 9)(x + 3)$   
 $x^4 - 81$

e.  $(x^2 + y^2)(x^2 - y^2)$   
 $x^4 - y^4$

f.  $(x^2 + y^2)^2$   
 $x^4 + 2x^2y^2 + y^4$

g.  $(x - y)^2(x + y)^2$   
 $x^4 - 2x^2y^2 + y^4$

h.  $(x - y)^2(x^2 + y^2)^2(x + y)^2$   
 $x^8 - 2x^4y^4 + y^8$

2. Tasha used a clever method to expand and simplify  $(a + b + c)(a + b - c)$ . She grouped the addends together like this  $[(a + b) + c][(a + b) - c]$  and then expanded them to get the difference of two squares:

$$(a + b + c)(a + b - c) = [(a + b) + c][(a + b) - c] = (a + b)^2 - c^2 = a^2 + 2ab + b^2 - c^2.$$

a. Is Tasha's method correct? Explain why or why not.

*Yes, Tasha is correct. Expanding in the traditional way gives the same result.*

$$\begin{aligned}(a + b + c)(a + b - c) &= (a + b + c)a + (a + b + c)b - (a + b + c)c \\ &= a^2 + ba + ca + ab + b^2 + cb - ac - bc - c^2 \\ &= a^2 + 2ab + b^2 - c^2\end{aligned}$$

b. Use a version of her method to find  $(a + b + c)(a - b - c)$ .

$$(a + (b + c))(a - (b + c)) = a^2 - (b + c)^2 = a^2 - b^2 - 2bc - c^2$$

c. Use a version of her method to find  $(a + b - c)(a - b + c)$ .

$$(a + (b - c))(a - (b - c)) = a^2 - (b - c)^2 = a^2 - b^2 + 2bc - c^2$$

3. Use the difference of two squares identity to factor each of the following expressions.

a.  $x^2 - 81$   
 $(x - 9)(x + 9)$

b.  $(3x + y)^2 - (2y)^2$   
 $(3x - y)(3x + 3y) = 3(3x - y)(x + y)$

c.  $4 - (x - 1)^2$   
 $(3 - x)(1 + x)$

d.  $(x + 2)^2 - (y + 2)^2$   
 $(x - y)(x + y + 4)$

4. Show that the expression  $(x + y)(x - y) - 6x + 9$  may be written as the difference of two squares, and then factor the expression.

$$(x + y)(x - y) - 6x + 9 = x^2 - y^2 - 6x + 9 = (x^2 - 6x + 9) - y^2 = (x - 3)^2 - y^2 = (x - 3 - y)(x - 3 + y)$$

5. Show that  $(x + y)^2 - (x - y)^2 = 4xy$  for all real numbers  $x$  and  $y$ .

$$(x + y)^2 - (x - y)^2 = [(x + y) - (x - y)][(x + y) + (x - y)] = (2y)(2x) = 4x$$

6. Prove that a triangle with side lengths  $2xy$ ,  $x^2 - y^2$ , and  $x^2 + y^2$  with  $x > y > 0$  is a right triangle.

*The proof should look like the proof in Example 1 but with  $y$  instead of 1.*

7. Complete the table below to find Pythagorean triples (the first row is done for you).

$x$	$y$	$x^2 - y^2$	$2xy$	$x^2 + y^2$	Check: Is it a Pythagorean Triple?
2	1	3	4	5	Yes: $3^2 + 4^2 = 25$ , $5^2 = 25$
3	1	8	6	10	Yes: $8^2 + 6^2 = 100$ , $10^2 = 100$
3	2	5	12	13	Yes: $5^2 + 12^2 = 169$ , $13^2 = 169$
4	1	15	8	17	Yes: $15^2 + 8^2 = 289$ , $17^2 = 289$
4	2	12	16	20	Yes: $12^2 + 16^2 = 400$ , $20^2 = 400$
4	3	7	24	25	Yes: $7^2 + 24^2 = 625$ , $25^2 = 625$
5	1	24	10	26	Yes: $24^2 + 10^2 = 676$ , $26^2 = 676$

8. Answer the following parts about the triple (9, 12, 15).

- a. Show that (9, 12, 15) is a Pythagorean triple.

*We see that  $9^2 + 12^2 = 81 + 144 = 225$ , and  $15^2 = 225$  so  $9^2 + 12^2 = 15^2$ .*

- b. Prove that neither (9, 12, 15) nor (12, 9, 15) can be found by choosing a pair of integers  $x$  and  $y$  with  $x > y$  and computing  $(x^2 - y^2, 2xy, x^2 + y^2)$ .

*(Hint: What are the possible values of  $x$  and  $y$  if  $2xy = 12$ ? What about if  $2xy = 9$ ?)*

*Proof: Since 9 is odd and  $2xy$  is even, there are no integer values of  $x$  and  $y$  that satisfy  $2xy = 9$ . Thus, our formula cannot generate the triple (12, 9, 15). Now suppose  $x$  and  $y$  are integers such that  $2xy = 12$ . Thus  $xy = 6$  and  $x > y$ . There are only two possibilities: either  $x = 6$  and  $y = 1$ , or  $x = 3$  and  $y = 2$ . In the first case, our formula generates the triple  $(6^2 - 1^2, 2 \cdot 6 \cdot 1, 6^2 + 1^2) = (35, 12, 37)$ . In the second case, our formula generates the triple  $(3^2 - 2^2, 2 \cdot 3 \cdot 2, 3^2 + 2^2) = (5, 12, 13)$ . Thus, there is no way to generate the triple (9, 12, 15) using this method, even though it is a Pythagorean triple.*

- c. Wouldn't it be nice if all Pythagorean triples were generated by the expressions  $x^2 - y^2, 2xy, x^2 + y^2$ ? Research Pythagorean triples on the Internet to discover what is known to be true about generating all Pythagorean triples using these three expressions.

*All Pythagorean triples are some multiple of a Pythagorean triple generated using the three expressions. For example, while (9, 12, 15) is not generated by the three expressions, it is a multiple of a Pythagorean triple (3, 4, 5), which is generated by the three expressions.*

9. Follow the steps below to prove the identity  $(a^2 + b^2)(x^2 + y^2) = (ax - by)^2 + (bx + ay)^2$ .

- a. Multiply  $(a^2 + b^2)(x^2 + y^2)$ .

$$(a^2 + b^2)(x^2 + y^2) = a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2$$

- b. Square both binomials in  $(ax - by)^2 + (bx + ay)^2$  and collect like terms.

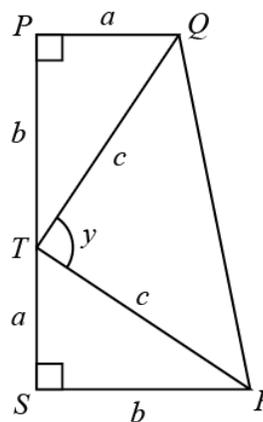
$$\begin{aligned} (ax - by)^2 + (bx + ay)^2 &= a^2x^2 - 2abxy + b^2y^2 + b^2x^2 + 2abxy + a^2y^2 \\ &= a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2 \end{aligned}$$

- c. Use your answers from part (a) and part (b) to prove the identity.

$$\begin{aligned} (a^2 + b^2)(x^2 + y^2) &= a^2x^2 + a^2y^2 + b^2x^2 + b^2y^2 \\ &= (ax - by)^2 + (bx + ay)^2 \end{aligned}$$

10. Many U.S. presidents took great delight in studying mathematics. For example, President James Garfield, while still a congressman, came up with a proof of the Pythagorean Theorem based upon the ideas presented below.

In the diagram, two congruent right triangles with side lengths  $a$ ,  $b$ , and hypotenuse  $c$ , are used to form a trapezoid  $PQRS$  composed of three triangles.



- a. Explain why  $\angle QTR$  is a right angle.

*Since  $\angle TSR$  is a right angle, the measures of  $\angle STR$  and  $\angle SRT$  sum to  $90^\circ$ , so  $\angle STR$  and  $\angle SRT$  are complementary angles. Since  $\triangle TSR \cong \triangle QPT$  by SSS triangle congruence, we have  $\angle SRT \cong \angle PTQ$ . Thus,  $\angle PTQ$  and  $\angle STR$  must also be complementary. By the angle sum properties,*

$$m\angle QTR + \underbrace{m\angle PTQ + m\angle STR}_{90^\circ} = 180^\circ$$

so that

$$m\angle QTR + 90^\circ = 180^\circ$$

and we have shown that  $m\angle QTR = 90^\circ$ . Thus,  $\angle QTR$  is a right angle.

- b. What are the areas of  $\triangle STR$ ,  $\triangle PTQ$ , and  $\triangle QTR$  in terms of  $a$ ,  $b$ , and  $c$ ?

*We see that  $A(\triangle STR) = \frac{1}{2}ab$ ,  $A(\triangle PTQ) = \frac{1}{2}ab$ , and because  $\angle QTR$  is a right angle,  $A(\triangle QTR) = \frac{1}{2}c^2$ .*

- c. Using the formula for the area of a trapezoid, what is the total area of trapezoid  $PQRS$  in terms of  $a$  and  $b$ ?

$$A(PQRS) = \frac{1}{2}(a + b)(a + b)$$

- d. Set the sum of the areas of the three triangles from part (b) equal to the area of the trapezoid you found in part (c), and simplify the equation to derive a relationship between  $a$ ,  $b$ , and  $c$ . Conclude that a right triangle with legs of length  $a$  and  $b$  and hypotenuse of length  $c$  must satisfy the relationship  $a^2 + b^2 = c^2$ .

*Equate areas:*

$$\frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{2}c^2 = \frac{1}{2}(a + b)(a + b),$$

$$ab + \frac{1}{2}c^2 = \frac{1}{2}(a^2 + 2ab + b^2).$$

*Multiply both sides by 2,*

$$2ab + c^2 = a^2 + 2ab + b^2,$$

*and subtract  $2ab$  from both sides,*

$$c^2 = a^2 + b^2.$$



## Lesson 11: The Special Role of Zero in Factoring

### Student Outcomes

- Students find solutions to polynomial equations where the polynomial expression is not factored into linear factors.
- Students construct a polynomial function that has a specified set of zeros with stated multiplicity.

### Lesson Notes

This lesson focuses on the first part of standard **A-APR.B.3**, identifying zeros of polynomials presented in factored form. Although the terms root and zero are interchangeable, for consistency only the term zero is used throughout this lesson and in later lessons. The second part of the standard, using the zeros to construct a rough graph of a polynomial function, is delayed until Lesson 14. The ideas that begin in this lesson continue in Lesson 19, in which students will be able to associate a zero of a polynomial function to a factor in the factored form of the associated polynomial as a consequence of the Remainder Theorem, and culminate in Lesson 39, in which students apply the Fundamental Theorem of Algebra to factor polynomial expressions completely over the complex numbers.

### Classwork

#### Opening Exercise (12 minutes)

##### Opening Exercise

Find all solutions to the equation  $(x^2 + 5x + 6)(x^2 - 3x - 4) = 0$ .

The main point of this opening exercise is for students to recognize and then formalize that the statement “If  $ab = 0$ , then  $a = 0$  or  $b = 0$ ” applies not only when  $a$  and  $b$  are numbers or linear functions (which we used when solving a quadratic equation), but also applies to cases where  $a$  and  $b$  are polynomial functions of any degree.

In small groups, let students discuss ways to solve this equation. Walk around the room and offer advice such as, “Have you considered factoring each quadratic expression? What do you get?” As soon as one group factors both quadratic expressions, or when three minutes have passed, show, or let that group show, the factorization on the board.

$$\frac{(x+2)(x+3)}{x^2+5x+6} \cdot \frac{(x-4)(x+1)}{x^2-3x-4} = 0$$

- What are the solutions to this equation?
  - $-2, -3, 4, -1$

#### Scaffolding:

Here is an alternative opening activity that may better illuminate the special role of zero.

- For each equation, list some possible values for  $x$  and  $y$ .
  - $xy = 10, xy = 1,$   
 $xy = -1, xy = 0$
- What do you notice? Does one equation tell you more information than others?

- Why?
  - If  $x$  is any number other than  $-2, -3, 4, -1$ , then each factor is a nonzero number, i.e.,  $x + 2 \neq 0, x + 3 \neq 0$ , etc. However, the multiplication of four nonzero numbers is nonzero, so that value of  $x$  cannot be a solution. Therefore, the only possible solutions are  $-2, -3, 4$ , and  $-1$ . It is easy to confirm that these are indeed solutions by substituting them each into the equation individually.
- Why are these numbers also solutions to the original equation?
  - Because the expression  $(x + 2)(x + 3)(x - 4)(x + 1)$  is equivalent to  $(x^2 + 5x + 6)(x^2 - 3x - 4)$ .

Now let's study the solutions to  $x^2 + 5x + 6 = 0$  and  $x^2 - 3x - 4 = 0$  separately.

- What are the solutions to  $x^2 + 5x + 6 = 0$ ?
  - $-2, -3$
- What are the solutions to  $x^2 - 3x - 4 = 0$ ?
  - $4, -1$
- Relate the solutions of the equation  $(x^2 + 5x + 6)(x^2 - 3x - 4) = 0$  to the solutions of the compound statement, " $x^2 + 5x + 6 = 0$  or  $x^2 - 3x - 4 = 0$ ."
  - They are the same.
- Given two polynomial functions  $p$  and  $q$  of any degree, the solution set of the equation  $p(x)q(x) = 0$  is the union of the solution set of  $p(x) = 0$  and the solution set of  $q(x) = 0$ . Can you explain why?

Lead students in a discussion of the following proof:

- Suppose  $a$  is a solution to the equation  $p(x)q(x) = 0$ ; that is, it is a number that satisfies  $p(a)q(a) = 0$ . Since  $p(a)$  is a number and  $q(a)$  is a number, one or both of them must be zero, by the zero product property that states, "If the product of two numbers is zero, then at least one of the numbers is zero." Therefore,  $p(a) = 0$  or  $q(a) = 0$ , which means  $a$  is a solution to the compound statement, " $p(x) = 0$  or  $q(x) = 0$ ."
- Now let's prove the other direction and show that if  $a$  is a solution to the compound statement, then it is a solution to the equation  $p(x)q(x) = 0$ . This direction is also easy: Suppose  $a$  is a number such that either  $p(a) = 0$  or  $q(a) = 0$ . In the first case,  $p(a)q(a) = 0 \cdot q(a) = 0$ . In the second case,  $p(a)q(a) = p(a) \cdot 0 = 0$ . Hence, in either case,  $a$  is a solution to the equation  $p(x)q(x) = 0$ .

Students may have difficulty understanding the distinction between the equations  $p(x)q(x) = 0$  and  $p(a)q(a) = 0$ . Help students understand that  $p(x)q(x) = 0$  is an equation in a variable  $x$ , while  $p(a)$  is the value of the function  $p$  when it is evaluated at the number. Thus,  $p(a)q(a)$  is a number. For example, if  $p$  and  $q$  are the quadratic polynomials in the Opening Exercise, and we are considering the case when  $a$  is 5, then  $p(5) = 56$  and  $q(5) = 6$ . Therefore, 5 cannot be a solution to the equation  $p(x)q(x) = 0$ .

- Communicate to students that they can use the statement below to break problems into simpler parts:
 

Given any two polynomial functions  $p$  and  $q$ , the set of solutions to the equation  $p(x)q(x) = 0$  can be found by solving  $p(x) = 0$ , solving  $q(x) = 0$ , and combining the solutions into one set.

Ask your students to try the following exercise on their own.

MP.1

## Exercise 1 (2 minutes)

## Exercise 1

1. Find the solutions of  $(x^2 - 9)(x^2 - 16) = 0$ .

*The solutions to  $(x^2 - 9)(x^2 - 16)$  are the solutions of  $x^2 - 9 = 0$  combined with the solutions of  $x^2 - 16 = 0$ . These solutions are  $-3, 3, -4,$  and  $4$ .*

In the next example, we look at a polynomial equation for which we already know a solution. The goal of this example and the discussion that follows is to use a solution to the equation  $f(x) = 0$  to further factor the polynomial  $f$ . In doing so, we end with a description of the zeros of a function, a concept first introduced in Algebra I, Module 4.

## Example 1 (8 minutes)

## Example 1

Suppose we know that the polynomial equation  $4x^3 - 12x^2 + 3x + 5 = 0$  has three real solutions and that one of the factors of  $4x^3 - 12x^2 + 3x + 5$  is  $(x - 1)$ . How can we find all three solutions to the given equation?

Steer the discussion to help students conjecture that

$$4x^3 - 12x^2 + 3x + 5 = (x - 1)(\text{some quadratic polynomial}).$$

Since  $(x - 1)$  is a factor and we know how to divide polynomials, we can find the quadratic polynomial by dividing:

$$\frac{4x^3 - 12x^2 + 3x + 5}{x - 1} = 4x^2 - 8x - 5.$$

Now we know that  $4x^3 - 12x^2 + 3x + 5 = (x - 1)(4x^2 - 8x - 5)$ , and we also know that  $4x^2 - 8x - 5$  is a quadratic polynomial that has linear factors  $(2x + 1)$  and  $(2x - 5)$ .

Therefore,  $4x^3 - 12x^2 + 3x + 5 = 0$  has the same solutions as  $(x - 1)(4x^2 - 8x - 5) = 0$ , which has the same solutions as

$$(x - 1)(2x + 1)(2x - 5) = 0.$$

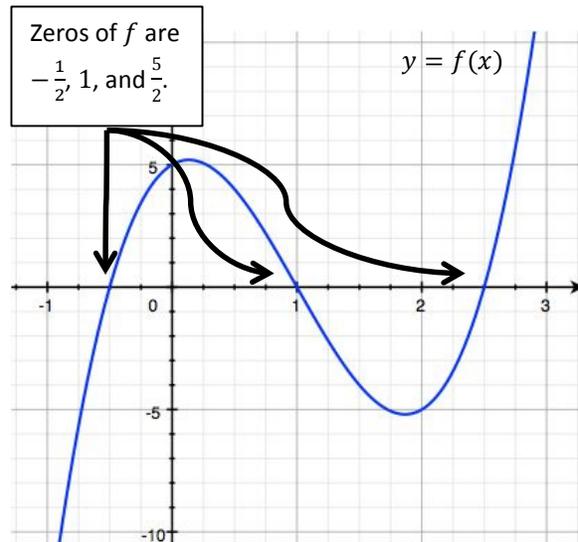
In this factored form, the solutions of  $f(x) = 0$  are readily apparent:  $-\frac{1}{2}$ ,  $1$ , and  $\frac{5}{2}$ .

*Scaffolding:*

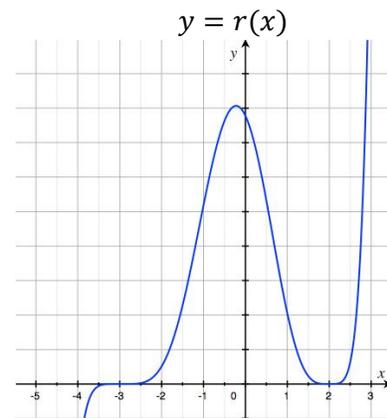
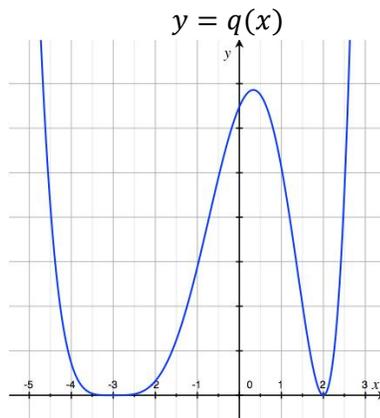
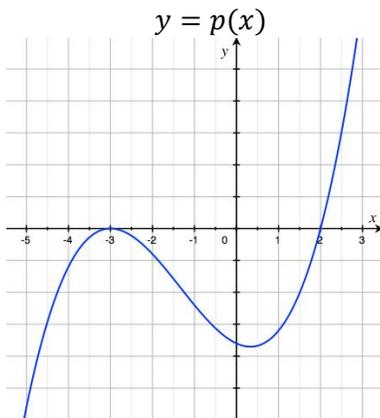
Allow students to generate ideas about how the linear factors affect the behavior of the graph so that they can use the graph of a function to identify zeros.

**Discussion (8 minutes)**

- In Example 1 above, we saw that factoring the polynomial into linear factors helped us to find solutions to the original polynomial equation  $4x^3 - 12x^2 + 3x + 5 = 0$ .
- There is a corresponding notion for the zeros of a function. Let  $f$  be a function whose domain is a subset of the real numbers and whose range is a subset of the real numbers. A *zero (or root) of the function  $f$*  is a number  $c$  such that  $f(c) = 0$ .
- The zeros of the function  $f(x) = 4x^3 - 12x^2 + 3x + 5$  are the  $x$ -intercepts of the graph of  $f$ : these are  $-\frac{1}{2}$ ,  $1$ , and  $\frac{5}{2}$ .
- By definition, a zero of a polynomial function  $f$  is a solution to the equation  $f(x) = 0$ . If  $(x - a)$  is a factor of a polynomial function  $f$ , then  $f(a) = 0$  and  $a$  is a zero of  $f$ .



However, consider the polynomial functions  $p(x) = (x - 2)(x + 3)^2$ ,  $q(x) = (x - 2)^2(x + 3)^4$ , and  $r(x) = (x - 2)^4(x + 3)^5$ . Because  $p(2) = 0$ ,  $q(2) = 0$ , and  $r(2) = 0$ , the number 2 is a zero of  $p$ ,  $q$ , and  $r$ . Likewise,  $-3$  is also a zero of  $p$ ,  $q$ , and  $r$ . Even though these polynomial functions have the same zeros, they are not the same function; they do not even have the same degree!



We would like to be able to distinguish between the zeros of these two polynomial functions. If we write out all of the factors for  $p$ ,  $q$ , and  $r$ , we see that

$$p(x) = (x - 2)(x + 3)(x + 3)$$

$$q(x) = (x - 2)(x - 2)(x + 3)(x + 3)(x + 3)(x + 3)$$

$$r(x) = (x - 2)(x - 2)(x - 2)(x - 2)(x + 3)(x + 3)(x + 3)(x + 3)(x + 3)$$

- We notice that  $(x - 2)$  is a factor of  $p$  once, and  $(x + 3)$  is a factor of  $p$  twice. Thus, we say that 2 is a zero of  $p$  of multiplicity 1, and  $-3$  is a zero of  $p$  of multiplicity 2. Zeros of multiplicity 1 are usually just referred to as zeros, without mentioning the multiplicity.
- What are the zeros of  $q$ , with their multiplicities?
  - For  $q$ , 2 is a zero of multiplicity 2, and  $-3$  is a zero of multiplicity 4.
- What are the zeros of  $r$ , with their multiplicities?
  - For  $r$ , 2 is a zero of multiplicity 4, and  $-3$  is a zero of multiplicity 5.
- Can you look at the factored form of a polynomial equation and identify the zeros with their multiplicities? Explain how you know.
  - Yes. Each linear factor  $(ax - b)^m$  of the polynomial will produce a zero  $\frac{b}{a}$  with multiplicity  $m$ .
- Can multiplicity be negative? Can it be zero? Can it be a fraction?
  - No. Multiplicity is the count of the number of times a factor appears in a factored polynomial expression. Polynomials can only have positive integer exponents, so a factor must have positive integer exponents. Thus, the multiplicity of a zero must be a positive integer.

Note: In Lesson 14, students will use the zeros of a polynomial function together with their multiplicities to create a graph of the function, and in Lesson 19, students will use the zeros of a polynomial function with their multiplicities to construct the equation of the function.

### Exercises 2–5 (8 minutes)

#### Exercises 2–5

2. Find the zeros of the following polynomial functions, with their multiplicities.

a.  $f(x) = (x + 1)(x - 1)(x^2 + 1)$

$-1$  with multiplicity 1

$1$  with multiplicity 1

b.  $g(x) = (x - 4)^3(x - 2)^8$

$4$  with multiplicity 3

$2$  with multiplicity 8

c.  $h(x) = (2x - 3)^5$

$\frac{3}{2}$  with multiplicity 5

d.  $k(x) = (3x + 4)^{100}(x - 17)^4$

$-\frac{4}{3}$  with multiplicity 100

$17$  with multiplicity 4

3. Find a polynomial function that has the following zeros and multiplicities. What is the degree of your polynomial?

Zero	Multiplicity
2	3
-4	1
6	6
-8	10

$$p(x) = (x - 2)^3(x + 4)(x - 6)^6(x + 8)^{10}$$

The degree of  $p$  is 20.

4. Is there more than one polynomial function that has the same zeros and multiplicities as the one you found in Exercise 3?

Yes. Consider  $q(x) = (x^2 + 5)(x - 2)^3(x + 4)(x - 6)^6(x + 8)^{10}$ . Since there are no real solutions to  $x^2 + 5 = 0$ , adding this factor does not produce a new zero. Thus  $p$  and  $q$  have the same zeros and multiplicities but are different functions.

5. Can you find a rule that relates the multiplicities of the zeros to the degree of the polynomial function?

Yes. If  $p$  is a polynomial function of degree  $n$ , then the sum of the multiplicities of all of the zeros is less than or equal to  $n$ . If  $p$  can be factored into linear terms, then the sum of the multiplicities of all of the zeros is exactly equal to  $n$ .

### Closing (2 minutes)

Ask students to summarize the key ideas of the lesson, either in writing or with a neighbor. Consider posing the questions below.

- Part of the lesson today has been that given two polynomials,  $p$  and  $q$ , we can determine solutions to  $p(x)q(x) = 0$  by solving both  $p(x) = 0$  and  $q(x) = 0$ , even if they are high degree polynomials. If  $p$  and  $q$  are polynomial functions that do not have any real number zeros, do you think the equation  $p(x)q(x) = 0$  still has real number solutions? Can you give an example of two such functions?
  - If  $p(x) \neq 0$  for all real numbers  $x$  and  $q(x) \neq 0$  for all real numbers  $x$ , then there is no possible way to have  $p(x)q(x) = 0$ .
  - For example: If  $p(x) = x^2 + 1$  and  $q(x) = x^4 + 1$ , then the equation  $(x^2 + 1)(x^4 + 1) = 0$  has no real solutions.

The following vocabulary was introduced in Algebra I (please see Module 3 and Module 4 in Algebra I). While you should not have to teach these terms explicitly, it may still be a good idea to go through them with your class.

#### Relevant Vocabulary Terms

In the definitions below, the symbol  $\mathbb{R}$  stands for the set of real numbers.

**Function:** A function is a correspondence between two sets,  $X$  and  $Y$ , in which each element of  $X$  is assigned to one and only one element of  $Y$ .

The set  $X$  in the definition above is called the *domain of the function*. The *range (or image)* of the function is the subset of  $Y$ , denoted  $f(X)$ , that is defined by the following property:  $y$  is an element of  $f(X)$  if and only if there is an  $x$  in  $X$  such that  $f(x) = y$ .

If  $f(x) = x^2$  where  $x$  can be any real number, then the domain is all real numbers (denoted  $\mathbb{R}$ ), and the range is the set of non-negative real numbers.

**Polynomial Function:** Given a polynomial expression in one variable, a *polynomial function in one variable* is a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that for each real number  $x$  in the domain,  $f(x)$  is the value found by substituting the number  $x$  into all instances of the variable symbol in the polynomial expression and evaluating.

It can be shown that if a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a polynomial function, then there is some non-negative integer  $n$  and collection of real numbers  $a_0, a_1, a_2, \dots, a_n$  with  $a_n \neq 0$  such that the function satisfies the equation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0,$$

for every real number  $x$  in the domain, which is called the *standard form of the polynomial function*. The function  $f(x) = 3x^3 + 4x^2 + 4x + 7$ , where  $x$  can be any real number, is an example of a function written in standard form.

**Degree of a Polynomial Function:** The *degree of a polynomial function* is the degree of the polynomial expression used to define the polynomial function.

The degree of  $f(x) = 8x^3 + 4x^2 + 7x + 6$  is 3, but the degree of  $g(x) = (x + 1)^2 - (x - 1)^2$  is 1 because when  $g$  is put into standard form, it is  $g(x) = 4x$ .

**Constant Function:** A *constant function* is a polynomial function of degree 0. A constant function is of the form  $f(x) = c$ , for a constant  $c$ .

**Linear Function:** A *linear function* is a polynomial function of degree 1. A linear function is of the form  $f(x) = ax + b$ , for constants  $a$  and  $b$  with  $a \neq 0$ .

**Quadratic Function:** A *quadratic function* is a polynomial function of degree 2. A quadratic function is in *standard form* if it is written in the form  $f(x) = ax^2 + bx + c$ , for constants  $a, b, c$  with  $a \neq 0$  and any real number  $x$ .

**Cubic Function:** A *cubic function* is a polynomial function of degree 3. A cubic function is of the form  $f(x) = ax^3 + bx^2 + cx + d$ , for constants  $a, b, c, d$  with  $a \neq 0$ .

**Zeros or Roots of a Function:** A *zero (or root)* of a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a number  $x$  of the domain such that  $f(x) = 0$ . A zero of a function is an element in the solution set of the equation  $f(x) = 0$ .

#### Lesson Summary

Given any two polynomial functions  $p$  and  $q$ , the solution set of the equation  $p(x)q(x) = 0$  can be quickly found by solving the two equations  $p(x) = 0$  and  $q(x) = 0$  and combining the solutions into one set.

The number  $a$  is zero of a polynomial function  $p$  with multiplicity  $m$  if the factored form of  $p$  contains  $(x - a)^m$ .

#### Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 11: The Special Role of Zero in Factoring

### Exit Ticket

A polynomial function  $p$  can be factored into seven factors:  $(x - 3)$ ,  $(x + 1)$ , and 5 factors of  $(x - 2)$ . What are its zeros with multiplicity, and what is the degree of the polynomial? Explain how you know.

## Exit Ticket Sample Solutions

A polynomial function  $p$  can be factored into seven factors:  $(x - 3)$ ,  $(x + 1)$ , and 5 factors of  $(x - 2)$ . What are its zeros with multiplicity, and what is the degree of the polynomial? Explain how you know.

**Zeros:** 3 with multiplicity 1;  $-1$  with multiplicity 1; 2 with multiplicity 5

*The polynomial has degree seven. There are seven linear factors as given above, so  $p(x) = (x - 3)(x + 1)(x - 2)^5$ . If the factors were multiplied out, the leading term would be  $x^7$ , so the degree of  $p$  is 7.*

## Problem Set Sample Solutions

For Problems 1–4, find all solutions to the given equations.

1.  $(x - 3)(x + 2) = 0$

3,  $-2$

2.  $(x - 5)(x + 2)(x + 3) = 0$

5,  $-2$ ,  $-3$

3.  $(2x - 4)(x + 5) = 0$

2,  $-5$

4.  $(2x - 2)(3x + 1)(x - 1) = 0$

1,  $-\frac{1}{3}$ , 1

5. Find four solutions to the equation  $(x^2 - 9)(x^4 - 16) = 0$ .

2,  $-2$ , 3,  $-3$

6. Find the zeros with multiplicity for the function  $p(x) = (x^3 - 8)(x^5 - 4x^3)$ .

*We can factor  $p$  to give  $p(x) = x^3(x - 2)(x^2 + 2x + 4)(x - 2)(x + 2) = x^3(x - 2)^2(x + 2)(x^2 + 2x + 4)$ . Then 0 is a zero of multiplicity 3,  $-2$  is a zero of multiplicity 1, and 2 is a zero of multiplicity 2.*

7. Find two different polynomial functions that have zeros at 1, 3, and 5 of multiplicity 1.

$p(x) = (x - 1)(x - 3)(x - 5)$  and  $q(x) = (x^2 + 1)(x - 1)(x - 3)(x - 5)$

8. Find two different polynomial functions that have a zero at 2 of multiplicity 5 and a zero at  $-4$  of multiplicity 3.

$p(x) = (x - 2)^5(x + 4)^3$  and  $q(x) = (x^2 + 1)(x - 2)^5(x + 4)^3$

9. Find three solutions to the equation  $(x^2 - 9)(x^3 - 8) = 0$ .

*From Lesson 6, we know that  $(x - 2)$  is a factor of  $(x^3 - 8)$ , so three solutions are 3,  $-3$ , and 2.*

10. Find two solutions to the equation  $(x^3 - 64)(x^5 - 1) = 0$ .

*From Lesson 6, we know that  $(x - 4)$  is a factor of  $(x^3 - 64)$ , and  $(x - 1)$  is a factor of  $(x^5 - 1)$ , so two solutions are 1 and 4.*

11. If  $p, q, r, s$  are non-zero numbers, find the solutions to the equation  $(px + q)(rx + s) = 0$  in terms of  $p, q, r, s$ .

*Setting each factor equal to zero gives solutions  $-\frac{q}{p}$  and  $-\frac{s}{r}$ .*

Use the identity  $a^2 - b^2 = (a - b)(a + b)$  to solve the equations given in Problems 12–13.

12.  $(3x - 2)^2 = (5x + 1)^2$

*Using algebra, we have  $(3x - 2)^2 - (5x + 1)^2 = 0$ . Applying the difference of squares formula, we have  $((3x - 2) - (5x + 1))((3x - 2) + (5x + 1)) = 0$ . Combining like terms gives  $(-2x - 3)(8x - 1) = 0$ , so the solutions are  $-\frac{3}{2}$  and  $\frac{1}{8}$ .*

13.  $(x + 7)^2 = (2x + 4)^2$

*Using algebra, we have  $(x + 7)^2 - (2x + 4)^2 = 0$ . Then  $((x + 7) - (2x + 4))((x + 7) + (2x + 4)) = 0$ , so we have  $(-x + 3)(3x + 11) = 0$ . Thus the solutions are  $-\frac{11}{3}$  and 3.*

14. Consider the polynomial function  $P(x) = x^3 + 2x^2 + 2x - 5$ .

- a. Divide  $P$  by the divisor  $(x - 1)$  and rewrite in the form  $P(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$ .

$$P(x) = (x - 1)(x^2 + 3x + 5) + 0$$

- b. Evaluate  $P(1)$ .

$$P(1) = 0$$

15. Consider the polynomial function  $Q(x) = x^6 - 3x^5 + 4x^3 - 12x^2 + x - 3$ .

- a. Divide  $Q$  by the divisor  $(x - 3)$  and rewrite in the form  $Q(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$ .

$$Q(x) = (x - 3)(x^5 + 4x^2 + 1) + 0$$

- b. Evaluate  $Q(3)$ .

$$Q(3) = 0$$

16. Consider the polynomial function  $R(x) = x^4 + 2x^3 - 2x^2 - 3x + 2$ .

- a. Divide  $R$  by the divisor  $(x + 2)$  and rewrite in the form  $R(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$ .

$$R(x) = (x + 2)(x^3 - 2x + 1) + 0$$

- b. Evaluate  $R(-2)$ .

$$R(-2) = 0$$

17. Consider the polynomial function  $S(x) = x^7 + x^6 - x^5 - x^4 + x^3 + x^2 - x - 1$ .

- a. Divide  $S$  by the divisor  $(x + 1)$  and rewrite in the form  $S(x) = (\text{divisor})(\text{quotient}) + \text{remainder}$ .

$$S(x) = (x + 1)(x^6 - x^4 + x^2 - 1) + 0$$

- b. Evaluate  $S(-1)$ .

$$S(-1) = 0$$

18. Make a conjecture based on the results of Questions 14–17.

*It seems that the zeros  $a$  of a polynomial function correspond to factors  $(x - a)$  in the equation of the polynomial.*



## Topic B:

## Factoring—Its Use and Its Obstacles

N-Q.A.2, A-SSE.A.2, A-APR.B.2, A-APR.B.3, A-APR.D.6, F-IF.C.7c

<b>Focus Standard:</b>	N-Q.A.2	Define appropriate quantities for the purpose of descriptive modeling.*
	A-SSE.A.2	Use the structure of an expression to identify ways to rewrite it. <i>For example, see <math>x^4 - y^4</math> as <math>(x^2)^2 - (y^2)^2</math>, thus recognizing it as a difference of squares that can be factored as <math>(x^2 - y^2)(x^2 + y^2)</math>.</i>
	A-APR.B.2	Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number $a$ , the remainder on division by $x - a$ is $p(a)$ , so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$ .
	A-APR.B.3	Identify zeros of polynomials when suitable factorizations are available, and use the zeros to construct a rough graph of the function defined by the polynomial.
	A-APR.D.6	Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$ , where $a(x)$ , $b(x)$ , $q(x)$ , and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$ , using inspection, long division, or, for the more complicated examples, a computer algebra system.
	F-IF.C.7	Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.* c. Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.
<b>Instructional Days:</b>	10	
	<b>Lesson 12:</b>	Overcoming Obstacles in Factoring (S) <sup>1</sup>
	<b>Lesson 13:</b>	Mastering Factoring (P)
	<b>Lesson 14:</b>	Graphing Factored Polynomials (S)
	<b>Lesson 15:</b>	Structure in Graphs of Polynomial Functions (P)
	<b>Lessons 16–17:</b>	Modeling with Polynomials—An Introduction (M, M)
	<b>Lesson 18:</b>	Overcoming a Second Obstacle in Factoring—What If There Is a Remainder? (P)

<sup>1</sup> Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson

**Lesson 19:** The Remainder Theorem (S)

**Lessons 20–21:** Modeling Riverbeds with Polynomials (M, M)

Armed with a newfound knowledge of the value of factoring, students develop their facility with factoring and then apply the benefits to graphing polynomial equations in Topic B. In Lessons 12–13, students are presented with the first obstacle to solving equations successfully. While dividing a polynomial by a given factor to find a missing factor is easily accessible, factoring without knowing one of the factors is challenging. Students recall the work with factoring done in Algebra I and expand on it to master factoring polynomials with degree greater than two, emphasizing the technique of factoring by grouping.

In Lessons 14–15, students find that another advantage to rewriting polynomial expressions in factored form is how easily a polynomial function written in this form can be graphed. Students read word problems to answer polynomial questions by examining key features of their graphs. They notice the relationship between the number of times a factor is repeated and the behavior of the graph at that zero (i.e., when a factor is repeated an even number of times, the graph of the polynomial will touch the  $x$ -axis and “bounce” back off, whereas when a factor occurs only once or an odd number of times, the graph of the polynomial at that zero will “cut through” the  $x$ -axis). In these lessons, students will compare hand plots to graphing-calculator plots and zoom in on the graph to examine its features more closely.

In Lessons 16–17, students encounter a series of more serious modeling questions associated with polynomials, developing their fluency in translating between verbal, numeric, algebraic, and graphical thinking. One example of the modeling questions posed in this lesson is how to find the maximum possible volume of a box created from a flat piece of cardboard with fixed dimensions.

In Lessons 18–19, students are presented with their second obstacle: “What if there is a remainder?” They learn the Remainder Theorem and apply it to further understand the connection between the factors and zeros of a polynomial and how this relates to the graph of a polynomial function. Students explore how to determine the smallest possible degree for a depicted polynomial and how information such as the value of the  $y$ -intercept will be reflected in the equation of the polynomial.

The topic culminates with two modeling lessons (Lessons 20–21) involving approximating the area of the cross-section of a riverbed to model the volume of flow. The problem description includes a graph of a polynomial equation that could be used to model the situation, and students are challenged to find the polynomial equation itself.



## Lesson 12: Overcoming Obstacles in Factoring

### Student Outcomes

- Students will factor certain forms of polynomial expressions by using the structure of the polynomials.

### Lesson Notes

Students have factored polynomial expressions in earlier lessons and in earlier courses. In this lesson, we explore further techniques for factoring polynomial expressions, including factoring by completing the square, by applying the quadratic formula and by grouping. We apply these techniques to solve polynomial equations.

The idea of the greatest common factor (GCF) is important to this lesson. The teacher may want to consider displaying a GCF poster on the classroom wall for reference. Consider using some of the problem set exercises during the lesson to supplement the examples included here.

### Classwork

#### Opening (4 minutes)

Consider the following polynomial equation.

$$(x^2 - 4x + 3)(x^2 + 4x - 5) = 0$$

Discuss the following questions in pairs or small groups:

- What is the degree of this polynomial? How do you know?
- How many solutions to this equation should there be? How do you know?
- How might you begin to solve this equation?

#### Scaffolding:

Ask struggling students to first solve the equation  $(x - 3)(x - 1) = 0$ . Point out that they have used the zero product property, and ask how that property applies to the given problem.

We can solve this equation by factoring because we can solve each of the equations

$$x^2 - 4x + 3 = 0$$

$$x^2 + 4x - 5 = 0.$$

There is no need to solve the whole way through; students completed a problem like this in Lesson 11. The idea is that students see that this can be done relatively quickly. The factored form of the original equation is

$$(x^2 - 4x + 3)(x^2 + 4x - 5) = (x - 1)(x - 3)(x - 1)(x + 5) = (x - 1)^2(x - 3)(x + 5) = 0,$$

and the three solutions are 1, 3, and  $-5$ .

However, consider the next example.

#### Example 1 (8 minutes)

##### Example 1

Find all real solutions to the equation  $(x^2 - 6x + 3)(2x^2 - 4x - 7) = 0$ .

Allow students the opportunity to struggle with factoring these expressions, discuss with their neighbors, and reach the conclusion that neither expression can be factored with integer coefficients.

- We have discovered an obstacle to factoring. The expressions  $x^2 - 6x + 3$  and  $2x^2 - 4x - 7$  do not factor as readily as the examples from the previous lesson. Does anybody recall how we might factor them?

Students have completed the square in both Geometry and Algebra I, so give them an opportunity to recall the process.

- When a quadratic expression is not easily factorable, we can either apply a technique called *completing the square*, or we can use the *quadratic formula*. Let's factor the first expression by completing the square.
- We first create some space between the  $x$  term and the constant term:

$$x^2 - 6x + \underline{\quad} - \underline{\quad} + 3 = 0.$$

- The next step is the key step. Take half of the coefficient of the  $x$  term, square that number, and add and subtract it in the space we created:

$$\begin{aligned} x^2 - 6x + (-3)^2 - (-3)^2 + 3 &= 0 \\ x^2 - 6x + 9 - 9 + 3 &= 0. \end{aligned}$$

Discuss the following questions with the class, and give them the opportunity to justify this step.

- Why did we choose 9? Why did we both add and subtract 9? How does this help us solve the equation?
  - Adding 9 creates a perfect square trinomial in the first three terms.
  - Adding and subtracting 9 means that we have not changed the value of the expression on the left side of the equation.
  - Adding and subtracting the 9 creates a perfect square trinomial  $x^2 - 6x + 9 = (x - 3)^2$ .
- We cannot just add a number to an expression without changing its value. By adding 9 and subtracting 9, we have essentially added 0 using the additive identity property, which leads to an equivalent expression on the left hand side of the equation and thus preserves solutions of the equations.
- This process creates a structure that allows us to factor the first three terms of the expression on the left side of the equation and then solve for the variable.

$$\begin{aligned} \underbrace{x^2 - 6x + 9}_{\text{Perfect square trinomial}} - 9 + 3 &= 0 \\ (x - 3)^2 - 6 &= 0 \end{aligned}$$

- Solving for  $x$ :

$$\begin{aligned} (x - 3)^2 &= 6 \\ x - 3 &= \sqrt{6} \quad \text{or} \quad x - 3 = -\sqrt{6} \\ x &= 3 + \sqrt{6} \quad \text{or} \quad x = 3 - \sqrt{6}. \end{aligned}$$

- Thus, we have found two solutions by setting the first quadratic expression equal to zero, completing the square, and solving the factored equation. Since the leading coefficient of  $x^2 - 6x + 3$  is 1, we know from our work in Algebra I that the factored form is

$$x^2 - 6x + 3 = (x - (3 + \sqrt{6}))(x - (3 - \sqrt{6})).$$

*Scaffolding:*

We can use the tabular method to give a visual representation of the process of completing the square. For example, the polynomial  $x^2 - 6x$  can be represented as follows to illustrate the "missing" term.

	$x$	$-3$	
$x^2$	$-3x$		$x$
$-3x$	?		$3$

- Let's repeat the process with the second equation. What is the first step to completing the square?

$$2x^2 - 4x - 7 = 0$$

Allow students an opportunity to suggest the first step to completing the square.

- We can only complete the square when the leading coefficient is 1, so our first step is to factor out the 2.

$$2\left(x^2 - 2x - \frac{7}{2}\right) = 0$$

- Now we can complete the square with the expression inside the parentheses.

$$2\left(x^2 - 2x + \underline{\quad} - \underline{\quad} + \frac{7}{2}\right) = 0$$

$$2\left(x^2 - 2x + (-1)^2 - (-1)^2 - \frac{7}{2}\right) = 0$$

$$2\left(x^2 - 2x + 1 - \frac{9}{2}\right) = 0$$

$$2\left((x - 1)^2 - \frac{9}{2}\right) = 0$$

- Next, we divide both sides by 2.

$$(x - 1)^2 - \frac{9}{2} = 0$$

- Finally, we solve for  $x$ .

$$(x - 1)^2 = \frac{9}{2}$$

$$x = 1 + \sqrt{\frac{9}{2}} \quad \text{or} \quad x = 1 - \sqrt{\frac{9}{2}}$$

$$x = 1 + \frac{3\sqrt{2}}{2} \quad \text{or} \quad x = 1 - \frac{3\sqrt{2}}{2}$$

- Thus, we have found two more solutions to our original fourth degree equation. We then have

$$\begin{aligned} 2x^2 - 4x - 7 &= 2\left(x^2 - 2x - \frac{7}{2}\right) \\ &= 2\left(x - \left(1 + \frac{3\sqrt{2}}{2}\right)\right)\left(x - \left(1 - \frac{3\sqrt{2}}{2}\right)\right). \end{aligned}$$

- Notice that we needed to multiply the factors by 2 to make the leading coefficients match.
- Finally, we have the factored form of our original polynomial equation:

$$(x^2 - 6x + 3)(2x^2 - 4x - 7) = 0 \text{ in factored form}$$

$$2(x - (3 + \sqrt{6}))(x - (3 - \sqrt{6}))\left(x - \left(1 + \frac{3\sqrt{2}}{2}\right)\right)\left(x - \left(1 - \frac{3\sqrt{2}}{2}\right)\right) = 0.$$

- Thus, the solutions to the equation  $(x^2 - 6x + 3)(2x^2 - 4x - 7) = 0$  are the four values  $3 + \sqrt{6}$ ,  $3 - \sqrt{6}$ ,  $1 + \frac{3\sqrt{2}}{2}$ , and  $1 - \frac{3\sqrt{2}}{2}$ .
- Similarly, we could have applied the quadratic formula to find the solutions to each quadratic equation in the previous example. Recall the quadratic formula.
  - The two solutions to the quadratic equation  $ax^2 + bx + c = 0$  are  $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$  and  $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$ .

**Exercise 1 (6 minutes)**

**Exercise 1**

Factor and find all real solutions to the equation  $(x^2 - 2x - 4)(3x^2 + 8x - 3) = 0$ .

Ask half of the students to apply the quadratic formula to solve  $x^2 - 2x - 4 = 0$  and the other half to apply the quadratic formula to solve  $3x^2 + 8x - 3 = 0$ .

The quadratic formula gives solutions  $1 + \sqrt{5}$  and  $1 - \sqrt{5}$  for the first equation and  $-3$  and  $\frac{1}{3}$  for the second equation.

Since  $1 + \sqrt{5}$  and  $1 - \sqrt{5}$  are the two solutions to  $x^2 - 2x - 4 = 0$  found by the quadratic formula, we know from work in Algebra I that  $(x - (1 + \sqrt{5}))(x - (1 - \sqrt{5})) = x^2 - 2x - 4$ . However, we need to be more careful when using the solutions to factor the second quadratic expression. The leading coefficient of  $(x + 3)(x - \frac{1}{3}) = x^2 + \frac{8}{3}x - 1$  is 1, and the leading coefficient of  $3x^2 + 8x - 3$  is 3, so we need to multiply our factors by 3:

$$3x^2 + 8x - 3 = 3(x + 3)\left(x - \frac{1}{3}\right).$$

Thus, the factored form of the original equation is

$$(x^2 - 2x - 4)(3x^2 + 8x - 3) = 3(x - (1 + \sqrt{5}))(x - (1 - \sqrt{5}))(x + 3)\left(x - \frac{1}{3}\right) = 0,$$

and the four solutions to  $(x^2 - 2x - 4)(3x^2 + 8x - 3) = 0$  are  $1 + \sqrt{5}$ ,  $1 - \sqrt{5}$ ,  $-3$ , and  $\frac{1}{3}$ .

To summarize, if we have a fourth degree polynomial already factored into two quadratic expressions, we can try to factor the entire polynomial by completing the square on one or both quadratic expressions, or by using the quadratic formula to find the roots of the quadratic polynomials and then constructing the factored form of each quadratic polynomial.

**Discussion (6 minutes)**

- We have overcome the obstacle of difficult-to-factor quadratic expressions. Let's look next at the obstacles encountered when attempting to solve a third degree polynomial equation such as the following:

$$x^3 + 3x^2 - 9x - 27 = 0.$$

- How might we begin to solve this equation?

*Scaffolding:*

Allow students to generate ideas. Advanced students may benefit from significant independent time to attempt to solve the equation (MP.1).

Allow students an opportunity to brainstorm as a class, in pairs, or in table groups. Students may note that coefficients are powers of 3 but may not be sure how that helps. Let them know they are seeing something important that we may be able to use. Stronger students might even try to group the components.

- While we have made some interesting observations, we have not quite found a way to factor this expression. What if we know that  $x + 3$  is one factor?

If the students do not come up with polynomial division, you can point them in that direction through a numerical example: Suppose we want the factors of 210, and we know that one factor is 3. How do we find the other factors?

Have the students perform the polynomial division

$$x + 3 \overline{) x^3 + 3x^2 - 9x - 27}$$

to find additional factors. Students may also use the tabular method discussed in earlier lessons in this module.

Handwritten work showing polynomial long division:

$$\begin{array}{r}
 x^2 + 0x - 9 \\
 x + 3 \overline{) x^3 + 3x^2 - 9x - 27} \\
 \underline{x^3 + 3x^2} \phantom{- 9x - 27} \\
 - 9x - 27 \\
 \underline{- 9x - 27} \\
 0
 \end{array}$$

So  $\frac{x^3 + 3x^2 - 9x - 27}{x + 3} = x^2 - 9$

and  $x^3 + 3x^2 - 9x - 27 = (x + 3)(x^2 - 9)$

- Since  $x^3 + 3x^2 - 9x - 27 = (x + 3)(x^2 - 9)$ , we know that  $x^3 + 3x^2 - 9x - 27 = (x + 3)(x - 3)(x + 3) = (x + 3)^2(x - 3)$ .
- By the zero product property, the solutions to  $x^3 + 3x^2 - 9x - 27 = 0$  are  $-3$  and  $3$ .
- But, how do we start if we don't know any of the factors in advance?

**Example 2 (6 minutes)**

**Example 2**

Find all solutions to  $x^3 + 3x^2 - 9x - 27 = 0$  by factoring the equation.

- Let's start with our original equation  $x^3 + 3x^2 - 9x - 27 = 0$ . Is there a greatest common factor (GCF) for all four terms on the left hand side we can factor out?
  - No, the GCF is 1.

*Scaffolding:*

If needed, have students compute the GCF for some numerical examples such as

- GCF 75, 100 is 25
- GCF 72, 16, 24 is 8
- GCF 10, 35, 63 is 1.

- Let's group the terms of the left hand side as follows:

$$x^3 + 3x^2 - 9x - 27 = (x^3 + 3x^2) - (9x + 27).$$

- Can we factor out a GCF from each set of parentheses independently?
  - Yes,  $x^2$  can be factored out of the first piece and 9 out of the second.

Factor the GCF out of each part. Have students do as much of this work as possible.

$$x^3 + 3x^2 - 9x - 27 = x^2(x + 3) - 9(x + 3).$$

MP.7

- Do you notice anything interesting about the right side of the above equation?
  - I noticed that  $x + 3$  is a common factor.
- Since both terms have a factor of  $(x + 3)$ , we have found a quantity that can be factored out.

$$x^3 + 3x^2 - 9x - 27 = (x + 3)(x^2 - 9)$$

- And as we saw above, we can take this one step further.

$$x^3 + 3x^2 - 9x - 27 = (x + 3)(x + 3)(x - 3)$$

- Because of the zero property, the original problem is now easy to solve because  $x^3 + 3x^2 - 9x - 27 = 0$  exactly when  $(x + 3)^2(x - 3) = 0$ . What are the solutions to the original equation?
  - The solutions to  $x^3 + 3x^2 - 9x - 27 = 0$  are  $x = -3$  and  $x = 3$ .
- The process you just completed is often called *factoring by grouping*, and it works only on certain 3<sup>rd</sup> degree polynomial expressions, such as  $x^3 + 3x^2 - 9x - 27$ .

**Exercise 2 (4 minutes)**

Allow students to work in pairs or small groups on these exercises. Realize that there are two ways to group the terms that will result in the same factored expression. Circulate around the room while students are working, and take note of any groups that are using a different approach. At the end of these exercises, ask students who grouped differently to share their method, and discuss as a class.

**Exercise 2**

Find all real solutions to  $x^3 - 5x^2 - 4x + 20 = 0$ .

$x^3 - 5x^2 - 4x + 20 = 0$ $x^2(x - 5) - 4(x - 5) = 0$ $(x - 5)(x^2 - 4) = 0$ $(x - 5)(x - 2)(x + 2) = 0$	$x^3 - 5x^2 - 4x + 20 = 0$ $x(x^2 - 4) - 5(x^2 - 4) = 0$ $(x - 5)(x^2 - 4) = 0$ $(x - 5)(x - 2)(x + 2) = 0$
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*Thus, the solutions are 5, 2, and -2.*

**Exercise 3 (4 minutes)****Exercise 3**

Find all real solutions to  $x^3 - 8x^2 - 2x + 16 = 0$ .

$$x^3 - 8x^2 - 2x + 16 = 0$$

$$x^2(x - 8) - 2(x - 8) = 0$$

$$(x - 8)(x^2 - 2) = 0$$

Thus, the solutions are  $8$ ,  $\sqrt{2}$ , and  $-\sqrt{2}$ .

$$x^3 - 8x^2 - 2x + 16 = 0$$

$$x(x^2 - 2) - 8(x^2 - 2) = 0$$

$$(x - 8)(x^2 - 2) = 0$$

**Closing (2 minutes)**

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.

**Lesson Summary**

In this lesson, we learned some techniques to use when faced with factoring polynomials and solving polynomial equations.

- If a fourth degree polynomial can be factored into two quadratic expressions, then each quadratic expression might be factorable either using the quadratic formula or by completing the square.
- Some third degree polynomials can be factored using the technique of factoring by grouping.

**Exit Ticket (5 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 12: Overcoming Obstacles in Factoring

### Exit Ticket

Solve the following equation and explain your solution method.

$$x^3 + 7x^2 - x - 7 = 0$$

## Exit Ticket Sample Solutions

Solve the following equation and explain your solution method.

$$\begin{aligned}x^3 + 7x^2 - x - 7 &= 0 \\x^2(x + 7) - (x + 7) &= 0 \\(x + 7)(x^2 - 1) &= 0 \\(x + 7)(x - 1)(x + 1) &= 0\end{aligned}$$

*The solutions are  $-7$ ,  $1$ , and  $-1$ . The equation was solved by factoring by grouping. I grouped the four terms into two groups, and then factored the GCF from each group. I then factored out the common term  $(x + 7)$  from each group to find the factored form of the equation. I then applied the zero product property to find the solutions to the equation.*

## Problem Set Sample Solutions

1. Solve each of the following equations by completing the square.

- |    |                                      |  |
|----|--------------------------------------|--|
| a. | $x^2 - 6x + 2 = 0$                   | $3 + \sqrt{7}, 3 - \sqrt{7}$   |
| b. | $x^2 - 4x = -1$                      | $2 + \sqrt{3}, 2 - \sqrt{3}$   |
| c. | $x^2 + x - \frac{3}{4} = 0$          | $\frac{1}{2}, -\frac{3}{2}$  |
| d. | $3x^2 - 9x = -6$                     | $2, 1$   |
| e. | $(2x^2 - 5x + 2)(3x^2 - 4x + 1) = 0$ | $2, \frac{1}{2}, 1, \frac{1}{3}$   |
| f. | $x^4 - 4x^2 + 2 = 0$                 | $\sqrt{2 + \sqrt{2}}, -\sqrt{2 + \sqrt{2}}, \sqrt{2 - \sqrt{2}}, -\sqrt{2 - \sqrt{2}}$ |

2. Solve each of the following equations using the quadratic formula.

- |    |                                       |  |
|----|---------------------------------------|--|
| a. | $x^2 - 5x - 3 = 0$                    | $\frac{5}{2} + \frac{\sqrt{37}}{2}, \frac{5}{2} - \frac{\sqrt{37}}{2}$           |
| b. | $(6x^2 - 7x + 2)(x^2 - 5x + 5) = 0$   | $\frac{1}{2}, \frac{2}{3}, \frac{1}{2}(5 + \sqrt{5}), \frac{1}{2}(5 - \sqrt{5})$ |
| c. | $(3x^2 - 13x + 14)(x^2 - 4x + 1) = 0$ | $2, \frac{7}{3}, 2 + \sqrt{3}, 2 - \sqrt{3}$                                     |

3. Not all of the expressions in the equations below can be factored using the techniques discussed so far in this course. First, determine if the expression can be factored with real coefficients. If so, factor the expression and find all real solutions to the equation.

a.  $x^2 - 5x - 24 = 0$

*Can be factored:*  $(x - 8)(x + 3) = 0$ .

*Solutions:* 8, -3

b.  $3x^2 + 5x - 2 = 0$

*Can be factored:*  $(3x - 1)(x + 2) = 0$ .

*Solutions:*  $\frac{1}{3}$ , -2

c.  $x^2 + 2x + 4 = 0$

*Cannot be factored with real number coefficients.*

d.  $x^3 + 3x^2 - 2x + 6 = 0$

*Cannot be factored with real number coefficients.*

e.  $x^3 + 3x^2 + 2x + 6 = 0$

*Can be factored:*  $(x + 3)(x^2 + 2) = 0$ .

*Solution:* -3

f.  $2x^3 + x^2 - 6x - 3 = 0$

*Can be factored:*  $(2x + 1)(x - \sqrt{3})(x + \sqrt{3}) = 0$ .

*Solutions:*  $-\frac{1}{2}$ ,  $\sqrt{3}$ ,  $-\sqrt{3}$

g.  $8x^3 - 12x^2 + 2x - 3 = 0$

*Can be factored:*  $(2x - 3)(4x^2 + 1) = 0$ .

*Solution:*  $\frac{3}{2}$

h.  $6x^3 + 8x^2 + 15x + 20 = 0$

*Can be factored:*  $(3x + 4)(2x^2 + 5) = 0$ .

*Solution:*  $-\frac{4}{3}$

i.  $4x^3 + 2x^2 - 36x - 18 = 0$

*Can be factored:*  $2(2x + 1)(x - 3)(x + 3) = 0$ .

*Solutions:*  $-\frac{1}{2}$ , 3, -3

j.  $x^2 - \frac{1}{2}x - \frac{15}{2} = 0$

*Can be factored:*  $(x + \frac{5}{2})(x - 3) = 0$ .

*Solutions:*  $-\frac{5}{2}$ , 3

4. Solve the following equations by bringing all terms to one side of the equation and factoring out the greatest common factor.

a.  $(x - 2)(x - 1) = (x - 2)(x + 1)$

$$(x - 2)(x + 1) - (x - 2)(x - 1) = 0$$

$$(x - 2)(x + 1 - (x - 1)) = 0$$

$$(x - 2)(2) = 0$$

$$x = 2$$

*So, the only solution to  $(x - 2)(x - 1) = (x - 2)(x + 1)$  is 2.*

b.  $(2x + 3)(x - 4) = (2x + 3)(x + 5)$

$$(2x + 3)(x - 4) - (2x + 3)(x + 5) = 0$$

$$(2x + 3)(x - 4 - (x + 5)) = 0$$

$$(2x + 3)(-9) = 0$$

$$x = -\frac{3}{2}$$

*So, the only solution to  $(2x + 3)(x - 4) = (2x + 3)(x + 5)$  is  $-\frac{3}{2}$ .*

c.  $(x-1)(2x+3) = (x-1)(x+2)$

$$(x-1)(2x+3) - (x-1)(x+2) = 0$$

$$(x-1)(2x+3 - (x+2)) = 0$$

$$(x-1)(x+1) = 0$$

$$x = 1 \text{ or } x = -1$$

*The solutions to  $(x-1)(2x+3) = (x-1)(x+2)$  are 1 and -1.*

d.  $(x^2+1)(3x-7) = (x^2+1)(3x+2)$

$$(x^2+1)(3x-7) - (x^2+1)(3x+2) = 0$$

$$(x^2+1)(3x-7 - (3x+2)) = 0$$

$$(x^2+1)(-9) = 0$$

$$x^2+1 = 0$$

*There are no real number solutions to  $(x^2+1)(3x-7) = (x^2+1)(3x+2)$ .*

e.  $(x+3)(2x^2+7) = (x+3)(x^2+8)$

$$(x+3)(2x^2+7) - (x+3)(x^2+8) = 0$$

$$(x+3)(2x^2+7 - (x^2+8)) = 0$$

$$(x+3)(x^2-1) = 0$$

$$(x+3)(x-1)(x+1) = 0$$

*The three solutions to  $(x+3)(2x^2+7) = (x+3)(x^2+8)$  are -3, -1, and 1.*

5. Consider the expression  $x^4 + 1$ . Since  $x^2 + 1$  does not factor with real number coefficients, we might expect that  $x^4 + 1$  also does not factor with real number coefficients. In this exercise, we will investigate the possibility of factoring  $x^4 + 1$ .

- a. Simplify the expression  $(x^2 + 1)^2 - 2x^2$ .

$$(x^2 + 1)^2 - 2x^2 = x^4 + 1$$

- b. Factor  $(x^2 + 1)^2 - 2x^2$  as a difference of squares.

$$(x^2 + 1)^2 - 2x^2 = (x^2 + 1 - \sqrt{2}x)(x^2 + 1 + \sqrt{2}x)$$

- c. Is it possible to factor  $x^4 + 1$  with real number coefficients? Explain.

$$\text{Yes. } x^4 + 1 = (x^2 + 1 - \sqrt{2}x)(x^2 + 1 + \sqrt{2}x)$$

*In an equivalent but more conventional form, we have*

$$x^4 + 1 = (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1).$$



## Lesson 13: Mastering Factoring

### Student Outcomes

- Students will use the structure of polynomials to identify factors.

### Lesson Notes

In previous lessons in this module, students have practiced the techniques of factoring by completing the square, by applying the quadratic formula, and by grouping. In this lesson, students are going to look for structure in more complicated polynomial expressions that will allow factorization. But first, students are going to review several factoring techniques; some they learned about in the last lesson, and others they learned about in previous classes.

### Opening Exercise (8 minutes)

MP.7

In this exercise, students should begin to factor polynomial expressions by first analyzing their structure, a skill that is developed throughout the lesson. Suggest that students work on their own for five minutes and then compare answers with a neighbor; allow students to help each other out for an additional three minutes, if needed.

#### Opening Exercise

Factor each of the following expressions. What similarities do you notice between the examples in the left column and those on the right?

a. $x^2 - 1$ $(x - 1)(x + 1)$	b. $9x^2 - 1$ $(3x - 1)(3x + 1)$
c. $x^2 + 8x + 15$ $(x + 5)(x + 3)$	d. $4x^2 + 16x + 15$ $(2x + 5)(2x + 3)$
e. $x^2 - y^2$ $(x - y)(x + y)$	f. $x^4 - y^4$ $(x^2 - y^2)(x^2 + y^2)$

Students should notice that the structure of each of the factored polynomials is the same; for example, the factored forms of part (a) and part (b) are nearly the same, except that part (b) contains  $3x$  in place of the  $x$  in part (a). In parts (c) and (d), the factored form of part (d) contains  $2x$ , where there is only an  $x$  in part (c). The factored form of part (f) is nearly the same as the factored form of part (e), with  $x^2$  replacing  $x$  and  $y^2$  replacing  $y$ .

**Discussion (2 minutes)**

The difference of two squares formula,

$$a^2 - b^2 = (a + b)(a - b),$$

can be used to factor an expression even when the two squares are not obvious.

Consider the following examples.

*Scaffolding:*

If students are unfamiliar with the difference of squares formula, work through a table of numeric examples and ask them to look for patterns.

$a$	$b$	$a + b$	$a - b$	$a^2 - b^2$
3	1	4	2	8
4	1	5	3	15
5	2	7	3	21
6	4	10	2	20

Students may benefit from teacher modeling.

**Example 1 (3 minutes)**

**Example 1**

Write  $9 - 16x^4$  as the product of two factors.

$$\begin{aligned} 9 - 16x^4 &= (3)^2 - (4x^2)^2 \\ &= (3 - 4x^2)(3 + 4x^2) \end{aligned}$$

**Example 2 (3 minutes)**

**Example 2**

Factor  $4x^2y^4 - 25x^4z^6$ .

$$\begin{aligned} 4x^2y^4 - 25x^4z^6 &= (2xy^2)^2 - (5x^2z^3)^2 \\ &= (2xy^2 + 5x^2z^3)(2xy^2 - 5x^2z^3) \\ &= [x(2y^2 + 5xz^3)][x(2y^2 - 5xz^3)] \\ &= x^2(2y^2 + 5xz^3)(2y^2 - 5xz^3) \end{aligned}$$

Have students discuss with each other the structure of each polynomial expression in the previous two examples and how it helps us to factor the expressions.

*There are two terms that are subtracted, and each term can be written as the square of an expression.*

**Example 3 (3 minutes)**

Consider the quadratic polynomial expression  $9x^2 + 12x - 5$ . We can factor this expression by considering  $3x$  as a single quantity as follows:

$$9x^2 + 12x - 5 = (3x)^2 + 4(3x) - 5.$$

Ask students to suggest the next step in factoring this expression.

Now, if we rename  $u = 3x$ , we have a quadratic expression of the form  $u^2 + 4u - 5$ , which we can factor:

$$u^2 + 4u - 5 = (u - 1)(u + 5).$$

Replacing  $u$  by  $3x$ , we have the following form of our original expression:

$$9x^2 + 12x - 5 = (3x - 1)(3x + 5).$$

**Exercise 1 (4 minutes)**

Allow students to work in pairs or small groups on the following exercises.

**Exercise 1**

1. Factor the following expressions:

a.  $4x^2 + 4x - 63$

$$4x^2 + 4x - 63 = (2x)^2 + 2(2x) - 63$$

$$= (2x + 9)(2x - 7)$$

b.  $12y^2 - 24y - 15$

$$12y^2 - 24y - 15 = 3(4y^2 - 8y - 5)$$

$$= 3((2y)^2 - 4(2y) - 5)$$

$$= 3(2y + 1)(2y - 5)$$

**Example 4 (10 minutes)**

We can use the example of factoring  $x^3 - 8$  to scaffold the discussion of factoring  $x^3 + 8$ . Students should be pretty familiar by now with factors of  $x^3 - 8$ . Let them try the problem on their own to check their understanding.

- Suppose we want to factor  $x^3 - 8$ .
- Ask students if they see anything interesting about this expression.

If they do not notice it, guide them toward both pieces being perfect cubes.

- We can rewrite  $x^3 - 8$  as  $x^3 - 2^3$ .
- Guess a factor.
  - *Anticipate that they will suggest  $x - 2$  and  $x + 2$  as possible factors, or guide them to these suggestions.*
- Ask half of the students to divide  $(x^3 - 8) \div (x - 2)$  and the other half to divide  $(x^3 - 8) \div (x + 2)$ . They should discover that  $x - 2$  is a factor, but  $x + 2$  is not.

$$\begin{array}{r}
 x^2 + 2x + 4 \\
 x-2 \overline{) x^3 + 0x^2 + 0x - 8} \\
 \underline{x^3 - 2x^2} \phantom{+ 0x - 8} \\
 2x^2 + 0x - 8 \\
 \underline{2x^2 - 4x} \phantom{- 8} \\
 4x - 8 \\
 \underline{4x - 8} \\
 0
 \end{array}$$

$$\begin{array}{r}
 x^2 - 2x + 4 \\
 x+2 \overline{) x^3 + 0x^2 + 0x - 8} \\
 \underline{x^3 + 2x^2} \phantom{+ 0x - 8} \\
 -2x^2 + 0x - 8 \\
 \underline{-2x^2 - 4x} \phantom{- 8} \\
 4x - 8 \\
 \underline{4x + 8} \\
 -16
 \end{array}$$

- Use the results of the previous step to factor  $x^3 - 8$ .
  - $x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

- Repeat the above process for  $x^3 - 27$ .
  - $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$
- Make a conjecture about a rule for factoring  $x^3 - a^3$ .
  - $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- Verify the conjecture: Multiply out  $(x - a)(x^2 + ax + a^2)$  to establish the identity for factoring a difference of cubes.

While we can factor a difference of squares such as the expression  $x^2 - 9$ , we cannot similarly factor a sum of squares such as  $x^2 + 9$ . Do we run into a similar problem when trying to factor a sum of cubes such as  $x^3 + 8$ ?

Again, ask students to propose potential factors of  $x^3 + 8$ . Lead students to  $x + 2$  if they do not guess it automatically.

Work through the polynomial long division for  $(x^3 + 8) \div (x + 2)$  as shown.

$$\begin{array}{r}
 x^2 - 2x + 4 \\
 \hline
 x+2 \overline{) x^3 + 0x^2 + 0x + 8} \\
 \underline{x^3 + 2x^2} \phantom{+ 0x + 8} \\
 -2x^2 + 0x + 8 \\
 \underline{-2x^2 - 4x} \phantom{+ 8} \\
 4x + 8 \\
 \underline{4x + 8} \\
 0
 \end{array}$$

- Conclude that  $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$ .
- Make a conjecture about a rule for factoring  $x^3 + a^3$ .
  - $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$
- Verify the conjecture: Multiply out the expression  $(x + a)(x^2 - ax + a^2)$  to establish the identity for factoring a sum of cubes.

**Exercises 2–4 (5 minutes)**

**Exercises 2–4**

Factor each of the following, and show that the factored form is equivalent to the original expression.

2.  $a^3 + 27$   
 $(a + 3)(a^2 - 3a + 9)$

3.  $x^3 - 64$   
 $(x - 4)(x^2 + 4x + 16)$

4.  $2x^3 + 128$   
 $2(x^3 + 64) = 2(x + 4)(x^2 - 4x + 16)$

*Scaffolding:*  
 Ask advanced students to generate their own factoring problems using the structure of  $a^3 + b^3$  or  $a^3 - b^3$ .

**Closing (2 minutes)**

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.

**Lesson Summary**

In this lesson we learned additional strategies for factoring polynomials.

- The difference of squares identity  $a^2 - b^2 = (a - b)(a + b)$  can be used to factor more advanced binomials.
- Trinomials can often be factored by looking for structure and then applying our previous factoring methods.
- Sums and differences of cubes can be factored by the formulas

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2).$$

**Exit Ticket (5 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 13: Mastering Factoring

### Exit Ticket

1. Factor the following expression and verify that the factored expression is equivalent to the original.

$$4x^2 - 9a^6$$

2. Factor the following expression and verify that the factored expression is equivalent to the original

$$16x^2 - 8x - 3$$

## Exit Ticket Sample Solutions

1. Factor the following expression and verify that the factored expression is equivalent to the original:  $4x^2 - 9a^6$ .

$$\begin{aligned}(2x - 3a^3)(2x + 3a^3) &= 4x^2 + 6a^3x - 6a^3x - 9a^6 \\ &= 4x^2 - 9a^6\end{aligned}$$

2. Factor the following expression and verify that the factored expression is equivalent to the original:  $16x^2 - 8x - 3$ .

$$\begin{aligned}(4x - 3)(4x + 1) &= 16x^2 + 4x - 12x - 3 \\ &= 16x^2 - 8x - 3\end{aligned}$$

## Problem Set Sample Solutions

1. If possible, factor the following expressions using the techniques discussed in this lesson.

- |    |                             |  |
|----|-----------------------------|--|
| a. | $25x^2 - 25x - 14$          | $(5x - 7)(5x + 2)$                       |
| b. | $9x^2y^2 - 18xy + 8$        | $(3xy - 4)(3xy - 2)$                     |
| c. | $45y^2 + 15y - 10$          | $5(3y + 2)(3y - 1)$                      |
| d. | $y^6 - y^3 - 6$             | $(y^3 - 3)(y^3 + 2)$                     |
| e. | $x^3 - 125$                 | $(x - 5)(x^2 + 5x + 25)$                 |
| f. | $2x^4 - 16x$                | $2x(x - 2)(x^2 + 2x + 4)$                |
| g. | $9x^2 - 25y^4z^6$           | $(3x - 5y^2z^3)(3x + 5y^2z^3)$           |
| h. | $36x^6y^4z^2 - 25x^2z^{10}$ | $x^2z^2(6x^2y^2 - 5z^4)(6x^2y^2 + 5z^4)$ |
| i. | $4x^2 + 9$                  | <i>Cannot be factored.</i>               |
| j. | $x^4 - 36$                  | $(x - \sqrt{6})(x + \sqrt{6})(x^2 + 6)$  |
| k. | $1 + 27x^9$                 | $(1 + 3x^3)(1 - 3x^3 + 9x^6)$            |
| l. | $x^3y^6 + 8z^3$             | $(xy^2 + 2z)(x^2y^4 - 2xy^2z + 4z^2)$    |

2. Consider the polynomial expression  $y^4 + 4y^2 + 16$ .

a. Is  $y^4 + 4y^2 + 16$  factorable using the methods we have seen so far?

*No. This will not factor into the form  $(y^2 + a)(y^2 + b)$  using any of our previous methods.*

b. Factor  $y^6 - 64$  first as a difference of cubes, then factor completely:  $(y^2)^3 - 4^3$ .

$$\begin{aligned} y^6 - 64 &= (y^2 - 4)(y^4 + 4y^2 + 16) \\ &= (y - 2)(y + 2)(y^4 + 4y^2 + 16) \end{aligned}$$

c. Factor  $y^6 - 64$  first as a difference of squares, then factor completely:  $(y^3)^2 - 8^2$ .

$$\begin{aligned} y^6 - 64 &= (y^3 - 8)(y^3 + 8) \\ &= (y - 2)(y^2 + 2y + 4)(y + 2)(y^2 - 2y + 4) \\ &= (y - 2)(y + 2)(y^2 - 2y + 4)(y^2 + 2y + 4) \end{aligned}$$

d. Explain how your answers to parts (b) and (c) provide a factorization of  $y^4 + 4y^2 + 16$ .

*Since  $y^6 - 64$  can be factored two different ways, those factorizations are equal. Thus we have*

$$(y - 2)(y + 2)(y^4 + 4y^2 + 16) = (y - 2)(y + 2)(y^2 - 2y + 4)(y^2 + 2y + 4).$$

*If we specify that  $y \neq 2$  and  $y \neq -2$ , we can cancel the common terms from both sides:*

$$(y^4 + 4y^2 + 16) = (y^2 - 2y + 4)(y^2 + 2y + 4).$$

*Multiplying this out, we see that*

$$\begin{aligned} (y^2 - 2y + 4)(y^2 + 2y + 4) &= y^4 + 2y^3 + 4y^2 - 2y^3 - 4y^2 - 8y + 4y^2 + 8y + 16 \\ &= y^4 + 4y^2 + 16 \end{aligned}$$

*for every value of  $y$ .*

e. If a polynomial can be factored as either a difference of squares or a difference of cubes, which formula should you apply first, and why?

*Based on this example, a polynomial should first be factored as a difference of squares and then as a difference of cubes. This will produce factors of lower degree.*

3. Create expressions that have a structure that allows them to be factored using the specified identity. Be creative and produce challenging problems!

a. Difference of squares

$$x^{14}y^4 - 225z^{10}$$

b. Difference of cubes

$$27x^9y^6 - 1$$

c. Sum of cubes

$$x^6z^3 + 64y^{12}$$



## Lesson 14: Graphing Factored Polynomials

### Student Outcomes

- Students will use the factored forms of polynomials to find zeros of a function.
- Students will use the factored forms of polynomials to sketch the components of graphs between zeros.

### Lesson Notes

In this lesson, we use the factored form of polynomials to identify important aspects of the graphs of polynomial functions and, therefore, important aspects of the situations they model. Using the factored form, we will identify zeros of the polynomial (and thus  $x$ -intercepts of the graph of the polynomial function) and see how to sketch a graph of the polynomial functions by examining what happens between the  $x$ -intercepts. We also introduce the concepts of relative minima and maxima and determining the possible degree of the polynomial by noting the number of relative extrema in a graph. We include definitions of relevant vocabulary at the end of the lesson for reference so that each definition can be used appropriately.

MP.5  
&  
MP.7

The use of a graphing utility is recommended for some examples in this lesson to encourage students to focus on understanding the structure of the polynomials without the tedium of repeated graphing by hand.

### Opening Exercise (9 minutes)

Prompt the students to answer part (a) of the Opening Exercise independently or in pairs before continuing with the scaffolded questions.

#### Opening Exercise

An engineer is designing a roller coaster for younger children and has tried some functions to model the height of the roller coaster during the first 300 yards. She came up with the following function to describe what she believes would make a fun start to the ride:

$$H(x) = -3x^4 + 21x^3 - 48x^2 + 36x,$$

where  $H(x)$  is the height of the roller coaster (in yards) when the roller coaster is  $100x$  yards from the beginning of the ride. Answer the following questions to help determine at which distances from the beginning of the ride the roller coaster is at its lowest height.

- Does this function describe a roller coaster that would be fun to ride? Explain.  
*Yes, the roller coaster quickly goes to the top then drops you down. This looks like a fun ride.*  
*OR*  
*No, I don't like roller coasters that climb steeply, and this one goes nearly straight up.*
- Can you see any obvious  $x$ -values from the equation where the roller coaster is at height 0?  
*The height is zero when  $x$  is 0 because at that value each term is equal to 0.*

#### Scaffolding:

- Consider beginning the class by reviewing graphs of simpler functions modeling "simple roller coasters," such as  $G(x) = -x^2 + 4x$ .
- A more visual approach may be taken by first describing and analyzing the graph of  $H$  before connecting each concept to the algebra associated with the function. Pose questions such as "When is the roller coaster going up? Going down? How many times does the roller coaster touch the bottom?"

MP.3

MP.5

MP.3

- c. Using a graphing utility, graph the function  $H$  on the interval  $0 \leq x \leq 3$ , and identify when the roller coaster is 0 yards off the ground.

*The lowest points of the graph on  $0 \leq x \leq 3$  are when the  $x$ -value satisfies  $H(x) = 0$ , which occurs when  $x$  is 0, 2, and 3.*

- d. What do the  $x$ -values you found in part (c) mean in terms of distance from the beginning of the ride?

*The distances represent 0 yards, 200 yards, and 300 yards, respectively.*

- e. Why do roller coasters always start with the largest hill first?

*So they can build up speed from gravity to help propel the cars through the rest of the track.*

- f. Verify your answers to part (c) by factoring the polynomial function  $H$ .

*Some students may need some hints or guidance with factoring.*

$$\begin{aligned} H(x) &= -3x^4 + 21x^3 - 48x^2 + 36x \\ &= -3x(x^3 - 7x^2 + 16x - 12) \end{aligned}$$

*From the graph, we suspect that  $(x - 3)$  is a factor; using long division, we obtain*

$$\begin{aligned} H(x) &= -3x(x - 3)(x^2 - 4x + 4) \\ &= -3x(x - 3)(x - 2)(x - 2) \\ &= -3x(x - 3)(x - 2)^2. \end{aligned}$$

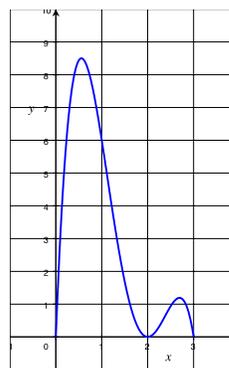
*The solutions to the equation  $H(x) = 0$  are 0, 2, and 3. Therefore, the roller coaster is at the bottom at 0 yards, 200 yards, and 300 yards from the start of the ride.*

- g. How do you think the engineer came up with the function for this model?

*Let students discuss this question in groups or as a whole class. The following conclusion should be made: To start at height 0 yards and end 300 yards later at height 0 yards, she multiplied  $x$  by  $x - 3$  (to create zeros at 0 and 3). To create the bottom of the hill at 200 yards, she multiplied this function by  $(x - 2)^2$ . She needed to multiply by  $-3$  to guarantee the roller coaster shape and to adjust the overall height of the roller coaster.*

- h. What is wrong with this roller coaster model at distance 0 yards and 300 yards? Why might this not initially bother the engineer when she is first designing the track?

*The model appears to abruptly start at 0 yards and abruptly end at 300 yards. In fact, the roller coaster looks as if it will crash into the ground at 300 yards! The engineer may be planning to "smooth" out the track later at 0 yards and 300 yards after she has selected the overall shape of the roller coaster.*



**Discussion (4 minutes)**

By manipulating a polynomial function into its factored form, we can identify the zeros of the function as well as identify the general shape of the graph. Thinking about the Opening Exercise, what else can we say about the polynomial function and its graph?

**Scaffolding:**

Encourage struggling learners to graph the original and the factored forms using a graphing utility to confirm that they are the same.

- The degree of the polynomial function  $H$  is 4. How can you find the degree of the function from its factored form?
  - Add the highest degree term from each factor:
    - $-3$  is a degree 0 factor
    - $x$  is degree 1 factor
    - $x - 3$  is degree 1 factor
    - $(x - 2)^2$  is a degree 2 factor, since  $(x - 2)^2 = (x - 2)(x - 2)$ .
  - Thus,  $0 + 1 + 1 + 2 = 4$ .
- How many  $x$ -intercepts does the graph of the polynomial function have?
  - For this graph, there are three:  $(0,0)$ ,  $(2,0)$ , and  $(3,0)$ .

**Scaffolding:**

- For advanced learners, consider challenging students to construct a variety of functions to meet different criteria (such as three factors and no  $x$ -intercepts or four factors with two  $x$ -intercepts).
- Students may enjoy challenging each other by trying to guess the equation that goes with the graph of their classmates.

You may want to include a discussion that the zeros of a function correspond to the  $x$ -intercepts of the graph of the function.

- Note that there are four factors, but only three  $x$ -intercepts. Why is that?
  - Two of the factors are the same.

Remind students that the  $x$ -intercepts of a graph  $y = f(x)$  are solutions to the equation  $f(x) = 0$ . Values of  $r$  that satisfy  $f(r) = 0$  are called zeros (or roots) of the function. Some of these zeros may be repeated.

- Can you make one change to the polynomial function such that the new graph would have four  $x$ -intercepts?
  - Change one of the  $(x - 2)$  factors to  $(x - 1)$ , for example.

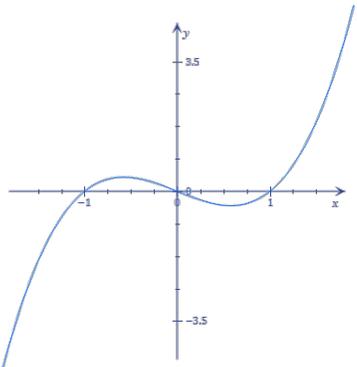
**Example 1 (10 minutes)**

We are now going to examine a few polynomial functions in factored form and compare the zeros of the function to the graph of the function on the calculator. Help students with part (a), and ask them to do part (b) on their own.

**Example 1**

Graph each of the following polynomial functions. What are the function's zeros (counting multiplicities)? What are the solutions to  $f(x) = 0$ ? What are the  $x$ -intercepts of the graph of the function? How does the degree of the polynomial function compare to the  $x$ -intercepts of the graph of the function?

a.  $f(x) = x(x - 1)(x + 1)$

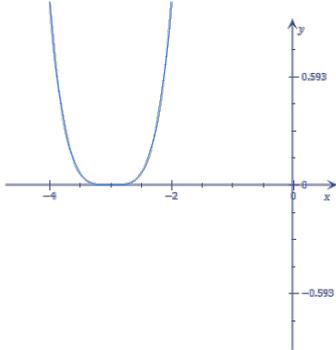


<b>Zeros:</b>	$-1, 0, 1$
<b>Solutions to <math>f(x) = 0</math>:</b>	$-1, 0, 1$
<b><math>x</math>-intercepts:</b>	$-1, 0, 1$

*The degree is 3, which is the same as the number of  $x$ -intercepts.*

Before graphing the next equation, ask students where they think the graph of  $f$  will cross the  $x$ -axis and how the repeated factor will affect the graph. After graphing, students may need to trace near  $x = -3$  depending on the graphing window to obtain a clear picture of the  $x$ -intercept.

b.  $f(x) = (x + 3)(x + 3)(x + 3)(x + 3)$



**Zeros:**  $-3, -3, -3, -3$  (repeated zero)

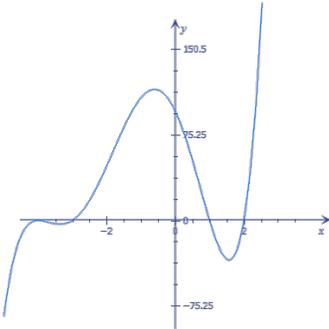
**Solutions to  $f(x) = 0$ :**  $-3$

**$x$ -intercept:**  $-3$

*The degree is 4, which is greater than the number of  $x$ -intercepts.*

By now, students should have an idea of what to expect in part (c). It may be worth noting the differences in the end behavior of the graphs, which will be explored further in Lesson 15. Discuss the degree of each polynomial.

c.  $f(x) = (x - 1)(x - 2)(x + 3)(x + 4)(x + 4)$



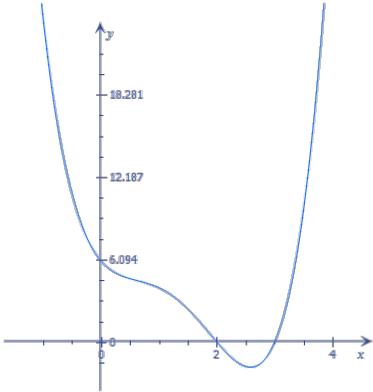
**Zeros:**  $-4, -4, -3, 1, 2$

**Solutions to  $f(x) = 0$ :**  $-4, -3, 1, 2$

**$x$ -intercepts:**  $-4, -3, 1, 2$

*The degree is 5, which is greater than the number of  $x$ -intercepts.*

d.  $f(x) = (x^2 + 1)(x - 2)(x - 3)$



**Zeros:**  $2, 3$

**Solutions to  $f(x) = 0$ :**  $2, 3$

**$x$ -intercepts:**  $2, 3$

*The degree is 4, which is greater than the number of  $x$ -intercepts.*

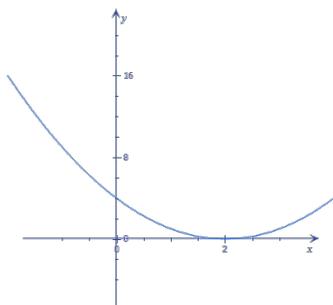
- Why is the factor  $x^2 + 1$  never zero and how does this affect the graph of  $f$ ?

(At this point in the module, all polynomial functions are defined from the real numbers to the real numbers; hence, the functions can have only real number zeros. We will extend polynomial functions to the domain of complex numbers later, and then it will be possible to have complex solutions.)

- For real numbers  $x$ , the value of  $x^2$  is always greater than or equal to zero, so  $x^2 + 1$  will always be strictly greater than zero. Thus,  $x^2 + 1 \neq 0$  for all real numbers  $x$ . Since there can be no  $x$ -intercept from this factor, the graph of  $f$  can have at most two  $x$ -intercepts.

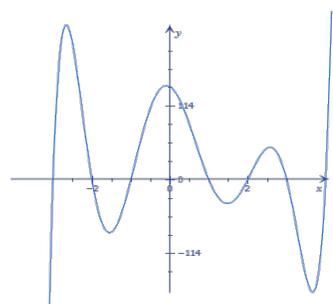
If there is time, consider graphing the functions for parts (e)–(h) on the board and asking students to match the functions to the graphs. Encourage students to use a graphing utility to graph their guesses, talk about the differences between guesses and the actual graph, and what may cause them in each case.

e.  $f(x) = (x - 2)^2$



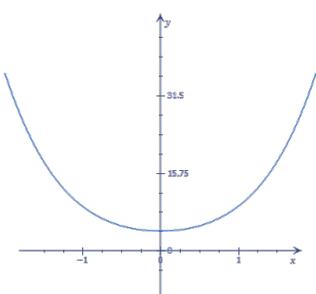
**Zeros:** 2, 2  
**Solutions to  $f(x) = 0$ :** 2  
 **$x$ -intercepts:** 2  
*The degree is 2, which is greater than the number of  $x$ -intercepts.*

f.  $f(x) = (x - 1)(x + 1)(x - 2)(x + 2)(x - 3)(x + 3)(x - 4)$



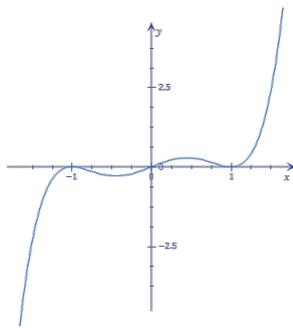
**Zeros:** 1, -1, 2, -2, 3, -3, 4  
**Solutions to  $f(x) = 0$ :** 1, -1, 2, -2, 3, -3, 4  
 **$x$ -intercepts:** 1, -1, 2, -2, 3, -3, 4  
*The degree is 7, which is equal to the number of  $x$ -intercepts.*

g.  $f(x) = (x^2 + 2)^2$



**Zeros:** None  
**Solutions to  $f(x) = 0$ :** No solutions  
 **$x$ -intercepts:** No  $x$ -intercepts  
*The degree is 4, which is greater than the number of  $x$ -intercepts.*

h.  $f(x) = (x + 1)^2(x - 1)^2x$



**Zeros:**  $-1, -1, 1, 1, 0$

**Solutions to  $f(x) = 0$ :**  $-1, 0, 1$

**$x$ -intercepts:**  $-1, 0, 1$

**The degree is 5, which is greater than the number of  $x$ -intercepts.**

### Discussion (1 minutes)

Ask students to summarize what they have learned so far, either in writing or with a partner. Check for understanding of the concepts, and help students reach the following conclusions if they do not do so on their own.

- The  $x$ -intercepts in the graph of a function correspond to the solutions to the equation  $f(x) = 0$  and correspond to the number of distinct zeros of the function (but the  $x$ -intercepts do not help us to determine the multiplicity of a given zero).
- The graph of a polynomial function of degree  $n$  has at most  $n$   $x$ -intercepts but may have fewer.
- A polynomial function whose graph has  $m$   $x$ -intercepts is at least a degree  $m$  polynomial.

### Example 2 (8 minutes)

Lead the students through the questions in order to arrive at a sketch of the final graph. The main point of this exercise is that if we know the  $x$ -intercepts of a polynomial function, we can sketch a fairly accurate graph of the function by just checking to see if the function is positive or negative at a few points. We are not graphing by plotting points and connecting the dots but by applying what we know of polynomial functions.

Give time for students to work through parts (a) and (b) in pairs or small groups before continuing with the discussion in parts (c)-(i). When sketching the graph in part (j), it is important to let the students know that we cannot pinpoint exactly the high and low points on the graph—the relative maximum and minimum values. For this reason, omit a scale on the  $y$ -axis in the sketch.

#### Example 2

Consider the function  $f(x) = x^3 - 13x^2 + 44x - 32$ .

- a. Use the fact that  $x - 4$  is a factor of  $f$  to factor this polynomial.

*Using polynomial division and then factoring,  $f(x) = (x - 4)(x^2 - 9x + 8) = (x - 4)(x - 8)(x - 1)$ .*

- b. Find the  $x$ -intercepts for the graph of  $f$ .

*The  $x$ -intercepts are 1, 4, and 8.*

- c. At which  $x$ -values can the function change from being positive to negative or from negative to positive?

*Only at the  $x$ -intercepts 1, 4, and 8.*

- d. To sketch a graph of  $f$ , we need to consider whether the function is positive or negative on the four intervals  $x < 1$ ,  $1 < x < 4$ ,  $4 < x < 8$ , and  $x > 8$ . Why is that?

*The function can only change sign at the  $x$ -intercepts; therefore, on each of those intervals, the graph will always be above or always be below the axis.*

- e. How can we tell if the function is positive or negative on an interval between  $x$ -intercepts?

*Evaluate the function at a single point in that interval. Since the function is either always positive or always negative between  $x$ -intercepts, checking a single point will indicate behavior on the entire interval.*

- f. For  $x < 1$ , is the graph above or below the  $x$ -axis? How can you tell?

*Since  $f(0) = -32$  is negative, the graph is below the  $x$ -axis for  $x < 1$ .*

- g. For  $1 < x < 4$ , is the graph above or below the  $x$ -axis? How can you tell?

*Since  $f(2) = 12$  is positive, the graph is above the  $x$ -axis for  $1 < x < 4$ .*

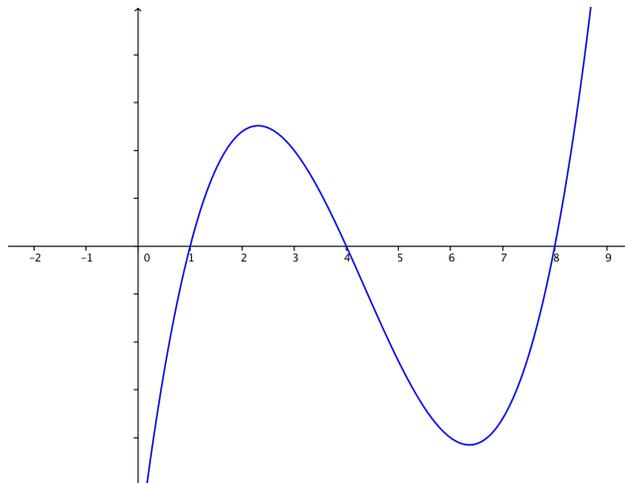
- h. For  $4 < x < 8$ , is the graph above or below the  $x$ -axis? How can you tell?

*Since  $f(5) = -12$  is negative, the graph is below the  $x$ -axis for  $4 < x < 8$ .*

- i. For  $x > 8$ , is the graph above or below the  $x$ -axis? How can you tell?

*Since  $f(10) = 108$  is positive, the graph is above the  $x$ -axis for  $x > 8$ .*

- j. Use the information generated in parts (f)–(i) to sketch a graph of  $f$ .



- k. Graph  $y = f(x)$  on the interval from  $[0, 9]$  using a graphing utility, and compare your sketch with the graph generated by the graphing utility.

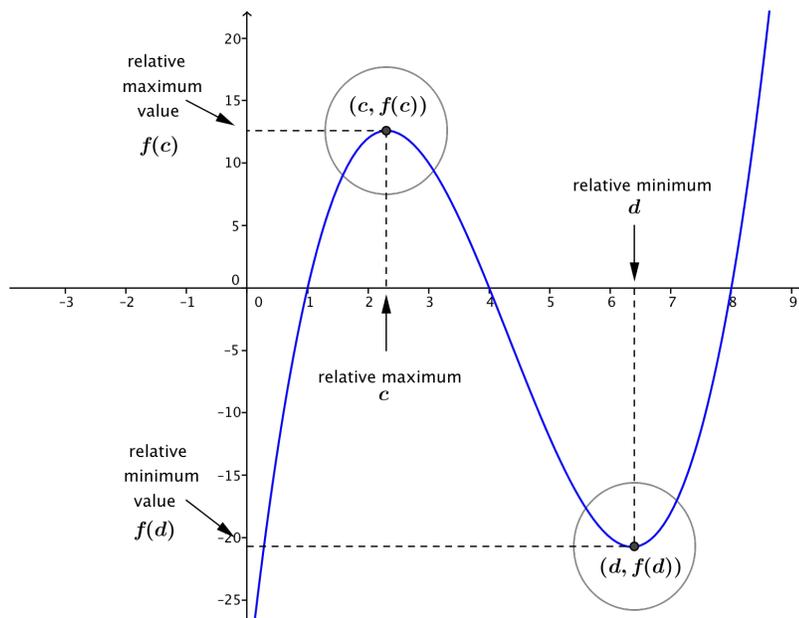
Discussion (6 minutes)

- Let’s examine the interval  $[1,4]$  in the graph of  $f(x) = x^3 - 13x^2 + 44x - 32$  above. Is there a number  $c$  in that interval where the value  $f(c)$  is greater than or equal to any other value of the function on that interval? Do we know exactly where that is?
  - *There is a value of  $c$  such that  $f(c)$  that is greater than or equal to the other values. It seems that  $2 < c < 2.5$ , but we do not know its exact value.*

You might mention that we can find  $c$  exactly using calculus, but this is a topic for another class. For now, you can point out that we can always find relative maxima or relative minima of quadratic functions—they occur at the  $x$ -value of the vertex.

- We call the number  $c$  a **relative maximum**. The *relative maximum value*,  $f(c)$ , may not be the greatest overall value of the function, but there is a circle around  $(c, f(c))$  so that there is no point on the graph higher than  $(c, f(c))$  in that circle.
- Similarly, a **relative minimum** is a number  $d$  in the domain of  $f$  such that there is a circle around the point  $(d, f(d))$  so that there is no point on the graph lower than  $(d, f(d))$  in that circle. The *relative minimum value* is  $f(d)$ .
- Show the relative maxima and relative minima on the graph. The terminology can be confusing, but the image below clarifies the distinction between the relative maximum and the relative minimum value. Point out that there are values of the function that are larger than  $f(c)$ , such as  $f(9)$ , but that  $f(c)$  is the highest value among the “neighbors” of  $c$ .

**Scaffolding:**  
 For English Language Learners, the term “relative” may need some additional instruction and practice to help differentiate it from other uses of this word.  
 It may help to think of the other points in the interval containing the relative maxima as all being related, and of all the relatives present,  $c$  is the value that gives the highest function value.

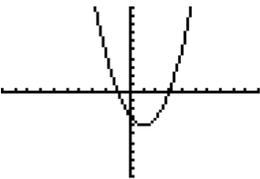


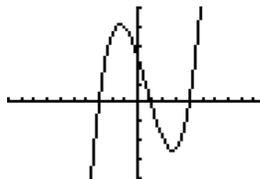
The precise definitions of relative maxima and relative minima are in the glossary of terms for this lesson. These definitions are new to students, so it is worth going over them at the end of the lesson. The main thing to point out to your students is that a relative maximum or minimum actually refers to the  $x$ -value, not the ordered pair on the graph of the function or the value of the function at the relative maximum or minimum. The  $y$ -value is called the relative maximum (or minimum) value of the function.

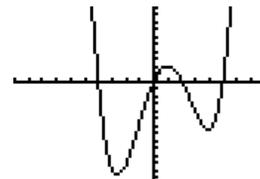
**Discussion**

For any particular polynomial, can we determine how many relative maxima or minima there are? Consider the following polynomial functions in factored form and their graphs.

$f(x) = (x + 1)(x - 3)$	$g(x) = (x + 3)(x - 1)(x - 4)$	$h(x) = (x)(x + 4)(x - 2)(x - 5)$
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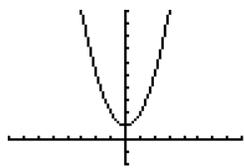
Degree of each polynomial:	3	4
Number of $x$ -intercepts in each graph:	3	4
Number of relative maxima or minima in each graph:	2	3

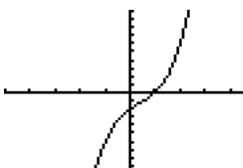
What observations can we make from this information?

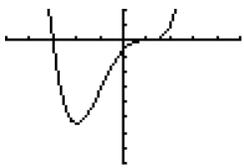
*The number of relative maxima and minima is one less than the degree and one less than the number of  $x$ -intercepts.*

Is this true for every polynomial? Consider the examples below.

$r(x) = x^2 + 1$	$s(x) = (x^2 + 2)(x - 1)$	$t(x) = (x + 3)(x - 1)(x - 1)(x - 1)$
------------------	---------------------------	---------------------------------------







Degree of each polynomial:	3	4
Number of $x$ -intercepts in each graph:	1	2
Number of relative maxima or minima in each graph:	0	1

What observations can we make from this information?

*The observations made in the previous examples do not hold for these examples, so it is difficult to determine from the degree of the polynomial function the number of relative maxima or minima in the graph of the function. What we can say is that for a degree  $n$  polynomial function, there are at most  $n - 1$  relative maxima or relative minima. You can also think about the information you can get from a graph. If a graph of a polynomial function has  $n$  relative maxima or minima, you can say that the degree of the polynomial is at least  $n + 1$ .*

## Closing (1 minute)

- By looking at the factored form of a polynomial, we can identify important characteristics of the graph such as  $x$ -intercepts and relative maxima and minima of the function, which in turn allow us to develop a sketch of the graph.
- A polynomial of degree  $n$  may have up to  $n$   $x$ -intercepts.
- A polynomial of degree  $n$  may have up to  $n - 1$  relative maxima and minima.

## Relevant Vocabulary

**Increasing/Decreasing:** Given a function  $f$  whose domain and range are subsets of the real numbers and  $I$  is an interval contained within the domain, the function is called *increasing on the interval  $I$*  if

$$f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

It is called *decreasing on the interval  $I$*  if

$$f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \text{ in } I.$$

**Relative Maximum:** Let  $f$  be a function whose domain and range are subsets of the real numbers. The function has a *relative maximum at  $c$*  if there exists an open interval  $I$  of the domain that contains  $c$  such that

$$f(x) \leq f(c) \text{ for all } x \text{ in the interval } I.$$

If  $c$  is a relative maximum, then the value  $f(c)$  is called the *relative maximum value*.

**Relative Minimum:** Let  $f$  be a function whose domain and range are subsets of the real numbers. The function has a *relative minimum at  $c$*  if there exists an open interval  $I$  of the domain that contains  $c$  such that

$$f(x) \geq f(c) \text{ for all } x \text{ in the interval } I.$$

If  $c$  is a relative minimum, then the value  $f(c)$  is called the *relative minimum value*.

**Graph of  $f$ :** Given a function  $f$  whose domain  $D$  and the range are subsets of the real numbers, the graph of  $f$  is the set of ordered pairs in the Cartesian plane given by

$$\{(x, f(x)) \mid x \in D\}.$$

**Graph of  $y = f(x)$ :** Given a function  $f$  whose domain  $D$  and the range are subsets of the real numbers, the *graph of  $y = f(x)$*  is the set of ordered pairs  $(x, y)$  in the Cartesian plane given by

$$\{(x, y) \mid x \in D \text{ and } y = f(x)\}.$$

## Lesson Summary

A polynomial of degree  $n$  may have up to  $n$   $x$ -intercepts and up to  $n - 1$  relative maximum/minimum points.

A relative maximum is the  $x$ -value  $c$  that produces the highest point on a graph of  $f$  in a circle around  $(c, f(c))$ . That highest value  $f(c)$  is a relative maximum value.

A relative minimum is the  $x$ -value  $d$  that produces the lowest point on a graph of  $f$  in a circle around  $(d, f(d))$ . That lowest value  $f(d)$  is a relative minimum value.

## Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 14: Graphing Factored Polynomials

### Exit Ticket

Sketch a graph of the function  $f(x) = x^3 + x^2 - 4x - 4$  on the interval  $[-2, 2]$  by finding the zeros and determining the sign of the function between zeros. Explain how the structure of the equation helps guide your sketch.

Exit Ticket Sample Solutions

Sketch a graph of the function  $f(x) = x^3 + x^2 - 4x - 4$  by finding the zeros and determining the sign of the function between zeros. Explain how the structure of the equation helps guide your sketch.

$$f(x) = (x + 1)(x + 2)(x - 2)$$

**Zeros:**  $-1, -2, 2$

**For  $x < -2$ :**  $f(-3) = -10$ , so the graph is below the  $x$ -axis on this interval.

**For  $-2 < x < -1$ :**  $f(-1.5) = 0.875$ , so the graph is above the  $x$ -axis on this interval.

**For  $-1 < x < 2$ :**  $f(0) = -4$ , so the graph is below the  $x$ -axis on this interval.

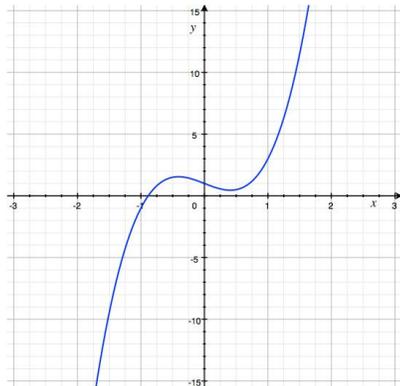
**For  $x > 2$ :**  $f(3) = 20$ , so the graph is above the  $x$ -axis on this interval.

Problem Set Sample Solutions

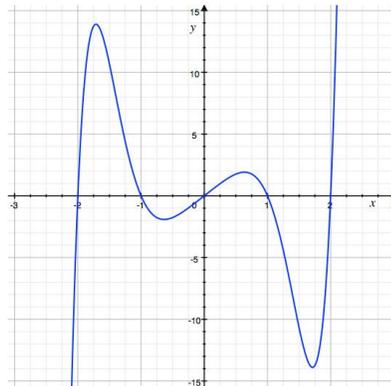
1. For each function below, identify the largest possible number of  $x$ -intercepts and the largest possible number of relative maximum and minimum points based on the degree of the polynomial. Then use a calculator or graphing utility to graph the function and find the actual number of  $x$ -intercepts and relative maximum/minimum points.

- $f(x) = 4x^3 - 2x + 1$
- $g(x) = x^7 - 4x^5 - x^3 + 4x$
- $h(x) = x^4 + 4x^3 + 2x^2 - 4x + 2$

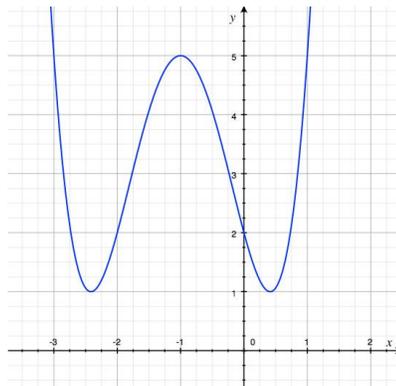
Function	Largest number of $x$ -intercepts	Largest number of relative max/min	Actual number of $x$ -intercepts	Actual number of relative max/min
a. $f(x)$	3	2	1	2
b. $g(x)$	7	6	5	4
c. $h(x)$	4	3	0	3



a.  $f(x) = 4x^3 - 2x + 1$



b.  $g(x) = x^7 - 4x^5 - x^3 + 4x$



c.  $h(x) = x^4 + 4x^3 + 2x^2 - 4x + 2$

2. Sketch a graph of the function  $f(x) = \frac{1}{2}(x + 5)(x + 1)(x - 2)$  by finding the zeros and determining the sign of the values of the function between zeros.

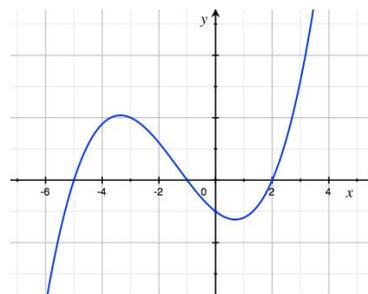
The zeros are  $-5$ ,  $-1$ , and  $2$ .

For  $x < -5$ :  $f(-6) = -20$ , so the graph is below the  $x$ -axis for  $x < -5$ .

For  $-5 < x < -1$ :  $f(-3) = 10$ , so the graph is above the  $x$ -axis for  $-5 < x < -1$ .

For  $-1 < x < 2$ :  $f(0) = -5$ , so the graph is below the  $x$ -axis for  $-1 < x < 2$ .

For  $x > 2$ :  $f(3) = 16$ , so the graph is above the  $x$ -axis for  $x > 2$ .



3. Sketch a graph of the function  $f(x) = -(x + 2)(x - 4)(x - 6)$  by finding the zeros and determining the sign of the values of the function between zeros.

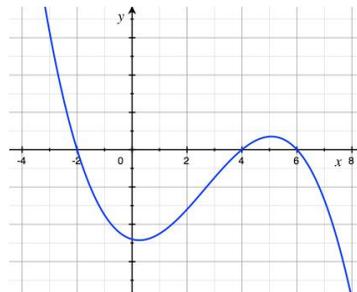
The zeros are  $-2, 4,$  and  $6$ .

For  $x < -2$ :  $f(-3) = 63$ , so the graph is above the  $x$ -axis for  $x < -2$ .

For  $-2 < x < 4$ :  $f(0) = -48$ , so the graph is below the  $x$ -axis for  $-2 < x < 4$ .

For  $4 < x < 6$ :  $f(5) = 7$ , so the graph is above the  $x$ -axis for  $4 < x < 6$ .

For  $x > 6$ :  $f(7) = -27$ , so the graph is below the  $x$ -axis for  $x > 6$ .



4. Sketch a graph of the function  $f(x) = x^3 - 2x^2 - x + 2$  by finding the zeros and determining the sign of the values of the function between zeros.

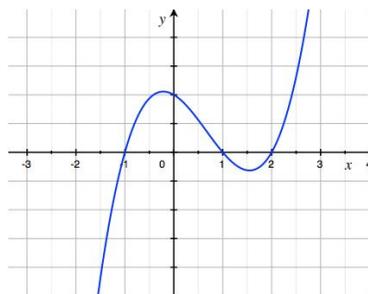
We can factor by grouping to find  $f(x) = (x^2 - 1)(x - 2)$ . The zeros are  $-1, 1,$  and  $2$ .

For  $x < -1$ :  $f(-2) = -12$ , so the graph is below the  $x$ -axis for  $x < -1$ .

For  $-1 < x < 1$ :  $f(0) = 2$ , so the graph is above the  $x$ -axis for  $-1 < x < 1$ .

For  $1 < x < 2$ :  $f\left(\frac{3}{2}\right) = -\frac{5}{8}$ , so the graph is below the  $x$ -axis for  $1 < x < 2$ .

For  $x > 2$ :  $f(3) = 8$ , so the graph is above the  $x$ -axis for  $x > 2$ .



5. Sketch a graph of the function  $f(x) = x^4 - 4x^3 + 2x^2 + 4x - 3$  by determining the sign of the values of the function between the zeros  $-1, 1,$  and  $3$ .

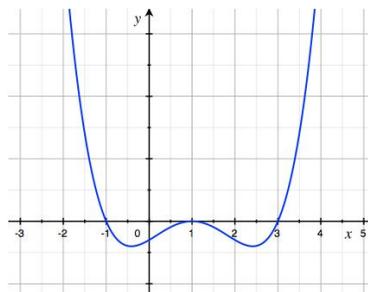
We are told that the zeros are  $-1, 1,$  and  $3$ .

For  $x < -1$ :  $f(-2) = 45$ , so the graph is above the  $x$ -axis for  $x < -1$ .

For  $-1 < x < 1$ :  $f(0) = -3$ , so the graph is below the  $x$ -axis for  $-1 < x < 1$ .

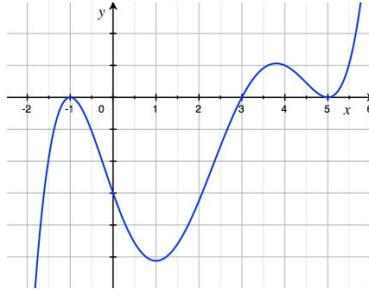
For  $1 < x < 3$ :  $f(2) = -3$ , so the graph is below the  $x$ -axis for  $1 < x < 3$ .

For  $x > 3$ :  $f(4) = 45$ , so the graph is above the  $x$ -axis for  $x > 3$ .



6. A function  $f$  has zeros at  $-1$ ,  $3$ , and  $5$ . We know that  $f(-2)$  and  $f(2)$  are negative, while  $f(4)$  and  $f(6)$  are positive. Sketch a graph of  $f$ .

*From the information given, the graph of  $f$  lies below the  $x$ -axis for  $x < -1$  and  $-1 < x < 3$  and that it touches the  $x$ -axis at  $-1$ . Similarly, we know that the graph of  $f$  lies above the  $x$ -axis for  $3 < x < 5$  and  $5 < x$  and that it touches the  $x$ -axis at  $5$ . We also know that the graph crosses the  $x$ -axis at  $3$ .*



7. The function  $h(t) = -16t^2 + 33t + 45$  represents the height of a ball tossed upward from the roof of a building 45 feet in the air after  $t$  seconds. Without graphing, determine when the ball will hit the ground.

**Factor:**  $h(t) = (t - 3)(-16t - 15)$

**Solve  $h(t) = 0$ :**  $(t - 3)(-16t - 15) = 0$

$t = 3$  seconds or  $t = -\frac{15}{16}$  seconds.

*The ball hits the ground at time 3 seconds; the solution  $-\frac{15}{16}$  does not make sense in the context of the problem.*



## Lesson 15: Structure in Graphs of Polynomial Functions

### Student Outcomes

- Students graph polynomial functions and describe end behavior based upon the degree of the polynomial.

### Lesson Notes

So far in this module, we have practiced factoring polynomials using several techniques and examined how we can use the factored form of the polynomial to identify interesting characteristics of the graphs of these functions. In this lesson, we are going to continue exploring graphs of polynomial functions in order to identify how the degree of the polynomial influences the end behavior of these graphs. We will also discuss how to identify  $y$ -intercepts of the graphs of polynomial functions and present an opportunity for students to construct viable arguments and critique the reasoning of others in the Opening Exercise (MP.3).

### Opening Exercise (8 minutes)

#### Opening Exercise

Sketch the graph of  $f(x) = x^2$ . What will the graph of  $g(x) = x^4$  look like? Sketch it on the same coordinate plane. What will the graph of  $h(x) = x^6$  look like?

Have students recall and sketch the graph of  $f(x) = x^2$ . Discuss the characteristics of the graph, where the  $x$ -intercept is, and why the graph stays above the  $x$ -axis on either side of the  $x$ -intercept.

In pairs or in groups, have them discuss or write what they think the graph of  $g(x) = x^4$  will look like and how they think it compares to the graph of  $f(x) = x^2$ . Once they do so, they should sketch their idea of the graph of  $g$  on top of the graph of  $f$ . Discuss with students what they have sketched and emphasize the similarities between the two graphs.

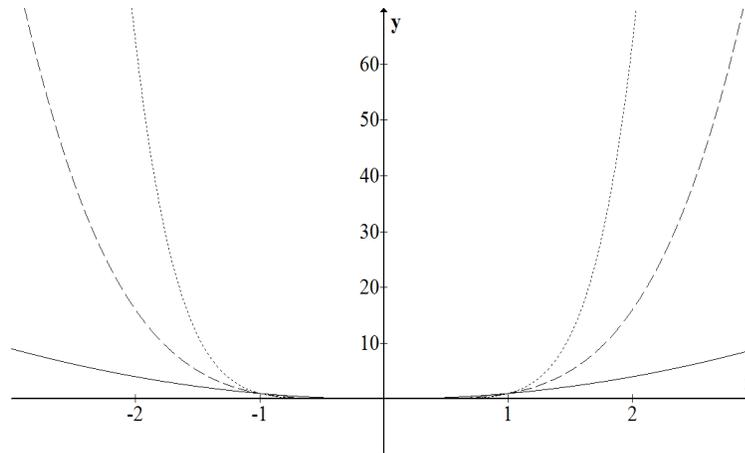
*Since  $g(x) = (x^2)^2$ ,  $g(x)$  will increase faster as  $x$  increases than  $f(x)$  does. Both graphs pass through  $(0, 0)$ . The basic shapes are the same, but near the origin the graph of  $g$  is flatter than the graph of  $f$ .*

MP.3

Finally, in pairs or in groups, have students discuss or write what they think the graph of  $h(x) = x^6$  will look like and how they think it will compare to graphs of  $f$  and  $g$ . Once they do so, students should sketch on the same graph the previous two graphs. Again, discuss graphs with students and emphasize the similarities between graphs.

*Since  $h(x) = x^2 \cdot x^2 \cdot x^2$ , the graph of  $h$  again passes through the origin. Since we are squaring and multiplying by squares, the graph of  $h$  should look about the same as the graphs of  $f$  and  $g$  but increase even faster and be even flatter near the origin.*

- Using a graphing utility, have students graph all three functions simultaneously to confirm their sketches.



### Discussion (5 minutes)

Use the graphs from the Opening Exercise to frame the following discussion about end behavior.

Ask students to compare and describe the behavior of the value  $f(x)$  as the absolute value of  $x$  increases without bound. Introduce the term *end behavior* as a way to talk about the function and what happens to its graph beyond the bounded region of the coordinate plane that is drawn on paper. That is, the end behavior is way to describe what happens to the function as  $x$  approaches positive and negative infinity without having to draw the graph.

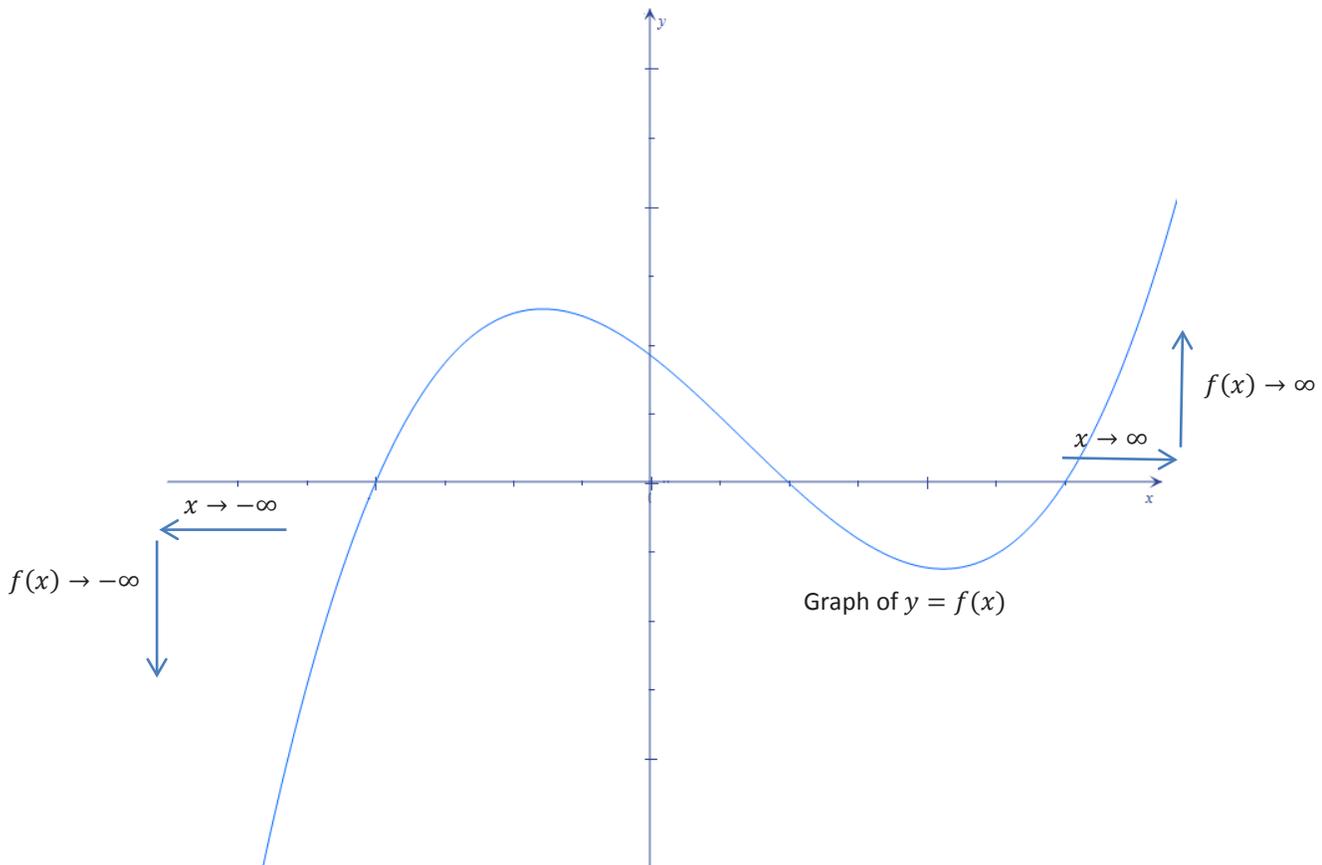
Note to teacher: It is important to note that end behavior cannot be given a precise mathematical definition until the concept of a limit is introduced in calculus. To get around this difficulty, most high school textbooks draw pictures and state things like, “As  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$ .” We will do this also, but it is important to carefully describe to your students the meaning of the phrase, “As  $x$  approaches positive infinity,” *before using the phrase (or its symbol version) to describe end behavior*. That is because the phrase appears to mean that the symbol  $x$  is literally “moving along the number line to the right.” Not true! Recall that a variable is just a placeholder for which a number can be substituted (think of a blank or box used in Grade 2 equations) and, therefore, does not actually move or vary.

The phrase, “As  $x \rightarrow \infty$ ,” can be profitably described as a *process* by which the user of the phrase thinks of repeatedly substituting larger and larger positive numbers in for  $x$ , each time performing whatever calculation is required by the problem for that number (which in this lesson is finding the value of the function).

This is how mathematicians often use the phrase even though the precise definition of limit removes any need to think of a limit as a process.

- End Behavior (description):** Let  $f$  be a function whose domain and range are subsets of the real numbers. The *end behavior of a function  $f$*  is a description of what happens to the values of the function
  - as  $x$  approaches positive infinity, and
  - as  $x$  approaches negative infinity.

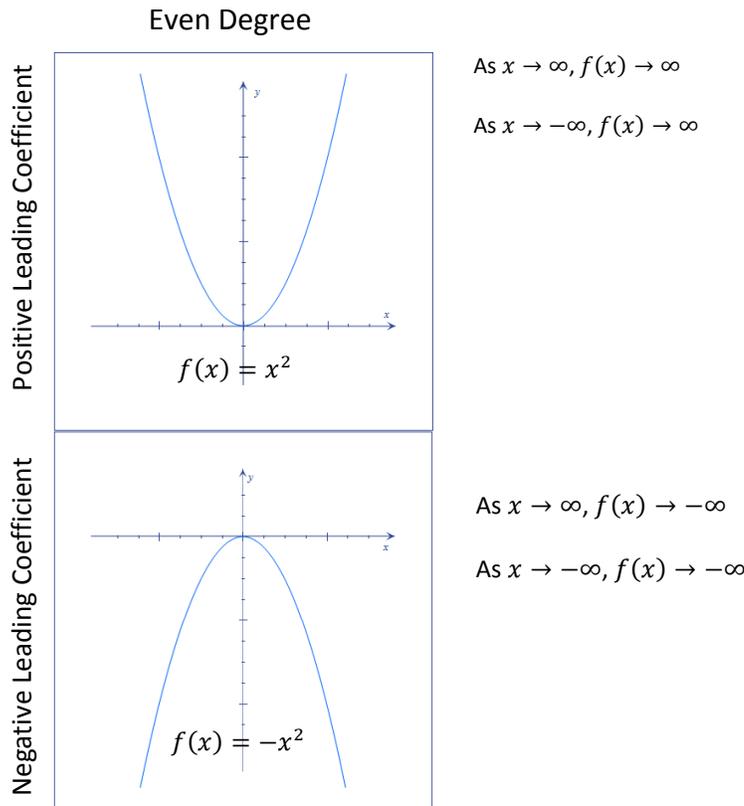
Help students understand the description of end behavior using the following picture.



As  $x \rightarrow -\infty$ ,  
 $f(x) \rightarrow -\infty$

As  $x \rightarrow \infty$ ,  
 $f(x) \rightarrow \infty$

Ask students to make a generalization about the end behavior of polynomials of even degree in writing individually or with a partner. They should conclude that an even degree polynomial function has the same end behavior as  $x \rightarrow \infty$  and as  $x \rightarrow -\infty$ . After students have generalized the end behavior, have them create their own graphic organizer like the following.



If students suspect that end behavior of a polynomial function with even degree will always increase, then suggest examining the graphs of  $f(x) = 1 - x^2$  and  $g(x) = -x^4$ .

**Example 1 (8 minutes)**

We are now going to look at a new set of functions but ask similar questions to those we asked in the Opening Exercise.

**Example 1**

Sketch the graph of  $f(x) = x^3$ . What will the graph of  $g(x) = x^5$  look like? Sketch this on the same coordinate plane. What will the graph of  $h(x) = x^7$  look like? Sketch this on the same coordinate plane.

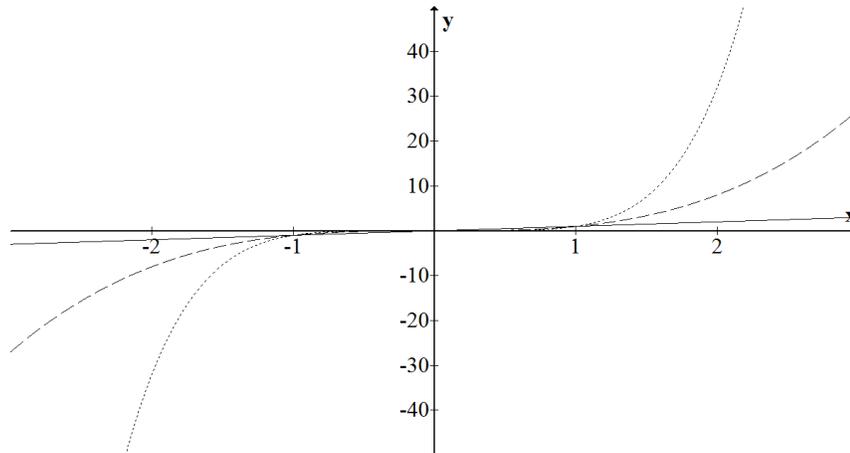
Have students recall and sketch the graph of  $f(x) = x^3$ . Discuss the characteristics of the graph, where the  $x$ -intercept is, and why the graph is above the  $x$ -axis for  $x > 0$  and below the  $x$ -axis for  $x < 0$ .

MP.3

In pairs or in groups, have students discuss or write what they think the graph of  $g(x) = x^5$  will look like and how it will relate to the graph of  $f(x) = x^3$ . They should sketch their results on top of the original graph of  $f$ . Discuss with students what they have sketched, and emphasize the similarities to the graph of  $f(x) = x^3$ .

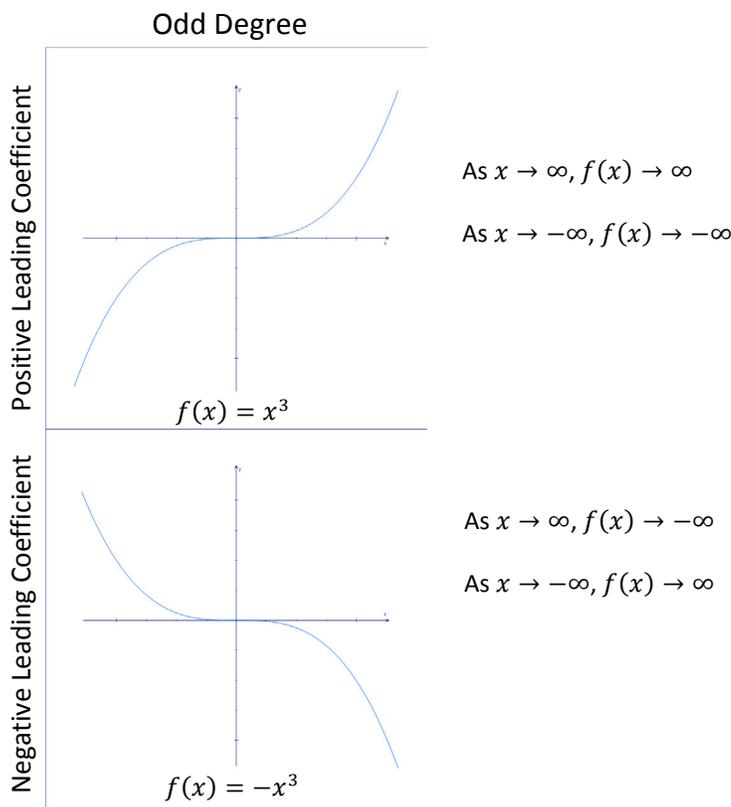
Finally, in pairs or in groups, have students discuss or write what they think the graph of  $h(x) = x^7$  will look like and how it will relate to the graphs of  $f$  and  $g$ . They should sketch on the same graph they used with the previous two graphs. Again, discuss the graphs with students, and emphasize the similarities between graphs.

Using a graphing utility, have students graph all three functions simultaneously to confirm their sketches.



Ask students to compare and describe the behavior of the value of  $f(x)$  as the absolute value of  $x$  increases without bound. Guide them to use the terminology of the end behavior of the function.

Ask students to make a generalization about the end behavior of polynomials of odd degree individually or with a partner. After students have generalized the end behavior, have them create their own graphic organizer like the following.



If students suspect that polynomial functions with odd degree always have the value of the function increase as  $x$  increases, then suggest examining a function with a negative leading coefficient, such as  $f(x) = 4 - x$  or  $g(x) = -x^3$ .

- How do these graphs differ from those in the Opening Exercise? Why are they different?
  - *Students may talk about how the Opening Exercise graphs stay above the  $x$ -axis while in this example the graphs cut through the  $x$ -axis. Guide students as necessary to concluding that the end behavior of even-degree polynomial functions is that both ends both approach positive infinity or both approach negative infinity while the end behavior of odd-degree polynomial is that the behavior as  $x \rightarrow \infty$  is opposite of the behavior as  $x \rightarrow -\infty$ .*

MP.8

### Exercise 1 (8 minutes)

Keeping the results of the examples above in mind, have students work with partners or in groups to answer the following questions.

#### Exercise 1

- a. Consider the following function,  $f(x) = 2x^4 + x^3 - x^2 + 5x + 3$ , with a mixture of odd and even degree terms. Predict whether its end behavior will be like the functions in the Opening Exercise or more like the functions from Example 1. Graph the function  $f$  using a graphing utility to check your prediction.  
*Students will see that this function acts more like the even-degree monomial functions from the Opening Exercise.*
- b. Consider the following function,  $f(x) = 2x^5 - x^4 - 2x^3 + 4x^2 + x + 3$ , with a mixture of odd and even degree terms. Predict whether its end behavior will be like the functions in the Opening Exercise or more like the functions from Example 1. Graph the function  $f$  using a graphing utility to check your prediction.  
*Students will see that this function acts more like odd-degree monomial functions from Example 1. They can draw a conclusion such as that the function will behave like the highest degree term.*
- c. Thinking back to our discussion of  $x$ -intercepts of graphs of polynomial functions from the previous lesson, sketch a graph of an even degree polynomial function that has no  $x$ -intercepts.  
*Students may draw the graph of a quadratic function that stays above the  $x$ -axis such as the graph of  $f(x) = x^2 + 1$ .*
- d. Similarly, can you sketch a graph of an odd degree polynomial function with no  $x$ -intercepts?  
*Have students work in pairs or groups and discover that because of the “cut through” nature of graphs of odd-degree polynomial function there will always be an  $x$ -intercept.*  
*Conclusion: Graphs of odd-powered polynomial functions will always have an  $x$ -intercept, which means that odd-degree polynomial functions will always have at least one zero (or root) and that polynomial functions of odd-degree always have opposite end behaviors as  $x \rightarrow \infty$  and  $x \rightarrow -\infty$ .*

Have students conclude that the graphs of odd-degree polynomial functions always have at least one  $x$ -intercept and so the functions always have at least one zero. The graphs of even-degree polynomial functions may or may not have  $x$ -intercepts.

**Exercise 2 (8 minutes)**

In this exercise, students use what they learned today about end behavior to determine whether or not the polynomial function used to model the data has an even or odd degree.

**Exercise 2**

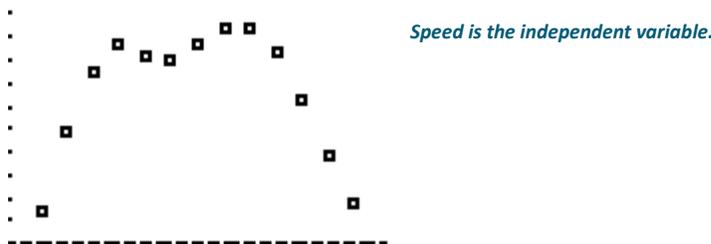
The Center for Transportation Analysis (CTA) studies all aspects of transportation in the United States, from energy and environmental concerns to safety and security challenges. A 1997 study compiled the following data of the fuel economy in miles per gallon (mpg) of a car or light truck at various speeds measured in miles per hour (mph). The data is compiled in the table below.

**Fuel Economy by Speed**

Speed (mph)	Fuel Economy (mpg)
15	24.4
20	27.9
25	30.5
30	31.7
35	31.2
40	31.0
45	31.6
50	32.4
55	32.4
60	31.4
65	29.2
70	26.8
75	24.8

Source: Transportation Energy Data Book, Table 4.28. <http://cta.ornl.gov/data/chapter4.shtml>

- a. Plot the data using a graphing utility. Which variable is the independent variable?



- b. This data can be modeled by a polynomial function. Determine if the function that models the data would have an even or odd degree.

*It seems we could model this data by an even degree polynomial function.*

- c. Is the leading coefficient of the polynomial that can be used to model this data positive or negative?

*The leading coefficient would be negative since the end behavior of this function is to approach negative infinity on both sides.*

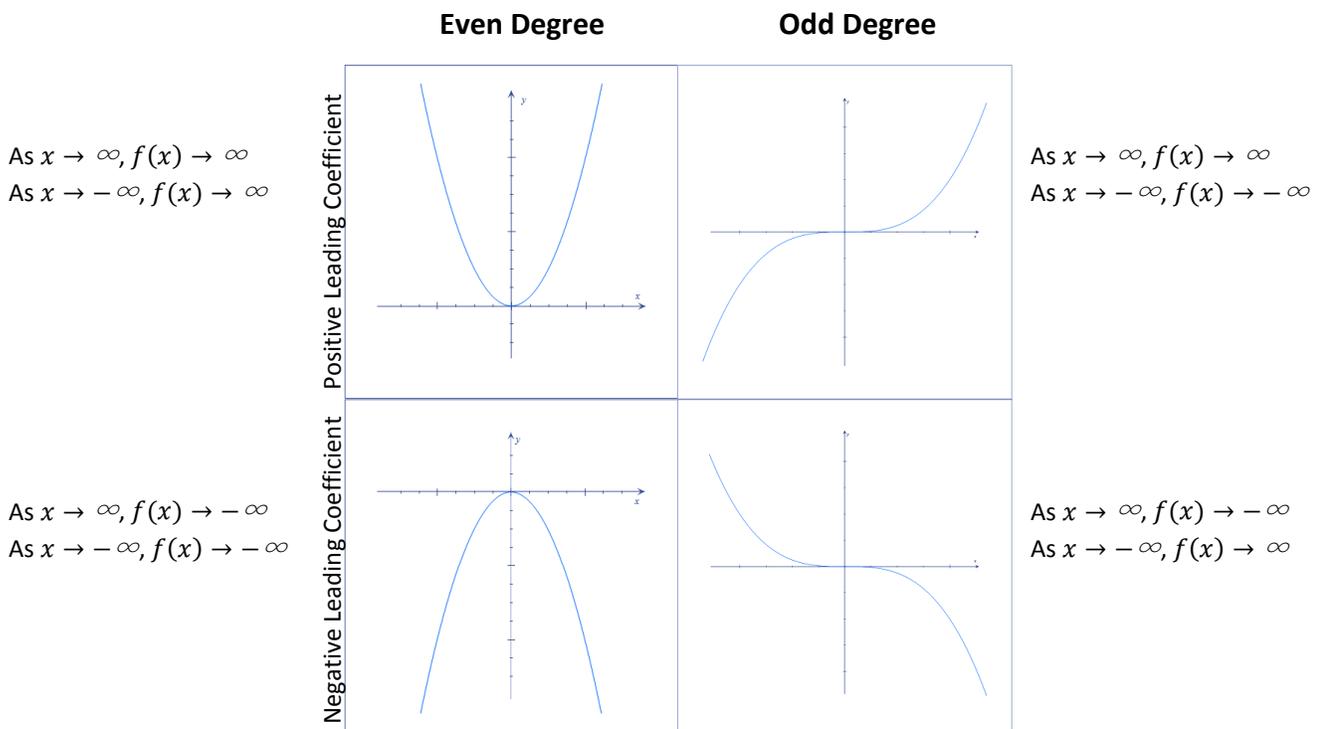
- d. List two possible reasons the data might have the shape that it does.

*Possible responses: Fuel economy improves up to a certain speed, but then wind resistance at higher speeds reduces fuel economy; the increased gas needed to go higher speeds reduces fuel economy.*

**Closing (3 minutes)**

- In this lesson, we explored the impact that an even- or odd-degree polynomial function has on its graph, where even-degree polynomials demonstrate the same end behavior as  $x \rightarrow \infty$  as it does as  $x \rightarrow -\infty$ , while odd-degree polynomials demonstrate opposite end behavior as  $x \rightarrow \infty$  as it does as  $x \rightarrow -\infty$ . Because of this fact, graphs of odd-degree polynomial functions will always intersect the  $x$ -axis; therefore, odd-degree polynomials have at least one zero or root.
- We also learned that it is the highest degree term of the polynomial function that determines if the graph exhibits odd-degree end behavior or even-degree end behavior. This makes sense because the highest degree term of a polynomial determines the degree of the polynomial.

Have students summarize the lesson either with a graphic organizer or a written summary. A graphic organizer is included below.



**Relevant Vocabulary**

**Even Function:** Let  $f$  be a function whose domain and range is a subset of the real numbers. The function  $f$  is called *even* if the equation  $f(x) = f(-x)$  is true for every number  $x$  in the domain.

Even-degree polynomial functions are sometimes even functions, like  $f(x) = x^{10}$ , and sometimes not, like  $g(x) = x^2 - x$ .

**Odd Function:** Let  $f$  be a function whose domain and range is a subset of the real numbers. The function  $f$  is called *odd* if the equation  $f(-x) = -f(x)$  is true for every number  $x$  in the domain.

Odd-degree polynomial functions are sometimes odd functions, like  $f(x) = x^{11}$ , and sometimes not, like  $h(x) = x^3 - x^2$ .

**Exit Ticket (5 minutes)**

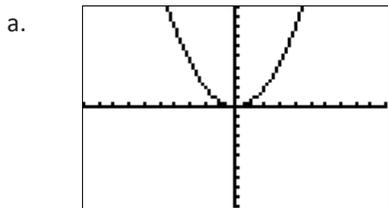
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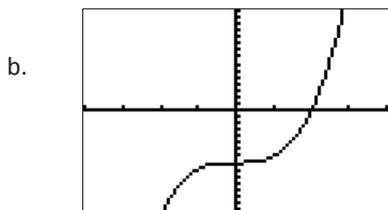
## Lesson 15: Structure in Graphs of Polynomial Functions

### Exit Ticket

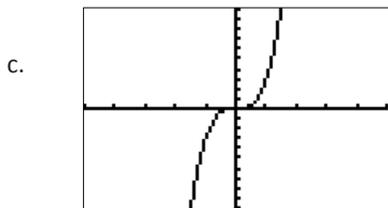
Without using a graphing utility, match each graph below in column 1 with the function in column 2 that it represents.



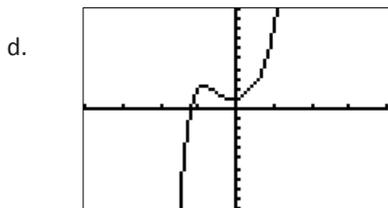
1.  $y = 3x^3$



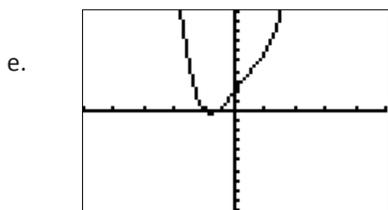
2.  $y = \frac{1}{2}x^2$



3.  $y = x^3 - 8$



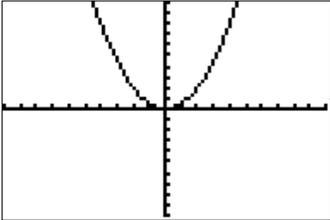
4.  $y = x^4 - x^3 + 4x + 2$

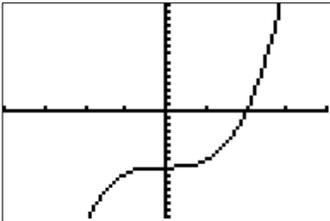


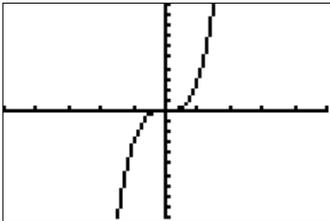
5.  $y = 3x^5 - x^3 + 4x + 2$

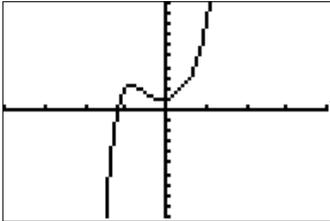
Exit Ticket Sample Solutions

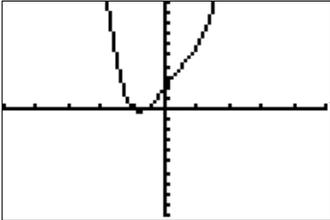
Without using a graphing utility, match each graph below in column 1 with the function in column 2 that it represents.

a. 

b. 

c. 

d. 

e. 

1.  $y = 3x^3$

2.  $y = \frac{1}{2}x^2$

3.  $y = x^3 - 8$

4.  $y = x^4 - x^3 + 4x + 2$

5.  $y = 3x^5 - x^3 + 4x + 2$

Problem Set Sample Solutions

1. Graph the functions from the Opening Exercise simultaneously using a graphing utility and zoom in at the origin.
  - a. At  $x = 0.5$ , order the values of the functions from least to greatest.  
 At  $x = 0.5$ ,  $x^6 < x^4 < x^2$ .
  - b. At  $x = 2.5$ , order the values of the functions from least to greatest.  
 At  $x = 2.5$ ,  $x^2 < x^4 < x^6$ .
  - c. Identify the  $x$ -value(s) where the order reverses. Write a brief sentence on why you think this switch occurs.  
 At  $x = 1$  and  $x = -1$ , the values of the functions are equal. Students may write that when a number between 0 and 1 is taken to higher even powers, it gets smaller, and when a number greater than 1 is taken to higher even powers, it gets larger, and when a negative number is raised to an even power it becomes positive. So for  $-1 < x < 0$  the behavior is the same as for  $0 < x < 1$ .

2. The National Agricultural Statistics Service (NASS) is an agency within the USDA that collects and analyzes data covering virtually every aspect of agriculture in the United States. The following table contains information on the amount (in tons) of the following vegetables produced in the U.S. from 1988–1994 for processing into canned, frozen, and packaged foods: lima beans, snap beans, beets, cabbage, sweet corn, cucumbers, green peas, spinach, and tomatoes.

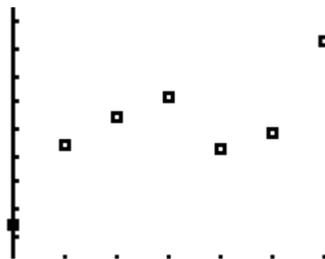
Vegetable Production by Year

Year	Vegetable Production (tons)
1988	11,393,320
1989	14,450,860
1990	15,444,970
1991	16,151,030
1992	14,236,320
1993	14,904,750
1994	18,313,150

Source: NASS Statistics of Vegetables and Melons, 1995, Table 191.

[http://www.nass.usda.gov/Publications/Ag\\_Statistics/1995-1996/agr95\\_4.pdf](http://www.nass.usda.gov/Publications/Ag_Statistics/1995-1996/agr95_4.pdf)

- a. Plot the data using a graphing utility.



- b. Determine if the data displays the characteristics of an odd- or even-degree polynomial function.  
 Looking at the end behavior, the data shows the characteristics of an odd-degree polynomial function.
- c. List two possible reasons the data might have the shape that it does.  
 Possible responses: Bad weather in 1992 and 1993; shifts in demand for fresh foods vs. processed.

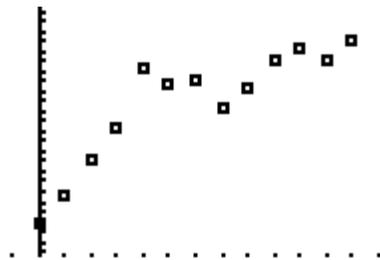
3. The U.S. Energy Information Administration (EIA) is responsible for collecting and analyzing information about energy production and use in the United States and for informing policy makers and the public about issues of energy, the economy, and the environment. The following table contains data from the EIA about natural gas consumption from 1950–2010, measured in millions of cubic feet.

U.S. Natural Gas Consumption by Year

Year	U.S. natural gas total consumption (millions of cubic feet)
1950	5.77
1955	8.69
1960	11.97
1965	15.28
1970	21.14
1975	19.54
1980	19.88
1985	17.28
1990	19.17
1995	22.21
2000	23.33
2005	22.01
2010	24.09

Source: U.S. Energy Information Administration. <http://www.eia.gov/dnav/ng/hist/n9140us2a.htm>

- a. Plot the data using a graphing utility.



- b. Determine if the data displays the characteristics of an odd- or even-degree polynomial function.

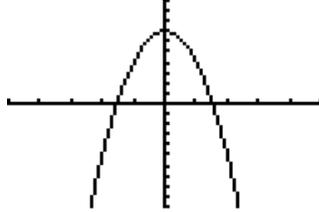
*Looking at the end behavior, the data shows the characteristics of an odd-degree polynomial function.*

- c. List two possible reasons the data might have the shape that it does.

*Possible responses: changes in supply, new sources and technology created new supplies, weather may impact usage.*

4. We use the term *even function* when a function  $f$  satisfies the equation  $f(-x) = f(x)$  for every number  $x$  in its domain. Consider the function  $f(x) = -3x^2 + 7$ . Note that the degree of the function is even, and each term is of an even degree (the constant term is degree 0).

- a. Graph the function using a graphing utility.



- b. Does this graph display any symmetry?

*Yes, it is symmetric about the y-axis.*

- c. Evaluate  $f(-x)$ .

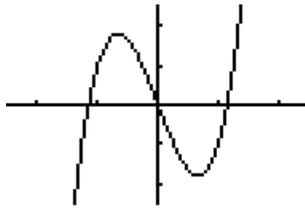
$$f(-x) = -3(-x)^2 + 7 = -3x^2 + 7$$

- d. Is  $f$  an even function? Explain how you know.

*Yes, because  $f(-x) = -3x^2 + 7 = f(x)$  for all real values of  $x$ .*

5. We use the term *odd function* when a function  $f$  satisfies the equation  $f(-x) = -f(x)$  for every number  $x$  in its domain. Consider the function  $f(x) = 3x^3 - 4x$ . The degree of the function is odd, and each term is of an odd degree.

- a. Graph the function using a graphing utility.



- b. Does this graph display any symmetry?

*Yes, but not the same as in part (a). This graph is symmetric about the origin. We can see this because if the graph is rotated 180° about the origin, it appears to be unchanged.*

- c. Evaluate  $f(-x)$ .

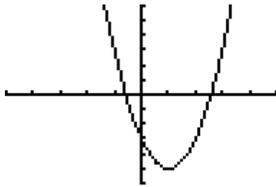
$$f(-x) = 3(-x)^3 - 4(-x) = -3x^3 + 4x$$

- d. Is  $f$  an odd function? Explain how you know.

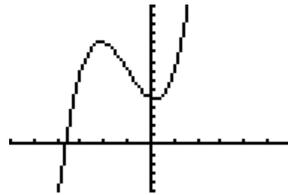
*Yes, we know because  $f(-x) = -3x^3 + 4x = -(3x^3 - 4x) = -f(x)$  for all real values of  $x$ .*

6. We have talked about  $x$ -intercepts of the graph of a function in both this lesson and the previous one. The  $x$ -intercepts correspond to the zeros of the function. Consider the following examples of polynomial functions and their graphs to determine an easy way to find the  $y$ -intercept of the graph of a polynomial function.

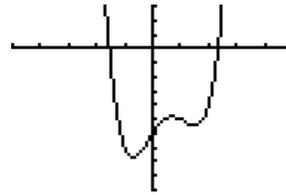
$$f(x) = 2x^2 - 4x - 3$$



$$f(x) = x^3 + 3x^2 - x + 5$$



$$f(x) = x^4 - 2x^3 - x^2 + 3x - 6$$



*The  $y$ -intercept is the value where the graph of a function  $f$  intersects the  $y$ -axis, if  $0$  is in the domain of  $f$ . Therefore, for a function  $f$  whose domain and range are a subset of the real numbers, the  $y$ -intercept is  $f(0)$ . For polynomial functions,  $f(0)$  is easy to determine—it is just the constant term when the polynomial function is written in standard form.*



## Lesson 16: Modeling with Polynomials—An Introduction

### Student Outcomes

- Students transition between verbal, numerical, algebraic, and graphical thinking in analyzing applied polynomial problems.

### Lesson Notes

Creating an open-topped box of maximum volume is a very common problem seen in calculus. The goal is to optimize resources by enclosing the most volume possible given the constraint of the size of the construction material; here we use paper. The dimensions given can be adjusted depending on the size of the paper chosen; hence, the dimensions are omitted from the figure on the student pages. This is the first part of a two-day lesson on modeling. Lesson 16 will focus more on students writing equations to model a situation.

### Classwork

#### Opening (5 minutes)

Each group has a piece of construction paper that measures 45.7 cm  $\times$  30.5 cm. Other sizes of paper may be used if necessary, but ensure that each group is using the same sized paper. Cut out congruent squares from each corner and fold the sides in order to create an open-topped box. Our goal is to create a box with the maximum possible volume.

MP.3

Ask students to make conjectures about what size cut will create the box with the largest volume. Demonstrate if desired using the applet <http://mste.illinois.edu/carvell/3dbox/>.

#### Mathematical Modeling Exercise (30 minutes)

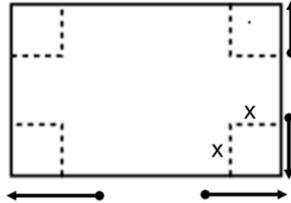
While the students work on their boxes, put the following table on the board. As students measure their boxes and calculate the volume, they should be recording the values in the table. Stop students once each group has recorded its values, and have the discussion below before allowing them to continue working.

##### Mathematical Modeling Exercise

You will be assigned to a group, which will create a box from a piece of construction paper. Each group will record its box's measurements and use said measurement values to calculate and record the volume of its box. Each group will contribute to the following class table on the board.

Group	Length	Width	Height	Volume
1				
2				
3				
4				

Using the given construction paper, cut out congruent squares from each corner and fold the sides in order to create an open-topped box as shown on the figure below.



1. Measure the length, width, and height of the box to the nearest tenth of a centimeter.

*Answers will vary. Sample answer:*

*Length:  $L = 35.7$  cm*

*Width:  $W = 20.5$  cm*

*Height:  $H = 5.0$  cm*

2. Calculate the volume.

*Answers will vary. Sample answer:*

*Volume:  $V = L \cdot W \cdot H = 3,659.25$  cm<sup>3</sup>*

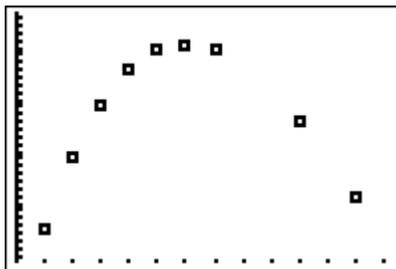
3. Have a group member record the values on the table on the board.

Discuss the results and compare them with the conjectures made before cutting the paper.

- Who was able to enclose the most volume?
- Why would our goal be to enclose the most volume?
  - *We are optimizing our resources by enclosing more volume than the other groups using the same size paper.*

Have students continue with the exercise.

4. Create a scatterplot of volume versus height using technology.



*Scaffolding:*  
Some students may have difficulty working with technology. Place them in a group with a student who can assist them through the steps.

5. What type of polynomial function could we use to model this data?

*Cubic or quadratic; we cannot tell from just this portion of the graph.*

6. Use the regression feature to find a function to model the data. Does a quadratic or a cubic regression provide a better fit to the data?

*Answers will vary based on the accuracy of the measurements, but the cubic regression should be a better fit.*

*Sample answer:  $V(x) = 4x^3 - 152.4x^2 + 1,398.8x$*

7. Find the maximum volume of the box.

*The maximum volume is  $3,770.4 \text{ cm}^3$*

8. What size square should be cut from each corner in order to maximize the volume?

*A  $6 \text{ cm} \times 6 \text{ cm}$  square should be cut from each corner.*

9. What are the dimensions of the box of maximum volume?

*The dimensions are  $33.7 \text{ cm} \times 18.5 \text{ cm} \times 6 \text{ cm}$*

- What are the possible values for the height of the box?
  - *From 0 to 15.25 cm*
- What is the domain of the volume function?
  - *The domain is the interval  $0 < x < 15.25$ .*
- Is constructing a box in such a way that its volume is maximized always the best option?
  - *No. A box may need to have particular dimensions (such as a shoe box).*

### Closing (5 minutes)

You can use the applet <http://www.mathopenref.com/calcbboxproblem.html> to summarize what the students discovered.

- Revisit their original conjecture either in writing or with a neighbor. Was it accurate? How would you change it now?

Have students share responses.

- Why would our goal be to maximize the volume?
  - *Maximizing resources, enclosing as much volume as possible using the least amount of material.*
- Is constructing a box in such a way that its volume is maximized always the best option?
  - *No. A box may need to have particular dimensions (such as a shoe box). In some cases, the base of the box may need to be stronger, so the material is more expensive. Minimizing cost may be different than maximizing the volume.*

### Exit Ticket (5 minutes)

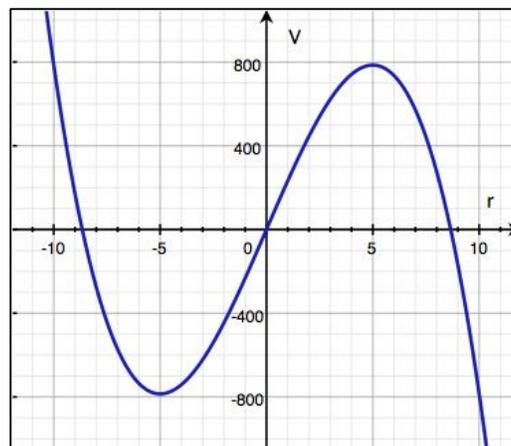
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## Lesson 16: Modeling with Polynomials—An Introduction

### Exit Ticket

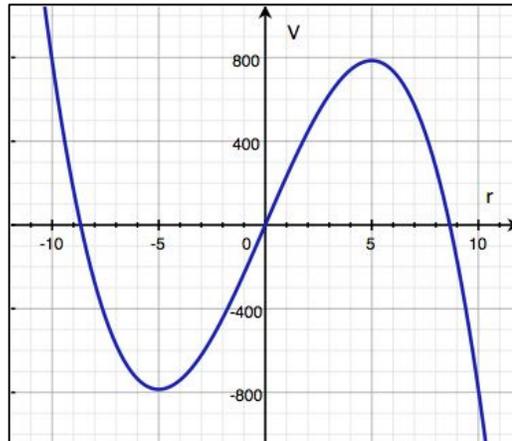
Jeannie wishes to construct a cylinder closed at both ends. The figure below shows the graph of a cubic polynomial function used to model the volume of the cylinder as a function of the radius if the cylinder is constructed using  $150\pi \text{ cm}^2$  of material. Use the graph to answer the questions below. Estimate values to the nearest half unit on the horizontal axis and to the nearest 50 units on the vertical axis.



1. What is the domain of the volume function? Explain.
2. What is the most volume that Jeannie's cylinder can enclose?
3. What radius yields the maximum volume?
4. The volume of a cylinder is given by the formula  $V = \pi r^2 h$ . Calculate the height of the cylinder that maximizes the volume.

## Exit Ticket Sample Solutions

Jeannie wishes to construct a cylinder closed at both ends. The figure below shows the graph of a cubic polynomial function used to model the volume of the cylinder as a function of the radius if the cylinder is constructed using  $150\pi \text{ cm}^2$  of material. Use the graph to answer the questions below. Estimate values to the nearest half unit on the horizontal axis and the nearest 50 units on the vertical axis.



1. What is the domain of the volume function? Explain.

*The domain is approximately  $0 \leq r \leq 8.5$  because a negative radius does not make sense, and a radius larger than 8.5 gives a negative volume, which also does not make sense.*

2. What is the most volume that Jeannie's cylinder can enclose?

*Approximately  $800 \text{ cm}^3$*

3. What radius yields the maximum volume?

*Approximately 5 cm*

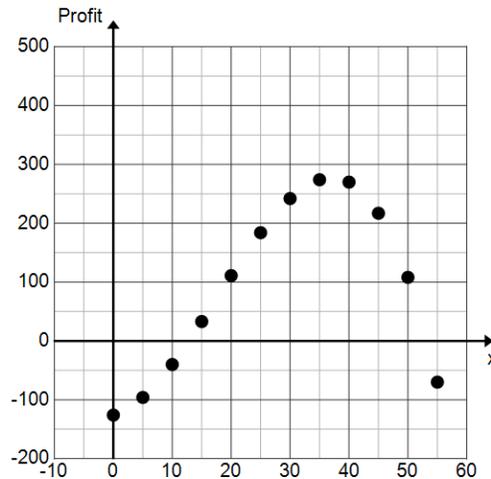
4. The volume of a cylinder is given by the formula  $V = \pi r^2 h$ . Calculate the height of the cylinder that maximizes the volume.

*Approximately 10.2 cm*

**Problem Set Sample Solutions**

For a bonus, ask your students what is meant by the caption on the t-shirt. (Hint that they can do a web search to find out.)

1. For a fundraiser, members of the math club decide to make and sell “Pythagoras may have been Fermat’s first problem but not his last!” t-shirts. They are trying to decide how many t-shirts to make and sell at a fixed price. They surveyed the level of interest of students around school and made a scatterplot of number of t-shirts sold ( $x$ ) versus profit shown below.

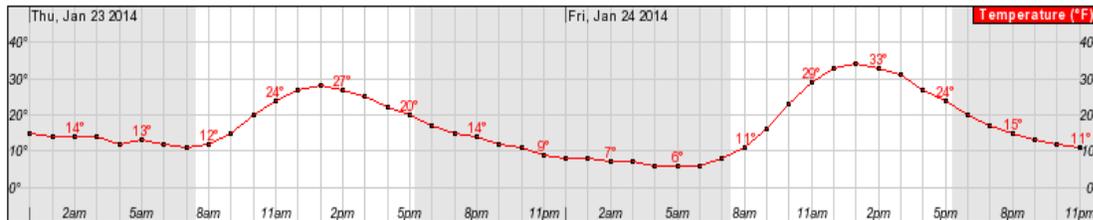


- a. Identify the  $y$ -intercept. Interpret its meaning within the context of this problem.  
*The  $y$ -intercept is approximately  $-125$ . The  $-125$  represents the money that they must spend on supplies in order to start making t-shirts. That is, they will lose \$125 if they sell 0 t-shirts.*
- b. If we model this data with a function, what point on the graph of that function represents the number of t-shirts they need to sell in order to break even? Why?  
*The break-even point is the first  $x$ -intercept of the graph of the function because at this point profit changes from negative to positive. When profit is 0, the club is breaking even.*
- c. What is the smallest number of t-shirts they can sell and still make a profit?  
*Approximately 12 or 13 t-shirts*
- d. How many t-shirts should they sell in order to maximize the profit?  
*Approximately 35 t-shirts*
- e. What is the maximum profit?  
*Approximately \$280*
- f. What factors would affect the profit?  
*The price of the t-shirts, the cost of supplies, the number of people who are willing to purchase a t-shirt*

g. What would cause the profit to start decreasing?

*Making more t-shirts than can be sold.*

2. The following graph shows the temperature in Aspen, Colorado during a 48-hour period beginning at midnight on Thursday, January 21, 2014. (Source: National Weather Service)



a. We can model the data shown with a polynomial function. What degree polynomial would be a reasonable choice?

*Since the graph has 4 turning points (2 relative minima, 2 relative maxima), a degree 5 polynomial could be used. Students could also argue that the final point is another minimum point and that a degree 6 polynomial could be used.*

b. Let  $T$  be the function that represents the temperature, in degrees Fahrenheit, as a function of time  $t$ , in hours. If we let  $t = 0$  correspond to midnight on Thursday, interpret the meaning of  $T(5)$ . What is  $T(5)$ ?

*The value  $T(5)$  represents the temperature at 5 am on Thursday. From the graph,  $T(5) = 13$ .*

c. What are the relative maximum points? Interpret their meanings.

*The relative maximum points are approximately  $T(13) = 28$  and  $T(37) = 34$ . These points represent the high temperature on Thursday and Friday and the times at which they occurred. The high on Thursday occurred at 1:00 (when  $t = 13$ ) and was  $28^\circ\text{F}$ . The high on Friday occurred at 1:00 (when  $t = 37$ ) and was  $34^\circ\text{F}$ .*



## Lesson 17: Modeling with Polynomials—An Introduction

### Student Outcomes

- Students interpret and represent relationships between two types of quantities with polynomial functions.

### Lesson Notes

In this lesson, students delve more deeply into modeling by writing polynomial equations that can be used to model a particular situation. Students are asked to interpret key features from a graph or table within a contextual situation (**F-IF.B.4**) and select the domain that corresponds to the appropriate graph or table (**F-IF.B.5**).

### Classwork

#### Opening Exercise (8 minutes)

Give students time to work independently on the Opening Exercise before discussing as a class.

#### Opening Exercise

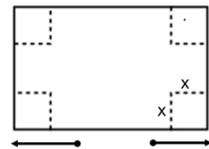
In Lesson 16, we created an open-topped box by cutting congruent squares from each corner of a piece of construction paper.

- a. Express the dimensions of the box in terms of  $x$ .

$$\text{Length: } L = 45.7 - 2x$$

$$\text{Width: } W = 30.5 - 2x$$

$$\text{Height: } H = x$$



- b. Write a formula for the volume of the box as a function of  $x$ . Give the answer in standard form.

$$V(x) = x(45.7 - 2x)(30.5 - 2x)$$

$$V(x) = 4x^3 - 152.4x^2 + 1393.85x$$

- How does this compare with the regression function found yesterday?
  - Answers will vary. Compare each parameter in the function.
- Which function is more accurate? Why?
  - The one found today. The one found yesterday depended on measurements that may not have been exact.

**Mathematical Modeling Exercises 1–13 (30 minutes)**

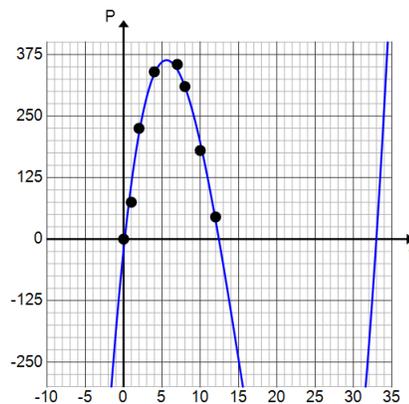
Allow students to work through the exercises in groups. Circulate the room to monitor students’ progress. Then, discuss results.

**Mathematical Modeling Exercises 1–13**

The owners of Dizzy Lizzy’s, an amusement park, are studying the wait time at their most popular roller coaster. The table below shows the number of people standing in line for the roller coaster  $t$  hours after Dizzy Lizzy’s opens.

$t$ (hours)	0	1	2	4	7	8	10	12
$P$ (people in line)	0	75	225	345	355	310	180	45

Jaylon made a scatterplot and decided that a cubic function should be used to model the data. His scatterplot and curve are shown below.



*Scaffolding:*

Have early finishers find a quadratic model for comparison. Which model seems to be a better fit? Why? Answers will vary, but one model would be  $P(t) = -10.134t(t - 12)$ . Students should see that the quadratic model does not seem to fit the data as well as the cubic model. They could confirm this using the regression feature of a graphing calculator.

1. Do you agree that a cubic polynomial function is a good model for this data? Explain.

*Yes. The curve passes through most of the points and seems to fit the data.*

2. What information would Dizzy Lizzy’s be interested in learning about from this graph? How could they determine the answer?

*The company should be interested in the time when the line is the longest and how many people are in line at that time. To find this out, they can find a model that could be used to predict the number of people in line at any time during the day. They could then estimate the maximum point from the graph.*

3. Estimate the time at which the line is the longest. Explain how you know.

*From the graph, the line is longest at 5.5 hours because the relative maximum value of the function occurs at 5.5 hours.*

4. Estimate the number of people in line at that time. Explain how you know.

*From the graph, there are roughly 372 people in line when  $t = 5.5$ ; that is the approximate relative maximum value of  $P$ .*

5. Estimate the  $t$ -intercepts of the function used to model this data.

*The  $t$ -intercepts are roughly 0, 12.5, and 33.*

6. Use the  $t$ -intercepts to write a formula for the function of the number of people in line,  $f$ , after  $t$  hours.

*$f(t) = ct(t - 12.5)(t - 33)$ , where  $c$  is a constant that has not yet been determined.*

7. Use the maximum point to find the leading coefficient. Explain your reasoning.

*Since we have estimated  $f(5.5) = 372$ , we can plug 5.5 into the function above, and we find that  $f(5.5) = 1058.75c = 372$ . Solving for  $c$ , we find that  $c \approx 0.35$ . The function that could model the data is then given by  $f(t) = 0.35t(t - 12.5)(t - 33)$ .*

8. What would be a reasonable domain for your function  $f$ ? Why?

*A reasonable domain for  $f$  would be  $0 \leq x \leq 12.5$  because the opening of the park corresponds to  $t = 0$ , and after 12.5 hours the park closes so there are no people waiting in line.*

9. Use your function  $f$  to calculate the number of people in line 10 hours after the park opens.

*The formula developed in Exercise 7 gives  $f(10) = 201$  people.*

10. Comparing the value calculated above to the actual value in the table, is your function  $f$  an accurate model for the data? Explain.

*The value of the function differs from the value from the table by about 21 people, so it is not a perfect fit for the data, but it is otherwise very close. It appears to overestimate the number of people in line.*

11. Use the regression feature of a graphing calculator to find a cubic function to model the data.

*The calculator gives  $g(t) = 0.43t^3 - 17.78t^2 + 156.63t - 24.16$ .*

12. Graph your function and the one generated by the graphing calculator and use the graphing calculator to complete the table. Round your answers to the nearest integer.

$t$ (hours)	0	1	2	4	7	8	10	12
$P$ (people in line)	0	75	225	345	355	310	180	45
$f$ (your equation)	0	129	228	345	350	315	201	44
$g$ (regression eqn.)	-24	115	221	345	349	311	194	38

13. Based on the results from the table, which model was more accurate at  $t = 2$  hours?  $t = 10$  hours?

*At  $t = 2$  hours, the function found by hand was more accurate. It was off by 3 people whereas the calculator function was off by 4 people. At  $t = 10$  hours, the graphing calculator function was more accurate. It was off by 14 people whereas the function found by hand was off by 21 people.*

MP.3

MP.2

### Closing (2 minutes)

- What type of functions were used to model the data? Were they good models?
  - *Cubic polynomial functions; yes, both functions were reasonably accurate.*
- What information did we use to find the function by hand?
  - *We used the  $x$ -intercepts and the relative maximum point.*
- Did we have to use the relative maximum specifically to find the leading coefficient?
  - *No. We could have chosen a different point on the curve.*
- How did polynomials help us solve a real-world problem?
  - *We were able to model the data using a polynomial function. The function allows us to estimate the number of people in line at any time  $t$  and also to estimate the time when the line is the longest and the maximum number of people are in line.*

### Exit Ticket (5 minutes)

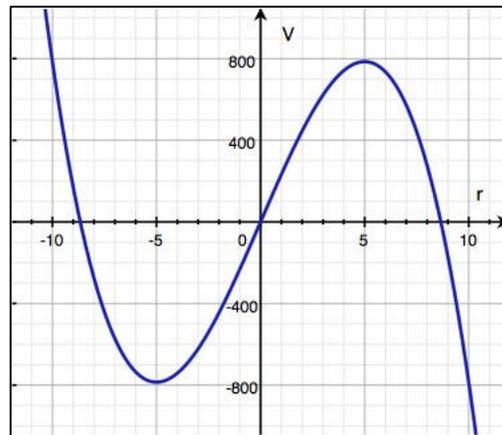
Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 17: Modeling with Polynomials—An Introduction

### Exit Ticket

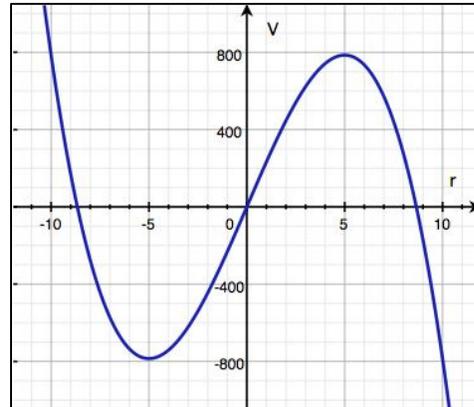
Jeannie wishes to construct a cylinder closed at both ends. The figure at right shows the graph of a cubic polynomial function used to model the volume of the cylinder as a function of the radius if the cylinder is constructed using  $150\pi \text{ cm}^3$  of material. Use the graph to answer the questions below. Estimate values to the nearest half unit on the horizontal axis and to the nearest 50 units on the vertical axis.



1. What are the zeros of the function?
2. What are the relative maximum and the relative maximum values of the function?
3. The equation of this function is  $V(r) = c(r^3 - 72.25r)$  for some real number  $c$ . Find the value of  $c$  so that this formula fits the graph.
4. Use the graph to estimate the volume of the cylinder with  $r = 2$  cm.
5. Use your formula to find the volume of the cylinder when  $r = 2$  cm. How close is the value from the formula to the value on the graph?

Exit Ticket Sample Solutions

Jeannie wishes to construct a cylinder closed at both ends. The figure at right shows the graph of a cubic polynomial function used to model the volume of the cylinder as a function of the radius if the cylinder is constructed using  $150\pi \text{ cm}^3$  of material. Use the graph to answer the questions below. Estimate values to the nearest half unit on the horizontal axis and to the nearest 50 units on the vertical axis.



1. What are the zeros of the function?

*Approximately 0, -8.5, and 8.5. (Students might round up to -9 and 9.)*

2. What are the relative maximum and the relative maximum values of the function?

*The relative maximum value is  $800 \text{ cm}^3$  at  $r = 5 \text{ cm}$  and the relative minimum value is  $-800 \text{ cm}^3$  at  $r = -5 \text{ cm}$ .*

3. The equation of this function is  $V(r) = c(r^3 - 72.25r)$  for some real number  $c$ . Find the value of  $c$  so that this formula fits the graph.

*Substituting  $r = 5 \text{ cm}$  and  $V(r) = 800 \text{ cm}^3$  and solving for  $c$  gives  $c \approx -3.4$ .*

4. Use the graph to estimate the volume of the cylinder with  $r = 2 \text{ cm}$ .

*Estimating from the graph, the volume of a cylinder of radius 2 cm is  $450 \text{ cm}^3$ .*

5. Use your formula to find the volume of the cylinder when  $r = 2 \text{ cm}$ . How close is the value from the formula to the value on the graph?

*Using the formula:  $V(2) = -3.4(2^3 - 72.25(2)) = 464.1$ . Therefore, the volume of the cylinder when  $r = 2 \text{ cm}$  is  $464.1 \text{ cm}^3$ . This value is close to the value of  $450 \text{ cm}^3$  found using the graph but not exact, particularly because we cannot read much detail from the graph.*

Problem Set Sample Solutions

Problem 2 requires the use of a graphing calculator. If students do not have the means to complete this, the last two parts could be done in class.

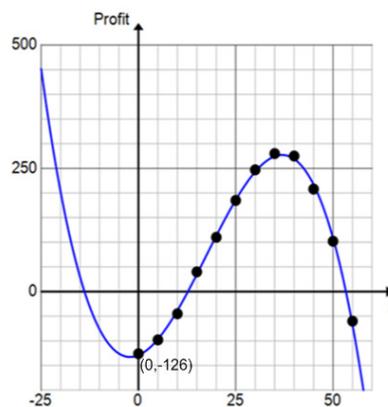
1. Recall the math club fundraiser from yesterday’s Problem Set. The club members would like to find a function to model their data, so Kylie draws a curve through the data points as shown below.

a. What type of function does it appear that she has drawn?

*Degree 3 polynomial (or cubic polynomial)*

b. The function that models the profit in terms of the number of t-shirts made has the form  $P(x) = c(x^3 - 53x^2 - 236x + 9828)$ . Use the vertical intercept labeled on the graph to find the value of the leading coefficient  $c$ .

*$c \approx -0.01282$ ,  
so  $P(x) = -0.01282(x^3 - 53x^2 - 236x + 9828)$*



- c. From the graph, estimate the profit if the math club sells 30 t-shirts.

*The profit is approximately \$250 if the club sells 30 t-shirts.*

- d. Use your function to estimate the profit if the math club sells 30 t-shirts.

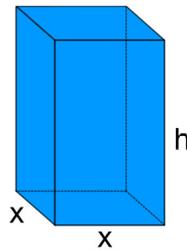
$P(30) = 230.14$ . *The equation predicts a profit of \$230.14.*

- e. Which estimate do you think is more reliable? Why?

*The estimate from the graph is probably more reliable because the equation required estimating the  $x$ -intercepts. If these estimates were off, it could have affected the equation.*

2. A box is to be constructed so that it has a square base and no top.

- a. Draw and label the sides of the box. Label the sides of the base as  $x$  and the height of the box as  $h$ .



- b. The surface area is  $108 \text{ cm}^2$ . Write a formula for the surface area  $S$  and then solve for  $h$ .

$$S = x^2 + 4xh = 108$$

$$h = \frac{108 - x^2}{4x}$$

- c. Write a formula for the function of the volume of the box in terms of  $x$ .

$$V(x) = x^2h = x^2 \left( \frac{108 - x^2}{4x} \right) = \frac{108x^2 - x^4}{4x} = \frac{108x - x^3}{4}$$

- d. Use a graphing utility to find the maximum volume of the box.

$$108 \text{ cm}^3$$

- e. What dimensions should the box be in order to maximize its volume?

$$6 \text{ cm} \times 6 \text{ cm} \times 3 \text{ cm}$$



# Lesson 18: Overcoming a Second Obstacle in Factoring— What If There Is a Remainder?

## Student Outcomes

- Students rewrite simple rational expressions in different forms, including representing remainders when dividing.

## Lesson Notes

Students have worked on dividing two polynomials using both the reverse tabular method and long division. In this lesson, they will continue that work but with quotients that have a remainder. In addition to the two methods of division already presented in this module, students will also use the method of inspection as stated in standard **A-APR.D.6**. The method of inspection is an opportunity to emphasize the mathematical practice of making use of structure (MP.7). The purpose of the Opening Exercise is to get students thinking about this idea of structure by leading them from writing rational numbers as mixed numbers to writing rational expressions as “mixed expressions.”

## Classwork

### Opening Exercise (3 minutes)

Have students work through the Opening Exercise briefly by themselves, and then summarize the exercise as a whole class by displaying all three methods. This will start students thinking about the different ways of rewriting an improper fraction as a mixed number. We will use methods 2 and 3 later today to write rational expressions as “mixed expressions.”

**Opening Exercise**

Write the rational number  $\frac{13}{4}$  as a mixed number.

**Method 1:**

**Method 2:**

$$\frac{13}{4} = \frac{12+1}{4} = \frac{12}{4} + \frac{1}{4} = 3 + \frac{1}{4} = 3\frac{1}{4}$$

**Method 3:**

**Scaffolding:**

- If students are struggling with the different methods of rewriting a rational number as a mixed number, provide them with extra examples such as the following.

$$\frac{10}{7} = 1\frac{3}{7}$$

$$\frac{26}{3} = 8\frac{2}{3}$$

**Example 1 (8 minutes)**

Work through the example as a class. Relate the process of inspection used in part (b) below to method 2 used in the Opening Exercise. Then, demonstrate how the quotient could have been found using the reverse tabular method or long division.

**Example 1**

**a. Find the quotient by factoring the numerator.**

$$\frac{x^2 + 3x + 2}{x + 2} = \frac{(x + 1)(x + 2)}{x + 2} = x + 1$$

**b. Find the quotient.**

$$\frac{x^2 + 3x + 5}{x + 2}$$

*See below.*

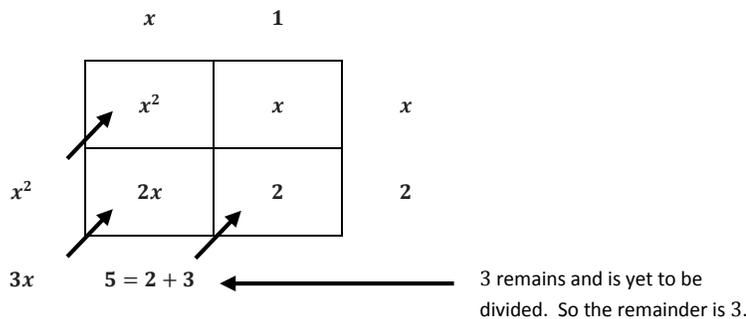
**Solutions for part (b):**

Method 1: Inspection

MP.7

- We already know that  $\frac{x^2+3x+2}{x+2} = x + 1$ , as long as  $x \neq -2$ . How could we use this fact to find  $\frac{x^2+3x+5}{x+2}$ ?
  - Since  $3 + 2 = 5$ , there must be 3 left over after performing division.
- How could we rewrite the problem in a way that is more convenient?
  - $\frac{x^2+3x+5}{x+2} = \frac{(x^2+3x+2)+3}{x+2} = \frac{x^2+3x+2}{x+2} + \frac{3}{x+2}$
- So, what are the quotient and remainder?
  - The quotient is  $x + 1$  with a remainder of 3.
- Since the 3 is left over and has not been divided by the  $x + 2$ , it is still written as a quotient.
  - $\frac{x^2+3x+5}{x+2} = \frac{x^2+3x+2+3}{x+2} = \frac{x^2+3x+2}{x+2} + \frac{3}{x+2} = (x + 1) + \frac{3}{x+2}$

Method 2: Reverse Tabular Method



$$\frac{x^2 + 3x + 5}{x + 2} = (x + 1) + \frac{3}{x + 2}$$

Method 3: Long division

Work the problem as a class using long division. Remind the students that we used the same process in method 3 of the Opening Exercise.

$$\begin{array}{r}
 x + 1 \\
 x + 2 \overline{) x^2 + 3x + 5} \\
 \underline{- x^2 - 2x} \phantom{+ 5} \\
 x + 5 \\
 \underline{- x - 2} \\
 3
 \end{array}$$

**Example 2 (7 minutes)**

Repeat the process for this example. Work through the process of inspection as a pair/share exercise. Then, ask the students to repeat the process using either the reverse tabular method or long division. Share the work from both methods.

**Example 2**

a. Find the quotient by factoring the numerator.

$$\begin{aligned}
 & \frac{x^3 - 8}{x - 2} \\
 &= \frac{(x - 2)(x^2 + 2x + 4)}{x - 2} \\
 &= x^2 + 2x + 4
 \end{aligned}$$

b. Find the quotient.

$$\frac{x^3 - 4}{x - 2}$$

- In pairs, see if you can determine how to use the quotient from (a) to find the quotient of  $\frac{x^3 - 4}{x - 2}$ .

Give students a couple of minutes to discuss and then elicit responses.

MP.7

- How did you rewrite the numerator?
  - $\frac{x^3 - 4}{x - 2} = \frac{x^3 - 8 + 4}{x - 2} = \frac{x^3 - 8}{x - 2} + \frac{4}{x - 2}$
- Why is this a useful way to rewrite the problem?
  - We know that  $\frac{x^3 - 8}{x - 2} = x^2 + 2x + 4$ .
- So what are the quotient and remainder of  $\frac{x^3 - 4}{x - 2}$ ?
  - The quotient is  $x^2 + 2x + 4$  with a remainder of 4.

Therefore,  $\frac{x^3 - 4}{x - 2} = (x^2 + 2x + 4) + \frac{4}{x - 2}$ .

Give students a couple of minutes to rework the problem using either the reverse tabular method or long division and then share student work.

	$x^2$	$2x$	$4$	
	$x^3$	$2x^2$	$4x$	$x$
$x^3$	$-2x^2$	$-4x$	$-8$	$-2$
$0x^2$	$0x$	$-4 = -8 + 4$		

← The remainder is 4.

$$\begin{array}{r}
 x^2 + 2x + 4 \\
 x - 2 \overline{) x^3 \phantom{+ 2x^2} - 4} \\
 \underline{-x^3 + 2x^2} \phantom{- 4} \\
 2x^2 \phantom{- 4x} \\
 \underline{-2x^2 + 4x} \phantom{- 4} \\
 4x - 4 \\
 \underline{-4x + 8} \\
 4
 \end{array}$$

- Which method is easier? Allow students to discuss advantages and disadvantages of the three methods.

**Exercise (5 minutes)**

Have students practice finding quotients by factoring the numerator using the cards on the page after next. Cut out the cards and hand each student a card. The students must move around the room and match their card with the same quotient. You could then have students stay in these pairs to work on the exercises.

**Exercises 1–10 (15 minutes)**

Allow students to work through the exercises either independently or in pairs. Some students may need to be reminded how to complete the square.

**Exercises 1–10**

Find each quotient by inspection.

1. $\frac{x+4}{x+1}$	2. $\frac{2x-7}{x-3}$	3. $\frac{x^2-21}{x+4}$
$1 + \frac{3}{x+1}$	$2 - \frac{1}{x-3}$	$(x-4) - \frac{5}{x+4}$

Find each quotient by using the reverse tabular method.

4. $\frac{x^2+4x+10}{x-8}$	5. $\frac{x^3-x^2+3x-1}{x+3}$	6. $\frac{x^2-2x-19}{x-1}$
$(x+12) + \frac{106}{x-8}$	$x^2 - 4x + 15 - \frac{46}{x+3}$	$(x-1) - \frac{20}{x-1}$

*Scaffolding:*

- Some students may have difficulty with inspection. Encourage them to use the reverse tabular method first, and then see if they can use that to rewrite the numerator.
- Early finishers can be given more challenging inspection problems such as the following.
 
$$\frac{x^2 - 5x + 9}{x - 1} = (x - 4) + \frac{5}{x - 1}$$

$$\frac{2x^2 - 5}{x - 3} = 2(x + 3) + \frac{13}{x - 3}$$

Find each quotient by using long division.

7.  $\frac{x^2 - x - 25}{x + 6}$

$(x - 7) + \frac{17}{x + 6}$

8.  $\frac{x^4 - 8x^2 + 12}{x + 2}$

$(x^3 - 2x^2 - 4x + 8) - \frac{4}{x + 2}$

9.  $\frac{4x^3 + 5x - 8}{2x - 5}$

$(2x^2 + 5x + 15) + \frac{67}{2x - 5}$

Rewrite the numerator in the form  $(x - h)^2 + k$  by completing the square. Then find the quotient.

10.  $\frac{x^2 + 4x - 9}{x + 2}$

$(x + 2) - \frac{13}{x + 2}$

The mental math exercises on the next page can be used for building fluency. All numerators factor nicely so that there are no remainders. The exercise can be timed and restrictions can be imposed (such as, “Only write your answer in the box next to the expression.”).

It is always a good idea to keep fluency exercises quick and stress-free for students. Here is one way to do that: Tell them that the activity will not be turned in for a grade, but they will be timed. Give them 2 minutes to write down as many answers as possible (using a stopwatch or a stopwatch feature on your phone). Afterwards, go through the solutions with them quickly; allow them only to mark the ones they did correct/incorrect—do not let them copy the correct answers down. Celebrate the student who got the greatest number correct, and then provide another 2–4 minutes for students to work on the remaining problems that they did not get right.

Mental Math Exercises

$\frac{x^2 - 9}{x + 3}$ $x - 3$	$\frac{x^2 - 4x + 3}{x - 1}$ $x - 3$	$\frac{x^2 - 16}{x + 4}$ $x - 4$	$\frac{x^2 - 3x - 4}{x + 1}$ $x - 4$
$\frac{x^3 - 3x^2}{x - 3}$ $x^2$	$\frac{x^4 - x^2}{x^2 - 1}$ $x^2$	$\frac{x^2 + x - 6}{x + 3}$ $x - 2$	$\frac{x^2 - 4}{x + 2}$ $x - 2$
$\frac{x^2 - 8x + 12}{x - 2}$ $x - 6$	$\frac{x^2 - 36}{x + 6}$ $x - 6$	$\frac{x^2 + 6x + 8}{x + 4}$ $x + 2$	$\frac{x^2 - 4}{x - 2}$ $x + 2$
$\frac{x^2 - x - 20}{x + 4}$ $x - 5$	$\frac{x^2 - 25}{x + 5}$ $x - 5$	$\frac{x^2 - 2x + 1}{x - 1}$ $x - 1$	$\frac{x^2 - 3x + 2}{x - 2}$ $x - 1$
$\frac{x^2 + 4x - 5}{x - 1}$ $x + 5$	$\frac{x^2 - 25}{x - 5}$ $x + 5$	$\frac{x^2 - 10x}{x}$ $x - 10$	$\frac{x^2 - 12x + 20}{x - 2}$ $x - 10$
$\frac{x^2 + 5x + 4}{x + 4}$ $x + 1$	$\frac{x^2 - 1}{x - 1}$ $x + 1$	$\frac{x^2 + 16x + 64}{x + 8}$ $x + 8$	$\frac{x^2 + 9x + 8}{x + 1}$ $x + 8$

**Closing (2 minutes)**

Consider asking students to respond in pairs or in writing.

- In the pair/share exercise, how did we use the structure of the expressions to help us to simplify them?
  - *We were able to factor the numerator. Since the numerator and denominator contained a common factor, we were able to simplify the expression.*
- How did we use structure in Exercise 10?
  - *We rewrote the expression by completing the square and then used inspection.*
- How does this use of structure help us when working with algebraic expressions?
  - *We can rewrite expressions into equivalent forms that may be more convenient.*
- What methods were used to find the quotients?
  - *Inspection, reverse tabular method, long division*
- What are some pros and cons of the methods?
  - *You may not see the answer when trying to divide by inspection but quick to do if you see the structure of the expression. Long division or tabular method can be time consuming but generally rely on a known process and not insight.*

**Exit Ticket (5 minutes)**



Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 18: Overcoming a Second Obstacle in Factoring—What If There Is a Remainder?

### Exit Ticket

1. Find the quotient of  $\frac{x-6}{x-8}$  by inspection.

2. Find the quotient of  $\frac{9x^3-12x^2+4}{x-2}$  by using either long division or the reverse tabular method.

Exit Ticket Sample Solutions

1. Find the quotient of  $\frac{x-6}{x-8}$  by inspection.

$$1 + \frac{2}{x-8}$$

2. Find the quotient of  $\frac{9x^3-12x^2+4}{x-2}$  by using either long division or the reverse tabular method.

$$(9x^2 + 6x + 12) + \frac{28}{x-2}$$

Problem Set Sample Solutions

1. For each pair of problems, find the first quotient by factoring the numerator. Then, find the second quotient by using the first quotient.

<p>a. <math>\frac{3x-6}{x-2}</math></p> <p>3</p>	<p><math>\frac{3x-9}{x-2}</math></p> <p><math>3 - \frac{3}{x-2}</math></p>
<p>b. <math>\frac{x^2-5x-14}{x-7}</math></p> <p><math>x+2</math></p>	<p><math>\frac{x^2-5x+2}{x-7}</math></p> <p><math>(x+2) + \frac{16}{x-7}</math></p>
<p>c. <math>\frac{x^3+1}{x+1}</math></p> <p><math>x^2-x+1</math></p>	<p><math>\frac{x^3}{x+1}</math></p> <p><math>(x^2-x+1) - \frac{1}{x+1}</math></p>
<p>d. <math>\frac{x^2-13x+36}{x-4}</math></p> <p><math>x-9</math></p>	<p><math>\frac{x^2-13x+30}{x-4}</math></p> <p><math>(x-9) - \frac{6}{x-4}</math></p>

Find each quotient by using the reverse tabular method.

<p>2. <math>\frac{x^3-9x^2+5x+2}{x-1}</math></p> <p><math>(x^2-8x-3) - \frac{1}{x-1}</math></p>	<p>3. <math>\frac{x^2+x+10}{x+12}</math></p> <p><math>(x-11) + \frac{142}{x+12}</math></p>
---	--

4.  $\frac{2x+6}{x-8}$

$2 + \frac{22}{x-8}$

5.  $\frac{x^2+8}{x+3}$

$(x-3) + \frac{17}{x+3}$

Find each quotient by using long division.

6.  $\frac{x^4-9x^2+10x}{x+2}$

$(x^3 - 2x^2 - 5x + 20) - \frac{40}{x+2}$

7.  $\frac{x^5-35}{x-2}$

$(x^4 + 2x^3 + 4x^2 + 8x + 16) - \frac{3}{x-2}$

8.  $\frac{x^2}{x-6}$

$(x+6) + \frac{36}{x-6}$

9.  $\frac{x^3+2x^2+8x+1}{x+5}$

$(x^2 - 3x + 23) - \frac{114}{x+5}$

10.  $\frac{x^3+2x+11}{x-1}$

$(x^2 + x + 3) + \frac{14}{x-1}$

11.  $\frac{x^4+3x^3-2x^2+6x-15}{x}$

$(x^3 + 3x^2 - 2x + 6) - \frac{15}{x}$

12. Rewrite the numerator in the form  $(x-h)^2 + k$  by completing the square. Then, find the quotient.

$$\frac{x^2 - 6x - 10}{x - 3}$$

$$x - 3 - \frac{19}{x - 3}$$



# Lesson 19: The Remainder Theorem

## Student Outcomes

- Students know and apply the Remainder Theorem and understand the role zeros play in the theorem.

## Lesson Notes

In this lesson, students are primarily working on exercises that lead them to the concept of the Remainder Theorem, the connection between factors and zeros of a polynomial, and how this relates to the graph of a polynomial function. Students should understand that for a polynomial function  $p$  and a number  $a$ , the remainder on division by  $x - a$  is the value  $p(a)$  and extend this to the idea that  $p(a) = 0$  if and only if  $(x - a)$  is a factor of the polynomial (A-APR.B.2). There should be lots of discussion after each exercise.

## Classwork

### Exercises 1–3 (5 minutes)

Assign different groups of students one of the three problems from this exercise. Have them complete their assigned problem and then have a student from each group put their solution on the board. This gets the division out of the way and allows students to start to look for a pattern without making the lesson too tedious.

#### Exercises 1–3

1. Consider the polynomial function  $f(x) = 3x^2 + 8x - 4$ .

a. Divide  $f$  by  $x - 2$ .

$$\begin{aligned} \frac{f(x)}{x-2} &= \frac{3x^2 + 8x - 4}{x-2} \\ &= (3x + 14) + \frac{24}{x-2} \end{aligned}$$

b. Find  $f(2)$ .

$$f(2) = 24$$

2. Consider the polynomial function  $g(x) = x^3 - 3x^2 + 6x + 8$ .

a. Divide  $g$  by  $x + 1$ .

$$\begin{aligned} \frac{g(x)}{x+1} &= \frac{x^3 - 3x^2 + 6x + 8}{x+1} \\ &= (x^2 - 4x + 10) - \frac{2}{x+1} \end{aligned}$$

b. Find  $g(-1)$ .

$$g(-1) = -2$$

#### Scaffolding:

If students are struggling, you may choose to replace the polynomials in Exercises 2 and 3 with easier polynomial functions.

Examples:

$$g(x) = x^2 - 7x - 11$$

a. Divide by  $x + 1$ .      b. Find  $g(-1)$ .

$$(x - 8) - \frac{3}{x + 1} \qquad g(-1) = -3$$

$$h(x) = 2x^2 + 9$$

a. Divide by  $x - 3$ .      b. Find  $h(3)$ .

$$(2x + 6) + \frac{27}{2x^2 + 9} \qquad h(3) = 27$$

3. Consider the polynomial function  $h(x) = x^3 + 2x - 3$ .

a. Divide  $h$  by  $x - 3$ . b. Find  $h(3)$ .

$$\frac{h(x)}{x-3} = \frac{x^3 + 2x - 3}{x-3} = (x^2 + 3x + 11) + \frac{30}{x-3}$$

$$h(3) = 30$$

**Discussion (7 minutes)**

MP.8

- What is  $f(2)$ ? What is  $g(-1)$ ? What is  $h(3)$ ?
  - $f(2) = 24$ ;  $g(-1) = -2$ ;  $h(3) = 30$
- Looking at the results of the quotient, what pattern do we see?
  - *The remainder is the value of the function.*
- Stating this in more general terms, what can we say about the connection between dividing a polynomial  $P$  by  $x - a$  and the value of  $P(a)$ ?
  - *The remainder found after dividing  $P$  by  $x - a$  will be the same value as  $P(a)$ .*
- Why would this be? Think about the quotient  $\frac{13}{3}$ . We could write this as  $13 = 4 \cdot 3 + 1$ , where 4 is the quotient and 1 is the remainder.
- Apply this same principle to Exercise 1. Write the following on the board and talk through it.

$$\frac{f(x)}{x-2} = \frac{3x^2 + 8x - 4}{x-2} = (3x + 14) + \frac{24}{x-2}$$

- How can we rewrite  $f$  using the equation above?
  - *Multiply both sides of the equation by  $x - 2$  to get  $f(x) = (3x + 14)(x - 2) + 24$ .*
- In general we can say that if you divide polynomial  $P$  by  $x - a$ , then the remainder must be a number; in fact, there is a (possibly non-zero degree) polynomial function  $q$  such that the equation,

$$P(x) = \underset{\substack{\uparrow \\ \text{quotient}}}{q(x)} \cdot (x - a) + \underset{\substack{\uparrow \\ \text{remainder}}}{r}$$

is true for all  $x$ .

- What is  $P(a)$ ?
  - $P(a) = q(a)(a - a) + r = q(a) \cdot 0 + r = 0 + r = r$ .

We have just proven the Remainder Theorem, which is formally stated in the box below.

**Remainder Theorem:**

Let  $P$  be a polynomial function in  $x$ , and let  $a$  be any real number. Then there exists a unique polynomial function  $q$  such that the equation

$$P(x) = q(x)(x - a) + P(a)$$

is true for all  $x$ . That is, when a polynomial is divided by  $(x - a)$ , the remainder is the value of the polynomial evaluated at  $a$ .

- Restate the Remainder Theorem in your own words to your partner.

While students are doing this, circulate and informally assess student understanding before asking students to share their responses as a class.

#### Exercise 4 (5 minutes)

Students may need more guidance through this exercise, but allow them to struggle with it first. After a few students have found  $k$ , share various methods used.

##### Exercise 4–6

4. Consider the polynomial  $P(x) = x^3 + kx^2 + x + 6$ .

- a. Find the value of  $k$  so that  $x + 1$  is a factor of  $P$ .

*In order for  $x + 1$  to be a factor of  $P$ , the remainder must be zero. Hence, since  $x + 1 = x - (-1)$ , we must have  $P(-1) = 0$ , so that  $0 = -1 + k - 1 + 6$ . Then  $k = -4$ .*

- b. Find the other two factors of  $P$  for the value of  $k$  found in part (a).

$$P(x) = (x + 1)(x^2 - 5x + 6) = (x + 1)(x - 2)(x - 3)$$

##### Scaffolding:

Challenge early finishers with this problem:

Given that  $x + 1$  and  $x - 1$  are factors of  $P(x) = x^4 + 2x^3 - 49x^2 - 2x + 48$ , write  $P$  in factored form.

Answer:

$$(x + 1)(x - 1)(x + 8)(x - 6)$$

#### Discussion (7 minutes)

- Remember that for any polynomial function  $P$  and real number  $a$ , the Remainder Theorem says that there exists a polynomial  $q$  so that  $P(x) = q(x)(x - a) + P(a)$ .
- What does it mean if  $a$  is a zero of a polynomial  $P$ ?
  - $P(a) = 0$ .
- So what does the Remainder Theorem say if  $a$  is a zero of  $P$ ?
  - There is a polynomial  $q$  so that  $P(x) = q(x)(x - a) + 0$ .
- How does  $(x - a)$  relate to  $P$  if  $a$  is a zero of  $P$ ?
  - If  $a$  is a zero of  $P$ , then  $(x - a)$  is a factor of  $P$ .
- How does the graph of a polynomial function  $y = P(x)$  correspond to the equation of the polynomial  $P$ ?
  - The zeros are the  $x$ -intercepts of the graph of  $P$ . If we know a zero of  $P$ , then we know a factor of  $P$ .
- If we know all of the zeros of a polynomial function, and their multiplicities, do we know the equation of the function?
  - Not necessarily. It is possible that the equation of the function contains some factors that cannot factor into linear terms.

We have just proven the Factor Theorem, which is a direct consequence of the Remainder Theorem.

##### Factor Theorem:

Let  $P$  be a polynomial function in  $x$ , and let  $a$  be any real number. If  $a$  is a zero of  $P$  then  $(x - a)$  is a factor of  $P$ .

- Give an example of a polynomial function with zeros of multiplicity 2 at 1 and 3.
  - $P(x) = (x - 1)^2(x - 3)^2$
- Give another example of a polynomial function with zeros of multiplicity 2 at 1 and 3.
  - $Q(x) = (x - 1)^2(x - 3)^2(x^2 + 1)$  or  $R(x) = 4(x - 1)^2(x - 3)^2$
- If we know the zeros of a polynomial, does the Factor Theorem tell us the exact formula for the polynomial?
  - *No. But, if we know the degree of the polynomial and the leading coefficient, we can often deduce the equation of the polynomial.*

**Exercise 5 (8 minutes)**

As students work through this exercise, circulate the room to make sure students have made the connection between zeros, factors, and  $x$ -intercepts. Question students to see if they can verbalize the ideas discussed in the prior exercise.

*Scaffolding:*

Encourage students who are struggling to work on part (a) using two methods:

- by finding  $P(1)$ , and
- by dividing  $P$  by  $x - 1$ .

This will help to reinforce the ideas discussed in Exercises 1 and 2.

5. Consider the polynomial  $P(x) = x^4 + 3x^3 - 28x^2 - 36x + 144$ .

a. Is 1 a zero of the polynomial  $P$ ?

*No.*

b. Is  $x + 3$  one of the factors of  $P$ ?

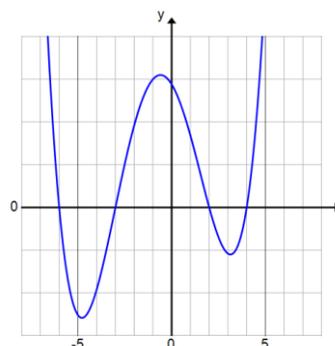
*Yes.  $P(-3) = 81 - 81 - 252 + 108 + 144 = 0$ .*

c. The graph of  $P$  is shown to the right. What are the zeros of  $P$ ?

*Approximately  $-6, -3, 2,$  and  $4$ .*

d. Write the equation of  $P$  in factored form.

*$P(x) = (x + 6)(x + 3)(x - 2)(x - 4)$*



- Is 1 a zero of the polynomial  $P$ ? How do you know?
  - *No.  $P(1) \neq 0$ .*
- What are two ways to determine the value of  $P(1)$ ?
  - *Fill in 1 for  $x$  into the function or divide  $P$  by  $x - 1$ .  $P(1)$  will be equal to the remainder.*
- Is  $x + 3$  a factor of  $P$ ? How do you know?
  - *Yes. Because  $P(-3) = 0$ , then when  $P$  is divided by  $x + 3$  the remainder is 0, which means that  $x + 3$  is a factor of the polynomial  $P$ .*
- How do you find the zeros of  $P$  from the graph?
  - *The zeros are the  $x$ -intercepts of the graph.*
- How do you find the factors?
  - *By using the zeros. If  $x = a$  is a zero, then  $x - a$  is a factor of  $P$ .*
- Multiply out the expression in part (d) to see that it is indeed the original polynomial function.

**Exercise 6 (6 minutes)**

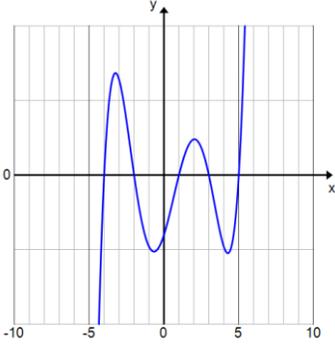
Allow students a few minutes to work on the problem and then share results.

6. Consider the graph of a degree 5 polynomial shown to the right, with  $x$ -intercepts  $-4$ ,  $-2$ ,  $1$ ,  $3$ , and  $5$ .

a. Write a formula for a possible polynomial function that the graph represents using  $c$  as constant factor.

$$P(x) = c(x + 4)(x + 2)(x - 1)(x - 3)(x - 5)$$

b. Suppose the  $y$ -intercept is  $-4$ . Adjust your function to fit the  $y$ -intercept by finding the constant factor  $c$ .

$$P(x) = \frac{1}{30}(x + 4)(x + 2)(x - 1)(x - 3)(x - 5)$$


- What information from the graph was needed to write the equation?
  - *The  $x$ -intercepts were needed to write the factors.*
- Why would there be more than one polynomial function possible?
  - *Because the factors could be multiplied by any constant and still produce a graph with the same  $x$ -intercepts.*
  - *Just as importantly—the graph only shows the behavior of the graph of a polynomial function between  $-10$  and  $10$ . It is possible that the function has many more zeros or has other behavior outside this window. Hence, we can only say that the polynomial we found is one possible example of a function whose graph looks like the picture.*
- Why can't we find the constant factor  $c$  by just knowing the zeros of the polynomial?
  - *The zeros only determine the graph of the polynomial up to the places where the graph passes through the  $x$ -intercepts. The constant factor can be used to vertically scale the graph of the polynomial to fit the depicted graph.*

**Closing (2 minutes)**

Have students summarize the results of the Remainder Theorem and the Factor Theorem.

- What is the connection between the remainder when a polynomial  $P$  is divided by  $x - a$  and the value of  $P(a)$ ?
  - *They are the same.*
- If  $x - a$  is factor, then \_\_\_\_\_.
  - *The number  $a$  is a zero of  $P$ .*
- If  $P(a) = 0$ , then \_\_\_\_\_.
  - *$(x - a)$  is a factor of  $P$ .*

## Lesson Summary

**Remainder Theorem:**

Let  $P$  be a polynomial function in  $x$ , and let  $a$  be any real number. Then there exists a unique polynomial function  $q$  such that the equation

$$P(x) = q(x)(x - a) + P(a)$$

is true for all  $x$ . That is, when a polynomial is divided by  $(x - a)$ , the remainder is the value of the polynomial evaluated at  $a$ .

**Factor Theorem:**

Let  $P$  be a polynomial function in  $x$ , and let  $a$  be any real number. If  $a$  is a zero of  $P$ , then  $(x - a)$  is a factor of  $P$ .

**Example:** If  $P(x) = x^2 - 3$  and  $a = 4$ , then  $P(x) = (x + 4)(x - 4) + 13$  where  $q(x) = x + 4$  and  $P(4) = 13$ .

**Example:** If  $P(x) = x^3 - 5x^2 + 3x + 9$ , then  $P(3) = 27 - 45 + 9 + 9 = 0$ , so  $(x - 3)$  is a factor of  $P$ .

## Exit Ticket (5 minutes)

Name \_\_\_\_\_

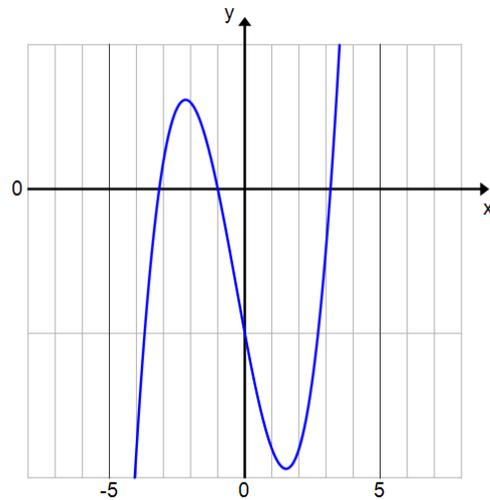
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## Lesson 19: The Remainder Theorem

### Exit Ticket

Consider the polynomial  $P(x) = x^3 + x^2 - 10x - 10$ .

- Is  $x + 1$  one of the factors of  $P$ ? Explain.
- The graph shown has  $x$ -intercepts at  $\sqrt{10}$ ,  $-1$ , and  $-\sqrt{10}$ . Could this be the graph of  $P(x) = x^3 + x^2 - 10x - 10$ ? Explain how you know.



Exit Ticket Sample Solutions

Consider polynomial  $P(x) = x^3 + x^2 - 10x - 10$ .

1. Is  $x + 1$  one of the factors? Explain.

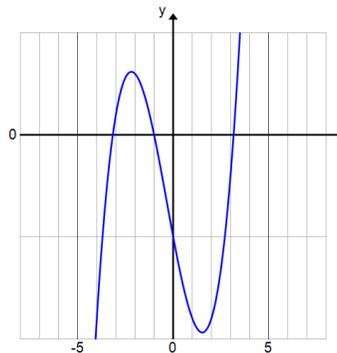
$$P(-1) = (-1)^3 + (-1)^2 - 10(-1) - 10 = -1 + 1 + 10 - 10 = 0$$

Yes,  $x + 1$  is a factor of  $P$  because  $P(-1) = 0$ . Or, using factoring by grouping, we have

$$P(x) = x^2(x + 1) - 10(x + 1) = (x + 1)(x^2 - 10).$$

2. The graph shown has  $x$ -intercepts at  $\sqrt{10}$ ,  $-1$ , and  $-\sqrt{10}$ . Could this be the graph of  $P(x) = x^3 + x^2 - 10x - 10$ ? Explain how you know.

Yes, this could be the graph of  $P$ . Since this graph has  $x$ -intercepts at  $\sqrt{10}$ ,  $-1$ , and  $-\sqrt{10}$ , the Factor Theorem says that  $(x - \sqrt{10})$ ,  $(x - 1)$ , and  $(x + \sqrt{10})$  are all factors of the equation that goes with this graph. Since  $(x - \sqrt{10})(x + \sqrt{10})(x - 1) = x^3 + x^2 - 10x - 10$ , the graph shown is quite likely to be the graph of  $P$ .



Problem Set Sample Solutions

1. Use the Remainder Theorem to find the remainder for each of the following divisions.

a.  $(x^2 + 3x + 1) \div (x + 2)$   
-1

b.  $(x^3 - 6x^2 - 7x + 9) \div (x - 3)$   
-39

c.  $(32x^4 + 24x^3 - 12x^2 + 2x + 1) \div (x + 1)$   
-5

d.  $(32x^4 + 24x^3 - 12x^2 + 2x + 1) \div (2x - 1)$   
Hint for part (d): Can you rewrite the division expression so that the divisor is in the form  $(x - c)$  for some constant  $c$ ?

4

2. Consider the polynomial  $P(x) = x^3 + 6x^2 - 8x - 1$ . Find  $P(4)$  in two ways.

$$P(4) = 4^3 + 6(4)^2 - 8(4) - 1 = 127$$

$$\frac{x^3 + 6x^2 - 8x - 1}{x - 4} \text{ has a remainder of } 127 \text{ so } P(4) = 127.$$

3. Consider the polynomial function  $P(x) = 2x^4 + 3x^2 + 12$ .

- a. Divide  $P$  by  $x + 2$  and rewrite  $P$  in the form (divisor)(quotient) + remainder.

$$P(x) = (x + 2)(2x^3 - 4x^2 + 11x - 22) + 56$$

- b. Find  $P(-2)$ .

$$P(-2) = (-2 + 2)(q(-2)) + 56 = 56$$

4. Consider the polynomial function  $P(x) = x^3 + 42$ .

a. Divide  $P$  by  $x - 4$  and rewrite  $P$  in the form (divisor)(quotient) + remainder.

$$P(x) = (x - 4)(x^2 + 4x + 16) + 106$$

b. Find  $P(4)$ .

$$P(4) = (4 - 4)(q(4)) + 106 = 106$$

5. Explain why for a polynomial function  $P$ ,  $P(a)$  is equal to the remainder of the quotient of  $P$  and  $x - a$ .

*The polynomial  $P$  can be rewritten in the form  $P(x) = (x - a)(q(x)) + r$ , where  $q(x)$  is the quotient function and  $r$  is the remainder. Then  $P(a) = (a - a)(q(a)) + r$ . Therefore,  $P(a) = r$ .*

6. Is  $x - 5$  a factor of the function  $f(x) = x^3 + x^2 - 27x - 15$ ? Show work supporting your answer.

*Yes, because  $f(5) = 0$  means that dividing by  $x - 5$  leaves a remainder of 0.*

7. Is  $x + 1$  a factor of the function  $f(x) = 2x^5 - 4x^4 + 9x^3 - x + 13$ ? Show work supporting your answer.

*No, because  $f(-1) = -1$  means that dividing by  $x + 1$  has a remainder of  $-1$ .*

8. A polynomial function  $p$  has zeros of 2, 2,  $-3$ ,  $-3$ ,  $-3$ , and 4. Find a possible formula for  $p$  and state its degree. Why is the degree of the polynomial not 3?

*One solution is  $p(x) = (x - 2)^2(x + 3)^3(x - 4)$ . The degree of  $p$  is 6. This is not a degree 3 polynomial function because the factor  $(x - 2)$  appears twice and the factor  $(x + 3)$  appears 3 times, while the factor  $(x - 4)$  appears once.*

9. Consider the polynomial function  $P(x) = x^3 - 8x^2 - 29x + 180$ .

a. Verify that  $P(9) = 0$ . Since  $P(9) = 0$ , what must one of the factors of  $P$  be?

$$P(9) = 9^3 - 8(9^2) - 29(9) + 180 = 0; x - 9$$

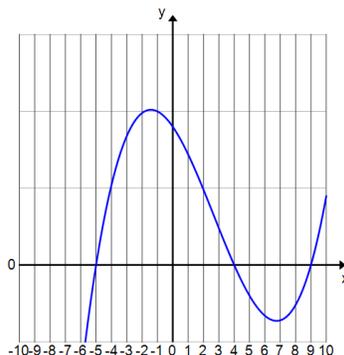
b. Find the remaining two factors of  $P$ .

$$P(x) = (x - 9)(x - 4)(x + 5)$$

c. State the zeros of  $P$ .

$$x = 9, 4, -5$$

d. Sketch the graph of  $P$ .



10. Consider the polynomial function  $P(x) = 2x^3 + 3x^2 - 2x - 3$ .

a. Verify that  $P(-1) = 0$ . Since,  $P(-1) = 0$ , what must one of the factors of  $P$  be?

$$P(-1) = 2(-1)^3 + 3(-1)^2 - 2(-1) - 3 = 0; x + 1$$

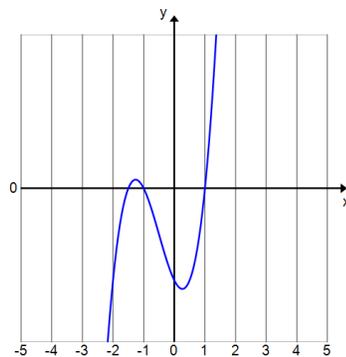
b. Find the remaining two factors of  $P$ .

$$P(x) = (x + 1)(x - 1)(2x + 3)$$

c. State the zeros of  $P$ .

$$x = -1, 1, -\frac{3}{2}$$

d. Sketch the graph of  $P$ .



11. The graph to the right is of a third degree polynomial function  $f$ .

a. State the zeros of  $f$ .

$$x = -10, -1, 2$$

b. Write a formula for  $f$  in factored form using  $c$  for the constant factor.

$$f(x) = c(x + 10)(x + 1)(x - 2)$$

c. Use the fact that  $f(-4) = -54$  to find the constant factor.

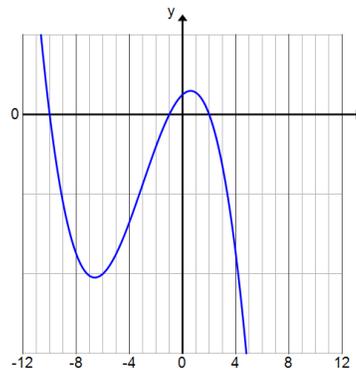
$$-54 = c(-4 + 10)(-4 + 1)(-4 - 2)$$

$$c = -\frac{1}{2}$$

$$f(x) = -\frac{1}{2}(x + 10)(x + 1)(x - 2)$$

d. Verify your equation by using the fact that  $f(1) = 11$ .

$$f(1) = -\frac{1}{2}(1 + 10)(1 + 1)(1 - 2) = -\frac{1}{2}(11)(2)(-1) = 11$$



12. Find the value of  $k$  so that  $(x^3 - kx^2 + 2) \div (x - 1)$  has remainder 8.

$$k = -5$$

13. Find the value  $k$  so that  $(kx^3 + x - k) \div (x + 2)$  has remainder 16.

$$k = -2$$

14. Show that  $x^{51} - 21x + 20$  is divisible by  $x - 1$ .

$$\text{Let } P(x) = x^{51} - 21x + 20.$$

$$\text{Then } P(1) = 1^{51} - 21(1) + 20 = 1 - 21 + 20 = 0.$$

Since  $P(1) = 0$ , the remainder of the quotient  $(x^{51} - 21x + 20) \div (x - 1)$  is 0.

Therefore,  $x^{51} - 21x + 20$  is divisible by  $x - 1$ .

15. Show that  $x + 1$  is a factor of  $19x^{42} + 18x - 1$ .

$$\text{Let } P(x) = 19x^{42} + 18x - 1.$$

$$\text{Then } P(-1) = 19(-1)^{42} + 18(-1) - 1 = 19 - 18 - 1 = 0.$$

Since  $P(-1) = 0$ ,  $x + 1$  must be a factor of  $P$ .

Note to Teacher: The following problems have multiple correct solutions. The answers provided here are polynomials with the lowest degree that meet the specified criteria. As an example, the answer to Exercise 16 is given as  $p(x) = (x + 2)(x - 1)$ , but the following are also correct responses:  $q(x) = 14(x + 2)(x - 1)$ ,  $r(x) = (x + 2)^4(x - 1)^8$ , and  $s(x) = (x^2 + 1)(x + 2)(x - 1)$ .

Write a polynomial function that meets the stated conditions.

16. The zeros are  $-2$  and  $1$ .

$$p(x) = (x + 2)(x - 1) \text{ or, equivalently, } p(x) = x^2 + x - 2$$

17. The zeros are  $-1$ ,  $2$ , and  $7$ .

$$p(x) = (x + 1)(x - 2)(x - 7) \text{ or, equivalently, } p(x) = x^3 - 8x^2 + 5x + 14$$

18. The zeros are  $-\frac{1}{2}$  and  $\frac{3}{4}$ .

$$p(x) = \left(x + \frac{1}{2}\right)\left(x - \frac{3}{4}\right) \text{ or, equivalently, } p(x) = x^2 - \frac{x}{4} - \frac{3}{8}$$

19. The zeros are  $-\frac{2}{3}$  and  $5$ , and the constant term of the polynomial is  $-10$ .

$$p(x) = (x - 5)(3x + 2) \text{ or, equivalently, } p(x) = 3x^2 - 13x - 10.$$

20. The zeros are  $2$  and  $-\frac{3}{2}$ , the polynomial has degree 3 and there are no other zeros.

$$p(x) = (x - 2)^2(2x + 3) \text{ or, equivalently, } p(x) = (x - 2)(2x + 3)^2$$



## Lesson 20: Modeling Riverbeds with Polynomials

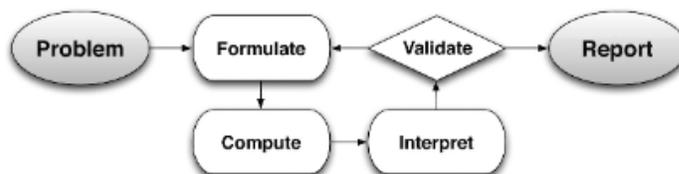
### Student Outcomes

- Students learn to fit polynomial functions to data values.

### Lesson Notes

In previous modeling lessons, students relied on the graphing calculator to find polynomial functions that fit a set of data. Lessons 20 and 21 comprise a two-lesson modeling exercise in which students model the shape of a riverbed in order to calculate flow rate of the river which can be used to determine if the river is vulnerable to flooding. In this first lesson, students use the Remainder Theorem as a way to fit a polynomial function to data points without relying on technology. In the second lesson, students will use technology, particularly Wolfram Alpha, to find the equation of the interpolating polynomial that best fits the data. The online calculations through Wolfram Alpha can be done as a demonstration for the class or individually if students have access to computers, or can be replaced by polynomial regression on a graphing calculator.

Students have some experience with modeling with polynomials from previous lessons in this module, but this is the first lesson sequence that takes them through the full modeling cycle as outlined in the modeling standard and as shown in the figure below. Students must first formulate a plan and develop a method for using the Remainder Theorem to fit a polynomial function to data. They must then use the model to compute the cross-sectional area of the riverbed and then volumetric flow of the water in the river. Students interpret their results and discuss how the results would be useful to the EPA.



### Classwork

#### Mathematical Modeling Exercise (7 minutes): Discussion

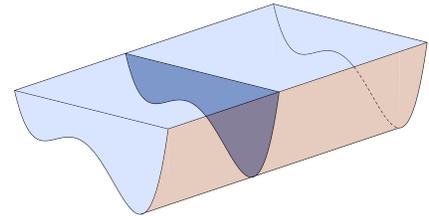
Have students read through the Mathematical Modeling Exercise. Have them discuss with a partner what questions need to be answered and brainstorm some ideas of how to answer them before having the discussion below. Tell the students that at the end of the next lesson, they will be expected to write a short report about the volumetric flow of the river. They will need to include their data and calculations in their report and should be preparing the report throughout the modeling exercise. Some pertinent vocabulary for this lesson includes the following:

Flow Rate (or volumetric flow rate): The volume of fluid that passes through a given surface per unit time.

Riverbed: The channel in which a river flows.

Cross-Section: A two-dimensional view of a slice through an object.

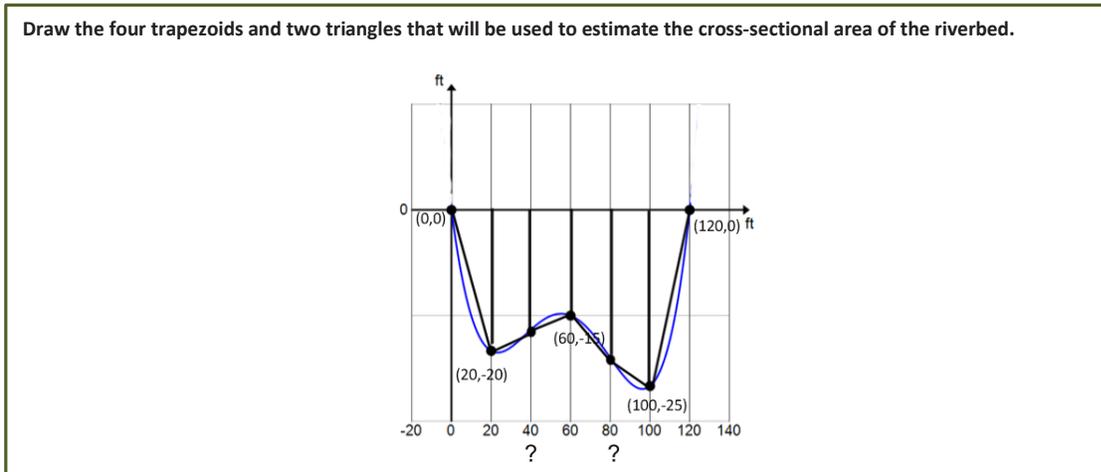
In order for students to formulate a plan for the modeling the shape of the riverbed, they must change their perspective from working with a three-dimensional idea of a river to the idea of working solely with a two-dimensional cross section. A figure such as the one to the right will help students see how the cross-sectional slice of the river relates to the river as a whole. The lower edge of the cross-section (shown in darker blue) is the curve that we need to find that goes through the given data points.



- Read through the statement of the Mathematical Modeling Exercise. What is our goal?
  - *We need to find the flow rate of water through the riverbed.*
- How can we estimate the flow rate of water?
  - *We need the area of the cross-section and the current speed of the water. Then the flow rate is the product of the cross-sectional area (in square feet) and the speed of the water (in feet per second).*
- To estimate the area of the cross-section, what information do we need?
  - *We need enough data points to make a good estimate.*
- To gather enough data points, we will need to make an assumption that the riverbed changes smoothly between the data points we have. With that assumption, we could estimate other data points by approximating the curve of the cross-section using a polynomial function.
- What is our first task?
  - *We need to find a polynomial function to fit the data.*
- Draw the graph of a polynomial function that passes through the 5 given data points.

**Mathematical Modeling Exercise**

The Environmental Protection Agency (EPA) is studying the flow of a river in order to establish flood zones. The EPA hired a surveying company to determine the flow rate of the river, measured as volume of water per minute. The firm set up a coordinate system and found the depths of the river at five locations as shown on the graph below. After studying the data, the firm decided to model the riverbed with a polynomial function and divide the area into six regions that are either trapezoidal or triangular so that the overall area can be easily estimated. The firm will need to approximate the depth of the river at two more data points in order to do this.



MP.1

- We have five data points. What information are we missing?
  - The depths at  $x = 40$  and  $x = 80$
- How can we find the missing values?
  - Find a polynomial function  $P$  that fits the given data and then use its equation to find  $P(40)$  and  $P(80)$ , which should approximate the depths at  $x = 40$  and  $x = 80$ .
- What is the lowest degree polynomial that could be used to model this data?
  - Degree four, because we have three relative maximum and minimum points.
- What constraints must our polynomial meet?
  - $P(0) = 0$ ,  $P(20) = -20$ ,  $P(60) = -15$ ,  $P(100) = -25$ , and  $P(120) = 0$

**Example 1 (12 minutes)**

Work this example out two ways. First, let students find the equation by setting up a system of equations (as in Lesson 1). Then, demonstrate how the Remainder Theorem can be used to find the polynomial.

- Before trying to find a polynomial function whose graph goes through all five points, let's try a simpler problem and find a polynomial whose graph goes through just three of the points:  $(0,28)$ ,  $(2,0)$ , and  $(8,12)$ .

Display these three points on a coordinate grid to use as reference through this discussion.

- Will the polynomial function that we find through these points be linear?
  - No. Looking at the graph we see that the points do not lie on a line.
- What do you think is the lowest degree polynomial we can find that will pass through the three points  $(0,28)$ ,  $(2,0)$ , and  $(8,12)$ ? (Remind students to look at the locations of these points in the plane as they decide on the lowest possible degree.)
  - It appears that we could fit a quadratic polynomial to these points; that is, they seem to lie on a parabola.
- Try to find a quadratic polynomial whose graph passes through these points. (Give students a hint to find a system of equations by plugging in  $x$  and  $P(x)$  values into the equation  $P(x) = ax^2 + bx + c$  as they did in Lesson 1.)

Allow students time to work and then discuss the process as a class.

**Example 1**

Find a polynomial  $P$  such that  $P(0) = 28$ ,  $P(2) = 0$ , and  $P(8) = 12$ .

Since  $P(0) = 28$ ,  $c = 28 \rightarrow P(x) = ax^2 + bx + 28$

Since  $P(2) = 0 \rightarrow 4a + 2b + 28 = 0$ , so  $2a + b = -14$  (1)

Since  $P(8) = 12 \rightarrow 64a + 8b + 28 = 12$ , so  $8a + b = -2$  (2)

Subtract (1) from (2)  $\rightarrow 6a = 12$ , so  $a = 2$

Substitute to find  $b$ .  $\rightarrow 2(2) + b = -14$ , so  $b = -18$

$P(x) = 2x^2 - 18x + 28$

Have students check their equation by finding  $P(0)$ ,  $P(2)$ , and  $P(8)$  and showing that  $P(0) = 28$ ,  $P(2) = 0$ , and  $P(8) = 12$ . Now work the problem again using the Remainder Theorem.

- We can also use the Remainder Theorem to find the quadratic polynomial  $P$  that satisfies  $P(0) = 28$ ,  $P(2) = 0$ , and  $P(8) = 12$ . We have already seen a technique that will find this quadratic polynomial, but the method using the Remainder Theorem generalizes to higher degree polynomials.

- Use the first data point:

Since  $P(0) = 28$ , the Remainder Theorem tells us that for some polynomial  $a$ ,

$$P(x) = (x - 0)a(x) + 28$$

$$= x a(x) + 28.$$

- What do we know about the degree of the polynomial  $a$ ? How do we know?
  - Since  $P$  will be a quadratic polynomial,  $a$  must be a linear polynomial.

- Use the second data point:

Since  $P(2) = 0$ , we have

$$P(2) = 2 a(2) + 28 = 0$$

$$2 a(2) = -28$$

$$a(2) = -14.$$

Thus, the Remainder Theorem tells us that for some polynomial  $b$ ,

$$a(x) = (x - 2)b(x) - 14.$$

- What do we know about the degree of the polynomial  $b$ ? How do we know?
  - Since  $a$  is a linear polynomial,  $b(x)$  must be a constant polynomial.
- Rewrite  $P$  in terms of  $b$ :

$$P(x) = x a(x) + 28$$

$$= x[(x - 2)b(x) - 14] + 28.$$

- Use the third data point:

Since  $P(8) = 12$ , we have

$$\begin{aligned} P(8) &= 8[(8 - 2)b(x) - 14] + 28 = 12 \\ 48b(x) - 104 + 28 &= 12 \\ 48b(x) &= 96 \\ b(x) &= 2. \end{aligned}$$

- Remember that we said the function  $b(x)$  would be constant, and we found that  $b(x) = 2$  for all values of  $x$ .
- Now we can rewrite the function  $P$ :

$$\begin{aligned} P(x) &= x[(x - 2)b(x) - 14] + 28 \\ &= x[(x - 2)2 - 14] + 28. \end{aligned}$$

- Do the three points satisfy this equation?

- $P(0) = 0[(0 - 2) \cdot 2 - 14] + 28 = 28$  ✓
- $P(2) = 2[(2 - 2) \cdot 2 - 14] + 28 = 0$  ✓
- $P(8) = 8[(8 - 2) \cdot 2 - 14] + 28 = 12$  ✓

- For a final check, write the polynomial we just found in standard form.

$$\begin{aligned} P(x) &= x[(x - 2)2 - 14] + 28 \\ &= 2x^2 - 18x - 28. \end{aligned}$$

- What if we wanted to fit a polynomial function to two data points? What is the lowest degree of a polynomial that will fit two data points?
  - *We know from geometry that two points determine a unique line, so a polynomial of degree one will fit two data points.*
- In the next Example we will fit a polynomial to four data points. What is the lowest degree of a polynomial that will fit four data points?
  - *Since two data points can be fit by a degree 1 polynomial, and three data points can be fit by a degree 2 polynomial, it makes sense that four data points can be fit by a degree 3 polynomial.*

### Example 2 (15 minutes)

The table below will help students organize their work as they proceed to find the cubic polynomial that fits four data points. Work through the first line of the table with the students, and then have the students work in groups to complete the table to determine the equation of the polynomial  $P$ . Point out to students that this would be a difficult task if we used systems of equations as we did first in Example 1. Have students present their work on the board.

MP.7

**Example 2**  
Find a degree 3 polynomial  $P$  such that  $P(-1) = -3$ ,  $P(0) = -2$ ,  $P(1) = -1$ , and  $P(2) = 6$ .

Function Value	Substitute data point into the current form of the equation for $P$ .	Apply the Remainder Theorem to $a$ , $b$ , or $c$	Rewrite the equation for $P$ in terms of $a$ , $b$ or $c$ .
$P(-1) = -3$			$P(x) = (x + 1)a(x) - 3$
$P(0) = -2$	$(0 + 1)a(0) - 3 = -2$	$a(x) = xb(x) + 1$	$P(x) = (x + 1)(xb(x) + 1) - 3$
$P(1) = -1$	$(1 + 1)(b(1) + 1) - 3 = -1$	$b(x) = (x - 1)c(x)$	$P(x) = (x + 1)(x(x - 1)c(x) + 1) - 3$
$P(2) = 6$	$(2 + 1)(2(2 - 1)c(2) + 1) - 3 = 6$	$c(x) = 1$	$P(x) = (x + 1)(x(x - 1) \cdot 1 + 1) - 3$

- Write the polynomial  $P$  in standard form.
  - $P(x) = (x + 1)(x^2 - x + 1) - 3$
  - $P(x) = x^3 - x^2 + x + x^2 - x + 1 - 3$
  - $P(x) = x^3 - 2$
- Verify that  $P$  satisfies the four given constraints.
  - $P(-1) = (-1)^3 - 2 = -3$
  - $P(0) = 0^3 - 2 = -2$
  - $P(1) = 1^3 - 2 = -1$
  - $P(2) = 2^3 - 2 = 6$

**Closing (5 minutes)**

- What methods have we used to find polynomial functions that fit given data?
  - *We plugged in data values to obtain a system of equations, and used the Remainder Theorem. In previous lessons, we used the graphing calculator.*
- If we want a polynomial to perfectly fit data points, how does the degree of the polynomial relate to the number of data points?
  - *The degree of the polynomial whose graph passes through a set of data points is one less than the number of data points.*

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.

**Lesson Summary**

A linear polynomial is determined by 2 points on its graph.  
 A degree 2 polynomial is determined by 3 points on its graph.  
 A degree 3 polynomial is determined by 4 points on its graph.  
 A degree 4 polynomial is determined by 5 points on its graph.  
 The Remainder Theorem can be used to find a polynomial  $P$  whose graph will pass through a given set of points.

**Exit Ticket (6 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 20: Modeling Riverbeds with Polynomials

### Exit Ticket

Use the Remainder Theorem to find a quadratic polynomial  $P$  so that  $P(1) = 5$ ,  $P(2) = 12$ , and  $P(3) = 25$ . Give your answer in standard form.

## Exit Ticket Sample Solutions

Use the Remainder Theorem to find a quadratic polynomial  $P$  so that  $P(1) = 5$ ,  $P(2) = 12$ , and  $P(3) = 25$ . Give your answer in standard form.

Since  $P(1) = 5$ , there is a linear polynomial  $a$  so that  $P(x) = (x - 1)a(x) + 5$ .

Since  $P(2) = 12$ , we have  $P(2) = (2 - 1)a(2) + 5 = 12$ , so  $a(2) = 7$ .

Because  $a(2) = 7$ , there exists a constant  $b$  so that  $a(x) = (x - 2)b + 7$ .

Substituting into  $P(x)$  gives  $P(x) = (x - 1)[(x - 2)b + 7]$ .

Since  $P(3) = 25$ , we have  $P(3) = (3 - 1)[(3 - 2)b + 7] = 25$ . Solving for  $b$  gives  $b = 3$ .

It follows that  $P(x) = (x - 1)[(x - 2)3 + 7]$ , and in standard form we have  $P(x) = 3x^2 - 2x + 4$ .

## Problem Set Sample Solutions

1. Suppose a polynomial  $P$  is such that  $P(2) = 5$  and  $P(3) = 12$ .

a. What is the largest degree polynomial that can be uniquely determined given the information?

*Degree one*

b. Is this the only polynomial that satisfies  $P(2) = 5$  and  $P(3) = 12$ ?

*No, there are an infinite number of polynomials that pass through those two points. However, two points will determine a unique equation for a linear function.*

c. Use the Remainder Theorem to find the polynomial  $P$  of least degree that satisfies the two points given.

$$P(x) = 7x - 9$$

d. Verify that your equation is correct by demonstrating that it satisfies the given points.

$$P(2) = 7(2) - 9 = 5$$

$$P(3) = 7(3) - 9 = 12$$

2. Write a quadratic equation  $P$  such that  $P(0) = -10$ ,  $P(5) = 0$ , and  $P(7) = 18$  using the specified method.

a. Setting up a system of equations

$$P(x) = x^2 - 3x - 10$$

b. Using the Remainder Theorem

$$P(x) = x^2 - 3x - 10$$

3. Find a degree three polynomial function  $P$  such that  $P(-1) = 0$ ,  $P(0) = 2$ ,  $P(2) = 12$ , and  $P(3) = 32$ . Use the table below to organize your work. Write your answer in standard form, and verify by showing that each point satisfies the equation.

Function Value	Substitute data point into the current form of the equation for $P$ .	Apply the Remainder Theorem to $a$ , $b$ , or $c$	Rewrite the equation for $P$ in terms of $a$ , $b$ or $c$ .
$P(-1) = 0$			$P(x) = (x + 1)a(x)$
$P(0) = 2$	$a(0) = 2$	$a(x) = xb(x) + 2$	$P(x) = (x + 1)(xb(x) + 2)$
$P(2) = 12$	$3(2b(2)) + 2 = 12$ $b(2) = 1$	$b(x) = (x - 2)c(x) + 1$	$P(x) = (x + 1)[x[(x - 2)c(x) + 1] + 2]$
$P(3) = 32$	$4[3(c(3) + 1)] + 2 = 32$ $c(3) = 1$	$c(3) = 1$	$P(x) = (x + 1)[x[(x - 2) + 1] + 2]$

$$P(x) = x^3 + x + 2$$

$$P(-1) = -1 - 1 + 2 = 0 \quad \checkmark$$

$$P(0) = 0 + 0 + 2 = 2 \quad \checkmark$$

$$P(2) = 8 + 2 + 2 = 12 \quad \checkmark$$

$$P(3) = 27 + 3 + 2 = 32 \quad \checkmark$$

4. The method used in Problem 3 is based on the Lagrange Interpolation method. Research Joseph-Louis Lagrange and write a paragraph about his mathematical work.

*Answers will vary.*



## Lesson 21: Modeling Riverbeds with Polynomials

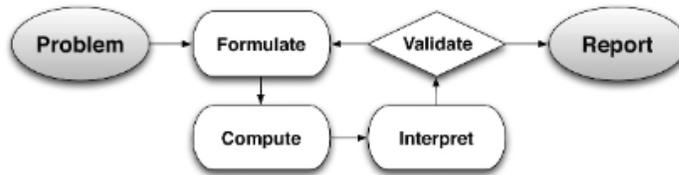
### Student Outcomes

- Students model a cross-section of a riverbed with a polynomial function and estimate fluid flow with their algebraic model.

### Lesson Notes

This is the second half of a two-day modeling lesson. In previous modeling lessons, students relied on the graphing calculator to find polynomial functions that fit a set of data, and in the previous lesson the polynomial was found algebraically. In this lesson, we use the website Wolfram Alpha ([www.wolframalpha.com](http://www.wolframalpha.com)) to find the polynomial to fit the data. If students do not have access to computers or the internet during class, the polynomial can be found using quartic regression on a graphing calculator.

The previous lesson introduced the problem of modeling the shape of a riverbed and computing the volumetric flow. In this lesson, students can actually do the modeling, with the help of technology, and then interpret the results and create a report detailing their findings, thus completing the modeling cycle.



### Classwork

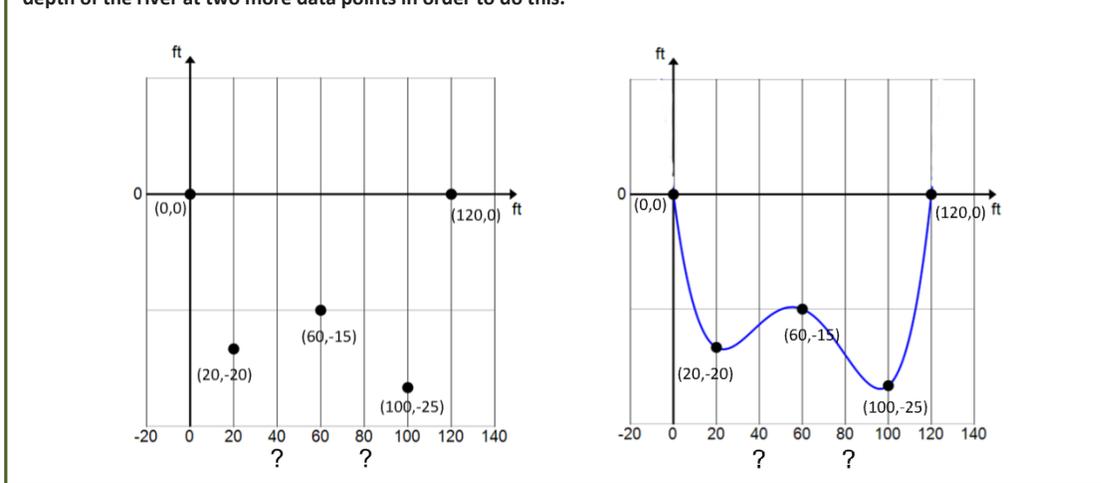
#### Opening (3 minutes)

Review the Mathematical Modeling Exercise with the students. In this lesson, the students will

- Find a function that models the shape of the riverbed based on the five data points given in the graph below.
- Approximate the area of the cross-sectional region using triangles and trapezoids.
- Calculate the volumetric flow rate of the water, in gallons per minute.
- Create a report of the findings.

**Mathematical Modeling Exercise**

The Environmental Protection Agency (EPA) is studying the flow of a river in order to establish flood zones. The EPA hired a surveying company to determine the flow rate of the river, measured as volume of water per minute. The firm set up a coordinate system and found the depths of the river at five locations as shown on the graph below. After studying the data, the firm decided to model the riverbed with a polynomial function and divide the area into six regions that are either trapezoidal or triangular so that the overall area can be easily estimated. The firm will need to approximate the depth of the river at two more data points in order to do this.



**Mathematical Modeling Exercises 1–5 (32 minutes)**

Now return to the Mathematical Modeling Exercise.

- How many data points are we given? What is the lowest degree of a polynomial that passes through these points?
  - *We were given five data points so we can model this data using a fourth degree polynomial function.*
- We are going to use Wolfram Alpha to find the fourth degree polynomial that fits the data. It will follow a procedure based on the Remainder Theorem that we used in the previous two examples to find the equation.

MP.5

*Scaffolding:*  
If students are overwhelmed with the new technology along with new content, most graphing calculators will perform polynomial regression up to degree 4, so you might have students use that more familiar technological tool.

Go to [www.wolframalpha.com](http://www.wolframalpha.com). Type in the following command:

$$\text{Interpolating polynomial} \{ \{(0,0), (20, -20), (60, -15), (100, -25), (120,0)\}, x \}.$$

Students can find  $P(40)$  and  $P(80)$  either by substituting into the function displayed or by editing the command on Wolfram Alpha as follows:

$$\text{Interpolating polynomial} \{ \{(0,0), (20, -20), (60, -15), (100, -25), (120,0)\}, 40 \}$$

$$\text{Interpolating polynomial} \{ \{(0,0), (20, -20), (60, -15), (100, -25), (120,0)\}, 80 \}.$$

Allow students time to work through the remainder of the exercise either individually or in groups. The conversion from cubic feet per minute to gallons per minute can also be done using Wolfram Alpha.

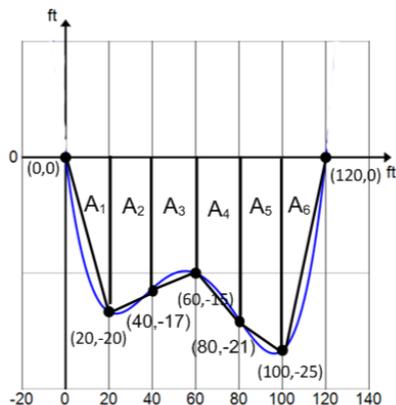
- Find a polynomial  $P$  that fits the five given data points.

$$P(x) = \frac{17}{3840000}x^4 - \frac{33}{32000}x^3 + \frac{751}{9600}x^2 - \frac{35}{16}x$$

- Use the polynomial to estimate the depth of the river at  $x = 40$  and  $x = 80$ .

$$P(40) = -17 \text{ and } P(80) = -21$$

- Estimate the area of the cross section.



$$\begin{aligned} A &= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 \\ &= 200 \text{ ft}^2 + 370 \text{ ft}^2 + 320 \text{ ft}^2 + 360 \text{ ft}^2 + 460 \text{ ft}^2 + 250 \text{ ft}^2 \\ &= 1,960 \text{ ft}^2 \end{aligned}$$

Suppose that the river flow speed was measured to be an average speed of  $176 \frac{\text{ft}}{\text{min}}$  at the cross section.

- What is the volumetric flow of the water (the volume of water per minute)?

$$(1,960 \text{ ft}^2) \left( 176 \frac{\text{ft}}{\text{min}} \right) = 344,960 \frac{\text{ft}^3}{\text{min}}$$

- Convert the flow to gallons per minute. [Note: 1 cubic foot  $\approx$  7.48052 gallons.]

$$2,580,480 \frac{\text{gallons}}{\text{min}}$$

- How could the surveyors measure the speed of the water?
  - They could time an object (like a ball) floating over a set distance.
- What factors would need to be considered when measuring the flow?
  - Water may flow faster below the surface. The flow rate may vary from the edge of the river to the middle of the river and around obstacles such as rocks.
- Remember that the EPA is interested in identifying flood-prone areas. How might the information you have gathered help the EPA?
  - If the normal volumetric flow of the river has been recorded by the EPA, then they can record the normal water levels in the river use that knowledge to predict what level of volumetric flow would result in flooding. In cases of heavy rain, the EPA could identify if an area is likely to flood due to the increased volumetric flow.

**Closing (5 minutes)**

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson.

**Exit Ticket (5 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 21: Modeling Riverbeds with Polynomials

### Exit Ticket

Explain the process you used to estimate the volumetric flow of the river, from accumulating the data to calculating the flow of water.

## Exit Ticket Sample Solutions

Explain the process you used to estimate the volumetric flow of the river, from accumulating the data to calculating the flow of water.

*We were given data points that represented the depth of the river at various points in the cross-section. We used Wolfram Alpha to find a polynomial  $P$  that passed through those given points. We used the polynomial  $P$  to find more data points. Then we approximated the area of the cross-section using triangles and trapezoids; figures whose area we know how to calculate. Once we had an approximate area of the cross-section, we multiplied that area by the speed of the water across the cross-section to find the amount of volumetric flow in units of cubic feet per minute. The last step was to convert cubic feet to gallons to get the amount of volumetric flow in units of gallons per minute.*

## Problem Set Sample Solutions

Problem 2 requires the use of a computer. This could be completed in class if students do not have access to computers.

1. As the leader of the surveying team, write a short report to the EPA on your findings from the in-class exercises. Be sure to include data and calculations.

*After collecting data at the site, we decided that the cross-sectional area could be approximated using trapezoids. In order to increase the accuracy of our area approximation, more data points were needed. We chose to use the data collected to model the riverbed using a degree 4 polynomial function. We used the computational knowledge engine Wolfram Alpha to find the polynomial that fit the data, which is  $P(x) = \frac{17}{3840000}x^4 - \frac{33}{32000}x^3 + \frac{751}{9600}x^2 - \frac{35}{16}x$ . Using the polynomial  $P$ , we were able to estimate enough data points to calculate the cross-sectional area and determined that it was  $1,960 \text{ ft}^2$ . Using this information and the average speed of the water at the cross-section, which was  $76 \frac{\text{ft}}{\text{min}}$ , we were able to compute the volumetric flow of the river at that cross-section. We determined that the volumetric flow is approximately  $344,960 \frac{\text{ft}^3}{\text{min}}$ , which is  $2,580,480 \frac{\text{gallons}}{\text{min}}$ .*

2. Suppose that depths of the riverbed were measured for a different cross-section of the river.

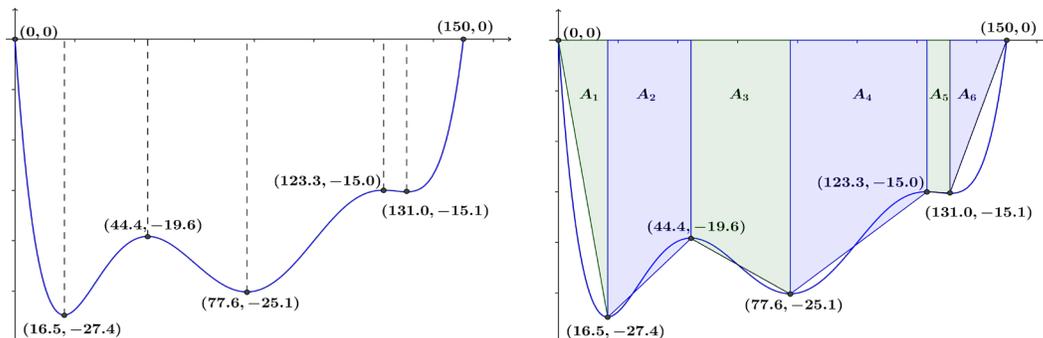
- a. Use Wolfram Alpha to find the interpolating polynomial  $Q$  with values:

$$Q(0) = 0, \quad Q(16.5) = -27.4, \quad Q(44.4) = -19.6, \quad Q(77.6) = -25.1,$$

$$Q(123.3) = -15.0, \quad Q(131.1) = -15.1, \quad Q(150) = 0.$$

$$Q(x) = \frac{7}{8,789,062,500}x^6 - \frac{11}{29,296,875}x^5 + \frac{191}{2,812,500}x^4 - \frac{11}{1875}x^3 + \frac{2759}{11,250}x^2 - \frac{329}{75}x$$

b. Sketch the cross-section of the river and estimate its area.



Area estimates:

$$A_1 = \frac{1}{2}(27.4)(16.5) = 226.05$$

$$A_2 = \frac{1}{2}(27.4 + 19.6)(44.4 - 16.5) = 655.65$$

$$A_3 = \frac{1}{2}(25.1 + 19.6)(77.6 - 44.4) = 742.02$$

$$A_4 = \frac{1}{2}(15.0 + 25.1)(123.3 - 77.6) = 916.285$$

$$A_5 = \frac{1}{2}(15.1 + 15.0)(131.0 - 123.3) = 115.885$$

$$A_6 = \frac{1}{2}(15.1)(150.0 - 131.0) = 143.45$$

So, the total area can be estimated by

$$A = A_1 + A_2 + A_3 + A_4 + A_5 + A_6 = 2799.34.$$

c. Suppose that the speed of the water was measured at  $124 \frac{\text{ft}}{\text{min}}$ . What is the approximate volumetric flow in this section of the river, measured in gallons per minute?

The volumetric flow is approximately  $2799.34 \text{ ft}^2 \left(124 \frac{\text{ft}}{\text{min}}\right) \approx 347,118 \frac{\text{ft}^3}{\text{min}}$ . Converting to gallons per minute, this is

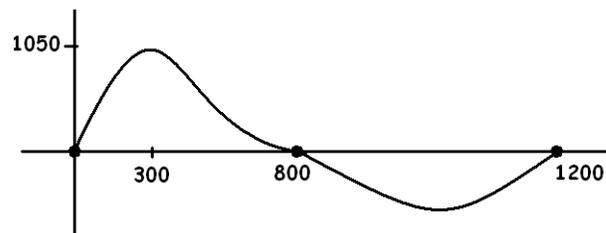
$$\left(347,118 \frac{\text{ft}^3}{\text{min}}\right) \left(7.48052 \frac{\text{gallons}}{\text{ft}^3}\right) \approx 2,596,623 \text{ gallons/min.}$$

Name \_\_\_\_\_

Date \_\_\_\_\_

1. Geographers sit at a café discussing their field work site, which is a hill and a neighboring riverbed. The hill is approximately 1,050 ft. high, 800 ft. wide, with peak about 300 ft. east of the western base of the hill. The river is about 400 ft. wide. They know the river is shallow, no more than about 20 ft. deep.

They make the following crude sketch on a napkin, placing the profile of the hill and riverbed on a coordinate system with the horizontal axis representing ground level.



The geographers do not have any computing tools with them at the café, so they decide to use pen and paper to compute a cubic polynomial that approximates this profile of the hill and riverbed.

- a. Using only a pencil and paper, write a cubic polynomial function  $H$  that could represent the curve shown (here,  $x$  represents the distance, in feet, along the horizontal axis from the western base of the hill, and  $H(x)$  is the height, in feet, of the land at that distance from the western base). Be sure that your formula satisfies  $H(300) = 1050$ .

- b. For the sake of convenience, the geographers make the assumption that the deepest point of the river is halfway across the river (recall that the river is no more than 20-ft. deep). Under this assumption, would a cubic polynomial provide a suitable model for this hill and riverbed? Explain.

2. Luke notices that by taking any three consecutive integers, multiplying them together, and adding the middle number to the result, the answer always seems to be the middle number cubed.

For example:

$$3 \times 4 \times 5 + 4 = 64 = 4^3$$
$$4 \times 5 \times 6 + 5 = 125 = 5^3$$
$$9 \times 10 \times 11 + 10 = 1000 = 10^3$$

- a. To prove his observation, Luke writes  $(n + 1)(n + 2)(n + 3) + (n + 2)$ . What answer is he hoping to show this expression equals?

- b. Lulu, upon hearing of Luke's observation, writes her own version with  $n$  as the middle number. What does her formula look like?

- c. Use Lulu's expression to prove that adding the middle number to the product of any three consecutive numbers is sure to equal that middle number cubed.
3. A cookie company packages its cookies in rectangular prism boxes designed with square bases that have both a length and width of 4 in. less than the height of the box.
- a. Write a polynomial that represents the volume of a box with height  $x$  inches.
- b. Find the dimensions of the box if its volume is 128 cubic inches.

- c. After solving this problem, Juan was very clever and invented the following strange question:

*A building, in the shape of a rectangular prism with a square base, has on its top a radio tower. The building is 25 times as tall as the tower, and the side-length of the base of the building is 100 ft. less than the height of the building. If the building has a volume of 2-million cubic feet, how tall is the tower?*

Solve Juan's problem.

A Progression Toward Mastery

Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem <u>OR</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	a  N-Q.A.2 A-APR.B.2 A-APR.B.3 F-IF.C.7c	Student identifies zeros on graph.	Student uses zeros to write a factored cubic polynomial for $H(x)$ without a leading coefficient.	Student uses given condition $H(300) = 1050$ to find $a$ -value (leading coefficient).	Student writes a complete cubic model for $H(x)$ in factored form with correct $a$ -value (leading coefficient).
	b  N-Q.A.2 A-APR.B.2 A-APR.B.3 F-IF.C.7c	Student finds the midpoint of the river.	Student evaluates $H(x)$ using the midpoint. The exact answer is not needed, only approximation.	Student determines if a cubic model is suitable for this hill and riverbed.	Student justifies answer using $H(\text{midpoint})$ in explanation.
2	a  A-SSE.A.2 A-APR.C.4	Student does not indicate any expression involving $n$ raised to an exponent of 3.	Student uses a base involving $n$ being raised to an exponent of 3 in the answer but does not choose a base of $(n + 2)$ .	Student writes $(n + 2)^3$ without including parentheses to indicate all of $(n + 2)$ is being cubed (i.e., $n + 2^3$ ). <u>OR</u> Student makes another error that shows general understanding but is technically incorrect.	Student writes the correct answer, $(n + 2)^3$ .
	b–c  A-SSE.A.2 A-APR.C.4	Student does not answer parts (b)–(c). <u>OR</u> Student provides incorrect or incomplete answers.	Student answers part (b) incorrectly but uses correct algebra in showing equivalence to $n^3$ <u>OR</u>	Student answers part (b) correctly as $(n - 1)(n)(n + 1) + n = n^3$ but makes minor errors in showing equivalence to $n^3$ .	Student answers correctly as $(n - 1)(n)(n + 1) + n = n^3$ and correctly multiplies the left side and then combines like

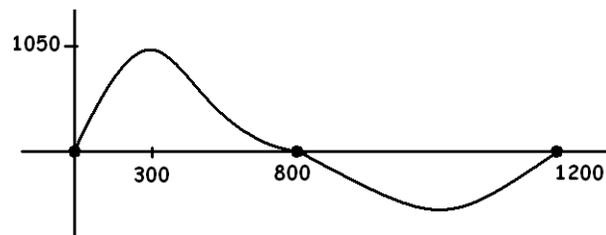
			Student answers part (b) correctly, but student makes major errors or is unable to show its equivalence to $n^3$ .		terms to show equivalence to $n^3$ .
3	<p><b>a–d</b></p> <p><b>N-Q.A.2</b>  <b>A-SSE.A.2</b>  <b>A-APR.B.2</b>  <b>A-APR.B.3</b>  <b>A-REI.A.1</b>  <b>A-REI.B.4b</b></p>	Student determines an expression for $V(x)$ .	Student sets $V(x)$ equal to given volume.	Student solves the equation understanding that only real values are possible solutions for the dimensions of a box.	Student states the three dimensions of the box with proper units.
	<p><b>c</b></p> <p><b>N-Q.A.2</b>  <b>A-SSE.A.2</b>  <b>A-APR.B.2</b>  <b>A-APR.B.3</b>  <b>A-REI.A.1</b>  <b>A-REI.B.4b</b></p>	Student determines an expression for $V(h)$ and sets it equal to the given volume, but does not solve the equation.	Student writes the equation as a polynomial equation, but is unable to find any solutions to the equation.	<p>Student finds one solution to the equation but is unable to use it to factor the polynomial expression to find the other potential solutions.</p> <p><u>OR</u>                      Student recognizes the simplified equation as the same as the equation in the previous part and states the solution from previous part immediately (but as the real number 8, not as a height measurement 8 feet).</p>	<p>Student finds the solutions to the equation, determines which solution is valid, and states the correct answer as the height measurement 8 feet.</p> <p><u>OR</u>                      Student recognizes the simplified equation as the same as the equation in the previous part and states the solution from previous part immediately using the correct units, i.e., 8 feet.</p>

Name \_\_\_\_\_

Date \_\_\_\_\_

1. Geographers sit at a café discussing their field work site, which is a hill and a neighboring riverbed. The hill is approximately 1,050 ft. high, 800 ft. wide, with peak about 300 ft. east of the western base of the hill. The river is about 400 ft. wide. They know the river is shallow, no more than about 20 ft. deep.

They make the following crude sketch on a napkin, placing the profile of the hill and riverbed on a coordinate system with the horizontal axis representing ground level.



The geographers do not have any computing tools with them at the café, so they decide to use pen and paper to compute a cubic polynomial that approximates this profile of the hill and riverbed.

- a. Using only a pencil and paper, write a cubic polynomial function  $H$  that could represent the curve shown (here,  $x$  represents the distance, in feet, along the horizontal axis from the western base of the hill, and  $H(x)$  is the height, in feet, of the land at that distance from the western base). Be sure that your formula satisfies  $H(300) = 1050$ .

We have  $H(x) = c(x-800)(x-1200)$ . For  $H(300)$  equal 1050,  
we need  $1050 = c(300)(-500)(-900)$

$$5 \times 21 \times 10 = c \times 3 \times 5 \times 9 \times 10^6$$

$$7 = c \times 9 \times 10^5$$

$$c = \frac{7}{9 \times 10^5}$$

$$\text{So } H(x) = \frac{7}{9 \times 10^5} x(x-800)(x-1200)$$

- b. For the sake of convenience, the geographers make the assumption that the deepest point of the river is halfway across the river (recall that the river is no more than 20-ft. deep). Under this assumption, would a cubic polynomial provide a suitable model for this hill and riverbed? Explain.

Halfway across the river is  $x = 1000$ , and

$$H(1000) = \frac{7}{9 \times 10^5} (1000)(200)(-200) = -\frac{7 \times 2 \times 2 \times 100}{9} = -\frac{2800}{9}$$

Notice that  $\frac{2800}{9} > \frac{2700}{9} = 300$ , so this model says that the river is over 300 ft. deep!  
This is not a good model for a shallow river, no more than 20 ft. deep.

2. Luke notices that by taking any three consecutive integers, multiplying them together, and adding the middle number to the result, the answer always seems to be the middle number cubed.

For example:

$$3 \times 4 \times 5 + 4 = 64 = 4^3$$

$$4 \times 5 \times 6 + 5 = 125 = 5^3$$

$$9 \times 10 \times 11 + 10 = 1000 = 10^3$$

- a. To prove his observation, Luke writes  $(n + 1)(n + 2)(n + 3) + (n + 2)$ . What answer is he hoping to show this expression equals?

$$(n + 2)^3$$

- b. Lulu, upon hearing of Luke's observation, writes her own version with  $n$  as the middle number. What does her formula look like?

$$(n - 1)n(n + 1) + n = n^3$$

- c. Use Lulu's expression to prove that adding the middle number to the product of any three consecutive numbers is sure to equal that middle number cubed.

We need to show  $(n-1)n(n+1)$  equals  $n^3$ .

Now,  $(n-1)n(n+1) + n = n(n-1)(n+1) + n$

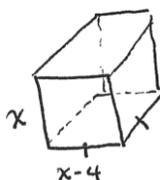
$$= n(n^2 - 1) + n$$

$$= n^3 - n + n$$

$$= n^3$$

It does!

3. A cookie company packages its cookies in rectangular prism boxes designed with square bases that have both a length and width of 4 in. less than the height of the box.
- a. Write a polynomial that represents the volume of a box with height  $x$  inches.



$$V = x(x-4)^2$$

- b. Find the dimensions of the box if its volume is 128 cubic inches.

$$128 = x(x-4)^2$$

$$0 = (x^2 - 8x + 16)x - 128$$

$$0 = x^3 - 8x^2 + 16x - 128$$

$$0 = (x^2 + 16)(x - 8)$$

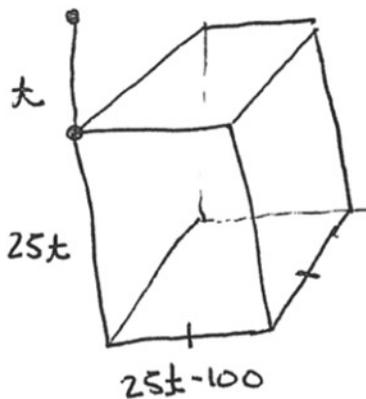
So either  $x - 8 = 0$  giving  $x = 8$   
 or  $x^2 + 16 = 0$ . But this second equation has no real solutions.

So  $x = 8$  is the only solution, and the dimensions of the box are  $4'' \times 4'' \times 8''$ .

- c. After solving this problem, Juan was very clever and invented the following strange question:

*A building, in the shape of a rectangular prism with a square base, has on its top a radio tower. The building is 25 times as tall as the tower, and the side-length of the base of the building is 100 ft. less than the height of the building. If the building has a volume of 2-million cubic feet, how tall is the tower?*

Solve Juan's problem.



$$V = 25t(25t - 100)^2 = 2,000,000$$

$$25t(25(t - 4))^2 = 2,000,000$$

$$25^3 t(t - 4)^2 = 2,000,000$$

$$t(t - 4)^2 = 128$$

Since this is the same equation as part b, we know the solution is  $t = 8$ .

The tower is 8 ft. tall.



## Topic C:

# Solving and Applying Equations—Polynomial, Rational, and Radical

A-APR.D.6, A-REI.A.1, A-REI.A.2, A-REI.B.4b, A-REI.C.6, A-REI.C.7, G-GPE.A.2

<b>Focus Standard:</b>	A-APR.D.6	Rewrite simple rational expressions in different forms; write $a(x)/b(x)$ in the form $q(x) + r(x)/b(x)$ , where $a(x)$ , $b(x)$ , $q(x)$ , and $r(x)$ are polynomials with the degree of $r(x)$ less than the degree of $b(x)$ , using inspection, long division, or, for the more complicated examples, a computer algebra system.
	A-REI.A.1	Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.
	A-REI.A.2	Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.
	A-REI.B.4	Solve quadratic equations in one variable. <ol style="list-style-type: none"> <li>Solve quadratic equations by inspection (e.g., for <math>x^2 = 49</math>), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as <math>a \pm bi</math> for real numbers <math>a</math> and <math>b</math>.</li> </ol>
	A-REI.C.6	Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.
	A-REI.C.7	Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. For example, find the points of intersection between the line $y = -3x$ and the circle $x^2 + y^2 = 3$ .
	G-GPE.A.2	Derive the equation of a parabola given a focus and directrix.

<b>Instructional Days:</b>	14
<b>Lesson 22:</b>	Equivalent Rational Expressions (S) <sup>1</sup>
<b>Lesson 23:</b>	Comparing Rational Expressions (S)
<b>Lesson 24:</b>	Multiplying and Dividing Rational Expressions (P)
<b>Lesson 25:</b>	Adding and Subtracting Rational Expressions (P)
<b>Lesson 26:</b>	Solving Rational Equations (P)
<b>Lesson 27:</b>	Word Problems Leading to Rational Equations (P)
<b>Lesson 28:</b>	A Focus on Square Roots (P)
<b>Lesson 29:</b>	Solving Radical Equations (P)
<b>Lesson 30:</b>	Linear Systems in Three Variables (P)
<b>Lesson 31:</b>	Systems of Equations (E)
<b>Lesson 32:</b>	Graphing Systems of Equations (S)
<b>Lesson 33:</b>	The Definition of a Parabola (S)
<b>Lesson 34:</b>	Are all Parabolas Congruent? (P)
<b>Lesson 35:</b>	Are all Parabolas Similar? (S)

In Topic C, students continue to build upon the reasoning used to solve equations and their fluency in factoring polynomial expressions. In Lesson 22, students expand their understanding of the division of polynomial expressions to rewriting simple rational expressions (**A-APR.D.6**) in equivalent forms. In Lesson 23, students learn techniques for comparing rational expressions numerically, graphically, and algebraically. In Lessons 24–25, students learn to rewrite simple rational expressions by multiplying, dividing, adding, or subtracting two or more expressions. They begin to connect operations with rational numbers to operations on rational expressions. The practice of rewriting rational expressions in equivalent forms in Lessons 22–25 is carried over to solving rational equations in Lessons 26 and 27. Lesson 27 also includes working with word problems that require the use of rational equations. In Lessons 28–29, we turn to radical equations. Students learn to look for extraneous solutions to these equations as they did for rational equations.

In Lessons 30–32, students solve and graph systems of equations including systems of one linear equation and one quadratic equation and systems of two quadratic equations. Next, in Lessons 33–35, students study the definition of a parabola as they first learn to derive the equation of a parabola given a focus and a directrix and later to create the equation of the parabola in vertex form from the coordinates of the vertex and the location of either the focus or directrix. Students build upon their understanding of rotations and translations from Geometry as they learn that any given parabola is congruent to the one given by the equation  $y = ax^2$  for some value of  $a$  and that all parabolas are similar.

<sup>1</sup> Lesson Structure Key: **P**-Problem Set Lesson, **M**-Modeling Cycle Lesson, **E**-Exploration Lesson, **S**-Socratic Lesson



## Lesson 22: Equivalent Rational Expressions

### Student Outcomes

- Students define rational expressions and write them in equivalent forms.

### Lesson Notes

In this module, we have been working with polynomials and polynomial functions. In elementary school, students mastered arithmetic operations with integers before advancing to performing arithmetic operations with rational numbers. Just as a rational number is built from integers, a rational expression is built from polynomial expressions. A precise definition of a rational expression is included at the end of the lesson.

Informally, a rational expression is any expression that is made by a finite sequence of addition, subtraction, multiplication, and division operations on polynomials. After algebraic manipulation, a rational expression can always be written as  $\frac{P}{Q}$ , where  $P$  is any polynomial and  $Q$  is any polynomial except the zero polynomial. Remember that constants, such as 2, and variables, such as  $x$ , count as polynomials, so the rational numbers are also considered to be rational expressions. Standard **A-APR.C.6** focuses on rewriting rational expressions in equivalent forms, and in the next three lessons, we apply that standard to write complicated rational expressions in the simplified form  $\frac{P}{Q}$ . However, the prompt “simplify the rational expression” does not only mean putting expressions in the form  $\frac{P}{Q}$  but also any form that is conducive to solving the problem at hand. The skills developed in Lessons 22–25 are necessary prerequisites for addressing standard **A-REI.A.2**, solving rational equations, which is the focus of Lessons 26 and 27.

### Classwork

#### Opening Exercise (8 minutes)

The Opening Exercise serves two purposes: (1) to reactivate prior knowledge of equivalent fractions, and (2) as a review for students who struggle with fractions. We want students to see that the process they use to reduce a fraction to lowest terms is the same they will use to reduce a rational expression to lowest terms. To begin, pass out 2–3 notecard-sized slips of paper to each student or pair of students.

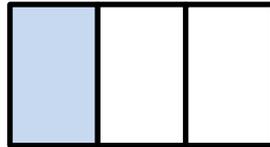
- We are going to start with a review of how to visualize equivalent fractions.

#### Opening Exercise

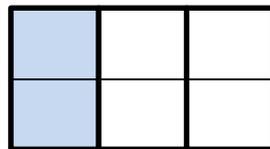
On your own or with a partner, write two fractions that are equivalent to  $\frac{1}{3}$ , and use the slips of paper to create visual models to justify your response.

Use the following to either walk through the exercise for scaffolding or as an example of likely student responses.

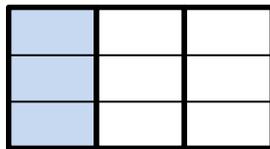
- We can use the following area model to represent the fraction  $\frac{1}{3}$ . Because the three boxes have the same area, shading one of the three boxes shows that  $\frac{1}{3}$  of the area in the figure is shaded.



- Now, if we draw a horizontal line dividing the columns in half, we have six congruent rectangles, two of which are shaded so that  $\frac{2}{6}$  of the area in the figure is shaded.



- In the figure below, we have now divided the original rectangle into nine congruent sub-rectangles, three of which are shaded so that  $\frac{3}{9}$  of the area in the figure is shaded.



- Let's suppose that the area of the original rectangle is 1. In walking the class through the example, point out that the shaded area in the first figure is  $\frac{1}{3}$ , the shaded area in the second figure is  $\frac{2}{6}$ , and the shaded area in the third figure is  $\frac{3}{9}$ . Since the area of the shaded regions are the same in all three figures, we see that  $\frac{1}{3} = \frac{2}{6} = \frac{3}{9}$ . Thus,  $\frac{1}{3}$ ,  $\frac{2}{6}$ , and  $\frac{3}{9}$  are equivalent fractions.

*Scaffolding:*

Students who are already comfortable with fractions can instead reduce the following rational expressions to lowest terms.

$$\frac{5}{15}, \frac{27}{36}, \frac{\sqrt{75}}{5}, \frac{\pi^4}{\pi^2}$$

In any case, do not spend too much time on these exercises; instead, use them as a bridge to reducing rational expressions that contain variables.

If students have come up with different equivalent fractions, then incorporate those into the discussion of equivalent areas, noting that the shaded regions are the same for every student.

MP.8

- Now, what if we were to choose any positive integer  $n$  and draw lines across our figure so that the columns are divided into  $n$  pieces of the same size? What is the area of the shaded region?

Give students time to think and write, and ask them to share their answers with a partner. Anticipate that students will express the generalization in words or suggest either  $\frac{1}{3}$  or  $\frac{n}{3n}$ . Both are correct and, ideally, both will be suggested.

- Thus, we have the rule:

If  $a$ ,  $b$ , and  $n$  are integers with  $n \neq 0$  and  $b \neq 0$ , then

$$\frac{na}{nb} = \frac{a}{b}.$$

*Scaffolding:*  
Students may also express the generalization in words.

The result summarized in the box above is also true for real numbers  $a$ ,  $b$ , and  $n$  as well as for polynomial and rational expressions.

- Then  $\frac{2}{6} = \frac{2(1)}{2(3)} = \frac{1}{3}$  and  $\frac{3}{9} = \frac{3(1)}{3(3)} = \frac{1}{3}$ .
- We say that a rational number is simplified, or reduced to lowest terms, when the numerator and denominator do not have a factor in common. Thus, while  $\frac{1}{3}$ ,  $\frac{2}{6}$ , and  $\frac{3}{9}$  are equivalent, only  $\frac{1}{3}$  is in lowest terms.

**Discussion (10 minutes)**

- Which of the following are rational numbers, and which are not?

$$\frac{3}{4}, 3.14, \pi, \frac{5}{0}, -\sqrt{17}, 23, \frac{1 + \sqrt{5}}{2}, -1, 6.022 \times 10^{23}, 0$$

- Rational:*  $\frac{3}{4}, 3.14, 23, -1, 6.022 \times 10^{23}, 0$
- Not rational:*  $\pi, \frac{5}{0}, -\sqrt{17}, \frac{1 + \sqrt{5}}{2}$

Of the numbers that were not rational, were they all irrational numbers?

- No. Since division by zero is undefined,  $\frac{5}{0}$  is neither rational nor irrational.*

- Today we learn about rational expressions, which are related to the polynomials we've been studying. Just as the integers are the foundational building blocks of rational numbers, polynomial expressions are the foundational building blocks for rational expressions. Based on what we know about rational numbers, give an example of what you think a rational expression is.

*Scaffolding:*  
Relate the new ideas of rational expressions back to the more familiar ideas of rational numbers throughout this lesson.

Ask students to write down an example and share it with their partner or small group. Allow groups to debate and present one of the group's examples to the class.

- Recall that a rational number is a number that we can write as  $\frac{p}{q}$ , where  $p$  and  $q$  are integers and  $q$  is nonzero. We can consider a new type of expression, called a *rational expression*, which is made from polynomials by adding, subtracting, multiplying, and dividing them. Any rational expression can be expressed as  $\frac{P}{Q}$ , where  $P$  and  $Q$  are polynomial expressions and  $Q$  is not zero, even though it may not be presented in this form originally.

Remind students that numbers are also polynomial expressions, which means that rational numbers are included in the set of rational expressions.

- The following are examples of rational expressions. Notice that we need to exclude values of the variables that make the denominators zero so that we do not divide by zero.
- $\frac{31}{47}$ 
  - *The denominator is never zero, so we do not need to exclude any values.*
- $\frac{ab^2}{3a-2b}$ 
  - *We need  $3a \neq 2b$ .*
- $\frac{5x+1}{3x^2+4}$ 
  - *The denominator is never zero, so we do not need to exclude any values.*
- $\frac{3}{b^2-7}$ 
  - *We need  $b \neq \sqrt{7}$  and  $b \neq -\sqrt{7}$ .*

Have students create a Frayer model in their notebooks, such as the one provided. Circulate around the classroom to informally assess student understanding. Since a formal definition of rational expressions has not yet been given, there is some leeway on the description and characteristics sections, but make sure that nothing they have written is incorrect. Ask students to share their characteristics, examples, and non-examples to populate a class model on the board.

<p>Description</p> <p><i>An expression that can be written as <math>\frac{P}{Q}</math>, where <math>P</math> and <math>Q</math> are polynomials and <math>Q</math> is not zero.</i></p>	<p>Characteristics</p> <p><i>Follows similar rules as rational numbers do.</i></p>
<p>Rational Expression</p>	
<p>Examples</p> <p><math>\frac{3}{5} \frac{x^2-4x}{x+1}</math> with <math>x \neq -1</math></p> <p><math>\frac{x^2y}{2}, \frac{a^2+b^2}{(a+2)(a-1)}</math> with <math>a \neq -2, 1</math></p>	<p>Non-examples</p> <p><math>\frac{x+1}{0}</math> (cannot divide by 0)</p> <p><math>\frac{2^x}{3x}</math> (<math>2^x</math> is not a polynomial)</p>

It is important to note that the excluded values of the variables remain even after simplification. This is because the two expressions would not be equal if the variables were allowed to take on these values. Discuss with a partner when the following are not equivalent and why:

- $\frac{2x}{3x}$  and  $\frac{2}{3}$ 
  - *These are equivalent everywhere except at  $x = 0$ . At  $x = 0$ ,  $\frac{2x}{3x}$  is undefined, whereas  $\frac{2}{3}$  is equal to  $\frac{2}{3}$ .*

**Scaffolding:**

Encourage struggling students to plug in various values of the variables to see that the expressions *are* equivalent for almost all values of the variables. But for values in which the denominator of one expression is equal to zero, they are not equivalent.

- $\frac{3x(x-5)}{4(x-5)}$  and  $\frac{3x}{4}$ 
  - At  $x = 5$ ,  $\frac{3x(x-5)}{4(x-5)}$  is undefined, whereas  $\frac{3x}{4} = \frac{3(5)}{4} = \frac{15}{4}$ .
- $\frac{x-3}{x^2-x-6}$  and  $\frac{1}{x+2}$ 
  - At  $x = 3$ ,  $\frac{x-3}{x^2-x-6}$  is undefined, whereas  $\frac{1}{x+2} = \frac{1}{3+2} = \frac{1}{5}$ .
- Summarize with your partner or in writing any conclusions you can draw about equivalent rational expressions. Circulate around the classroom to assess understanding.

**Example 1 (6 minutes)**

**Example 1**

Consider the following rational expression:  $\frac{2(a-1)-2}{6(a-1)-3a}$ . Turn to your neighbor and discuss the following: For what values of  $a$  is the expression undefined?

Sample the students' answers. When they suggest that the denominator cannot be zero, give the class a minute to work out that the denominator is zero when  $a = 2$ .

$$\begin{aligned} & \frac{2(a-1)-2}{6(a-1)-3a} \\ & 6(a-1)-3a=0 \\ & 6a-6-3a=0 \\ & 3a-6=0 \\ & a=2 \end{aligned}$$

- Let's reduce the rational expression  $\frac{2(a-1)-2}{6(a-1)-3a}$  with  $a \neq 2$  to lowest terms. Since no common factor is visible in the given form of the expression, we first simplify the numerator and denominator by distributing and combining like terms.

$$\begin{aligned} \frac{2(a-1)-2}{6(a-1)-3a} &= \frac{2a-2-2}{6a-6-3a} \\ &= \frac{2a-4}{3a-6} \end{aligned}$$

- Next, we factor the numerator and denominator, and divide both by any common factors. This step shows clearly why we had to specify that  $a \neq 2$ .

$$\begin{aligned} \frac{2(a-1)-2}{6(a-1)-3a} &= \frac{2a-4}{3a-6} \\ &= \frac{2(a-2)}{3(a-2)} \\ &= \frac{2}{3} \end{aligned}$$

**Scaffolding:**

Students may need to be reminded that although  $(a - 1)$  appears in the numerator and denominator, it is not a common *factor* to the numerator and denominator, and thus, we cannot simplify the expression by dividing by  $(a - 1)$ .

- As long as  $a \neq 2$ , we see that  $\frac{2(a-1)-2}{6(a-1)-3a}$  and  $\frac{2}{3}$  are equivalent rational expressions.

If we allow  $a$  to take on the value of 2, then  $\frac{2(a-1)-2}{6(a-1)-3a}$  is undefined. However, the expression  $\frac{2}{3}$  is always defined so these expressions are not equivalent.

### Exercise 1 (10 minutes)

Allow students to work on the following exercises in pairs.

#### Exercise 1

Reduce the following rational expressions to lowest terms, and identify the values of the variable(s) that must be excluded to prevent division by zero.

- a.  $\frac{2(x+1)+2}{(2x+3)(x+1)-1}$   
 $\frac{2(x+1)+2}{(2x+3)(x+1)-1} = \frac{2x+4}{2x^2+5x+2} = \frac{2(x+2)}{(2x+1)(x+2)} = \frac{2}{2x+1}$ , for  $x \neq -2$  and  $x \neq -\frac{1}{2}$ .
- b.  $\frac{x^2-x-6}{5x^2+10x}$   
 $\frac{x^2-x-6}{5x^2+10x} = \frac{(x+2)(x-3)}{5x(x+2)} = \frac{x-3}{5x}$ , for  $x \neq 0$  and  $x \neq -2$ .
- c.  $\frac{3-x}{x^2-9}$   
 $\frac{3-x}{x^2-9} = \frac{-(x-3)}{(x-3)(x+3)} = -\frac{1}{x+3}$ , for  $x \neq 3$  and  $x \neq -3$ .
- d.  $\frac{3x-3y}{y^2-2xy+x^2}$   
 $\frac{3x-3y}{y^2-2xy+x^2} = \frac{-3(y-x)}{(y-x)(y-x)} = -\left(\frac{3}{y-x}\right)$ , for  $y \neq x$ .

### Closing (5 minutes)

The precise definition of a rational expression is presented here for teacher reference and may be shared with students at your discretion. Discussion questions for closing this lesson follow the definition. Notice the similarity between the definition of a rational expression given here and the definition of a polynomial expression given in the closing of Lesson 5 earlier in this module.

**Rational Expression:** A rational expression is either a numerical expression or a variable symbol or the result of placing two previously generated rational expressions into the blanks of the addition operator ( $\_ + \_$ ), the subtraction operator ( $\_ - \_$ ), the multiplication operator ( $\_ \times \_$ ), or the division operator ( $\_ \div \_$ ).

Have students discuss the following questions with a partner and write down their conclusions. Circulate around the room to assess their understanding.

- How do you reduce a rational expression of the form  $\frac{P}{Q}$  to lowest terms?
  - Factor the polynomial expressions in the numerator and denominator, and divide any common factors from both the numerator and denominator.
- How do you know which values of the variable(s) to exclude for a rational expression?
  - Any value of the variable(s) that makes the denominator zero at any point of the process must be excluded.

**Lesson Summary**

- If  $a$ ,  $b$ , and  $n$  are integers with  $n \neq 0$  and  $b \neq 0$ , then
$$\frac{na}{nb} = \frac{a}{b}.$$
- The rule for rational expressions is the same as the rule for integers but requires the domain of the rational expression to be restricted (i.e., no value of the variable that can make the denominator of the original rational expression zero is allowed).

**Exit Ticket (6 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 22: Equivalent Rational Expressions

### Exit Ticket

1. Find an equivalent rational expression in lowest terms, and identify the value(s) of the variables that must be excluded to prevent division by zero.

$$\frac{x^2 - 7x + 12}{6 - 5x + x^2}$$

2. Determine whether or not the rational expressions  $\frac{x+4}{(x+2)(x-3)}$  and  $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)}$  are equivalent for  $x \neq -1$ ,  $x \neq -2$ , and  $x \neq 3$ . Explain how you know.

Exit Ticket Sample Solutions

1. Find an equivalent rational expression in lowest terms, and identify the value(s) of the variables that must be excluded to prevent division by zero.

If  $x \neq 3$  and  $x \neq 2$ , then we have

$$\frac{x^2 - 7x + 12}{6 - 5x + x^2} = \frac{(x - 4)(x - 3)}{(x - 3)(x - 2)} = \frac{x - 4}{x - 2}$$

2. Determine whether or not the rational expressions  $\frac{x+4}{(x+2)(x-3)}$  and  $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)}$  are equivalent for  $x \neq -1$ ,  $x \neq -2$ , and  $x \neq 3$ . Explain how you know.

Since  $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)} = \frac{(x+1)(x+4)}{(x+1)(x+2)(x-3)} = \frac{x+4}{(x+2)(x-3)}$  as long as  $x \neq -1$ ,  $x \neq -2$ , and  $x \neq 3$ , the rational expressions  $\frac{x+4}{(x+2)(x-3)}$  and  $\frac{x^2+5x+4}{(x+1)(x+2)(x-3)}$  are equivalent.

Problem Set Sample Solutions

1. Find an equivalent rational expression in lowest terms, and identify the value(s) of the variable that must be excluded to prevent division by zero.

- |    |                                  |                        |  |
|----|----------------------------------|------------------------|--|
| a. | $\frac{16n}{20n}$                | $\frac{4}{5}$          | $n \neq 0$                                       |
| b. | $\frac{x^3y}{y^4x}$              | $\frac{x^2}{y^3}$      | $x \neq 0$ and $y \neq 0$                        |
| c. | $\frac{2n-8n^2}{4n}$             | $\frac{1-4n}{2}$       | $n \neq 0$                                       |
| d. | $\frac{db+dc}{db}$               | $\frac{b+c}{b}$        | $b \neq 0$ and $d \neq 0$ .                      |
| e. | $\frac{x^2-9b^2}{x^2-2xb-3b^2}$  | $\frac{x+3b}{x+b}$     | $x \neq 3b$ and $x \neq -b$                      |
| f. | $\frac{3n^2-5n-2}{2n-4}$         | $\frac{3n+1}{2}$       | $n \neq 2$                                       |
| g. | $\frac{3x-2y}{9x^2-4y^2}$        | $\frac{1}{3x+2y}$      | $y \neq \frac{3}{2}x$ and $y \neq -\frac{3}{2}x$ |
| h. | $\frac{4a^2-12ab}{a^2-6ab+9b^2}$ | $\frac{4a}{a-3b}$      | $a \neq 3b$                                      |
| i. | $\frac{y-x}{x-y}$                | $-1$                   | $x \neq y$                                       |
| j. | $\frac{a^2-b^2}{b+a}$            | $a - b$                | $a \neq -b$                                      |
| k. | $\frac{4x-2y}{3y-6x}$            | $-\frac{2}{3}$         | $y \neq 2x$                                      |
| l. | $\frac{9-x^2}{(x-3)^3}$          | $-\frac{3+x}{(x-3)^2}$ | $x \neq 3$                                       |

m.  $\frac{x^2-5x+6}{8-2x-x^2}$        $-\frac{x-3}{4+x}$        $x \neq 2$  and  $x \neq -4$

n.  $\frac{a-b}{xa-xb-a+b}$        $\frac{1}{x-1}$        $x \neq 1$  and  $a \neq b$

o.  $\frac{(x+y)^2-9a^2}{2x+2y-6a}$        $\frac{x+y+3a}{2}$        $a \neq \frac{x+y}{3}$

p.  $\frac{8x^3-y^3}{4x^2-y^2}$        $\frac{4x^2+2xy+y^2}{2x+y}$        $y \neq 2x$  and  $y \neq -2x$

2. Write a rational expression with denominator  $6b$  that is equivalent to

a.  $\frac{a}{b}$   
 $\frac{6a}{6b}$

b. one-half of  $\frac{a}{b}$   
 $\frac{3a}{6b}$

c.  $\frac{1}{3}$   
 $\frac{2b}{6b}$

3. Remember that algebra is just another way to perform arithmetic but with variables replacing numbers.

a. Simplify the following rational expression:  $\frac{(x^2y)^2(xy)^3z^2}{(xy^2)^2yz}$ .

$$\frac{(x^2y)^2(xy)^3z^2}{(xy^2)^2yz} = \frac{x^4y^2 \cdot x^3y^3 \cdot z^2}{x^2y^4 \cdot yz} = \frac{x^7y^5z^2}{x^2y^5z} = x^5z$$

b. Simplify the following rational expression without using a calculator:  $\frac{12^2 \cdot 6^3 \cdot 5^2}{18^2 \cdot 15}$ .

$$\frac{12^2 \cdot 6^3 \cdot 5^2}{18^2 \cdot 15} = \frac{4^2 \cdot 3^2 \cdot 6^3 \cdot 5^2}{2^2 \cdot 9^2 \cdot 3 \cdot 5} = \frac{2^4 \cdot 3^2 \cdot 2^3 \cdot 3^3 \cdot 5^2}{2^2 \cdot 3^4 \cdot 3 \cdot 5} = \frac{2^7 \cdot 3^5 \cdot 5^2}{2^2 \cdot 3^5 \cdot 5} = 2^5 \cdot 5 = 32 \cdot 5 = 160$$

c. How are the calculations in parts (a) and (b) similar? How are they different? Which expression was easier to simplify?

*Both simplifications relied on using the rules of exponents. It was easier to simplify the algebraic expression in part (a) because we did not have to factor any numbers, such as 18, 15, and 12. However, if we substitute  $x = 2$ ,  $y = 3$ , and  $z = 5$ , these two expressions have the exact same structure. Algebra allows us to do this calculation more quickly.*

MP.7



## Lesson 23: Comparing Rational Expressions

### Student Outcomes

- Students compare rational expressions by writing them in different but equivalent forms.

### Lesson Notes

The skills developed in Lessons 22–25 are prerequisites for addressing standard **A-REI.A.2**, solving rational equations, which is the focus of Lessons 26 and 27. In this lesson, students extend comparisons of rational numbers to comparing rational expressions and using numerical, graphical, and algebraic analysis. Although students use graphing calculators to compare certain rational expressions, learning to graph rational functions is not the focus of this lesson.

### Classwork

#### Opening Exercise (10 minutes)

The Opening Exercise serves two purposes: (1) to reactivate prior knowledge of comparing fractions and (2) as a review for students who struggle with fractions. We want students to see that the same process is used to compare fractions and to compare rational expressions.

As done in the previous lesson, give students slips of notecard-sized paper on which to make visual arguments for which fraction is larger. Each student (or pair of students) should get at least two slips of paper. This exercise leads us to the graphical analysis employed in the last example of the lesson.

#### Opening Exercise

Use the slips of paper you have been given to create visual arguments for which fraction is larger.

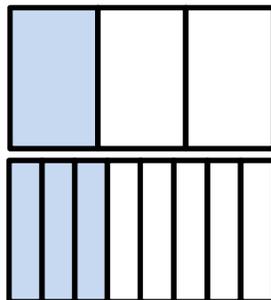
#### Scaffolding:

Students already comfortable with fractions may wish to only briefly review the visual representations; however, it is important for each student to be aware of the three methods of comparison, in general, graphical (or visual), numerical, and algebraic (by finding a common denominator).

In any case, do not spend too much time on these exercises, but instead use them as a bridge to comparing rational expressions that contain variables.

Ask students to make visual arguments as to whether  $\frac{1}{3}$  or  $\frac{3}{8}$  is larger. Use the following as either scaffolding for struggling students or as an example of student work.

- We can use the following area models to represent the fraction  $\frac{1}{3}$  and  $\frac{3}{8}$  as we did in Lesson 22.



- We see that these visual representations of the fractions give us strong evidence that  $\frac{1}{3} < \frac{3}{8}$ .
- Discuss with your neighbor another way to make a comparison between the two fractions and why we might not want to always rely on visual representations.
  - *Students should suggest finding decimal approximations of fractions and converting the fractions to equivalent fractions with common denominators. Reasons for not using visual representations may include the difficulty with fractions with large denominators or with irrational parts.*

Once students have had a chance to discuss alternative methods, ask them to choose one of the two methods to verify that the visual representations above are accurate.

- *Decimal approximations: We have  $\frac{1}{3} \approx 0.333$  and  $\frac{3}{8} = 0.375$ ; thus,  $\frac{1}{3} < \frac{3}{8}$ .*
- *Common denominators: We have  $\frac{1}{3} = \frac{8}{24}$  and  $\frac{3}{8} = \frac{9}{24}$ . Since  $\frac{8}{24} < \frac{9}{24}$ , we know that  $\frac{1}{3} < \frac{3}{8}$ .*
- Discuss with your partner the pros and cons of both methods before discussing as a class.
  - *Decimal approximations are quick with a calculator but may take a while if long division is needed. Many students prefer decimals to fractions, but they use approximations of the numbers instead of the exact values of the numbers. Common denominators use the actual numbers but are still working with fractions.*
- Just as we can determine whether two rational expressions are equivalent in a similar way as we can with rational numbers, we can extend our ideas of comparing rational numbers to comparing rational expressions.

**Exercises 1–5 (11 minutes)**

As the students work on Exercises 1–5, circulate through the class to assess their understanding.

**Exercises 1–5**

We will start by working with positive integers. Let  $m$  and  $n$  be positive integers. Work through the following exercises with a partner.

1. Fill out the following table.

$n$	$\frac{1}{n}$
1	1
2	$\frac{1}{2}$
3	$\frac{1}{3}$
4	$\frac{1}{4}$
5	$\frac{1}{5}$
6	$\frac{1}{6}$

2. Do you expect  $\frac{1}{n}$  to be bigger or smaller than  $\frac{1}{n+1}$ ? Do you expect  $\frac{1}{n}$  to be bigger or smaller than  $\frac{1}{n+2}$ ? Explain why.

*From the table, as  $n$  increases,  $\frac{1}{n}$  decreases. This means that if you add 1 to  $n$ , then you will get a smaller number.*

*That is,  $\frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+2}$ .*

3. Compare the rational expressions  $\frac{1}{n}$ ,  $\frac{1}{n+1}$ , and  $\frac{1}{n+2}$  for  $n = 1, 2$ , and  $3$ . Do your results support your conjecture from Exercise 2? Revise your conjecture if necessary.

*For  $n = 1$ ,  $\frac{1}{n} = 1$ ,  $\frac{1}{n+1} = \frac{1}{2}$ , and  $\frac{1}{n+2} = \frac{1}{3}$ , and we know  $1 > \frac{1}{2} > \frac{1}{3}$ .*

*For  $n = 2$ , we have  $\frac{1}{2}$ ,  $\frac{1}{3}$ , and  $\frac{1}{4}$ , and we know  $\frac{1}{2} > \frac{1}{3} > \frac{1}{4}$ .*

*For  $n = 3$ , we have  $\frac{1}{3}$ ,  $\frac{1}{4}$ , and  $\frac{1}{5}$ , and we know  $\frac{1}{3} > \frac{1}{4} > \frac{1}{5}$ .*

*This supports the conjecture that  $\frac{1}{n} > \frac{1}{n+1} > \frac{1}{n+2}$ .*

4. From your work in Exercises 1 and 2, generalize how  $\frac{1}{n}$  compares to  $\frac{1}{n+m}$ , where  $m$  and  $n$  are positive integers.

*Since  $m$  is a positive integer being added to  $n$ , the denominator will increase, which will decrease the value of the rational expression overall. That is,  $\frac{1}{n} > \frac{1}{n+m}$  for positive integers  $m$  and  $n$ .*

5. Will your conjecture change or stay the same if the numerator is 2 instead of 1? Make a conjecture about what happens when the numerator is held constant, but the denominator increases for positive numbers.

*It will stay the same because this would be the same as multiplying the inequality by 2, and multiplication by a positive number does not change the direction of the inequality. If the numerator is held constant and the denominator increases, you are dividing by a larger number, so you get a smaller number overall.*

### Example 1 (11 minutes)

- Suppose we want to compare the values of the rational expressions  $\frac{x+1}{x}$  and  $\frac{x+2}{x+1}$  for positive values of  $x$ . What are some ways to do this?

Ask students to suggest some methods of comparison. If needed, guide them to the ideas of using a numerical comparison through a table of values and a graphical comparison of the related rational functions  $y = \frac{x+1}{x}$  and  $y = \frac{x+2}{x+1}$  for  $x > 0$ .

- Let's start our comparison of  $\frac{x+1}{x}$  and  $\frac{x+2}{x+1}$  by looking at a table of values.

Have students complete the table below using their calculators and rounding to four decimal places.

**Example 1**

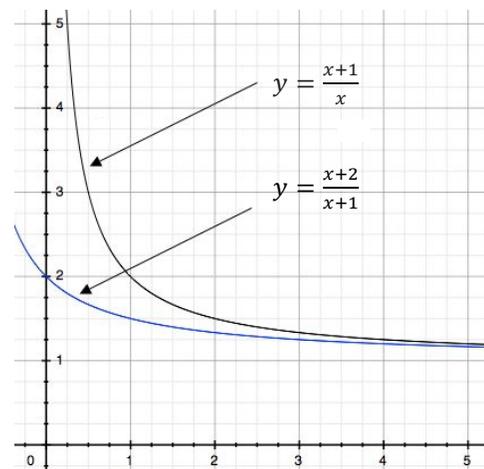
$x$	$\frac{x+1}{x}$	$\frac{x+2}{x+1}$
0.5	3	1.6667
1	2	1.5000
1.5	1.6667	1.4000
2	1.5	1.3333
5	1.2	1.1667
10	1.1	1.0909
100	1.01	1.0099

**Discussion**

- From the table of values, it appears that  $\frac{x+1}{x} > \frac{x+2}{x+1}$  for positive values of  $x$ . However, we have only checked 7 values of  $x$ , so we cannot yet say that this is the case for every positive value of  $x$ . How else can we compare the values of these two expressions?
  - Students should suggest graphing the functions  $y = \frac{x+1}{x}$  and  $y = \frac{x+2}{x+1}$ .

Have students graph the two functions  $y = \frac{x+1}{x}$  and  $y = \frac{x+2}{x+1}$  on their calculators, and ask them to share their observations. Does the graph verify the conclusions we drew from the table above?

- It seems from both the table of data and from the graph that  $\frac{x+1}{x} > \frac{x+2}{x+1}$  for positive values of  $x$ , but we have not shown it conclusively. How can we use algebra to determine if this inequality is always true?
  - We want to compare  $\frac{x+1}{x}$  and  $\frac{x+2}{x+1}$ .



Ask students what they need to do before comparing fractions such as  $\frac{3}{8}$  and  $\frac{2}{7}$ . Wait for someone to suggest that they need to find a common denominator.

- Let's take a step back and see how we would compare the two fractions  $\frac{3}{8}$  and  $\frac{2}{7}$ . First, we find the common denominator.

Wait for a student to volunteer that the common denominator is 56.

- Next, we rewrite each fraction as an equivalent fraction with denominator 56:

$$\frac{3}{8} = \frac{21}{56} \quad \text{and} \quad \frac{2}{7} = \frac{16}{56}.$$

- Since  $21 > 16$ , and 56 is a positive number, we know that  $\frac{21}{56} > \frac{16}{56}$ ; thus, we know that  $\frac{3}{8} > \frac{2}{7}$ .
- The process for comparing rational expressions is the same as the process for comparing fractions. As is always the case with inequalities, we need to be careful about changing the inequality if we multiply or divide by a negative number.

- What is the common denominator of the two expressions  $\frac{x+1}{x}$  and  $\frac{x+2}{x+1}$ ?

- $x(x + 1)$

- First, multiply the numerator and denominator of the first expression by  $(x + 1)$ :

$$\frac{x + 1}{x} = \frac{(x + 1)(x + 1)}{x(x + 1)}$$

$$\frac{x + 1}{x} = \frac{x^2 + 2x + 1}{x(x + 1)}.$$

- Next, multiply the numerator and denominator of the second expression by  $x$ :

$$\frac{x + 2}{x + 1} = \frac{x(x + 2)}{x(x + 1)}$$

$$\frac{x + 2}{x + 1} = \frac{x^2 + 2x}{x(x + 1)}.$$

- Clearly, we have

$$x^2 + 2x + 1 > x^2 + 2x,$$

and since  $x$  is always positive, we know that the denominator  $x(x + 1)$  is always positive. Thus, we see that

$$\frac{x^2 + 2x + 1}{x(x + 1)} > \frac{x^2 + 2x}{x(x + 1)},$$

so we have established that  $\frac{x+1}{x} > \frac{x+2}{x+1}$  for all positive values of  $x$ .

- For rational expressions, numerical and visual comparisons can provide evidence that one expression is larger than another for specified values of the variable. However, finding common denominators and doing the algebra to show that one is larger than the other is the conclusive way to show that the values of one rational expression are consistently larger than the values of another.

**Closing (6 minutes)**

Ask students to do a side-by-side comparison of the different methods for comparing rational numbers to the extended method used for comparing rational expressions.

Rational Numbers	Rational Expressions
<ul style="list-style-type: none"> <li>▪ <u>Visually</u>: Use area models or number lines to represent fractions and compare their relative sizes. Difficult with large numbers.</li> <li>▪ <u>Numerically</u>: Perform the division to find a decimal approximation to compare the sizes.</li> <li>▪ <u>Algebraically</u>: Find equivalent fractions with common denominators and compare their numerators.</li> </ul>	<ul style="list-style-type: none"> <li>▪ <u>Visually</u>: Use a graphing utility to graph functions representing each expression and compare their relative heights. Easy with technology but inconclusive.</li> <li>▪ <u>Numerically</u>: Compare several values of the functions to see their relative sizes. Straightforward but tells us even less than graphing.</li> <li>▪ <u>Algebraically</u>: Find equivalent fractions with common denominators and compare their numerators. Best way but special care needs to be taken with values that may be negative.</li> </ul>

- How do you compare two rational expressions of the form  $\frac{P}{Q}$ ?
  - *Before comparing the expressions, find equivalent rational expressions with the same denominator. Then we can compare the numerators for values of the variable that do not cause the positive/negative signs to switch. Numerical and graphical analysis may be used to help understand the relative sizes of the expressions.*

**Lesson Summary**

To compare two rational expressions, find equivalent rational expression with the same denominator. Then we can compare the numerators for values of the variable that do not cause the rational expression to change from positive to negative or vice versa.

We may also use numerical and graphical analysis to help understand the relative sizes of expressions.

**Exit Ticket (7 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 23: Comparing Rational Expressions

### Exit Ticket

Use the specified methods to compare the following rational expressions:  $\frac{x+1}{x^2}$  and  $\frac{1}{x}$  for  $x > 0$ .

1. Fill out the table of values.

$x$	$\frac{x+1}{x^2}$	$\frac{1}{x}$
1		
10		
25		
50		
100		
500		

2. Graph  $y = \frac{x+1}{x^2}$  and  $y = \frac{1}{x}$  for positive values of  $x$ .

3. Find the common denominator and compare numerators for positive values of  $x$ .

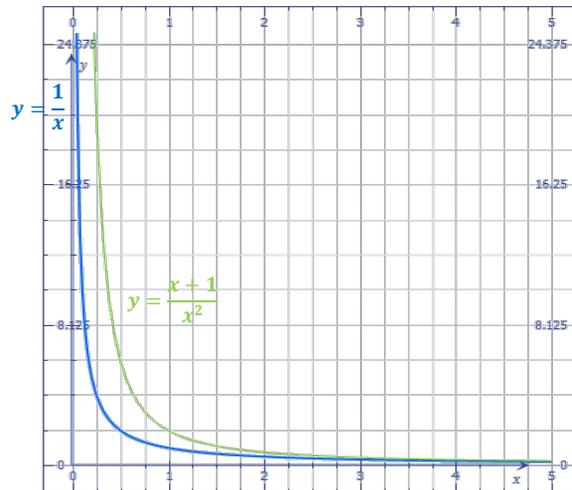
Exit Ticket Sample Solutions

Use the following methods to compare the following rational expressions:  $\frac{x+1}{x^2}$  and  $\frac{1}{x}$ .

1. Fill out the table of values.

$x$	$\frac{x+1}{x^2}$	$\frac{1}{x}$
1	$\frac{2}{1} = 2$	$\frac{1}{1} = 1$
10	$\frac{11}{100} = 0.11$	$\frac{1}{10} = 0.1$
25	$\frac{26}{625} = 0.0416$	$\frac{1}{25} = 0.04$
50	$\frac{51}{2500} = 0.0204$	$\frac{1}{50} = 0.02$
100	$\frac{101}{10000} = 0.0101$	$\frac{1}{100} = 0.01$
500	$\frac{501}{250000} = 0.002004$	$\frac{1}{500} = 0.002$

2. Graph  $y = \frac{x+1}{x^2}$  and  $y = \frac{1}{x}$  for positive values of  $x$ .



3. Find the common denominator and compare numerators for positive values of  $x$ .

The common denominator of  $x$  and  $x^2$  is  $x^2$ .

$$\frac{1}{x} = \frac{1}{x} \cdot \frac{x}{x} = \frac{x}{x^2}$$

$$\frac{x+1}{x^2} = \frac{x+1}{x^2}$$

For any value of  $x$ ,  $x^2$  is positive. Since

$$x + 1 > x,$$

we then have,

$$\frac{x+1}{x^2} > \frac{x}{x^2}$$

$$\frac{x+1}{x^2} > \frac{1}{x}.$$

### Problem Set Sample Solutions

1. For parts (a)–(d), rewrite each rational expression as an equivalent rational expression so that each expression has a common denominator.

a.  $\frac{3}{5}, \frac{9}{10}, \frac{7}{15}, \frac{7}{21}$

$$\frac{18}{30}, \frac{27}{30}, \frac{14}{30}, \frac{10}{30}$$

b.  $\frac{m}{sd}, \frac{s}{dm}, \frac{d}{ms}$

$$\frac{m^2}{msd}, \frac{s^2}{sdm}, \frac{d^2}{dms}$$

c.  $\frac{1}{(2-x)^2}, \frac{3}{(2x-5)(2-x)}$

$$\frac{(2x-5)}{(2-x)^2(2x-5)}, \frac{-3(2-x)}{(2x-5)(2-x)^2}$$

d.  $\frac{3}{x^2-x}, \frac{5}{x}, \frac{2x+2}{2x^2-2}$

$$\frac{3(x+1)}{x(x-1)(x+1)}, \frac{5(x-1)(x+1)}{x(x-1)(x+1)}, \frac{x(x+1)}{x(x-1)(x+1)}$$

2. If  $x$  is a positive number, for which values of  $x$  is  $x < \frac{1}{x}$ ?

Before we can compare two rational expressions, we need to express them as equivalent expressions with a common denominator. Since  $x \neq 0$ , we have  $x = \frac{x^2}{x}$ . Then  $x < \frac{1}{x}$  exactly when  $\frac{x^2}{x} < \frac{1}{x}$ , which happens when  $x^2 < 1$ . The only positive real number values of  $x$  that satisfy  $x^2 < 1$  are  $0 < x < 1$ .

3. Can we determine if  $\frac{y}{y-1} > \frac{y+1}{y}$  for all values  $y > 1$ ? Provide evidence to support your answer.

*Before we can compare two rational expressions, we need to express them as equivalent expressions with a common denominator. Since  $y > 1$ , neither denominator is ever zero. Then  $\frac{y}{y-1} = \frac{y^2}{y(y-1)}$  and  $\frac{y+1}{y} = \frac{(y+1)(y-1)}{y(y-1)} = \frac{y^2-1}{y(y-1)}$ . Since  $y^2 > y^2 - 1$  for all values of  $y$ , we know that  $\frac{y^2}{y(y-1)} > \frac{y^2-1}{y(y-1)}$ . Then we can conclude that  $\frac{y}{y-1} > \frac{y+1}{y}$  for all values  $y > 1$ .*

4. For positive  $x$ , determine when the following rational expressions have negative denominators:

a.  $\frac{3}{5}$

*5 is never less than 0.*

b.  $\frac{x}{5-2x}$

*$5 - 2x < 0$  when  $5 < 2x$ , which is equivalent to  $\frac{5}{2} < x$ .*

c.  $\frac{x+3}{x^2+4x+8}$

*For any real number  $x$ ,  $x^2 + 4x + 8$  is never negative. One way to see this is that  $x^2 + 4x + 8 = (x + 2)^2 + 4$ , which is the sum of two positive numbers.*

d.  $\frac{3x^2}{(x-5)(x+3)(2x+3)}$

*For positive  $x$ ,  $x + 3$  and  $2x + 3$  are always positive. The number  $x - 5$  is negative when  $x < 5$ , so the denominator is negative when  $x < 5$ .*

5. Consider the rational expressions  $\frac{x}{x-2}$  and  $\frac{x}{x-4}$ .

- a. Evaluate each expression for  $x = 6$ .

*If  $x = 6$ , then  $\frac{x}{x-2} = 1.5$  and  $\frac{x}{x-4} = 3$ .*

- b. Evaluate each expression for  $x = 3$ .

*If  $x = 3$ , then  $\frac{x}{x-2} = 3$  and  $\frac{x}{x-4} = -3$ .*

- c. Can you conclude that  $\frac{x}{x-2} < \frac{x}{x-4}$  for all positive values of  $x$ ? Explain how you know.

*No, because  $\frac{x}{x-2} > \frac{x}{x-4}$  when  $x = 3$ , it is not true that  $\frac{x}{x-2} < \frac{x}{x-4}$  for every positive value of  $x$ .*

MP.3

MP.2

- d. EXTENSION: Raphael claims that the calculation below shows that  $\frac{x}{x-2} < \frac{x}{x-4}$  for all values of  $x$ , where  $x \neq 2$  and  $x \neq 4$ . Where is the error in the calculation?

Starting with the rational expressions  $\frac{x}{x-2}$  and  $\frac{x}{x-4}$ , we need to first find equivalent rational expressions with a common denominator. The common denominator we will use is  $(x-4)(x-2)$ . We then have

$$\frac{x}{x-2} = \frac{x(x-4)}{(x-4)(x-2)}$$

$$\frac{x}{x-4} = \frac{x(x-2)}{(x-4)(x-2)}$$

Since  $x^2 - 4x < x^2 - 2x$  for  $x > 0$ , we can divide each expression by  $(x-4)(x-2)$ . We then have

$$\frac{x(x-4)}{(x-4)(x-2)} < \frac{x(x-2)}{(x-4)(x-2)}, \text{ and we can conclude that } \frac{x}{x-2} < \frac{x}{x-4} \text{ for all positive values of } x.$$

*The error in logic in this calculation is that the denominator  $(x-4)(x-2)$  is not always a positive number for all positive values of  $x$ . In fact, if  $2 < x < 4$  then  $(x-4)(x-2) < 0$ . Thus, even though  $x^2 - 4x < x^2 - 2x$  when  $x > 0$ , the inequality  $\frac{x^2-4x}{(x-4)(x-2)} < \frac{x^2-2x}{(x-4)(x-2)}$  is not valid for every positive value of  $x$ .*

6. Consider the populations of two cities within the same state where the large city's population is  $P$  and the small city's population is  $Q$ . For each of the following pairs, state which of the expressions has a larger value. Explain your reasoning in the context of the populations.

- a.  $P + Q$  and  $P$

*The value of  $P + Q$  is larger than  $P$ . The expression  $P + Q$  represents the total population of the two cities and  $P$  represents the population of the larger city. Since these quantities are populations of cities, we can assume they are greater than zero.*

- b.  $\frac{P}{P+Q}$  and  $\frac{Q}{P+Q}$

*The value of  $\frac{P}{P+Q}$  is larger. As stated in part (a),  $P + Q$  represents the total population of the two cities.*

*Hence,  $\frac{P}{P+Q}$  and  $\frac{Q}{P+Q}$  represent each city's respective fraction of the total population. Since  $P > Q$ ,*

$$\frac{P}{P+Q} > \frac{Q}{P+Q}$$

- c.  $2Q$  and  $P + Q$

*The value of  $P + Q$  is larger than the value of  $2Q$ . The population of the smaller of the two cities is represented by  $Q$ , so  $2Q$  represents a population twice the size of the smaller city, but  $2Q = Q + Q < P + Q$ .*

- d.  $\frac{P}{Q}$  and  $\frac{Q}{P}$

*The value of  $\frac{P}{Q}$  is larger. These expressions represent the ratio between the populations of the cities. For*

*instance, the larger city is  $\frac{P}{Q}$  times larger than the smaller city. Since  $P > Q$ ,  $\frac{P}{Q} > 1 > \frac{Q}{P}$ , we can say that*

*there are  $\frac{P}{Q}$  people in the larger city for every one person in the smaller city.*

e.  $\frac{P}{P+Q}$  and  $\frac{1}{2}$

The value of  $\frac{P}{P+Q}$  is larger. Since  $P$  is the population of the larger city, the first city represents more than half of the total.

f.  $\frac{P+Q}{P}$  and  $P - Q$

The value of  $P - Q$  is larger. The expression  $P - Q$  represents the difference in population between the two cities. The expression  $\frac{P+Q}{P}$  can represent the ratio of how much larger the total is compared to the population of the larger city, but we know that  $P$  represents more than half of the total; therefore,  $\frac{P+Q}{P}$  cannot be larger than 2. Without the context, we could not say that  $P - Q$  is larger than 2, but in the context of the problem, since  $P$  is the population of a large city and  $Q$  is the population of a small city,  $P - Q > 2$ . Thus,  $P - Q > \frac{P+Q}{P}$ .

g.  $\frac{P+Q}{2}$  and  $\frac{P+Q}{Q}$

The value of  $\frac{P+Q}{2}$  is larger. The sum divided by the number of cities represents the average population of the two cities and will be significantly higher than the ratio represented by  $\frac{P+Q}{Q}$ . Alternatively,  $Q$  is much larger than 2, so  $\frac{P+Q}{2} < \frac{P+Q}{Q}$ .

h.  $\frac{1}{P}$  and  $\frac{1}{Q}$

The value of  $\frac{1}{Q}$  is larger. The expression  $\frac{1}{Q}$  represents the proportion of the population of the second city a single citizen represents. Similarly for  $\frac{1}{P}$ , since the second city has a smaller population, each individual represents a larger proportion of the whole than in the first city.

MP.2



## Lesson 24: Multiplying and Dividing Rational Expressions

### Student Outcomes

- Students multiply and divide rational expressions and simplify using equivalent expressions.

### Lesson Notes

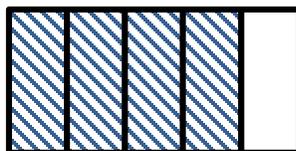
This lesson quickly reviews the process of multiplying and dividing rational numbers using techniques students already know and translates that process to multiplying and dividing rational expressions (MP.7). This enables students to develop techniques to solve rational equations in Lesson 26 (A-APR.D.6). This lesson also begins developing facility with simplifying complex rational expressions, which is important for later work in trigonometry. Teachers may consider treating the multiplication and division portions of this lesson as two separate lessons.

### Classwork

#### Opening Exercise (5 minutes)

Distribute notecard-sized slips of paper to students, and ask them to shade the paper to represent the result of  $\frac{2}{3} \cdot \frac{4}{5}$ . Circulate around the classroom to assess student proficiency. If many students are still struggling to remember the area model after the scaffolding, present the problem to them as shown. Otherwise, allow them time to do the multiplication on their own or with their neighbor and then progress to the question of the general rule.

- First, we represent  $\frac{4}{5}$  by dividing our region into five vertical strips of equal area and shading 4 of the 5 parts.



- Now we need to find  $\frac{2}{3}$  of the shaded area. So we divide the area horizontally into three parts of equal area, and then shade two of those parts.



- Thus,  $\frac{2}{3} \cdot \frac{4}{5}$  is represented by the region that is shaded twice. Since 8 out of 15 subrectangles are shaded twice, we have  $\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$ . With this in mind, can we create a general rule about multiplying rational numbers?

#### Scaffolding:

If students do not remember the area model for multiplication of fractions, have them discuss it with their neighbor. If necessary, use an example like  $\frac{1}{2} \cdot \frac{1}{2}$  to see if they can scale this to the problem presented.

If students are comfortable with multiplying rational numbers, omit the area model and ask them to determine the following products.

- $\frac{2}{3} \cdot \frac{3}{8} = \frac{1}{4}$
- $\frac{1}{4} \cdot \frac{5}{6} = \frac{5}{24}$
- $\frac{4}{7} \cdot \frac{8}{9} = \frac{32}{63}$

Allow students to come up with this “rule” based on the example and prior experience. Have them discuss their thoughts with their neighbor and write the rule.

If  $a$ ,  $b$ ,  $c$ , and  $d$  are integers with  $c \neq 0$  and  $d \neq 0$ , then

$$\frac{a}{c} \cdot \frac{b}{d} = \frac{ab}{cd}$$

The rule summarized above is also valid for real numbers.

**Discussion (2 minutes)**

- To multiply rational expressions, we follow the same procedure we use when multiplying rational numbers: we multiply together the numerators and multiply together the denominators. We finish by reducing the product to lowest terms.

If  $a$ ,  $b$ ,  $c$ , and  $d$  are rational expressions with  $b \neq 0$ ,  $d \neq 0$ , then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Lead students through Examples 1 and 2, and ask for their input at each step.

**Example 1 (4 minutes)**

Give students time to work on this problem and discuss their answers with a neighbor before proceeding to a whole class discussion.

**Example 1**  
 Make a conjecture about the product  $\frac{x^3}{4y} \cdot \frac{y^2}{x}$ . What will it be? Explain your conjecture and give evidence that it is correct.

- We begin by multiplying the numerators and denominators.

$$\frac{x^3}{4y} \cdot \frac{y^2}{x} = \frac{x^3y^2}{4yx}$$

- Identify the greatest common factor (GCF) of the numerator and denominator. The GCF of  $x^3y^2$  and  $4xy$  is  $xy$ .

$$\frac{x^3}{4y} \cdot \frac{y^2}{x} = \frac{(xy)x^2y}{4(xy)}$$

*Scaffolding:*

- To assist students in making the connection between rational numbers and rational expressions, show a side-by-side comparison of a numerical example from a previous lesson like the one shown.

$\frac{2}{3} \cdot \frac{4}{5} = \frac{8}{15}$	$\frac{x^3}{4y} \cdot \frac{y^2}{x} = ?$
--	--

- If students are struggling with this example, include some others, such as

- $\frac{x^2}{3} \cdot \frac{6}{x}$
- $\frac{y}{x^2} \cdot \frac{y^4}{x}$

- Finally, we divide the common factor  $xy$  from the numerator and denominator to find the reduced form of the product:

$$\frac{x^3}{4y} \cdot \frac{y^2}{x} = \frac{x^2y}{4}$$

Note that we are intentionally avoiding using the phrases “cancel  $xy$ ” or “cancel the common factor” in this lesson. We want to highlight that it is division that allows us to simplify these expressions. Ambiguous words like “cancel” can lead students to simplify  $\frac{\sin x}{x}$  to  $\sin$ —they “canceled” the  $x$ !

It is important to understand why we are allowed to divide the numerator *and* denominator by  $x$ . The rule  $\frac{na}{nb} = \frac{a}{b}$  works for rational expressions as well. Performing a simplification such as  $\frac{x}{x^3y} = \frac{1}{x^2y}$  requires doing the following steps:  $\frac{x}{x^3y} = \frac{x \cdot 1}{x \cdot x^2y} = \frac{x}{x} \cdot \frac{1}{x^2y} = 1 \cdot \frac{1}{x^2y} = \frac{1}{x^2y}$ .

### Example 2 (3 minutes)

Before walking students through the steps of this example, ask them to try to find the product using the ideas of the previous example.

#### Example 2

Find the following product:  $\left(\frac{3x-6}{2x+6}\right) \cdot \left(\frac{5x+15}{4x+8}\right)$ .

First, we can factor the numerator and denominator of each rational expression.

- Identify any common factors in the numerator and denominator.

$$\begin{aligned} \left(\frac{3x-6}{2x+6}\right) \cdot \left(\frac{5x+15}{4x+8}\right) &= \left(\frac{3(x-2)}{2(x+3)}\right) \cdot \left(\frac{5(x+3)}{4(x+2)}\right) \\ &= \frac{15(x-2)(x+3)}{8(x+3)(x+2)} \end{aligned}$$

The GCF of the numerator and denominator is  $x + 3$ .

Then, we can divide the common factor  $(x + 3)$  from the numerator and denominator and obtain the reduced form of the product.

$$\left(\frac{3x-6}{2x+6}\right) \cdot \left(\frac{5x+15}{4x+8}\right) = \frac{15(x-2)}{8(x+2)}$$

### Exercises 1–3 (5 minutes)

Students can work in pairs on the following three exercises. Circulate around the class to informally assess their understanding. For Exercise 1, listen for key points such as “factoring the numerator and denominator can help” and “multiplying rational expressions is similar to multiplying rational numbers.”

MP.7

Exercises 1–3

1. Summarize what you have learned so far with your neighbor.

*Answers will vary.*

2. Find the following product and reduce to lowest terms:  $\left(\frac{2x+6}{x^2+x-6}\right) \cdot \left(\frac{x^2-4}{2x}\right)$ .

$$\begin{aligned} \left(\frac{2x+6}{x^2+x-6}\right) \cdot \left(\frac{x^2-4}{2x}\right) &= \left(\frac{2(x+3)}{(x+3)(x-2)}\right) \cdot \left(\frac{(x-2)(x+2)}{2x}\right) \\ &= \frac{2(x+3)(x-2)(x+2)}{2x(x+3)(x-2)} \end{aligned}$$

*The factors 2, x + 3, and x – 2 can be divided from the numerator and the denominator in order to reduce the rational expression to lowest terms.*

$$\left(\frac{2x+6}{x^2+x-6}\right) \cdot \left(\frac{x^2-4}{2x}\right) = \frac{x+2}{x}$$

3. Find the following product and reduce to lowest terms:  $\left(\frac{4n-12}{3m+6}\right)^{-2} \cdot \left(\frac{n^2-2n-3}{m^2+4m+4}\right)$ .

$$\begin{aligned} \left(\frac{4n-12}{3m+6}\right)^{-2} \cdot \left(\frac{n^2-2n-3}{m^2+4m+4}\right) &= \left(\frac{3m+6}{4n-12}\right)^2 \cdot \left(\frac{n^2-2n-3}{m^2+4m+4}\right) \\ &= \frac{3^2(m+2)^2(n-3)(n+1)}{4^2(n-3)^2(m+2)^2} \\ &= \frac{9(n+1)}{16(n-3)} \end{aligned}$$

*Scaffolding:*

Students may need to be reminded how to interpret a negative exponent. If so, ask them to calculate these values.

- $3^{-2} = \frac{1}{3^2} = \frac{1}{9}$
- $\left(\frac{2}{5}\right)^{-3} = \left(\frac{5}{2}\right)^3 = \frac{5^3}{2^3} = \frac{125}{8}$
- $\left(\frac{x}{y^2}\right)^{-5} = \left(\frac{y^2}{x}\right)^5 = \frac{(y^2)^5}{x^5} = \frac{y^{10}}{x^5}$

Discussion (5 minutes)

Recall that division of numbers is equivalent to multiplication of the numerator by the reciprocal of the denominator. That is, for any two numbers  $a$  and  $b$ , where  $b \neq 0$ , we have

$$\frac{a}{b} = a \cdot \frac{1}{b}$$

where the number  $\frac{1}{b}$  is the multiplicative inverse of  $b$ . But, what if  $b$  is itself a fraction? How do we evaluate a quotient such as  $\frac{3}{5} \div \frac{4}{7}$ ?

- How do we evaluate  $\frac{3}{5} \div \frac{4}{7}$ ?

Have students work in pairs to answer this and then discuss.

By our rule above,  $\frac{3}{5} \div \frac{4}{7} = \frac{3}{5} \cdot \frac{1}{4/7}$ . But, what is the value of  $\frac{1}{4/7}$ ? Let  $x$  represent  $\frac{1}{4/7}$ , which is the multiplicative inverse of  $\frac{4}{7}$ . Then we have

$$\begin{aligned} x \cdot \frac{4}{7} &= 1 \\ 4x &= 7 \\ x &= \frac{7}{4} \end{aligned}$$

*Scaffolding:*

Students may be better able to generalize the procedure for dividing rational numbers by repeatedly dividing several examples, such as  $\frac{1}{2} \div \frac{3}{4}$ ,  $\frac{2}{3} \div \frac{7}{10}$ , and  $\frac{1}{5} \div \frac{2}{9}$ . After dividing several of these examples, ask students to generalize the process (MP.8).

Since we have shown that  $\frac{1}{4/7} = \frac{7}{4}$ , we can continue our calculation of  $\frac{3}{5} \div \frac{4}{7}$  as follows:

$$\begin{aligned} \frac{3}{5} \div \frac{4}{7} &= \frac{3}{5} \cdot \frac{1}{4/7} \\ &= \frac{3}{5} \cdot \frac{7}{4} \\ &= \frac{21}{20} \end{aligned}$$

This same process applies to dividing rational expressions, although we might need to perform the additional step of reducing the resulting rational expression to lowest terms. Ask students to generate the rule for division of rational numbers.

If  $a, b, c,$  and  $d$  are integers with  $b \neq 0, c \neq 0,$  and  $d \neq 0,$  then

$$\frac{a}{c} \div \frac{b}{d} = \frac{a}{c} \cdot \frac{d}{b}.$$

The result summarized in the box above is also valid for real numbers.

Now that we know how to divide rational numbers, how do we extend this to divide rational expressions?

- Dividing rational expressions follows the same procedure as dividing rational numbers: we multiply the first term by the reciprocal of the second. We finish by reducing the product to lowest terms.

If  $a, b, c,$  and  $d$  are rational expressions with  $b \neq 0, c \neq 0,$  and  $d \neq 0,$  then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

**Example 3 (3 minutes)**

As in Example 2, ask students to apply their knowledge of rational number division to rational expressions by working on their own or with a partner. Circulate to assist and assess understanding. Once students have made attempts to divide, use the scaffolded questions to develop the concept as necessary.

**Example 3**

Find the quotient and reduce to lowest terms:  $\frac{x^2-4}{3x} \div \frac{x-2}{2x}$ .

*Scaffolding:*  
A side-by-side comparison may help as before.

$\frac{3}{5} \div \frac{4}{7} = \frac{21}{20}$	$\frac{x^3}{4y} \div \frac{y^2}{x} = ?$
--	---

For struggling students, give

- $\frac{x^2y}{4} \div \frac{xy^2}{8} = \frac{2x}{y}$
- $\frac{3y^2}{z-1} \div \frac{12y^5}{(z-1)^2} = \frac{(z-1)}{4y^3}$ .

For advanced students, give

- $\frac{x-3}{x^2+x-2} \div \frac{x^2-x-6}{x-1} = \frac{1}{(x+2)^2}$
- $\frac{x^2-2x-24}{x^2-4} \div \frac{x^2+3x-4}{x^2+x-2} = \frac{x-6}{x-2}$ .

- First, we change the division of  $\frac{x^2-4}{3x}$  by  $\frac{x-2}{2x}$  into multiplication of  $\frac{x^2-4}{3x}$  by the multiplicative inverse of  $\frac{x-2}{2x}$ .

$$\frac{x^2-4}{3x} \div \frac{x-2}{2x} = \frac{x^2-4}{3x} \cdot \frac{2x}{x-2}$$

- Then, we perform multiplication as in the previous examples and exercises. That is, we factor the numerator and denominator and divide any common factors present in both the numerator and denominator.

$$\begin{aligned} \frac{x^2-4}{3x} \div \frac{x-2}{2x} &= \frac{(x-2)(x+2)}{3x} \cdot \frac{2x}{x-2} \\ &= \frac{2(x+2)}{3} \end{aligned}$$

**Exercise 4 (3 minutes)**

Allow students to work in pairs or small groups to evaluate the following quotient.

**Exercises 4–5**

4. Find the quotient and reduce to lowest terms:  $\frac{x^2-5x+6}{x+4} \div \frac{x^2-9}{x^2+5x+4}$

$$\begin{aligned} \frac{x^2-5x+6}{x+4} \div \frac{x^2-9}{x^2+5x+4} &= \frac{x^2-5x+6}{x+4} \cdot \frac{x^2+5x+4}{x^2-9} \\ &= \frac{(x-3)(x-2)}{x+4} \cdot \frac{(x+4)(x+1)}{(x-3)(x+3)} \\ &= \frac{(x-2)(x+1)}{(x+3)} \end{aligned}$$

**Discussion (4 minutes)**

What do we do when the numerator and denominator of a fraction are themselves fractions? We call a fraction that contains fractions a *complex fraction*. Remind students that the fraction bar represents division, so a complex fraction represents division between rational expressions.

Allow students the opportunity to simplify the following complex fraction.

$$\frac{12/49}{27/28}$$

Allow students to struggle with the problem before discussing solution methods.

$$\begin{aligned} \frac{12/49}{27/28} &= \frac{12}{49} \div \frac{27}{28} \\ &= \frac{12}{49} \cdot \frac{28}{27} \\ &= \frac{3 \cdot 4}{7 \cdot 7} \cdot \frac{4 \cdot 7}{3^3} \\ &= \frac{4^2}{7 \cdot 3^2} \\ &= \frac{16}{63} \end{aligned}$$

Notice that in simplifying the complex fraction above, we are merely performing division of rational numbers, and we already know how to do that. Since we already know how to divide rational expressions, we can also simplify rational expressions whose numerators and denominators are rational expressions.

### Exercise 5 (4 minutes)

Allow students to work in pairs or small groups to simplify the following rational expression.

5. Simplify the rational expression.

$$\frac{\frac{x+2}{x^2-2x-3}}{\frac{x^2-x-6}{x^2+6x+5}}$$

$$\frac{x+2}{x^2-2x-3} \div \frac{x^2-x-6}{x^2+6x+5}$$

$$= \frac{x+2}{x^2-2x-3} \cdot \frac{x^2+6x+5}{x^2-x-6}$$

$$= \frac{x+2}{(x-3)(x+1)} \cdot \frac{(x+5)(x+1)}{(x-3)(x+2)}$$

$$= \frac{x+5}{(x-3)^2}$$

*Scaffolding:*

For struggling students, give a simpler example, such as

$$\frac{\left(\frac{2x}{3y}\right)}{\left(\frac{6x}{4y^2}\right)} = \frac{2x}{3y} \div \frac{6x}{4y^2}$$

$$= \frac{2x}{3y} \cdot \frac{4y^2}{6x}$$

$$= \frac{4}{9}y$$

### Closing (3 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. In particular, ask students to articulate the processes for multiplying and dividing rational expressions and simplifying complex rational expressions either verbally or symbolically.

#### Lesson Summary

In this lesson we extended multiplication and division of rational numbers to multiplication and division of rational expressions.

- To multiply two rational expressions, multiply the numerators together and multiply the denominators together, and then reduce to lowest terms.
- To divide one rational expression by another, multiply the first by the multiplicative inverse of the second, and reduce to lowest terms.
- To simplify a complex fraction, apply the process for dividing one rational expression by another.

### Exit Ticket (4 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 24: Multiplying and Dividing Rational Expressions

### Exit Ticket

Perform the indicated operations and reduce to lowest terms.

1.  $\frac{x-2}{x^2+x-2} \cdot \frac{x^2-3x+2}{x+2}$

2.  $\frac{\left(\frac{x-2}{x^2+x-2}\right)}{\left(\frac{x^2-3x+2}{x+2}\right)}$

Exit Ticket Sample Solutions

Perform the indicated operations and reduce to lowest terms.

1.  $\frac{x-2}{x^2+x-2} \cdot \frac{x^2-3x+2}{x+2}$   $\frac{x-2}{x^2+x-2} \cdot \frac{x^2-3x+2}{x+2} = \frac{x-2}{(x-1)(x+2)} \cdot \frac{(x-1)(x-2)}{x+2}$   
 $= \frac{(x-2)^2}{(x+2)^2}$

2.  $\frac{\left(\frac{x-2}{x^2+x-2}\right)}{\left(\frac{x^2-3x+2}{x+2}\right)}$   $\frac{\left(\frac{x-2}{x^2+x-2}\right)}{\left(\frac{x^2-3x+2}{x+2}\right)} = \frac{x-2}{x^2+x-2} \div \frac{x^2-3x+2}{x+2}$   
 $= \frac{x-2}{x^2+x-2} \cdot \frac{x+2}{x^2-3x+2}$   
 $= \frac{x-2}{(x-1)(x+2)} \cdot \frac{x+2}{(x-2)(x-1)}$   
 $= \frac{1}{(x-1)^2}$

Problem Set Sample Solutions

1. Complete the following operations:

a. Multiply  $\frac{1}{3}(x-2)$  by 9. b. Divide  $\frac{1}{4}(x-8)$  by  $\frac{1}{12}$ . c. Multiply  $\frac{1}{4}\left(\frac{1}{3}x+2\right)$  by 12.  
 $3x-6$   $3x-24$   $x+6$

d. Divide  $\frac{1}{3}\left(\frac{2}{5}x-\frac{1}{5}\right)$  by  $\frac{1}{15}$ . e. Multiply  $\frac{2}{3}\left(2x+\frac{2}{3}\right)$  by  $\frac{9}{4}$ . f. Multiply  $0.03(4-x)$  by 100.  
 $2x-1$   $3x+1$   $12-3x$

2. Simplify each of the following expressions.

a.  $\left(\frac{a^3b^2}{c^2d^2} \cdot \frac{c}{ab}\right) \div \frac{a}{c^2d^3}$  b.  $\frac{a^2+6a+9}{a^2-9} \cdot \frac{3a-9}{a+3}$   
 $abcd$   $3$

c.  $\frac{6x}{4x-16} \div \frac{4x}{x^2-16}$  d.  $\frac{3x^2-6x}{3x+1} \cdot \frac{x+3x^2}{x^2-4x+4}$   
 $\frac{3(x+4)}{8}$   $\frac{3x^2}{x-2}$

e.  $\frac{2x^2-10x+12}{x^2-4} \cdot \frac{2+x}{3-x}$  f.  $\frac{a-2b}{a+2b} \div (4b^2-a^2)$   
 $-2$   $-\frac{1}{(a+2b)^2}$

g. 
$$\frac{d+c}{c^2+d^2} \div \frac{c^2-d^2}{d^2-dc}$$

$$-\frac{d}{c^2+d^2}$$

h. 
$$\frac{12a^2-7ab+b^2}{9a^2-b^2} \div \frac{16a^2-b^2}{3ab+b^2}$$

$$\frac{b}{4a+b}$$

i. 
$$\left(\frac{x-3}{x^2-4}\right)^{-1} \cdot \left(\frac{x^2-x-6}{x-2}\right)$$

$$(x+2)^2$$

j. 
$$\left(\frac{x-2}{x^2+1}\right)^{-3} \div \left(\frac{x^2-4x+4}{x^2-2x-3}\right)$$

$$\frac{(x-3)(x+1)(x^2+1)^3}{(x-2)^5}$$

k. 
$$\frac{6x^2-11x-10}{6x^2-5x-6} \cdot \frac{6-4x}{25-20x+4x^2}$$

$$-\frac{2}{2x-5}, \text{ or } \frac{2}{5-2x}$$

l. 
$$\frac{3x^3-3a^2x}{x^2-2ax+a^2} \cdot \frac{a-x}{a^3x+a^2x^2}$$

$$-\frac{3}{a^2}$$

3. Simplify the following complex rational expressions.

a. 
$$\frac{\left(\frac{4a}{6b^2}\right)}{\left(\frac{20a^3}{12b}\right)} \qquad \frac{2}{5a^2b}$$

b. 
$$\frac{\left(\frac{x-2}{x^2-1}\right)}{\left(\frac{x^2-4}{x-6}\right)} \qquad \frac{x-6}{(x+2)(x^2-1)}$$

c. 
$$\frac{\left(\frac{x^2+2x-3}{x^2+3x-4}\right)}{\left(\frac{x^2+x-6}{x+4}\right)} \qquad \frac{1}{x-2}$$

4. Suppose that  $x = \frac{t^2+3t-4}{3t^2-3}$  and  $y = \frac{t^2+2t-8}{2t^2-2t-4}$ , for  $t \neq 1$ ,  $t \neq -1$ ,  $t \neq 2$ , and  $t \neq -4$ . Show that the value of  $x^2y^{-2}$  does not depend on the value of  $t$ .

$$\begin{aligned} x^2y^{-2} &= \left(\frac{t^2+3t-4}{3t^2-3}\right)^2 \left(\frac{t^2+2t-8}{2t^2-2t-4}\right)^{-2} \\ &= \left(\frac{t^2+3t-4}{3t^2-3}\right)^2 \div \left(\frac{t^2+2t-8}{2t^2-2t-4}\right)^2 \\ &= \left(\frac{t^2+3t-4}{3t^2-3}\right)^2 \left(\frac{2t^2-2t-4}{t^2+2t-8}\right)^2 \\ &= \left(\frac{(t-1)(t+4)}{3(t-1)(t+1)}\right)^2 \left(\frac{2(t-2)(t+1)}{(t-2)(t+4)}\right)^2 \\ &= \frac{4(t-1)^2(t+4)^2(t-2)^2(t+1)^2}{9(t-1)^2(t+1)^2(t-2)^2(t+4)^2} \\ &= \frac{4}{9} \end{aligned}$$

Since  $x^2y^{-2} = \frac{4}{9}$ , the value of  $x^2y^{-2}$  does not depend on  $t$ .

5. Determine which of the following numbers is larger without using a calculator,  $\frac{15^{16}}{16^{15}}$  or  $\frac{20^{24}}{24^{20}}$ . (Hint: We can compare two positive quantities  $a$  and  $b$  by computing the quotient  $\frac{a}{b}$ . If  $\frac{a}{b} > 1$ , then  $a > b$ . Likewise, if  $0 < \frac{a}{b} < 1$ , then  $a < b$ .)

$$\begin{aligned} \frac{15^{16}}{16^{15}} \div \frac{20^{24}}{24^{20}} &= \frac{15^{16}}{16^{15}} \cdot \frac{24^{20}}{20^{24}} \\ &= \frac{3^{16} \cdot 5^{16}}{(2^4)^{15}} \cdot \frac{(2^3)^{20} \cdot 3^{20}}{(2^2)^{24} \cdot 5^{24}} \\ &= \frac{2^{60} \cdot 3^{36} \cdot 5^{16}}{2^{108} \cdot 5^{24}} \\ &= \frac{3^{36}}{2^{48} \cdot 5^8} \\ &= \frac{9^{18}}{2^{40} \cdot 10^8} \\ &= \left(\frac{9}{10}\right)^8 \cdot \left(\frac{9^{10}}{2^{40}}\right) \\ &= \left(\frac{9}{10}\right)^8 \cdot \left(\frac{9}{16}\right)^{10} \end{aligned}$$

Since  $\frac{9}{16} < 1$ , and  $\frac{9}{10} < 1$ , we know that  $\left(\frac{9}{10}\right)^8 \cdot \left(\frac{9}{16}\right)^{10} < 1$ . Thus,  $\frac{15^{16}}{16^{15}} \div \frac{20^{24}}{24^{20}} < 1$ , and we know that  $\frac{15^{16}}{16^{15}} < \frac{20^{24}}{24^{20}}$ .

6. (Optional) One of two numbers can be represented by the rational expression  $\frac{x-2}{x}$ , where  $x \neq 0$  and  $x \neq 2$ .

- a. Find a representation of the second number if the product of the two numbers is 1.

Let the second number be  $y$ . Then  $\left(\frac{x-2}{x}\right) \cdot y = 1$ , so we have

$$\begin{aligned} y &= 1 \div \left(\frac{x-2}{x}\right) \\ &= 1 \cdot \left(\frac{x}{x-2}\right) \\ &= \frac{x}{x-2}. \end{aligned}$$

- b. Find a representation of the second number if the product of the two numbers is 0.

Let the second number be  $z$ . Then  $\left(\frac{x-2}{x}\right) \cdot z = 0$ , so we have

$$\begin{aligned} z &= 0 \div \left(\frac{x-2}{x}\right) \\ &= 0 \cdot \left(\frac{x}{x-2}\right) \\ &= 0. \end{aligned}$$



## Lesson 25: Adding and Subtracting Rational Expressions

### Student Outcomes

- Students perform addition and subtraction of rational expressions.

### Lesson Notes

In this lesson, we review addition and subtraction of fractions using the familiar number line technique that students have seen in earlier grades. This leads to an algebraic explanation of how to add and subtract fractions and an opportunity to practice MP.7. We then move to the process for adding and subtracting rational expressions by converting to equivalent rational expressions with a common denominator. As in the past three lessons, we continue to draw parallels between arithmetic of rational numbers and arithmetic of rational expressions.

### Classwork

The four basic arithmetic operations are addition, subtraction, multiplication, and division. The previous lesson showed how to multiply and divide rational expressions. This lesson tackles the remaining operations of addition and subtraction of rational expressions, which are skills needed to address **A-APR.C.6**. As discussed in the previous lesson, we operate with rational expressions in the same way we work with rational numbers expressed as fractions. First, we will review the theory behind addition and subtraction of rational numbers.

### Exercise 1 (8 minutes)

First, remind students how to add fractions with the same denominator. Allow them to work through the following sum individually. The solution should be presented to the class either by the teacher or by a student because we are going to extend the process of adding fractions to the new process of adding rational expressions.

#### Scaffolding:

If students need practice adding and subtracting fractions with a common denominator, have them compute the following.

- $\frac{2}{5} + \frac{1}{5}$
- $\frac{5}{7} - \frac{3}{7}$
- $\frac{17}{24} - \frac{12}{24}$

#### Exercises 1–4

1. Calculate the following sum:  $\frac{3}{10} + \frac{6}{10}$ .

*One approach to this calculation is to factor out  $\frac{1}{10}$  from each term.*

$$\begin{aligned}\frac{3}{10} + \frac{6}{10} &= 3 \cdot \frac{1}{10} + 6 \cdot \frac{1}{10} \\ &= (3 + 6) \cdot \frac{1}{10} \\ &= \frac{9}{10}\end{aligned}$$

Ask students for help in stating the rule for adding and subtracting rational numbers with the same denominator.

If  $a$ ,  $b$ , and  $c$  are integers with  $b \neq 0$ , then

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$

The result in the box above is also valid for real numbers  $a$ ,  $b$ , and  $c$ .

- But what if the fractions have different denominators? Let's examine a technique to add the fractions  $\frac{2}{5}$  and  $\frac{1}{3}$ .
- Recall that when we first learned to add fractions, we represented them on a number line. Let's first look at  $\frac{2}{5}$ .



- And we want to add to this the fraction  $\frac{1}{3}$ .



- If we try placing these two segments next to each other, the exact location of the endpoint is difficult to identify.



- The units on the two original graphs do not match. We need to identify a common unit in order to identify the endpoint of the combined segments. We need to identify a number into which both denominators divide without remainder and write each fraction as an equivalent fraction with that number as denominator; such a number is known as a *common denominator*.
- Since 15 is a common denominator of  $\frac{2}{5}$  and  $\frac{1}{3}$ , we divide the interval  $[0, 1]$  into 15 parts of equal length. Now when we look at the segments of length  $\frac{2}{5}$  and  $\frac{1}{3}$  placed next to each other on the number line, we can see that the combined segment has length  $\frac{11}{15}$ .



- How can we do this without using the number line every time? The fraction  $\frac{2}{5}$  is equivalent to  $\frac{6}{15}$ , and the fraction  $\frac{1}{3}$  is equivalent to  $\frac{5}{15}$ . We then have

$$\begin{aligned} \frac{2}{5} + \frac{1}{3} &= \frac{6}{15} + \frac{5}{15} \\ &= \frac{11}{15} \end{aligned}$$

- Thus, when adding rational numbers, we have to find a common multiple for the two denominators and write each rational number as an equivalent rational number with the new common denominator. Then we can add the numerators together.

Have students discuss how to rewrite the original fraction as an equivalent fraction with the chosen common denominator. Discuss how the identity property of multiplication allows you to multiply the top and the bottom by the same number so that product of the original denominator and the number gives the chosen common denominator.

- Generalizing, let's add together two rational numbers  $\frac{a}{b}$  and  $\frac{c}{d}$ . The first step is to rewrite both fractions as equivalent fractions with the same denominator. A simple common denominator that could be used is the product of the original two denominators:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd}$$

- Once we have a common denominator, we can add the two expressions together, using our previous rule for adding two expressions with the same denominator:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

- We could use the same approach to develop a process for subtracting rational numbers:

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

- Now that we know to find a common denominator before adding or subtracting, we can state the general rule for adding and subtracting rational numbers. Notice that one common denominator that will always work is the product of the two original denominators.

If  $a$ ,  $b$ ,  $c$ , and  $d$  are integers with  $b \neq 0$  and  $d \neq 0$ , then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

As with our other rules developed in this and the previous lesson, the rule summarized in the box above is also valid for real numbers.

### Exercises 2–4 (5 minutes)

Ask the students to work in groups to write what they have learned in their notebooks or journals. Check in to assess their understanding. Then, have students work in pairs to quickly work through the following review exercises. Allow them to think about how to approach Exercise 4, which involves adding three rational expressions. There are multiple ways to approach this problem. They could generalize the process for two rational expressions, rearrange terms using the commutative property to combine the terms with the same denominator, and then add using the above process, or they could group the addends using the associative property and perform addition twice.

$$2. \quad \frac{3}{20} - \frac{4}{15}$$

$$\frac{3}{20} - \frac{4}{15} = \frac{9}{60} - \frac{16}{60} = -\frac{7}{60}$$

$$3. \frac{\pi}{4} + \frac{\sqrt{2}}{5}$$

$$\frac{\pi}{4} + \frac{\sqrt{2}}{5} = \frac{5\pi}{20} + \frac{4\sqrt{2}}{20} = \frac{5\pi + 4\sqrt{2}}{20}$$

$$4. \frac{a}{m} + \frac{b}{2m} - \frac{c}{m}$$

$$\frac{a}{m} + \frac{b}{2m} - \frac{c}{m} = \frac{2a}{2m} + \frac{b}{2m} - \frac{2c}{2m} = \frac{2a + b - 2c}{2m}$$

**Discussion (2 minutes)**

- Before we can add rational numbers or rational expressions, we need to convert to equivalent rational expressions with the same denominators. Finding such a denominator involves finding a common multiple of the original denominators. For example, 60 is a common multiple of 20 and 15. There are other common multiples, such as 120, 180, and 300, but smaller numbers are easier to work with.
- To add and subtract rational expressions, we follow the same procedure as when adding and subtracting rational numbers. First, we find a denominator that is a common multiple of the other denominators, and then we rewrite each expression as an equivalent rational expression with this new common denominator. We then apply the rule for adding or subtracting with the same denominator.

If  $a$ ,  $b$ , and  $c$  are rational expressions with  $b \neq 0$ , then

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b} \quad \text{and} \quad \frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}.$$

**Example 1 (10 minutes)**

Work through these examples as a class, getting input from the students at each step.

**Example 1**

Perform the indicated operations below and simplify.

$$a. \frac{a+b}{4} + \frac{2a-b}{5}$$

*A common multiple of 4 and 5 is 20, so we can write each expression as an equivalent rational expression with denominator 20. We have  $\frac{a+b}{4} = \frac{5a+5b}{20}$  and  $\frac{2a-b}{5} = \frac{8a-4b}{20}$ , so that*

$$\frac{a+b}{4} + \frac{2a-b}{5} = \frac{5a+5b}{20} + \frac{8a-4b}{20} = \frac{13a+b}{20}.$$

$$b. \frac{4}{3x} - \frac{3}{5x^2}$$

*A common multiple of  $3x$  and  $5x^2$  is  $15x^2$ , so we can write each expression as an equivalent rational expression with denominator  $15x^2$ . We have  $\frac{4}{3x} = \frac{20x}{15x^2}$  and  $\frac{3}{5x^2} = \frac{9}{15x^2}$ , so that*

$$\frac{4}{3x} - \frac{3}{5x^2} = \frac{20x}{15x^2} - \frac{9}{15x^2} = \frac{20x-9}{15x^2}.$$

c.  $\frac{3}{2x^2+2x} + \frac{5}{x^2-3x-4}$

Since  $2x^2 + 2x = 2x(x + 1)$  and  $x^2 - 3x - 4 = (x - 4)(x + 1)$ , a common multiple of  $2x^2 + 2x$  and  $x^2 - 3x - 4$  is  $2x(x + 1)(x - 4)$ . Then we have  $\frac{3}{2x^2+2x} + \frac{5}{x^2-3x-4} = \frac{3(x-4)}{2x(x+1)(x-4)} + \frac{5 \cdot 2x}{2x(x+1)(x-4)} = \frac{13x-12}{2x(x+1)(x-4)}$ .

**Exercises 5–8 (8 minutes)**

Have students work on these exercises in pairs or small groups.

**Exercises 5–8**

Perform the indicated operations for each problem below.

5.  $\frac{5}{x-2} + \frac{3x}{4x-8}$

A common multiple is  $4(x - 2)$ .

$$\frac{5}{x-2} + \frac{3x}{4x-8} = \frac{20}{4(x-2)} + \frac{3x}{4(x-2)} = \frac{3x+20}{4(x-2)}$$

6.  $\frac{7m}{m-3} + \frac{5m}{3-m}$

Notice that  $(3 - m) = -(m - 3)$ .

A common multiple is  $(m - 3)$ .

$$\frac{7m}{m-3} + \frac{5m}{3-m} = \frac{7m}{m-3} + \frac{-5m}{m-3} = \frac{7m}{m-3} - \frac{5m}{m-3} = \frac{2m}{m-3}$$

7.  $\frac{b^2}{b^2-2bc+c^2} - \frac{b}{b-c}$

A common multiple is  $(b - c)(b - c)$ .

$$\frac{b^2}{b^2-2bc+c^2} - \frac{b}{b-c} = \frac{b^2}{(b-c)(b-c)} - \frac{b^2-bc}{(b-c)(b-c)} = \frac{bc}{(b-c)^2}$$

8.  $\frac{x}{x^2-1} - \frac{2x}{x^2+x-2}$

A common multiple is  $(x - 1)(x + 1)(x + 2)$ .

$$\frac{x}{x^2-1} - \frac{2x}{x^2+x-2} = \frac{x}{(x-1)(x+1)} - \frac{2x}{(x-1)(x+2)} = \frac{x(x+2)}{(x-1)(x+1)(x+2)} - \frac{2x(x+1)}{(x-1)(x+1)(x+2)} = \frac{-x^2}{(x-1)(x+1)(x+2)}$$

**Example 2 (5 minutes)**

Complex fractions were introduced in the previous lesson with multiplication and division of rational expressions, but these examples require performing addition and subtraction operations prior to doing the division. Remind students that when rewriting a complex fraction as division of rational expressions, they should add parentheses to the expressions both in the numerator and denominator. Then they should work inside the parentheses first following the standard order of operations.

**Example 2**

Simplify the following expression.

$$\frac{\frac{b^2 + b - 1}{2b - 1} - 1}{4 - \frac{8}{b + 1}}$$

First, we can rewrite the complex fraction as a division problem, remembering to add parentheses.

$$\frac{\frac{b^2 + b - 1}{2b - 1} - 1}{4 - \frac{8}{b + 1}} = \left( \frac{b^2 + b - 1}{2b - 1} - 1 \right) \div \left( 4 - \frac{8}{b + 1} \right)$$

Remember that to divide rational expressions, we multiply by the reciprocal of the quotient. However, we first need to write each rational expression in an equivalent  $\frac{P}{Q}$  form. For this, we need to find common denominators.

$$\begin{aligned} \frac{b^2 + b - 1}{2b - 1} - 1 &= \frac{b^2 + b - 1}{2b - 1} - \frac{2b - 1}{2b - 1} \\ &= \frac{b^2 - b}{2b - 1} \end{aligned}$$

$$\begin{aligned} 4 - \frac{8}{b + 1} &= \frac{4(b + 1)}{b + 1} - \frac{8}{b + 1} \\ &= \frac{4b - 4}{b + 1} \\ &= \frac{4(b - 1)}{b + 1} \end{aligned}$$

Now, we can substitute these equivalent expressions into our calculation above and continue to perform the division as we did in Lesson 24.

$$\begin{aligned} \frac{\frac{b^2 + b - 1}{2b - 1} - 1}{4 - \frac{8}{b + 1}} &= \left( \frac{b^2 + b - 1}{2b - 1} - 1 \right) \div \left( 4 - \frac{8}{b + 1} \right) \\ &= \left( \frac{b^2 - b}{2b - 1} \right) \div \left( \frac{4(b - 1)}{b + 1} \right) \\ &= \left( \frac{b(b - 1)}{2b - 1} \right) \cdot \left( \frac{b + 1}{4(b - 1)} \right) \\ &= \frac{b(b + 1)}{4(2b - 1)} \end{aligned}$$

**Closing (2 minutes)**

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this opportunity to informally assess their understanding of the lesson. In particular, ask students to verbally or symbolically articulate the processes for adding and subtracting rational expressions.

MP.7

**Lesson Summary**

In this lesson, we extended addition and subtraction of rational numbers to addition and subtraction of rational expressions. The process for adding or subtracting rational expressions can be summarized as follows:

- Find a common multiple of the denominators to use as a common denominator.
- Find equivalent rational expressions for each expression using the common denominator.
- Add or subtract the numerators as indicated and simplify if needed.

**Exit Ticket (5 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 25: Adding and Subtracting Rational Expressions

### Exit Ticket

Perform the indicated operation.

1.  $\frac{3}{a+2} + \frac{4}{a-5}$

2.  $\frac{4r}{r+3} - \frac{5}{r}$

Exit Ticket Sample Solutions

Perform the indicated operation.

1.  $\frac{3}{a+2} + \frac{4}{a-5}$

$$\frac{3}{a+2} + \frac{4}{a-5} = \frac{3a-15}{(a+2)(a-5)} + \frac{4a+8}{(a+2)(a-5)}$$

$$= \frac{7a-7}{(a+2)(a-5)}$$

2.  $\frac{4r}{r+3} - \frac{5}{r}$

$$\frac{4r}{r+3} - \frac{5}{r} = \frac{4r^2}{r(r+3)} - \frac{5r+15}{r(r+3)}$$

$$= \frac{4r^2-5r-15}{r(r+3)}$$

Problem Set Sample Solutions

1. Write each sum or difference as a single rational expression.

a.  $\frac{7}{8} - \frac{\sqrt{3}}{5}$

$$\frac{35 - 8\sqrt{3}}{40}$$

b.  $\frac{\sqrt{5}}{10} + \frac{\sqrt{2}}{6} + 2$

$$\frac{3\sqrt{5} + 5\sqrt{2} + 60}{30}$$

c.  $\frac{4}{x} + \frac{3}{2x}$

$$\frac{11}{2x}$$

2. Write as a single rational expression.

a.  $\frac{1}{x} - \frac{1}{x-1}$

$$-\frac{1}{x(x-1)}$$

b.  $\frac{3x}{2y} - \frac{5x}{6y} + \frac{x}{3y}$

$$\frac{x}{y}$$

c.  $\frac{a-b}{a^2} + \frac{1}{a}$

$$\frac{2a-b}{a^2}$$

d.  $\frac{1}{p-2} - \frac{1}{p+2}$

$$\frac{4}{(p-2)(p+2)}$$

e.  $\frac{1}{p-2} + \frac{1}{2-p}$

$$0$$

f.  $\frac{1}{b+1} - \frac{b}{1+b}$

$$\frac{1-b}{b+1}$$

g.  $1 - \frac{1}{1+p}$   
 $\frac{p}{1+p}$

h.  $\frac{p+q}{p-q} - 2$   
 $\frac{3q-p}{p-q}$

i.  $\frac{r}{s-r} + \frac{s}{r+s}$   
 $\frac{r^2 + s^2}{(s-r)(r+s)}$

j.  $\frac{3}{x-4} + \frac{2}{4-x}$   
 $\frac{1}{x-4}$

k.  $\frac{3n}{n-2} + \frac{3}{2-n}$   
 $\frac{3n-3}{n-2}$

l.  $\frac{8x}{3y-2x} + \frac{12y}{2x-3y}$   
 $-4$

m.  $\frac{1}{2m-4n} - \frac{1}{2m+4n} - \frac{m}{m^2-4n^2}$   
 $-\frac{1}{m+2n}$

n.  $\frac{1}{(2a-b)(a-c)} + \frac{1}{(b-c)(b-2a)}$   
 $\frac{b-a}{(a-c)(b-c)(2a-b)}$

o.  $\frac{b^2+1}{b^2-4} + \frac{1}{b+2} + \frac{1}{b-2}$   
 $\frac{b^2 + 2b + 1}{(b-2)(b+2)}$

3. Simplify the following expressions.

a.  $\frac{\frac{1}{a} - \frac{1}{2a}}{\frac{4}{a}}$   
 $\frac{1}{8}$

b.  $\frac{\frac{5x}{2} + 1}{\frac{5x}{4} - \frac{1}{5x}}$   
 $\frac{10x}{5x-2}$

c.  $\frac{1 + \frac{4x+3}{x^2+1}}{1 - \frac{x+7}{x^2+1}}$   
 $\frac{x+2}{x-3}$

EXTENSION:

4. Suppose that  $x \neq 0$  and  $y \neq 0$ . We know from our work in this section that  $\frac{1}{x} \cdot \frac{1}{y}$  is equivalent to  $\frac{1}{xy}$ . Is it also true that  $\frac{1}{x} + \frac{1}{y}$  is equivalent to  $\frac{1}{x+y}$ ? Provide evidence to support your answer.

*No, the rational expressions  $\frac{1}{x} + \frac{1}{y}$  and  $\frac{1}{x+y}$  are not equivalent. Consider  $x = 2$  and  $y = 1$ . Then  $\frac{1}{x+y} = \frac{1}{2+1} = \frac{1}{3}$  but  $\frac{1}{x} + \frac{1}{y} = \frac{1}{2} + 1 = \frac{3}{2}$ . Since  $\frac{1}{3} \neq \frac{3}{2}$ , the expressions  $\frac{1}{x} + \frac{1}{y}$  and  $\frac{1}{x+y}$  are not equivalent.*

5. Suppose that  $x = \frac{2t}{1+t^2}$  and  $y = \frac{1-t^2}{1+t^2}$ . Show that the value of  $x^2 + y^2$  does not depend on the value of  $t$ .

$$\begin{aligned} x^2 + y^2 &= \left(\frac{2t}{1+t^2}\right)^2 + \left(\frac{1-t^2}{1+t^2}\right)^2 \\ &= \frac{4t^2}{(1+t^2)^2} + \frac{(1-t^2)^2}{(1+t^2)^2} \\ &= \frac{4t^2 + (1-2t^2+t^4)}{(1+t^2)^2} \\ &= \frac{1+2t^2+t^4}{1+2t^2+t^4} \\ &= 1 \end{aligned}$$

Since  $x^2 + y^2 = 1$ , the value of  $x^2 + y^2$  does not depend on the value of  $t$ .

6. Show that for any real numbers  $a$  and  $b$ , and any integers  $x$  and  $y$  so that  $x \neq 0$ ,  $y \neq 0$ ,  $x \neq y$ , and  $x \neq -y$ ,

$$\left(\frac{y}{x} - \frac{x}{y}\right)\left(\frac{ax + by}{x + y} - \frac{ax - by}{x - y}\right) = 2(a - b).$$

$$\begin{aligned} \left(\frac{y}{x} - \frac{x}{y}\right)\left(\frac{ax + by}{x + y} - \frac{ax - by}{x - y}\right) &= \left(\frac{y^2 - x^2}{xy}\right)\left(\frac{(ax + by)(x - y)}{(x + y)(x - y)} - \frac{(ax - by)(x + y)}{(x - y)(x + y)}\right) \\ &= \left(\frac{y^2 - x^2}{xy}\right)\left(\frac{ax^2 - axy + bxy - by^2}{x^2 - y^2} - \frac{ax^2 + axy - bxy - by^2}{x^2 - y^2}\right) \\ &= -\left(\frac{x^2 - y^2}{xy}\right)\left(\frac{-2axy + 2bxy}{x^2 - y^2}\right) \\ &= -\left(\frac{1}{xy}\right)\left(\frac{-2xy(a - b)}{1}\right) \\ &= 2(a - b) \end{aligned}$$

7. Suppose that  $n$  is a positive integer.

- a. Simplify the expression  $\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n+1}\right)$ .

$$\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n+1}\right) = \left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right) = \left(\frac{n+2}{n}\right)$$

- b. Simplify the expression  $\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n+1}\right)\left(1 + \frac{1}{n+2}\right)$ .

$$\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n+1}\right)\left(1 + \frac{1}{n+2}\right) = \left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right)\left(\frac{n+3}{n+2}\right) = \left(\frac{n+3}{n}\right)$$

- c. Simplify the expression  $\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n+1}\right)\left(1 + \frac{1}{n+2}\right)\left(1 + \frac{1}{n+3}\right)$ .

$$\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n+1}\right)\left(1 + \frac{1}{n+2}\right)\left(1 + \frac{1}{n+3}\right) = \left(\frac{n+1}{n}\right)\left(\frac{n+2}{n+1}\right)\left(\frac{n+3}{n+2}\right)\left(\frac{n+4}{n+3}\right) = \left(\frac{n+4}{n}\right)$$

- d. If this pattern continues, what is the product of  $n$  of these factors?

*If we have  $n$  of these factors, then the product will be*

$$\left(1 + \frac{1}{n}\right)\left(1 + \frac{1}{n+1}\right) \cdots \left(1 + \frac{1}{n+(n-1)}\right) = \frac{n+n}{n} = 2.$$

MP.7



## Lesson 26: Solving Rational Equations

### Student Outcomes

- Students solve rational equations, monitoring for the creation of extraneous solutions.

### Lesson Notes

In the preceding lessons, students learned to add, subtract, multiply, and divide rational expressions so that in this lesson we can solve equations involving rational expressions (**A-REI.A.2**). The skills developed in this lesson will be required to solve equations of the form  $f(x) = c$  for a rational function  $f$  and constant  $c$  in Lesson 27 and later in Module 3 (**F-BF.B.4a**).

There is more than one approach to solving a rational equation, and we explore two such methods in this section. The first method is to multiply both sides by the common denominator to clear fractions. The second method is to find equivalent forms of all expressions with a common denominator, set the numerators equal to each other, and solve the resulting equation. Either approach requires that we keep an eye out for extraneous solutions; in other words, values that appear to be a solution to the equation but cause division by zero and are, thus, not valid. Throughout our work with rational expressions, students will be analyzing the structure of the expressions in order to decide on their next algebraic steps (MP.7). Encourage students to check their answers by substituting their solutions back into each side of the equation separately.

### Classwork

#### Exercises 1–2 (8 minutes)

Allow students to solve this any way that they can and then discuss their answers. Focus on adding the fractions on the left and equating numerators or multiplying both sides by a common multiple. Indicate a practical technique of finding a common denominator. These first two exercises highlight MP.7, as the students must recognize the given expressions to be of the form  $\frac{a}{b} + \frac{c}{b}$  or  $\frac{a}{b} + \frac{c}{d}$ ; by expressing the equations in the simplified form  $\frac{A}{B} = \frac{C}{B}$ , they realize that we must have  $A = C$ .

#### Scaffolding:

Struggling students may benefit from first solving the equation  $\frac{x}{5} - \frac{2}{5} = \frac{1}{5}$ . More advanced students may try to solve

$$\frac{x-1}{x+2} = \frac{3}{4}$$

#### Exercises 1–2

Solve the following equations for  $x$ , and give evidence that your solutions are correct.

1.  $\frac{x}{2} + \frac{1}{3} = \frac{5}{6}$

*Combining the expressions on the left, we have  $\frac{3x}{6} + \frac{2}{6} = \frac{5}{6}$ , so  $\frac{3x+2}{6} = \frac{5}{6}$ ; therefore,  $3x + 2 = 5$ . Then,  $x = 1$ .*

*Or, using another approach:*

$6 \cdot \left(\frac{x}{2} + \frac{1}{3}\right) = 6 \cdot \left(\frac{5}{6}\right)$ , so  $3x + 2 = 5$ ; then,  $x = 1$ .

*The solution to this equation is 1. To verify, we see that when  $x = 1$ , we have  $\frac{x}{2} + \frac{1}{3} = \frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$ , so 1 is a valid solution.*

$$2. \quad \frac{2x}{9} + \frac{5}{9} = \frac{8}{9}$$

Since the expressions already have a common denominator, we see that  $\frac{2x}{9} + \frac{5}{9} = \frac{2x+5}{9}$ , so we need to solve

$$\frac{2x+5}{9} = \frac{8}{9}. \text{ It then follows that the numerators are equal, so } 2x + 5 = 8. \text{ Solving for } x \text{ gives } x = \frac{3}{2}. \text{ To verify, we}$$

see that when  $x = \frac{3}{2}$ , we have  $\frac{2x}{9} + \frac{5}{9} = \frac{2(\frac{3}{2})}{9} + \frac{5}{9} = \frac{3}{9} + \frac{5}{9} = \frac{8}{9}$ ; thus,  $\frac{3}{2}$  is a valid solution.

Remind students that two rational expressions with the same denominator are equal if the numerators are equal.

### Discussion (2 minutes)

Now that we know how to add, subtract, multiply, and divide rational expressions, we are going to use some of those basic operations to solve equations involving rational expressions. An equation involving rational expressions is called a *rational equation*. Keeping the previous exercise in mind, we will look at two different approaches to solving rational equations.

### Example 1 (6 minutes)

MP.1

Ask students to try to solve this challenge problem on their own. Have them discuss and explain their methods in groups or with neighbors. The teacher should circulate and lead a discussion of both methods once students have had a chance to try solving on their own.

#### Example 1

Solve the following equation:  $\frac{x+3}{12} = \frac{5}{6}$ .

**Equating Numerators Method:** Obtain expressions on both sides with the same denominator and equate numerators.

$$\begin{aligned} \frac{x+3}{12} &= \frac{5}{6} \cdot \frac{2}{2} \\ \frac{x+3}{12} &= \frac{10}{12} \end{aligned}$$

Thus,  $x + 3 = 10$ , and  $x = 7$ ; therefore, 7 is the solution to our original equation.

**Clearing Fractions Method:** Multiply both sides by a common multiple of the denominators to clear the fractions, and then solve the resulting equation.

$$\begin{aligned} 12 \cdot \left( \frac{x+3}{12} \right) &= 12 \cdot \left( \frac{5}{6} \right) \\ x + 3 &= 10 \end{aligned}$$

We can see, once again, that the solution is 7.

**Discussion (3 minutes)**

Ask students to discuss both methods used in the previous example. Which method do they prefer, and why? Does one method seem to be more efficient than the other? Have a few groups report their opinions to the class. At no time should students be required to use a particular method; just be sure they understand both approaches, and allow them to use whichever method seems more natural.

**Exercise 3 (6 minutes)**

Remind students that when we say a “solution” to an equation, we are talking about a value of the variable, usually  $x$ , that will result in a true number sentence. In Lesson 22, students learned that there are some values of the variable that are not allowed in order to avoid division by zero. Before students start working on the following exercise, ask them to identify the values of  $x$  that must be excluded. Wait for students to respond that we must have  $x \neq 0$  and  $x \neq 2$  before having them work with a partner on the following exercise.

**Exercises 3–7**

3. Solve the following equation:  $\frac{3}{x} = \frac{8}{x-2}$ .

**Method 1:** Convert both expressions to equivalent expressions with a common denominator. The common denominator is  $x(x-2)$ , so we use the identity property of multiplication to multiply the left side by  $\frac{x-2}{x-2}$  and the right side by  $\frac{x}{x}$ . This does not change the value of the expression on either side of the equation.

$$\begin{aligned} \left(\frac{x-2}{x-2}\right) \cdot \left(\frac{3}{x}\right) &= \left(\frac{x}{x}\right) \cdot \left(\frac{8}{x-2}\right) \\ \frac{3x-6}{x(x-2)} &= \frac{8x}{x(x-2)} \end{aligned}$$

Since the denominators are equal, we can see that the numerators must be equal; thus,  $3x - 6 = 8x$ . Solving for  $x$  gives a solution of  $-\frac{6}{5}$ . At the outset of this example, we noted that  $x$  cannot take on the value of 0 or 2, but there is nothing preventing  $x$  from taking on the value  $-\frac{6}{5}$ . Thus, we have found a solution. We can check our work. Substituting  $-\frac{6}{5}$  into  $\frac{3}{x}$  gives us  $\frac{3}{(-6/5)} = -\frac{5}{2}$ , and substituting  $-\frac{6}{5}$  into  $\frac{8}{x-2}$  gives us  $\frac{8}{(-6/5)-2} = -\frac{5}{2}$ . Thus, when  $x = -\frac{6}{5}$ , we have  $\frac{3}{x} = \frac{8}{x-2}$ ; therefore,  $-\frac{6}{5}$  is indeed a solution.

**Method 2:** Multiply both sides of the equation by the common denominator  $x(x-2)$ , and solve the resulting equation.

$$\begin{aligned} x(x-2) \left(\frac{3}{x}\right) &= x(x-2) \left(\frac{8}{x-2}\right) \\ 3(x-2) &= 8x \\ 3x-6 &= 8x \end{aligned}$$

From this point, we follow the same steps as we did in Method 1, and we get the same solution:  $-\frac{6}{5}$ .

MP.7

**Exercise 4 (6 minutes)**

Have students continue to work with partners to solve the following equation. Walk around the room and observe student progress; if necessary, offer the following hints and reminders:

- **Reminder:** Ask students to identify excluded values of  $a$ . Suggest that they factor the denominator  $a^2 - 4$ . They should discover that we must specify  $a \neq 2$  and  $a \neq -2$ .
- **Hint 1:** Ask students to identify a common denominator of the three expressions in the equation. They should respond with  $(a - 2)(a + 2)$ , or equivalently,  $a^2 - 4$ .
- **Hint 2:** What do we need to do with this common denominator? They should determine that they need to find equivalent rational expressions for each of the terms with denominator  $(a - 2)(a + 2)$ .

4. Solve the following equation for  $a$ :  $\frac{1}{a+2} + \frac{1}{a-2} = \frac{4}{a^2-4}$ .

*First, we notice that we must have  $a \neq 2$  and  $a \neq -2$ . Then, we apply Method 1:*

$$\begin{aligned} \left(\frac{a-2}{a-2}\right) \cdot \left(\frac{1}{a+2}\right) + \left(\frac{a+2}{a+2}\right) \cdot \left(\frac{1}{a-2}\right) &= \frac{4}{(a-2)(a+2)} \\ \frac{a-2}{(a-2)(a+2)} + \frac{a+2}{(a-2)(a+2)} &= \frac{4}{(a-2)(a+2)} \\ \frac{2a}{(a-2)(a+2)} &= \frac{4}{(a-2)(a+2)}. \end{aligned}$$

*Since the denominators are equal, we know that the numerators are equal; thus, we have  $2a = 4$ , which means that  $a = 2$ . Thus, the only solution to this equation is 2. However,  $a$  is not allowed to be 2 because if  $a = 2$ , then  $\frac{1}{a-2}$  is not defined. This means that the original equation,  $\frac{1}{a+2} + \frac{1}{a-2} = \frac{4}{a^2-4}$ , has no solution.*

**Scaffolding:**

Let students know that the word *extraneous* has meaning outside of the mathematics classroom; ask them to guess its definition, and then provide the actual definition.

**Extraneous:** Irrelevant or unrelated to the subject being dealt with.

Students may benefit from choral repetition, as well as a visual representation of this word.

Introduce the term *extraneous solution*. An invalid solution that may arise when we manipulate a rational expression is called an extraneous solution. An extraneous solution is a value that satisfies a transformed equation but does not satisfy the original equation.

**Exercises 5–7 (8 minutes)**

Give students a few minutes to discuss extraneous solutions with a partner. When do they occur, and how do you know when you have one? Extraneous solutions occur when one of the solutions found does not make a true number sentence when substituted into the original equation. The only way to know you have one is to note the values of the variable that will cause a part of the equation to be undefined. In this lesson, we are concerned with division by zero; in later lessons, we exclude values of the variable that would cause the square root of a negative number. Make sure that all students have an understanding of extraneous solutions before proceeding. Then, have them work in pairs on the following exercises.

5. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{4}{3x} + \frac{5}{4} = \frac{3}{x}$$

First, note that we must have  $x \neq 0$ .

Equating numerators:  $\frac{16}{12x} + \frac{15x}{12x} = \frac{36}{12x}$

Then, we have  $16 + 15x = 36$ , and the solution is  $x = \frac{4}{3}$ .

Clearing fractions:  $12x\left(\frac{4}{3x} + \frac{5}{4}\right) = 12x\left(\frac{3}{x}\right)$

Then, we have  $16 + 15x = 36$ , and the solution is  $x = \frac{4}{3}$ .

The solution  $\frac{4}{3}$  is valid since the only excluded value is 0.

6. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b-2}{b^2-9}$$

First, note that we must have  $x \neq 3$  and  $x \neq -3$ .

Equating numerators:  $\frac{7(b-3)}{(b-3)(b+3)} + \frac{5(b+3)}{(b-3)(b+3)} = \frac{10b-2}{(b-3)(b+3)}$

Matching numerators, we have  $7b - 21 + 5b + 15 = 10b - 2$ , which leads to  $2b = 4$ ; therefore,  $b = 2$ .

Clearing fractions:  $(b-3)(b+3)\left(\frac{7}{b+3} + \frac{5}{b-3}\right) = (b-3)(b+3)\left(\frac{10b-2}{b^2-9}\right)$

We have  $7(b-3) + 5(b+3) = 10b - 2$ , which leads to  $2b = 4$ ; therefore,  $b = 2$ .

The solution 2 is valid since the only excluded values are 3 and -3.

7. Solve the following equation. Remember to check for extraneous solutions.

$$\frac{1}{x-6} + \frac{x}{x-2} = \frac{4}{x^2-8x+12}$$

First, note that we must have  $x \neq 6$  and  $x \neq 2$ .

Equating numerators:

$$\begin{aligned} \frac{x-2}{(x-6)(x-2)} + \frac{x^2-6x}{(x-6)(x-2)} &= \frac{4}{(x-6)(x-2)} \\ x^2-5x-2 &= 4 \\ x^2-5x-6 &= 0 \\ (x-6)(x+1) &= 0 \end{aligned}$$

The solutions are 6 and -1.

Clearing fractions:

$$\begin{aligned} \left(\frac{1}{x-6} + \frac{x}{x-2}\right)(x-6)(x-2) &= \left(\frac{4}{(x-6)(x-2)}\right)(x-6)(x-2) \\ (x-2) + x(x-6) &= 4 \\ x^2-6x+x-2 &= 4 \\ x^2-5x-6 &= 0 \\ (x-6)(x+1) &= 0 \end{aligned}$$

The solutions are 6 and -1.

Because  $x$  is not allowed to be 6 in order to avoid division by zero, the solution 6 is extraneous; thus, -1 is the only solution to the given rational equation.

**Closing (2 minutes)**

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. In particular, ask students to explain how we identify extraneous solutions and why they arise when solving rational equations.

**Lesson Summary**

In this lesson, we applied what we have learned in the past two lessons about addition, subtraction, multiplication, and division of rational expressions to solve rational equations. An extraneous solution is a solution to a transformed equation that is not a solution to the original equation. For rational functions, extraneous solutions come from the excluded values of the variable.

Rational equations can be solved one of two ways:

1. Write each side of the equation as an equivalent rational expression with the same denominator and equate the numerators. Solve the resulting polynomial equation, and check for extraneous solutions.
2. Multiply both sides of the equation by an expression that is the common denominator of all terms in the equation. Solve the resulting polynomial equation, and check for extraneous solutions.

**Exit Ticket (4 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 26: Solving Rational Equations

### Exit Ticket

Find all solutions to the following equation. If there are any extraneous solutions, identify them and explain why they are extraneous.

$$\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b}{b^2-9}$$

Exit Ticket Sample Solutions

Find all solutions to the following equation. If there are any extraneous solutions, identify them and explain why they are extraneous.

$$\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b}{b^2-9}$$

First, note that we must have  $x \neq 3$  and  $x \neq -3$ .

Using the equating numerators method: 
$$\frac{7(b-3)}{(b-3)(b+3)} + \frac{5(b+3)}{(b-3)(b+3)} = \frac{10b}{(b-3)(b+3)}$$

Matching numerators, we have  $7b - 21 + 5b + 15 = 10b$ , which leads to  $12b - 6 = 10b$ ; therefore,  $b = 3$ .

However, since the excluded values are 3 and -3, the solution 3 is an extraneous solution, and there is no solution to

$$\frac{7}{b+3} + \frac{5}{b-3} = \frac{10b}{b^2-9}$$

Problem Set Sample Solutions

1. Solve the following equations and check for extraneous solutions.

a.  $\frac{x-8}{x-4} = 2$   
0

b.  $\frac{4x-8}{x-2} = 4$   
All real numbers except 2

c.  $\frac{x-4}{x-3} = 1$   
No solution

d.  $\frac{4x-8}{x-2} = 3$   
No solution

e.  $\frac{1}{2a} - \frac{2}{2a-3} = 0$   
 $-\frac{3}{2}$

f.  $\frac{3}{2x+1} = \frac{5}{4x+3}$   
-2

g.  $\frac{4}{x-5} - \frac{2}{5+x} = \frac{2}{x}$   
 $-\frac{5}{3}$

h.  $\frac{y+2}{3y-2} + \frac{y}{y-1} = \frac{2}{3}$   
 $\frac{5}{6}, -2$

i.  $\frac{3}{x+1} - \frac{2}{1-x} = 1$   
0, 5

j.  $\frac{4}{x-1} + \frac{3}{x} - 3 = 0$   
 $\frac{1}{3}, 3$

k.  $\frac{x+1}{x+3} - \frac{x-5}{x+2} = \frac{17}{6}$   
0,  $-\frac{55}{17}$

l.  $\frac{x+7}{4} - \frac{x+1}{2} = \frac{5-x}{3x-14}$   
5, 6

m.  $\frac{b^2-b-6}{b^2} - \frac{2b+12}{b} = \frac{b-39}{2b}$   
 $3, \frac{4}{3}$

n.  $\frac{1}{p(p-4)} + 1 = \frac{p-6}{p}$   
 $\frac{23}{6}$

o.  $\frac{1}{h+3} = \frac{h+4}{h-2} + \frac{6}{h-2}$   
-8, -4

p.  $\frac{m+5}{m^2+m} = \frac{1}{m^2+m} - \frac{m-6}{m+1}$   
4, 1

2. Create and solve a rational equation that has 0 as an extraneous solution.

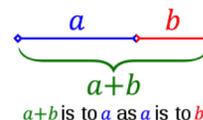
One such equation is  $\frac{1}{x-1} + \frac{1}{x} = \frac{1}{x-x^2}$ .

3. Create and solve a rational equation that has 2 as an extraneous solution.

One such equation is  $\frac{1}{x-2} + \frac{1}{x+2} = \frac{4}{x^2-4}$ .

EXTENSION:

4. Two lengths  $a$  and  $b$ , where  $a > b$ , are in *golden ratio* if the ratio of  $a + b$  is to  $a$  is the same as  $a$  is to  $b$ . Symbolically, this is expressed as  $\frac{a}{b} = \frac{a+b}{a}$ . We denote this common ratio by the Greek letter *phi* (pronounced “fee”) with symbol  $\varphi$ , so that if  $a$  and  $b$  are in common ratio, then  $\varphi = \frac{a}{b} = \frac{a+b}{a}$ . By setting  $b = 1$ , we find that  $\varphi = a$  and  $\varphi$  is the positive number that satisfies the equation  $\varphi = \frac{\varphi+1}{\varphi}$ . Solve this equation to find the numerical value for  $\varphi$ .



We can apply either method from the previous lesson to solve this equation.

$$\begin{aligned} \varphi &= \frac{\varphi + 1}{\varphi} \\ \varphi^2 &= \varphi + 1 \\ \varphi^2 - \varphi - 1 &= 0 \end{aligned}$$

Applying the quadratic formula, we have two solutions:

$$\varphi = \frac{1 + \sqrt{5}}{2} \text{ or } \varphi = \frac{1 - \sqrt{5}}{2}$$

Since  $\varphi$  is a positive number, and  $\frac{1 - \sqrt{5}}{2} < 0$ , we have  $\varphi = \frac{1 + \sqrt{5}}{2}$ .

5. Remember that if we use  $x$  to represent an integer, then the next integer can be represented by  $x + 1$ .
- a. Does there exist a pair of consecutive integers whose reciprocals sum to  $\frac{5}{6}$ ? Explain how you know.

Yes, 2 and 3 because  $\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6}$ .

- b. Does there exist a pair of consecutive integers whose reciprocals sum to  $\frac{3}{4}$ ? Explain how you know.

If  $x$  represents the first integer, then  $x + 1$  represents the next integer. Suppose  $\frac{1}{x} + \frac{1}{x+1} = \frac{3}{4}$ . Then,

$$\begin{aligned} \frac{1}{x} + \frac{1}{x+1} &= \frac{3}{4} \\ \frac{4(x+1) + 4x}{4x(x+1)} &= \frac{3x(x+1)}{4x(x+1)} \\ 8x + 4 &= 3x^2 + 3x \\ 3x^2 - 5x - 4 &= 0. \end{aligned}$$

The solutions to this quadratic equation are  $\frac{5 + \sqrt{73}}{6}$  and  $\frac{5 - \sqrt{73}}{6}$ , so there are no integers that solve this equation. Thus, there are no pairs of consecutive integers whose reciprocals sum to  $\frac{3}{4}$ .

- c. Does there exist a pair of consecutive *even* integers whose reciprocals sum to  $\frac{3}{4}$ ? Explain how you know.

If  $x$  represents the first integer, then  $x + 2$  represents the next even integer. Suppose  $\frac{1}{x} + \frac{1}{x+2} = \frac{3}{4}$ . Then,

$$\begin{aligned}\frac{1}{x} + \frac{1}{x+2} &= \frac{3}{4} \\ \frac{4(x+2) + 4x}{4x(x+2)} &= \frac{3x(x+2)}{4x(x+2)} \\ 8x + 8 &= 3x^2 + 6x \\ 3x^2 - 2x - 8 &= 0.\end{aligned}$$

The solutions to this quadratic equation are  $-\frac{4}{3}$  and 2; therefore, the only even integer  $x$  that solves the equation is 2. Then, 2 and 4 are consecutive even integers whose reciprocals sum to  $\frac{3}{4}$ .

- d. Does there exist a pair of consecutive *even* integers whose reciprocals sum to  $\frac{5}{6}$ ? Explain how you know.

If  $x$  represents the first integer, then  $x + 2$  represents the next even integer. Suppose  $\frac{1}{x} + \frac{1}{x+2} = \frac{5}{6}$ . Then,

$$\begin{aligned}\frac{1}{x} + \frac{1}{x+2} &= \frac{5}{6} \\ \frac{6(x+2) + 6x}{6x(x+2)} &= \frac{5x(x+2)}{6x(x+2)} \\ 12x + 12 &= 5x^2 + 10x \\ 5x^2 - 2x - 12 &= 0.\end{aligned}$$

The solutions to this quadratic equation are  $\frac{1+\sqrt{61}}{5}$  and  $\frac{1-\sqrt{61}}{5}$ , so there are no integers that solve this equation. Thus, there are no pairs of consecutive even integers whose reciprocals sum to  $\frac{5}{6}$ .



## Lesson 27: Word Problems Leading to Rational Equations

### Student Outcomes

- Students solve word problems using models that involve rational expressions.

### Lesson Notes

In the preceding lessons, students learned to add, subtract, multiply, and divide rational expressions and solve rational equations in order to develop the tools needed for solving application problems involving rational equations in this lesson (A-REI.A.2). Students will develop their problem-solving and modeling abilities by carefully reading the problem description and converting information into equations (MP.1), thus creating a mathematical model of the problem (MP.4).

### Classwork

#### Exercise 1 (13 minutes)

We now turn to some applied problems that we can model with rational equations, strengthening students' problem-solving and modeling experience in alignment with standards MP.1 and MP.4. We can solve these equations using the skills developed in previous lessons. Have students work in small groups to answer this set of four questions. At the end of the work time, ask different groups to present their solutions to the class. Suggest to students that they should: (a) read the problem aloud, (b) paraphrase and summarize the problem in their own words, (c) find an equation that models the situation, and (d) say how it represents the quantities involved. Check to make sure that students understand the problem before they begin trying to solve it.

You may want to encourage students, especially with this first problem, by suggesting that they use  $m$  for the unknown quantity and asking if they can arrive at an equation that relates this unknown quantity to the known quantities.

#### Scaffolding:

Students may benefit from having the problem read aloud and summarized. They should be encouraged to restate the problem in their own words to a partner.

If students are struggling, present the equation  $\frac{12+m}{20+m} = 0.75$ , and ask students how this models the situation.

Students who may be above grade level could be challenged to write their own word problems that result in rational equations.

#### Exercise 1

- Anne and Maria play tennis almost every weekend. So far, Anne has won 12 out of 20 matches.

- How many matches will Anne have to win in a row to improve her winning percentage to 75%?

*Suppose that Anne has already won 12 of 20 matches, and let  $m$  represent the number of additional matches she must win to raise her winning percentage to 75%. After playing and winning all of those additional  $m$  matches, she has won  $12 + m$  matches out of a total of  $20 + m$  matches played. Her winning percentage is then  $\frac{12+m}{20+m}$ , and we want to find the value of  $m$  that solves the equation*

$$\frac{12 + m}{20 + m} = 0.75.$$

Multiply both sides by  $20 + m$ .

$$12 + m = 0.75(20 + m)$$

$$12 + m = 15 + 0.75m$$

Solving for  $m$ :

$$0.25m = 3$$

$$m = 12$$

So, Anne would need to win 12 matches in a row in order to improve her winning percentage to 75%.

- b. How many matches will Anne have to win in a row to improve her winning percentage to 90%?

*This situation is similar to that for part (a), except that we want a winning percentage of 0.90, instead of 0.75. Again, we let  $m$  represent the number of matches Anne must win consecutively to bring her winning percentage up to 90%.*

$$\frac{12 + m}{20 + m} = 0.90$$

Solving for  $m$ :

$$12 + m = 0.90(20 + m)$$

$$12 + m = 18 + 0.90m$$

$$0.10m = 6$$

$$m = 60$$

*In order for Anne to bring her winning percentage up to 90%, she would need to win the next 60 consecutive matches.*

- c. Can Anne reach a winning percentage of 100%?

*Allow students to come to the conclusion that Anne will never reach a winning percentage of 100% because she has already lost 8 matches.*

- d. After Anne has reached a winning percentage of 90% by winning consecutive matches as in part (b), how many matches can she now lose in a row to have a winning percentage of 50%?

*Recall from part (b) that she had won 72 matches out of 80 to reach a winning percentage of 90%. We will now assume that she loses the next  $k$  matches in a row. Then, she will have won 72 matches out of  $80 + k$  matches, and we want to know the value of  $k$  that makes this a 50% win rate.*

$$\frac{72}{80 + k} = 0.50$$

Solving the equation:

$$72 = 0.50(80 + k)$$

$$72 = 40 + 0.50k$$

$$32 = 0.50k$$

$$64 = k$$

*Thus, after reaching a 90% winning percentage in 80 matches, Anne can lose 64 matches in a row to drop to a 50% winning percentage.*

MP.1  
&  
MP.4

MP.2

**Example 1 (5 minutes)**

Work this problem at the front of the room, but allow the class to provide input and steer the discussion. Depending on how students did with the first exercise, the teacher may lead a discussion of this problem as a class, ask students to work in groups, or ask students to work independently while targeting instruction with a small group that struggled on the first example.

**Example 1**

Working together, it takes Sam, Jenna, and Francisco two hours to paint one room. When Sam is working alone, he can paint one room in 6 hours. When Jenna works alone, she can paint one room in 4 hours. Determine how long it would take Francisco to paint one room on his own.

*Consider how much can be accomplished in one hour. Sam, Jenna, and Francisco together can paint half a room in one hour. If Sam can paint one room in 6 hours on his own, then in one hour he can paint  $\frac{1}{6}$  of the room. Similarly, Jenna can paint  $\frac{1}{4}$  of the room in one hour. We do not yet know how much Francisco can paint in one hour, so we will say he can paint  $\frac{1}{f}$  of the room. So, in one hour, Sam has painted  $\frac{1}{6}$  of the room, Jenna has painted  $\frac{1}{4}$  of the room, and all three together can paint  $\frac{1}{2}$  the room, leading to the following equation for how much can be painted in one hour:*

$$\frac{1}{6} + \frac{1}{4} + \frac{1}{f} = \frac{1}{2}.$$

*A common multiple of the denominators is  $12f$ . Multiplying both sides by  $12f$  gives us:*

$$\frac{12f}{6} + \frac{12f}{4} + \frac{12f}{f} = \frac{12f}{2}$$

$$2f + 3f + 12 = 6f,$$

*which leads us to the value of  $f$ :*

$$f = 12.$$

*So, Francisco can paint the room in 12 hours on his own.*

MP.4

**Exercise 2 (5 minutes)**

Remind students that distance equals rate times time ( $d = r \cdot t$ ) before having them work on this exercise in pairs or small groups. Be sure to have groups share their results before continuing to the next exercise.

Exercises 2–4

2. Melissa walks 3 miles to the house of a friend and returns home on a bike. She averages 4 miles per hour faster when cycling than when walking, and the total time for both trips is two hours. Find her walking speed.

Using the relationship  $d = r \cdot t$ , we have  $t = \frac{d}{r}$ . The time it takes for Melissa to walk to her friend's house is  $\frac{3}{r}$  and the time to cycle back is  $\frac{3}{r+4}$ . Thus, we can write an equation that describes the combined time for both trips:

$$\frac{3}{r} + \frac{3}{r+4} = 2.$$

A common multiple of the denominators is  $r(r+4)$ , so we multiply both sides of the equation by  $r(r+4)$ .

$$3(r+4) + 3r = 2r(r+4)$$

$$3r + 12 + 3r = 2r^2 + 8r$$

$$2r^2 + 2r - 12 = 0$$

$$2(r-2)(r+3) = 0$$

Thus,  $r = -3$  or  $r = 2$ . Since  $r$  represents Melissa's speed, it does not make sense for  $r$  to be negative. So, the only solution is 2, which means that Melissa's walking speed is 2 miles per hour.

MP.1  
&  
MP.4

Exercise 3 (10 minutes)

3. You have 10 liters of a juice blend that is 60% juice.

- a. How many liters of pure juice need to be added in order to make a blend that is 75% juice?

We start off with 10 liters of a blend containing 60% juice. Then, this blend contains  $0.60(10) = 6$  liters of juice in the 10 liters of mixture. If we add  $A$  liters of pure juice, then the concentration of juice in the blend is  $\frac{6+A}{10+A}$ . We want to know which value of  $A$  makes this blend 75% juice.

$$\frac{6+A}{10+A} = 0.75$$

$$6+A = 0.75(10+A)$$

$$6+A = 7.5 + 0.75A$$

$$0.25A = 1.5$$

$$A = 6$$

Thus, if we add 6 liters of pure juice, we have 16 liters of a blend that contains 12 liters of juice, meaning that the concentration of juice in this blend is 75%.

- b. How many liters of pure juice need to be added in order to make a blend that is 90% juice?

$$\frac{6+A}{10+A} = 0.90$$

$$6+A = 0.9(10+A)$$

$$6+A = 9 + 0.9A$$

$$3 = 0.1A$$

$$A = 30$$

Thus, if we add 30 liters of pure juice, we will have 40 liters of a blend that contains 36 liters of pure juice, meaning that the concentration of juice in this blend is 90%.

MP.4

- c. Write a rational equation that relates the desired percentage  $p$  to the amount  $A$  of pure juice that needs to be added to make a blend that is  $p\%$  juice, where  $0 < p < 100$ . What is a reasonable restriction on the set of possible values of  $p$ ? Explain your answer.

$$\frac{6 + A}{10 + A} = \frac{p}{100}$$

*We need to have  $60 < p < 100$  for the problem to make sense. We already have 60% juice; the percentage cannot decrease by adding more juice, and we can never have a mixture that is more than 100 percent juice.*

- d. Suppose that you have added 15 liters of juice to the original 10 liters. What is the percentage of juice in this blend?

$$\frac{p}{100} = \frac{6 + 15}{10 + 15} = 0.84$$

*So, the new blend contains 84% pure juice.*

- e. Solve your equation in part (c) for the amount  $A$ . Are there any excluded values of the variable  $p$ ? Does this make sense in the context of the problem?

$$\begin{aligned} \frac{6 + A}{10 + A} &= \frac{p}{100} \\ 100(6 + A) &= p(10 + A) \\ 600 + 100A &= 10p + pA \\ 100A - pA &= 10p - 600 \\ A(100 - p) &= 10p - 600 \\ A &= \frac{10p - 600}{100 - p} \end{aligned}$$

*We see from the equation for  $A$  that  $p \neq 100$ . This makes sense because we can never make a 100% juice solution since we started with a diluted amount of juice.*

MP.2

### Exercise 4 (5 minutes)

Allow students to work together in pairs or small groups for this exercise. This exercise is a bit different from the previous example in that the amount of acid comes from a diluted solution, not a pure solution. Be sure that students get the numerator set up correctly. (If there is not enough time to do the entire problem, have students set up the equations in class and finish solving them for homework.)

4. You have a solution containing 10% acid and a solution containing 30% acid.
- a. How much of the 30% solution must you add to 1 liter of the 10% solution to create a mixture that is 22% acid?

*If we add  $A$  liters of the 30% solution, then the new mixture is  $1 + A$  liters of solution that contains  $0.1 + 0.3A$  liters of acid. We want the final mixture to be 22% acid, so we need to solve the equation:*

$$\frac{0.1 + 0.3A}{1 + A} = 0.22.$$

*Solving this gives*

$$\begin{aligned} 0.1 + 0.3A &= 0.22(1 + A) \\ 0.1 + 0.3A &= 0.22 + 0.22A \\ 0.08A &= 0.12 \\ A &= 1.5. \end{aligned}$$

*Thus, if we add 1.5 liters of 30% acid solution to 1 liter of 10% acid solution, the result is 2.5 liters of 22% acid solution.*

- b. Write a rational equation that relates the desired percentage  $p$  to the amount  $A$  of 30% acid solution that needs to be added to 1 liter of 10% acid solution to make a blend that is  $p\%$  acid, where  $0 < p < 100$ . What is a reasonable restriction on the set of possible values of  $p$ ? Explain your answer.

$$\frac{0.1 + 0.3A}{1 + A} = \frac{p}{100}$$

*We must have  $10 < p < 30$  because if we blend a 10% acid solution and a 30% acid solution, the blend will contain an acid percentage between 10% and 30%.*

- c. Solve your equation in part (b) for  $A$ . Are there any excluded values of  $p$ ? Does this make sense in the context of the problem?

$$A = \frac{10 - p}{p - 30}$$

*We need to exclude 30 from the possible range of values of  $p$ , which makes sense in context because we can never reach a 30% acid solution since we started with a solution that was 10% acid.*

- d. If you have added some 30% acid solution to 1 liter of 10% acid solution to make a 26% acid solution, how much of the stronger acid did you add?

*The formula in part (c) gives  $A = \frac{10 - 26}{26 - 30}$ , therefore,  $A = 4$ . We added 4 liters of the 30% acid solution to the 1 liter of 10% acid solution to make a 26% acid mixture.*

MP.4

MP.2

### Closing (2 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson.

#### Lesson Summary

In this lesson, we developed the students' problem solving skills by asking them to carefully read a problem, rephrase it in a form comfortable for their own understanding, and convert fact sentences about unknown quantities into algebraic equations. Specifically, they used rational equations to model and solve some application problems and further developed their skills in working with rational expressions.

### Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 27: Word Problems Leading to Rational Equations

### Exit Ticket

Bob can paint a fence in 5 hours, and working with Jen, the two of them painted the fence in 2 hours. How long would it have taken Jen to paint the fence alone?

## Exit Ticket Sample Solutions

Bob can paint a fence in 5 hours, and working with Jen, the two of them painted the fence in 2 hours. How long would it have taken Jen to paint the fence alone?

Let  $x$  represent the time it would take Jen to paint the fence alone. Then, Bob can paint the entire fence in 5 hours; therefore, in one hour he can paint  $\frac{1}{5}$  of the fence. Similarly, Jen can paint  $\frac{1}{x}$  of the fence in one hour. We know that it took them two hours to complete the job, and together they can paint  $\frac{1}{2}$  of the fence in one hour. We then have to solve the equation:

$$\begin{aligned}\frac{1}{5} + \frac{1}{x} &= \frac{1}{2} \\ \frac{2x}{10x} + \frac{10}{10x} &= \frac{5x}{10x} \\ 2x + 10 &= 5x \\ x &= \frac{10}{3}.\end{aligned}$$

Thus, it would have taken Jen 3 hours and 20 minutes to paint the fence alone.

## Problem Set Sample Solutions

1. If 2 inlet pipes can fill a pool in one hour and 30 minutes, and one pipe can fill the pool in two hours and 30 minutes on its own, how long would the other pipe take to fill the pool on its own?

$$\frac{1}{2.5} + \frac{1}{x} = \frac{1}{1.5}$$

We find that  $x = 3.75$ ; therefore, it takes 3 hours and 45 minutes for the second pipe to fill the pool by itself.

2. If one inlet pipe can fill the pool in 2 hours with the outlet drain closed, and the same inlet pipe can fill the pool in 2.5 hours with the drain open, how long does it take the drain to empty the pool if there is no water entering the pool?

$$\frac{1}{2} - \frac{1}{x} = \frac{1}{2.5}$$

We find that  $x = 10$ ; therefore, it takes 10 hours for the drain to empty the pool by itself.

3. It takes 36 minutes less time to travel 120 miles by car at night than by day because the lack of traffic allows the average speed at night to be 10 miles per hour faster than in the daytime. Find the average speed in the daytime.

$$\frac{120}{t-36} = \frac{120}{t} + \frac{1}{6}$$

We find that  $t = 180$  minutes, which is 3 hours; therefore, the average speed in the daytime is  $\frac{120}{3} = 40$  miles per hour.

4. The difference in the average speed of two trains is 16 miles per hour. The slower train takes 2 hours longer to travel 170 miles than the faster train takes to travel 150 miles. Find the speed of the slower train.

$$\frac{150}{t} - \frac{170}{t+2} = 16$$

*We find that  $t = 3$  hours. The average speed of the slower train is 34 miles per hour.*

5. A school library spends \$80 a month on magazines. The average price for magazines bought in January was 70 cents more than the average price in December. Because of the price increase, the school library was forced to subscribe to 7 fewer magazines. How many magazines did the school library subscribe to in December?

$$\frac{80}{x+0.70} = \frac{80}{x} - 7$$

*The average price in December is  $x = \$2.50$ , so the school subscribed to 32 magazines in December.*

6. An investor bought a number of shares of stock for \$1,600. After the price dropped by \$10 per share, the investor sold all but 4 of her shares for \$1,120. How many shares did she originally buy?

$$\frac{1600}{x} = \frac{1120}{x-4} + 10$$

*Two possible answers:  $x = 32$  shares or  $x = 20$  shares*

7. Newton's law of universal gravitation,  $F = \frac{Gm_1m_2}{r^2}$ , measures the force of gravity between two masses  $m_1$  and  $m_2$ , where  $r$  is the distance between the centers of the masses, and  $G$  is the universal gravitational constant. Solve this equation for  $G$ .

$$G = \frac{Fr^2}{m_1m_2}$$

8. Suppose that  $t = \frac{x+y}{1-xy}$ .

- a. Show that when  $x = \frac{1}{a}$  and  $y = \frac{2a-1}{a+2}$ , the value of  $t$  does not depend on the value of  $a$ .

*When simplified, we find that  $t = 2$ ; therefore, the value of  $t$  does not depend on the value of  $a$ .*

- b. For which values of  $a$  do these relationships have no meaning?

*If  $a$  is 0, then  $x$  has no meaning. If  $a = -2$ , then  $y$  has no meaning.*

9. Consider the rational equation  $\frac{1}{R} = \frac{1}{x} + \frac{1}{y}$ .

- a. Find the value of  $R$  when  $x = \frac{2}{5}$  and  $y = \frac{3}{4}$ .

$$\frac{1}{R} = \frac{1}{2/5} + \frac{1}{3/4}$$

$$\text{So } R = \frac{6}{23}.$$

- b. Solve this equation for  $R$  and simplify.

*There are two approaches to solve this equation for  $R$ .*

*The first way is to perform the addition on the right:*

$$\begin{aligned} \frac{1}{R} &= \frac{1}{x} + \frac{1}{y} \\ &= \frac{y}{xy} + \frac{x}{xy} \\ &= \frac{x+y}{xy}. \end{aligned}$$

*In either case, we find that  $R = \frac{xy}{x+y}$ .*

*The second way is to take reciprocals of both sides, and then simplify:*

$$\begin{aligned} R &= \frac{1}{\frac{1}{x} + \frac{1}{y}} \\ &= \frac{1}{\frac{y}{xy} + \frac{x}{xy}} \\ &= \frac{1}{\frac{x+y}{xy}}. \end{aligned}$$

10. Consider an ecosystem of rabbits in a park that starts with 10 rabbits and can sustain up to 60 rabbits. An equation that roughly models this scenario is

$$P = \frac{60}{1 + \frac{5}{t+1}},$$

where  $P$  represents the rabbit population in year  $t$  of the study.

- a. What is the rabbit population in year 10? Round your answer to the nearest whole rabbit.

*If  $t = 10$ , then  $P = 41.25$ ; therefore, there are 41 rabbits in the park.*

- b. Solve this equation for  $t$ . Describe what this equation represents in the context of this problem.

$$t = \frac{60 - 6P}{P - 60}$$

*This equation represents the relationship between the number of rabbits,  $P$ , and the year,  $t$ . If we know how many rabbits we have,  $10 < P < 60$ , we will know how long it took for the rabbit population to grow that much. If the population is 10, then this equation says we are in year 0 of the study, which fits with the given scenario.*

- c. At what time does the population reach 50 rabbits?

*If  $P = 50$ , then  $t = \frac{60-300}{50-60} = -\frac{240}{-10} = 24$ ; therefore, the rabbit population is 50 in year 24 of the study.*

Extension:

11. Suppose that Huck Finn can paint a fence in 5 hours. If Tom Sawyer helps him paint the fence, they can do it in 3 hours. How long would it take for Tom to paint the fence by himself?

*Huck paints the fence in 5 hours, so his rate of fence painting is  $\frac{1}{5}$  fence per hour. Let  $T$  denote the percentage of the fence Tom can paint in an hour. Then*

$$\begin{aligned} 1 \text{ fence} &= \left( \left( \frac{1}{5} + T \right) \text{ fence per hour} \right) \cdot (3 \text{ hours}). \\ 3 &= \frac{1}{\frac{1}{5} + T} = \frac{1}{\frac{1}{5} + \frac{5T}{5}} = \frac{5}{1 + 5T} \\ 3(1 + 5T) &= 5 \\ 15T &= 2 \\ T &= \frac{2}{15} \end{aligned}$$

*So, Tom can paint  $\frac{2}{15}$  of the fence in an hour. Thus, Tom would take  $\frac{15}{2} = 7.5$  hours to paint the fence by himself.*

12. Huck Finn can paint a fence in 5 hours. After some practice, Tom Sawyer can now paint the fence in 6 hours.

a. How long would it take Huck and Tom to paint the fence together?

The amount of fence that Huck can paint per hour is  $\frac{1}{5}$ , and the amount that Tom can paint per hour is  $\frac{1}{6}$ . So, together they can paint  $\frac{1}{5} + \frac{1}{6}$  of the fence per hour. Suppose the entire job of painting the fence takes  $h$  hours. Then, the amount of the fence that is painted is  $h\left(\frac{1}{5} + \frac{1}{6}\right)$ . Since the entire fence is painted, we need to solve the equation  $h\left(\frac{1}{5} + \frac{1}{6}\right) = 1$ .

$$\begin{aligned} h\left(\frac{1}{5} + \frac{1}{6}\right) &= 1 \\ h &= \frac{1}{\frac{1}{5} + \frac{1}{6}} = \frac{30}{6 + 5} = \frac{30}{11} \end{aligned}$$

So, together they can paint the fence in  $\frac{30}{11}$  hours, which is 2 hours and 44 minutes.

b. Tom demands a half hour break while Huck continues to paint, and they finish the job together. How long does it take them to paint the fence?

Suppose the entire job of painting the fence takes  $h$  hours. Then, Huck paints at a rate of  $\frac{1}{5}$  of the fence per hour for  $h$  hours, so he paints  $\frac{h}{5}$  of the fence. Tom paints at a rate of  $\frac{1}{6}$  of the fence per hour for  $h - \frac{1}{2}$  hours, so he paints  $\frac{1}{6}\left(h - \frac{1}{2}\right)$  of the fence. Together, they paint the whole fence; so, we need to solve the following equation for  $h$ :

$$\begin{aligned} \frac{1}{5}h + \frac{1}{6}\left(h - \frac{1}{2}\right) &= 1 \\ \frac{1}{5}h + \frac{1}{6}h - \frac{1}{12} &= 1 \\ \frac{1}{5}h + \frac{1}{6}h &= \frac{13}{12} \\ 60\left(\frac{1}{5}h + \frac{1}{6}h\right) &= 60 \cdot \frac{13}{12} \\ 12h + 10h &= 65 \\ h &= \frac{65}{22}. \end{aligned}$$

Thus, it takes  $\frac{65}{22}$  hours, which is 2 hours 57 minutes, to paint the fence with Tom taking a  $\frac{1}{2}$  hour break.

c. Suppose that they have to finish the fence in  $3\frac{1}{2}$  hours. What's the longest break that Tom can take?

Suppose the entire job of painting the fence takes  $\frac{7}{2}$  hours, and Tom stops painting for  $b$  hours for his break. Then, Huck paints at a rate of  $\frac{1}{5}$  of the fence per hour for  $\frac{7}{2}$  hours, so he paints  $\frac{7}{10}$  of the fence. Tom paints at a rate of  $\frac{1}{6}$  of the fence per hour for  $\left(\frac{7}{2} - b\right)$  hours, so he paints  $\frac{1}{6}\left(\frac{7}{2} - b\right)$  of the fence. Together, they paint the whole fence; so, we need to solve the following equation for  $b$ :

$$\begin{aligned} \frac{7}{10} + \frac{1}{6}\left(\frac{7}{2} - b\right) &= 1 \\ \frac{7}{10} + \frac{7}{12} - \frac{b}{6} &= 1 \\ 60\left(\frac{7}{10} + \frac{7}{12} - \frac{b}{6}\right) &= 60 \\ 42 + 35 - 10b &= 60 \\ 42 + 35 - 60 &= 10b \\ b &= \frac{17}{10}. \end{aligned}$$

Thus, if Tom takes a break for  $\frac{17}{10}$  hours, which is 1 hour and 42 minutes, the fence will be painted in  $3\frac{1}{2}$  hours.



## Lesson 28: A Focus on Square Roots

### Student Outcomes

- Students solve simple radical equations and understand the possibility of extraneous solutions. They understand that care must be taken with the role of square roots so as to avoid apparent paradoxes.
- Students explain and justify the steps taken in solving simple radical equations.

### Lesson Notes

In the next two lessons, students work with radical expressions and equations. They extend their understanding of the idea that not all operations are invertible, which was explored in Algebra I and continued in the previous lessons on solving rational equations. Squaring both sides of an equation in some cases produces an extraneous solution. They also continue to work with fractional expressions and equations as seen in the previous lessons, but those expressions now contain radicals. This lesson highlights standards **A-REI.A.1**, which calls for students to be able to explain each step required to solve an equation, and **A-REI.A.2**, which calls for students to solve a radical equation and show how extraneous solutions might arise. It also addresses the standard MP.3 by building and analyzing arguments used in solving radical equations. In Example 2, students consider the difference between working with an *expression*, whose value should be preserved, and working with an *equation*, whose sides can be changed in value as long as the equality is preserved. This difference addresses the standard MP.7 because students are stepping back to get an overview of expressions and equations as objects subject to different rules.

### Classwork

#### Opening (1 minute)

Recall that working with radicals can be tricky, especially when negative numbers are involved. When solving a radical equation, one must always check the answers found to verify that they are indeed valid solutions to the equation. In some cases, extraneous solutions appear and must be eliminated. Recall that an *extraneous solution* is one that satisfies a transformed equation but not the original one.

#### Exercises 1–4 (7 minutes)

Give students a few minutes to work through the first four exercises, and then discuss the results as a whole class. Circulate the room to assess the students' understanding.

#### Scaffolding:

- If students are struggling, show a few simpler examples such as solving  $\sqrt{x} = 5$  or  $\sqrt{x} = -5$ .
- Another option would be to provide the following alternative model for students to complete.

Fill in the blanks to fully show and explain the solution process.

$$\begin{array}{r} \sqrt{x} - 6 = 4 \\ \underline{\sqrt{x} = 10} \\ \sqrt{x} = 10^2 \\ x = 100 \end{array} \quad \begin{array}{l} \text{Added 6 to both sides to isolate the radical} \\ \text{Square both sides} \end{array}$$

## Exercises 1–4

For Exercises 1–4, describe each step taken to solve the equation. Then, check the solution to see if it is valid. If it is not a valid solution, explain why.

1.  $\sqrt{x} - 6 = 4$

$$\sqrt{x} = 10 \quad \text{Add 6 to both sides}$$

$$x = 100 \quad \text{Square both sides}$$

$$\text{Check: } \sqrt{100} - 6 = 10 - 6 = 4$$

So 100 is a valid solution.

2.  $\sqrt[3]{x} - 6 = 4$

$$\sqrt[3]{x} = 10 \quad \text{Add 6 to both sides}$$

$$x = 1000 \quad \text{Cube both sides}$$

$$\text{Check: } \sqrt[3]{1000} - 6 = 10 - 6 = 4$$

So 1000 is a valid solution.

3.  $\sqrt{x} + 6 = 4$

$$\sqrt{x} = -2$$

$$x = 4$$

$$\text{Check: } \sqrt{4} + 6 = 2 + 6 = 8, \text{ and } 8 \neq 4, \text{ so } 4 \text{ is not a valid solution.}$$

4.  $\sqrt[3]{x} + 6 = 4$

$$\sqrt[3]{x} = -2$$

$$x = -8$$

$$\text{Check: } \sqrt[3]{-8} + 6 = -2 + 6 = 4, \text{ so } -8 \text{ is a valid solution.}$$

## Discussion

Consider each of the following questions, one at a time.

- What was the first step taken in solving the radical equations in Exercises 1 and 2?
  - *The radical was isolated.*
- What was the second step taken?
  - *Both sides were squared or cubed to eliminate the radical.*
- What happened in Exercise 3?
  - *The same steps were used to solve the equation as were used in Exercise 1, but this time the solution found did not work. There is no solution to the equation.*
- Why did that happen?

This is one of the focal points of the lesson. Ask students to answer in writing or discuss with a partner before sharing their answers with the rest of the class. In the discussion, emphasize that 4 is an *extraneous solution*; it is the solution to  $x = 4$  but not to the original equation.

- *For 4 to be a solution,  $\sqrt{4}$  would need to equal  $-2$ . Even though  $(-2)^2 = 4$ , we define  $\sqrt{4} = 2$  so that  $f(x) = \sqrt{x}$  takes on only one value for  $x \geq 0$  and is thus a function. As a result, the square root of a positive number is only the positive value. Therefore, 4 is an extraneous solution.*

- What other types of equations sometimes have extraneous solutions?
  - *Rational equations can have extraneous solutions that create zero in the denominator.*
- Why did the solution process work in Exercise 4?
  - *The cube root of a negative number is negative, so a cube root equation does not have the same issues with the negative numbers as a square root does.*

MP.3

**Example 1 (5 minutes)**

Work through the example to solidify the steps in solving a radical equation. Depending on how the students did with the first four exercises, you may want them to continue working with a partner, or you may want to work through this example with the whole class at once. Be sure the students can explain and justify the steps they are taking.

**Example 1**

Solve the radical equation. Be sure to check your solutions.

$$\sqrt{3x+5} - 2 = -1$$

**Solution:**  $\sqrt{3x+5} = 1$

$$3x + 5 = 1$$

$$3x = -4$$

$$x = -\frac{4}{3}$$

**Check:**  $\sqrt{3\left(-\frac{4}{3}\right) + 5} - 2 = \sqrt{-4 + 5} - 2 = \sqrt{1} - 2 = -1$ , so  $-\frac{4}{3}$  is a valid solution.

**Discussion**

- What was the first step you took?
  - *Isolated the radical*
- Why did you do that first?
  - *Isolating the radical allows it to be eliminated by squaring or cubing both sides of the equation.*
- What was the next step?
  - *Squared both sides*
- Why did you do that?
  - *The purpose was to eliminate the radical from the equation.*
- Even though we are solving a new type of equation, does this feel like a familiar process?
  - *Yes. When solving an equation, we work on undoing any operation by doing the inverse. To undo a square root we use the inverse, so we square the expression.*
- How do the steps we are following relate to your previous experiences with solving other types of equations?
  - *We are still following the basic process to solve an equation, which is to undo any operation on the same side as the variable by using the inverse operation.*
- Why is it important to check the solution?
  - *Sometimes extraneous solutions appear because the square root of a positive number or zero is never negative.*

Summarize (in writing or with a partner) what you have learned about solving radical equations. Be sure that you explain what to do when you get an invalid solution.

**Exercises 5–15 (15 minutes)**

Allow students time to work the problems individually and then check with a partner. Circulate around the room. Make sure students are checking for extraneous solutions.

**Exercises 5–15**

Solve each radical equation. Be sure to check your solutions.

5.  $\sqrt{2x - 3} = 11$

62

6.  $\sqrt[3]{6 - x} = -3$

33

7.  $\sqrt{x + 5} - 9 = -12$

No solution

8.  $\sqrt{4x - 7} = \sqrt{3x + 9}$

16

9.  $-12\sqrt{x - 6} = 18$

No solution

10.  $3\sqrt[3]{x + 2} = 12$

62

11.  $\sqrt{x^2 - 5} = 2$

3 and -3

12.  $\sqrt{x^2 + 8x} = 3$

-9 and 1

- Which exercises produced extraneous solutions?
  - Exercises 7 and 9
- Which exercises produced more than one solution? Why?
  - Exercises 11 and 12 because after eliminating the radical, the equation became a quadratic equation. Both solutions were valid when checked.
- Write an example of a radical equation that has an extraneous solution. Exchange with a partner and confirm that the example does in fact have an extraneous solution.

Multiply each expression.

13.  $(\sqrt{x} + 2)(\sqrt{x} - 2)$

 $x - 4$ 

14.  $(\sqrt{x} + 4)(\sqrt{x} + 4)$

 $x + 8\sqrt{x} + 16$ 

15.  $(\sqrt{x - 5})(\sqrt{x - 5})$

 $x - 5$ 

In the next example and exercises, we are working with rational expressions and equations that contain radicals. The purpose of these problems is to continue to build fluency working with radicals, to build on the work done in the previous lessons on rational expressions and equations, and to highlight MP.7, which calls for students to recognize and make use of structure in an expression.

**Example 2 (5 minutes)**

Work through the two examples as a class, making sure students are clear on the differences between working with an *expression* and working with an *equation*.

**Example 2**

Rationalize the denominator in each expression. That is, rewrite each expression so that the fraction has a rational expression in the denominator.

a.  $\frac{x-9}{\sqrt{x-9}}$

$$\frac{x-9}{\sqrt{x-9}} \cdot \frac{\sqrt{x-9}}{\sqrt{x-9}}$$

$$\frac{(x-9)\sqrt{x-9}}{x-9}$$

$$\frac{\sqrt{x-9}}{\sqrt{x-9}}$$

b.  $\frac{x-9}{\sqrt{x+3}}$

$$\frac{x-9}{\sqrt{x+3}} \cdot \frac{\sqrt{x-3}}{\sqrt{x-3}}$$

$$\frac{(x-9)(\sqrt{x-3})}{x-9}$$

$$\frac{\sqrt{x-3}}{\sqrt{x-3}}$$

- What do the directions mean by “rationalize the denominator”?
  - *To remove the radical from the denominator so that the denominator will be a rational expression.*
- How can we accomplish this goal in part (a)?
  - *Multiply the numerator and denominator by  $\sqrt{x-9}$ .*
- Why not just square the expression?
  - *We are working with an expression, not an equation. You cannot square the expression because you would be changing its value. You can multiply the numerator and denominator by  $\sqrt{x-9}$  because that is equivalent to multiplying by 1. It does not change the value of the expression.*
- Can we take the same approach in part (b)?
  - *No, multiplying by  $\sqrt{x+3}$  would not remove the radical from the denominator.*
- Based on Exercise 13, what number should we multiply numerator and denominator by in part (b) in order to make the denominator rational?
  - $\sqrt{x-3}$
- In these examples, what was accomplished by rationalizing the denominator?
  - *It allowed us to create an equivalent expression that is simpler.*
- Why would that be advantageous?
  - *It would be easier to work with if we were evaluating it for a particular value of  $x$ .*

MP.7

**Exercises 16–18 (5 minutes)**

Allow students time to work on the three problems and then debrief. Students may have taken different approaches on Exercise 17, such as squaring both sides first or rationalizing the denominator. Share a few different approaches and compare.

## Exercises 16–18

16. Rewrite  $\frac{1}{\sqrt{x}-5}$  in an equivalent form with a rational expression in the denominator.

$$\frac{\sqrt{x} + 5}{x - 25}$$

17. Solve the radical equation  $\frac{3}{\sqrt{x+3}} = 1$ . Be sure to check for extraneous solutions.

$$x = 6$$

18. Without solving the radical equation  $\sqrt{x+5} + 9 = 0$ , how could you tell that it has no real solution?

*The radical expression  $\sqrt{x+5}$  is positive or zero. In either case, adding 9 to it cannot give zero.*

## Scaffolding:

- If students are struggling with Exercise 17, have them approach the equation logically first rather than algebraically. If the output must equal 1 and the numerator is 3, what must the denominator equal?

## Closing (2 minutes)

Ask students to respond to these questions in writing or with a partner. Use this as an opportunity to informally assess the students' understanding.

- Explain to your neighbor how to solve a radical equation. What steps do you take and why?
  - *Isolate the radical and then eliminate it by raising both sides to an exponent. The radical is isolated so that both sides can be squared or cubed as a means of eliminating the radical.*
- How is solving a radical equation similar to solving other types of equations we have solved?
  - *We are isolating the variable by undoing any operation on the same side.*
- Why is it important to check the solutions?
  - *Remember that the square root of a number takes on only the positive value. When solving a radical equation involving a square root, squaring both sides of the equation in the process of solving may make the negative "disappear" and may create an extraneous solution.*

## Exit Ticket (5 minutes)



## Exit Ticket Sample Solutions

Consider the radical equation  $3\sqrt{6-x} + 4 = -8$

1. Solve the equation. Next to each step, write a description of what is being done.

$$\begin{array}{ll} 3\sqrt{6-x} = -12 & \text{Subtract 4 from both sides.} \\ \sqrt{6-x} = -4 & \text{Divide both sides by 3 in order to isolate the radical.} \\ 6-x = 16 & \text{Square both sides to eliminate the radical.} \\ x = -10 & \text{Subtract 6 from both sides and divide by } -1. \end{array}$$

2. Check the solution.

$$3\sqrt{6-(-10)} + 4 = 3\sqrt{16} + 4 = 3(4) + 4 = 16, \text{ and } 16 \neq -8, \text{ so } -10 \text{ is not a valid solution.}$$

3. Explain why the calculation in Problem 1 does not produce the solution to the equation.

*Because the square root of a positive number is positive,  $3\sqrt{6-x}$  will be positive. A positive number added to 4 cannot be  $-8$ .*

## Problem Set Sample Solutions

1. a. If  $\sqrt{x} = 9$ , then what is the value of  $x$ ?  
 $x = 81$
- b. If  $x^2 = 9$ , then what is the value of  $x$ ?  
 $x = 3$  or  $x = -3$
- c. Is there a value of  $x$  such that  $\sqrt{x+5} = 0$ ? If yes, what is the value? If no, explain why not.  
*Yes,  $x = -5$*
- d. Is there a value of  $x$  such that  $\sqrt{x} + 5 = 0$ ? If yes, what is the value? If no, explain why not.  
*No,  $\sqrt{x}$  will be a positive value or zero for any value of  $x$ , so the sum cannot equal 0. If  $x = 25$ , then  $\sqrt{25} + 5 = 10$ .*
2. a. Is the statement  $\sqrt{x^2} = x$  true for all  $x$ -values? Explain.  
*No, this statement is only true for  $x \geq 0$ . If  $x < 0$ , it is not true. For example, if  $x = -5$ ,  $\sqrt{(-5)^2} = \sqrt{25} = 5$ , then  $\sqrt{(-5)^2} \neq -5$ .*
- b. Is the statement  $\sqrt[3]{x^3} = x$  true for all  $x$ -values? Explain.  
*Yes, this statement is true for all  $x$ -values. For example, if  $x = 2$ , then  $\sqrt[3]{2^3} = 2$ . If  $x = -2$ , then  $\sqrt[3]{(-2)^3} = -2$ . Since the cube root of a positive number is positive, and the cube root of a negative number is negative, this statement is true for any value of  $x$ .*

Rationalize the denominator in each expression.

3.  $\frac{4-x}{2+\sqrt{x}}$   
 $2 - \sqrt{x}$

4.  $\frac{2}{\sqrt{x-12}}$   
 $\frac{2\sqrt{x-12}}{x-12}$

5.  $\frac{1}{\sqrt{x+3}-\sqrt{x}}$   
 $\frac{\sqrt{x+3} + \sqrt{x}}{3}$

Solve each equation and check the solutions.

6.  $\sqrt{x+6} = 3$   
 $x = 3$

7.  $2\sqrt{x+3} = 6$   
 $x = 6$

8.  $\sqrt{x+3} + 6 = 3$   
*No solution*

9.  $\sqrt{x+3} - 6 = 3$   
 $x = 78$

10.  $16 = 8 + \sqrt{x}$   
 $x = 64$

11.  $\sqrt{3x-5} = 7$   
 $x = 18$

12.  $\sqrt{2x-3} = \sqrt{10-x}$   
 $x = \frac{13}{3}$

13.  $3\sqrt{x+2} + \sqrt{x-4} = 0$   
*No solution*

14.  $\frac{\sqrt{x+9}}{4} = 3$   
 $x = 135$

15.  $\frac{12}{\sqrt{x+9}} = 3$   
 $x = 7$

16.  $\sqrt{x^2+9} = 5$   
 $x = 4$  or  $x = -4$

17.  $\sqrt{x^2-6x} = 4$   
 $x = 8$  or  $x = -2$

18.  $\frac{5}{\sqrt{x-2}} = 5$   
 $x = 3$

19.  $\frac{5}{\sqrt{x-2}} = 5$   
 $x = 9$

20.  $\sqrt[3]{5x-3} + 8 = 6$   
 $x = -1$

21.  $\sqrt[3]{9-x} = 6$   
 $x = -207$

22. Consider the inequality  $\sqrt{x^2+4x} > 0$ . Determine whether each  $x$ -value is a solution to the inequality.

a.  $x = -10$   
*Yes*

b.  $x = -4$   
*No*

c.  $x = 10$   
*Yes*

d.  $x = 4$   
*Yes*

23. Show that  $\frac{a-b}{\sqrt{a}-\sqrt{b}} = \sqrt{a} + \sqrt{b}$  for all values of  $a$  and  $b$  such that  $a > 0$  and  $b > 0$  and  $a \neq b$ .

*If we multiply the numerator and denominator of  $\frac{a-b}{\sqrt{a}-\sqrt{b}}$  by  $\sqrt{a} + \sqrt{b}$  to rationalize the denominator, then we have*

$$\frac{a-b}{\sqrt{a}-\sqrt{b}} = \frac{a-b}{\sqrt{a}-\sqrt{b}} \cdot \frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}} = \frac{(a-b)(\sqrt{a} + \sqrt{b})}{a-b} = \sqrt{a} + \sqrt{b}.$$

24. Without actually solving the equation, explain why the equation  $\sqrt{x+1} + 2 = 0$  has no solution.

*The value of  $\sqrt{x+1}$  must be positive which is then added to 2. The sum of two positive numbers is positive; therefore, the sum cannot equal 0.*



## Lesson 29: Solving Radical Equations

### Student Outcomes

- Students develop facility in solving radical equations.

### Lesson Notes

In the previous lesson, students were introduced to the notion of solving radical equations and checking for extraneous solutions (**A-REI.A.2**). Students continue this work by looking at radical equations that contain variables on both sides. The main point to stress to students is that radical equations become polynomial equations through exponentiation. So we really have not left the notion of polynomials that we have been studying throughout this module. This lesson also provides opportunities to emphasize MP.7 (look for and make use of structure).

### Classwork

#### Discussion (5 minutes)

Before beginning the lesson, remind students of past experiences by providing the following scenario, which illustrates when an operation performed to both sides of an equation has changed the set of solutions.

Carlos and Andrea were solving the equation  $x^2 + 2x = 0$ . Andrea says that there are two solutions, 0 and  $-2$ . Carlos says the only solution is  $-2$  because he divided both sides by  $x$  and got  $x + 2 = 0$ . Who is correct and why?

- Do both 0 and  $-2$  satisfy the original equation?
  - Yes. If we replace  $x$  with either 0 or  $-2$ , the answer is 0.*
- What happened when Carlos divided both sides of the equation by  $x$ ?
  - He changed the solutions from 0 and  $-2$  to simply  $-2$ . He lost one solution to the equation.*
- What does this say about the solution of equations after we have performed algebraic operations on both sides?
  - Performing algebraic steps may alter the set of solutions to the original equation.*

Now, Carlos and Andrea are solving the equation  $\sqrt{x} = -3$ . Andrea says the solution is 9 because she squared both sides and got  $x = 9$ . Carlos says there is no solution. Who is correct? Why?

- Was Andrea correct to square both sides?
  - Yes. To eliminate a radical from an equation, we raise both sides to an exponent.*
- Is she correct that the solution is 9?
  - No. Carlos is correct. If we let  $x = 9$ , then we get  $\sqrt{9} = 3$ , and  $3 \neq -3$ , so 9 is not a solution.*

MP.3

#### Scaffolding

- Use several examples to illustrate that if  $a > 0$ , then an equation of the form  $\sqrt{x} = -a$  will not have a solution (e.g.,  $\sqrt{x} = -4$ ,  $\sqrt{x} = -5$ ).
- Extension: Write an equation that has an extraneous solution of  $x = 50$ .

- What is the danger in squaring both sides of an equation?
  - *It sometimes produces an equation whose solution set is not equivalent to that of the original equation. If both sides of  $\sqrt{x} = -3$  are squared, the equation  $x = 9$  is produced, but 9 is not a solution to the original equation. The original equation has no solution.*
- Because of this danger, what is the final essential step of solving a radical equation?
  - *Checking the solution or solutions to ensure that an extraneous solution was not produced by the step of squaring both sides.*
- How could we have predicted that the equation would have no solution?
  - *The square root of a number is never equal to a negative value, so there is no  $x$ -value so that  $\sqrt{x} = -3$ .*

**Example 1 (5 minutes)**

MP.1

While this problem is difficult, students should attempt to solve it on their own first, by applying their understandings of radicals. Students should be asked to verify the solution they come up with and describe their solution method. Discuss Example 1 as a class once they have worked on it individually.

**Example 1**

Solve the equation  $6 = x + \sqrt{x}$ .

$$6 - x = \sqrt{x}$$

$$(6 - x)^2 = \sqrt{x}^2$$

$$36 - 12x + x^2 = x$$

$$x^2 - 13x + 36 = 0$$

$$(x - 9)(x - 4) = 0$$

*The solutions are 9 and 4.*

**Check  $x = 9$ :**  $9 + \sqrt{9} = 9 + 3 = 12$   
 $6 \neq 12$

**Check  $x = 4$ :**  $4 + \sqrt{4} = 4 + 2 = 6$

*So, 9 is not a solution.*

*The only solution is 4.*

- How does this equation differ from the ones from yesterday’s lesson?
  - *There are two  $x$ ’s; one inside and one outside of the radical.*
- Explain how you were able to determine the solution to the equation above.
  - *Isolate the radical and square both sides. Solve the resulting equation.*
- Did that change the way in which the equation was solved?
  - *Not really. We still eliminated the radical by squaring both sides.*
- What type of equation were we left with after squaring both sides?
  - *A quadratic polynomial equation*
- Why did 9 fail to work as a solution?
  - *Because the square root of 9 takes the positive value of 3*

**Exercises 1–4 (13 minutes)**

Allow students time to work the problems independently and then pair up to compare solutions. Use this time to informally assess student understanding by examining their work. Display student responses, making sure that students checked for extraneous solutions.

**Exercises 1–4**

Solve.

<p>1. <math>3x = 1 + 2\sqrt{x}</math>  <i>The only solution is 1.</i>  <i>Note that <math>\frac{1}{9}</math> is an extraneous solution.</i></p>	<p>2. <math>3 = 4\sqrt{x} - x</math>  <i>The two solutions are 9 and 1.</i></p>
<p>3. <math>\sqrt{x+5} = x - 1</math>  <i>The only solution is 4.</i>  <i>Note that <math>-1</math> is an extraneous solution.</i></p>	<p>4. <math>\sqrt{3x+7} + 2\sqrt{x-8} = 0</math>  <i>There are no solutions.</i></p>

- When solving Exercise 1, what solutions did you find? What happened when you checked these solutions?
  - *The solutions found were  $\frac{1}{9}$  and 1. Only 1 satisfies the original equation, so  $\frac{1}{9}$  is an extraneous solution.*
- Did Exercise 2 have any extraneous solutions?
  - *No. Both solutions satisfied the original equation.*
- Looking at Exercise 4, could we have predicted that there would be no solution?
  - *Yes. The only way the two square roots could add to zero would be if both of them produced a zero, meaning that  $3x + 7 = 0$  and  $x - 8 = 0$ . Since  $x$  cannot be both  $-\frac{7}{3}$  and 8, both radicals cannot be simultaneously zero. Thus, at least one of the square roots will be positive, and they cannot sum to zero.*

MP.7

**Example 2 (5 minutes)**

What do we do when there is no way to isolate the radical? What is going to be the easiest way to square both sides? Give students time to work on Example 2 independently. Point out that even though we had to square both sides twice we were still able to rewrite the equation as a polynomial.

MP.7

*Scaffolding:*

- What if we had squared both sides of the equation as it was presented? Have early finishers work out the solution this way and share with the class.

**Example 2**

Solve the equation  $\sqrt{x} + \sqrt{x+3} = 3$ .

$\begin{aligned} \sqrt{x+3} &= 3 - \sqrt{x} \\ (\sqrt{x+3})^2 &= (3 - \sqrt{x})^2 \\ x + 3 &= 9 - 6\sqrt{x} + x \\ 1 &= \sqrt{x} \\ 1 &= x \end{aligned}$	<p><i>Check:</i></p> $\sqrt{1} + \sqrt{1+3} = 1 + 2 = 3$ <p><i>So the solution is 1.</i></p>
---	--

**Exercises 5–6 (7 minutes)**

Allow students time to work the problems independently and then pair up to compare solutions. Circulate to assess understanding. Consider targeted instruction with a small group of students while others are working independently. Display student responses, making sure that students check for extraneous solutions.

**Exercises 5–6**

Solve the following equations.

5.  $\sqrt{x-3} + \sqrt{x+5} = 4$

4

6.  $3 + \sqrt{x} = \sqrt{x+81}$

144

**Closing (5 minutes)**

Ask students to respond to these questions in writing or with a partner. Use these responses to informally assess their understanding of the lesson.

- How did these equations differ from the equations seen in yesterday's lesson?
  - *Most of them contained variables on both sides of the equation or a variable outside of the radical.*
- How were they similar to the equations from yesterday's lesson?
  - *They were solved using the same process of squaring both sides. Even though they were more complicated, the equations could still be rewritten as a polynomial equation and solved using the same process seen throughout this module.*
- Ask the students to summarize the lesson. Give an example where  $a^n = b^n$  but  $a \neq b$ .
  - $(-3)^2 = 3^2$  but  $-3 \neq 3$ .

**Lesson Summary**

If  $a = b$  and  $n$  is an integer, then  $a^n = b^n$ . However, the converse is not necessarily true. The statement  $a^n = b^n$  does not imply that  $a = b$ . Therefore, it is necessary to check for extraneous solutions when both sides of an equation are raised to an exponent.

**Exit Ticket (5 minutes)**



Exit Ticket Sample Solutions

1. Solve  $\sqrt{2x + 15} = x + 6$ . Verify the solution(s).

$$2x + 15 = x^2 + 12x + 36$$

$$0 = x^2 + 10x + 21$$

$$0 = (x + 3)(x + 7)$$

The solutions are  $-3$  and  $-7$ .

Check  $x = -3$ :  $\sqrt{2(-3) + 15} = \sqrt{9} = 3$   
 $-3 + 6 = 3$

Check  $x = -7$ :  $\sqrt{2(-7) + 15} = \sqrt{1} = 1$   
 $-7 + 6 = -1$

So,  $-3$  is a valid solution. Since  $-1 \neq 1$ , we see that  $-1$  is an extraneous solution.

Therefore, the only solution to the original equation is  $-3$ .

2. Explain why it is necessary to check the solutions to a radical equation.

Squaring both sides in some cases produces an equation whose solution set is not equivalent to that of the original equation. In the problem above,  $x = -7$  does not satisfy the equation.

Problem Set Sample Solutions

Solve.

1. $\sqrt{2x - 5} - \sqrt{x + 6} = 0$ 11	2. $\sqrt{2x - 5} + \sqrt{x + 6} = 0$ No solution
3. $\sqrt{x - 5} - \sqrt{x + 6} = 2$ No solution	4. $\sqrt{2x - 5} - \sqrt{x + 6} = 2$ 43
5. $\sqrt{x + 4} = 3 - \sqrt{x}$ $\frac{25}{36}$	6. $\sqrt{x + 4} = 3 + \sqrt{x}$ No solution
7. $\sqrt{x + 3} = \sqrt{5x + 6} - 3$ 6	8. $\sqrt{2x + 1} = x - 1$ 4
9. $\sqrt{x + 12} + \sqrt{x} = 6$ 4	10. $2\sqrt{x} = 1 - \sqrt{4x - 1}$ $\frac{1}{4}$
11. $2x = \sqrt{4x - 1}$ $\frac{1}{2}$	12. $\sqrt{4x - 1} = 2 - 2x$ $\frac{1}{2}$

13.  $x + 2 = 4\sqrt{x - 2}$   
6

14.  $\sqrt{2x - 8} + \sqrt{3x - 12} = 0$   
4

15.  $x = 2\sqrt{x - 4} + 4$   
4, 8

16.  $x - 2 = \sqrt{9x - 36}$   
5, 8

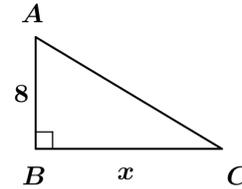
17. Consider the right triangle  $ABC$  shown to the right, with  $AB = 8$  and  $BC = x$ .

a. Write an expression for the length of the hypotenuse in terms of  $x$ .

$AC = \sqrt{64 + x^2}$

b. Find the value of  $x$  for which  $AC - AB = 9$ .

*The solutions to the mathematical equation  $\sqrt{64 + x^2} - 8 = 9$  are  $-15$  and  $15$ . Since lengths must be positive,  $-15$  is an extraneous solution, and  $x = 15$ .*



18. Consider the triangle  $ABC$  shown to the right where  $AD = DC$  and  $\overline{BD}$  is the altitude of the triangle.

a. If the length of  $\overline{BD}$  is  $x$  cm and the length of  $\overline{AC}$  is 18 cm, write an expression for the lengths of  $\overline{AB}$  and  $\overline{BC}$  in terms of  $x$ .

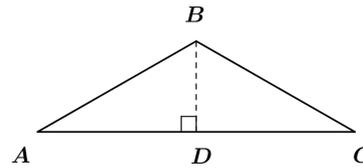
$AB = BC = \sqrt{81 + x^2}$  cm

b. Write an expression for the perimeter of  $\triangle ABC$  in terms of  $x$ .

$(2\sqrt{81 + x^2} + 18)$  cm

c. Find the value of  $x$  for which the perimeter of  $\triangle ABC$  is equal to 38 cm.

$\sqrt{19}$  cm





## Lesson 30: Linear Systems in Three Variables

### Student Outcomes

- Students solve linear systems in three variables algebraically.

### Lesson Notes

Students solved systems of linear equations in two variables using substitution and elimination in Grade 8 and then encountered the topic again in Algebra I when talking about solving systems of linear equalities and inequalities. In this lesson, we begin with a quick review of elimination to solve a linear system in two variables along with one application problem before moving onto solving a system of equations in three variables using algebraic techniques.

### Classwork

#### Opening (2 minutes)

This lesson transitions from solving linear 2-by-2 equations as in Algebra I to solving systems of equations involving linear and nonlinear equations in two variables in the next two lessons. These nonlinear systems will be solved algebraically using substitution or by graphing each equation and finding points of intersection, if any. This lesson helps remind students how to solve linear systems of equations and introduces them to 3-by-3 systems of linear equations (which will be later analyzed using matrices in Precalculus).

#### Exercises 1–3 (8 minutes)

##### Exercises 1–3

Determine the value of  $x$  and  $y$  in the following systems of equations.

1.  $2x + 3y = 7$   
 $2x + y = 3$

$$x = \frac{1}{2}, y = 2$$

2.  $5x - 2y = 4$   
 $-2x + y = 2$

$$x = 8, y = 18$$

After this review of using elimination to solve a system, guide students through the set-up of the following problem, and then let them solve using the techniques reviewed in Exercises 1 and 2.

3. A scientist wants to create 120 ml of a solution that is 30% acidic. To create this solution, she has access to a 20% solution and a 45% solution. How many milliliters of each solution should she combine to create the 30% solution?

Solve this problem using a system of two equations in two variables.

**Solution:**

**Milliliters of 20% solution:  $x$  ml**

**Milliliters of 45% solution:  $y$  ml**

**Write one equation to represent the total amounts of each solution needed:**

$$x + y = 120$$

**Since 30% of 120 ml is 36, we can write one equation to model the acidic portion:**

$$0.20x + 0.45y = 36$$

**Writing these two equations as a system:**

$$x + y = 120$$

$$0.20x + 0.45y = 36$$

**To solve, multiply both sides of the top equation by either 0.20 to eliminate  $x$  or 0.45 to eliminate  $y$ . The following work is for eliminating  $x$ :**

$$0.20(x + y) = 0.20(120)$$

$$0.20x + 0.45y = 40$$

**which gives**

$$0.20x + 0.20y = 24$$

$$0.20x + 0.45y = 36.$$

**Replacing the top equation with the difference between the bottom equation and top equation results in a new system with the same solutions:**

$$0.25y = 12$$

$$0.20x + 0.45y = 36$$

**The top equation can quickly be solved for  $y$ ,**

$$y = 48,$$

**and substituting  $y = 48$  back into the original first equation allows us to find  $x$ :**

$$x + 48 = 120$$

$$x = 72$$

**Thus, we need 48 ml of the 45% solution and 72 ml of the 20% solution.**

**Discussion (5 minutes)**

- In the previous examples we see how to solve a system of linear equations in two variables using elimination methods. However, what if we have three variables? For example, what are the solutions to the following system of equations?

$$2x + 3y - z = 5$$

$$4x - y - z = -1$$

Allow students time to work together and struggle with this system and realize that they cannot find a unique solution. Include the following third equation and ask students if they can solve it now.

$$x + 4y + z = 12$$

Give students an opportunity to consider solutions or other ideas on how to begin the process of solving this system. After considering their suggestions and providing feedback, guide them through the process in the example below.

*Scaffolding:*

To help students, ask them if they can eliminate two of the variables from either equation (they cannot). Have a discussion around what that means (the graph of the solution set is a line, not a

MP.1

**Example 1 (9 minutes)****Example 1**

Determine the values for  $x$ ,  $y$ , and  $z$  in the following system:

$$2x + 3y - z = 5 \quad (1)$$

$$4x - y - z = -1 \quad (2)$$

$$x + 4y + z = 12 \quad (3)$$

Suggest numbering the equations as shown above to help organize the process.

- Eliminate  $z$  from equations (1) and (2) by subtraction:

$$2x + 3y - z = 5$$

$$4x - y - z = -1$$

$$-2x + 4y = 6$$

- Our goal is to find two equations in two unknowns. Thus, we will also eliminate  $z$  from equations (2) and (3) by adding as follows:

$$4x - y - z = -1$$

$$x + 4y + z = 12$$

$$5x + 3y = 11$$

- Our new system of three equations in three variables has two equations with only two variables in them:

$$-2x + 4y = 6$$

$$4x - y - z = -1$$

$$5x + 3y = 11$$

- These two equations now give us two equations in two variables, which we reviewed how to solve in Exercises 1–2.

$$-2x + 4y = 6$$

$$5x + 3y = 11$$

At this point, you can let students solve this individually or with partners, or guide them through the process if necessary.

- To get matching coefficients, we need to multiply both equations by a constant:

$$5(-2x + 4y) = 5(6) \quad \rightarrow \quad -10x + 20y = 30$$

$$2(5x + 3y) = 2(11) \quad \rightarrow \quad 10x + 6y = 22$$

- Replacing the top equation with the sum of the top and bottom equations together gives the following:

$$26y = 52$$

$$10x + 6y = 22$$

- The new top equation can be solved for  $y$ :

$$y = 2$$

- Replace  $y = 2$  in one of the equations to find  $x$ :

$$5x + 3(2) = 11$$

$$5x + 6 = 11$$

$$5x = 5$$

$$x = 1$$

- Replace  $x = 1$  and  $y = 2$  in any of the original equations to find  $z$ :

$$2(1) + 3(2) - z = 5$$

$$2 + 6 - z = 5$$

$$8 - z = 5$$

$$z = 3$$

- The solution,  $x = 1$ ,  $y = 2$ , and  $z = 3$ , can be written compactly as an ordered triple of numbers  $(1, 2, 3)$ .

You might want to point out to your students that the point  $(1, 2, 3)$  can be thought of as a point in a three-dimensional coordinate plane, and that it is, like a two-by-two system of equations, the intersection point in three-space of the three planes given by the graphs of each equation. These concepts are not the point of this lesson, so addressing them is optional.

Point out that a linear system involving three variables requires three equations in order for the solution to possibly be a single point.

The following problems provide examples of situations that require solving systems of equations in three variables.

Exercise 4 (8 minutes)

Exercise 4

Given the system below, determine the values of  $r$ ,  $s$ , and  $u$  that satisfy all three equations.

$$r + 2s - u = 8$$

$$s + u = 4$$

$$r - s - u = 2$$

*Adding the second and third equation together produces the equation  $r = 6$ . Substituting this into the first equation and adding it to the second gives  $6 + 3s = 12$ , so that  $s = 2$ . Replacing  $s$  with 2 in the second equation gives  $u = 2$ . The solution to this system of equations is  $(6, 2, 2)$ .*

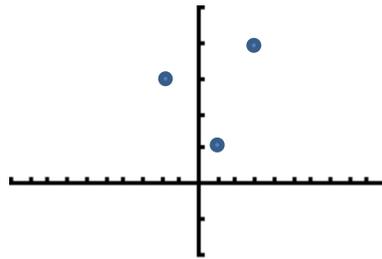
Exercise 5 (6 minutes)

Exercise 5

Find the equation of the form  $y = ax^2 + bx + c$  that satisfies the points  $(1, 6)$ ,  $(3, 20)$ , and  $(-2, 15)$ .

*$a = 2$ ,  $b = -1$ ,  $c = 5$ ; therefore, the quadratic equation is  $y = 2x^2 - x + 5$ .*

Students may need help setting this up. A graph of the points may help.



- Since we know three ordered pairs, we can create three equations.

$$6 = a + b + c$$

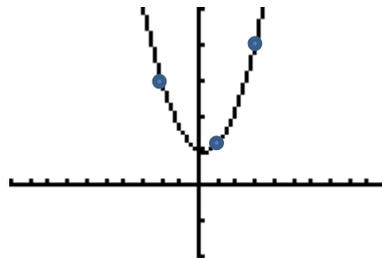
$$20 = 9a + 3b + c$$

$$15 = 4a - 2b + c$$

MP.7

Ask students to explain where the three equations came from. Then have them use the technique from Example 1 to solve this system.

Have students use a graphing utility to graph the equation using the coefficient solutions to confirm the answer.



**Closing (2 minutes)**

- Having solved systems of two linear equations, we see in the lesson that in order to solve a linear system in three variables, we need three equations. How many equations might we need to solve a system with four variables? Five?

**Exit Ticket (5 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 30: Linear Systems in Three Variables

### Exit Ticket

For the following system, determine the values of  $p$ ,  $q$ , and  $r$  that satisfy all three equations:

$$2p + q - r = 8$$

$$q + r = 4$$

$$p - q = 2$$

Exit Ticket Sample Solutions

For the following system, determine the values of  $p$ ,  $q$ , and  $r$  that satisfy all three equations:

$$2p + q - r = 8$$

$$q + r = 4$$

$$p - q = 2$$

$p = 4, q = 2, r = 2$ , or equivalently  $(4, 2, 2)$

Problem Set Sample Solutions

Solve the following systems.

1.  $x + y = 3$   
 $y + z = 6$   
 $x + z = 5$

$x = 1, y = 2, z = 4$  or  $(1, 2, 4)$

3.  $2a + 4b + c = 5$   
 $a - 4b = -6$   
 $2b + c = 7$

$a = -2, b = 1, c = 5$  or  $(-2, 1, 5)$

5.  $r + 3s + t = 3$   
 $2r - 3s + 2t = 3$   
 $-r + 3s - 3t = 1$

$r = 3, s = \frac{1}{3}, t = -1$  or  $(3, \frac{1}{3}, -1)$

7.  $x = 3(y - z)$   
 $y = 5(z - x)$   
 $x + y = z + 4$

$x = 3, y = 5, z = 4$  or  $(3, 5, 4)$

9.  $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 5$   
 $\frac{1}{x} + \frac{1}{y} = 2$   
 $\frac{1}{x} - \frac{1}{z} = -2$

$x = 1, y = 1, z = \frac{1}{3}$  or  $(1, 1, \frac{1}{3})$

2.  $r = 2(s - t)$   
 $2t = 3(s - r)$   
 $r + t = 2s - 3$

$r = 2, s = 4, t = 3$ , or  $(2, 4, 3)$

4.  $2x + y - z = -5$   
 $4x - 2y + z = 10$   
 $2x + 3y + 2z = 3$

$x = \frac{1}{2}, y = -2, z = 4$  or  $(\frac{1}{2}, -2, 4)$

6.  $x - y = 1$   
 $2y + z = -4$   
 $x - 2z = -6$

$x = -2, y = -3, z = 2$  or  $(-2, -3, 2)$

8.  $p + q + 3r = 4$   
 $2q + 3r = 7$   
 $p - q - r = -2$

$p = 2, q = 5, r = -1$  or  $(2, 5, -1)$

10.  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 6$   
 $\frac{1}{b} + \frac{1}{c} = 5$   
 $\frac{1}{a} - \frac{1}{b} = -1$

$a = 1, b = \frac{1}{2}, c = \frac{1}{3}$  or  $(1, \frac{1}{2}, \frac{1}{3})$

11. Find the equation of the form  $y = ax^2 + bx + c$  whose graph passes through the points  $(1, -1)$ ,  $(3, 23)$ , and  $(-1, 7)$ .

$$y = 4x^2 - 4x - 1$$

12. Show that for any number  $t$ , the values  $x = t + 2$ ,  $y = 1 - t$ , and  $z = t + 1$  are solutions to the system of equations below.

$$x + y = 3$$

$$y + z = 2$$

(in this situation, we say that  $t$  parameterizes the solution set of the system.)

$$x + y = (t + 2) + (1 - t) = 3$$

$$y + z = (1 - t) + (t + 1) = 2$$

13. Some rational expressions can be written as the sum of two or more rational expressions whose denominators are the factors of its denominator (called a *partial fraction decomposition*). Find the partial fraction decomposition for the following example by filling in the blank to make the equation true for all  $n$  except 0 and  $-1$ .

$$\frac{1}{n(n+1)} = \frac{\quad}{n} - \frac{1}{n+1}$$

Adding  $\frac{1}{n+1}$  to both sides of the equations, we have  $\frac{1}{n(n+1)} + \frac{1}{n+1} = \frac{1}{n(n+1)} + \frac{n}{n(n+1)} = \frac{(n+1)}{n(n+1)} = \frac{1}{n}$ , so

$\frac{1}{n(n+1)} + \frac{1}{n+1} = \frac{1}{n}$ , and equivalently we have  $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ . Thus, the blank should contain a 1.

14. A chemist needs to make 40 ml of a 15% acid solution. He has a 5% acid solution and a 30% acid solution on hand. If he uses the 5% and 30% solutions to create the 15% solution, how many ml of each will he need?

*He will need 24 ml of the 5% solution and 16 ml of the 30% solution.*

15. An airplane makes a 400 mile trip against a head wind in 4 hours. The return trip takes 2.5 hours, the wind now being a tail wind. If the plane maintains a constant speed with respect to still air, and the speed of the wind is also constant and does not vary, find the still-air speed of the plane and the speed of the wind.

*The speed of the plane in still wind is 130 mph, and the speed of the wind is 30 mph.*

16. A restaurant owner estimates that she needs in small change the same number of dimes as pennies and nickels together and the same number of pennies as nickels. If she gets \$26 worth of pennies, nickels, and dimes, how should they be distributed?

*She will need 200 dimes (\$20 worth), 100 nickels (\$5 worth), and 100 pennies (\$1 worth) for a total of \$26.*



## Lesson 31: Systems of Equations

### Student Outcomes

- Students solve systems of linear equations in two variables and systems of a linear and a quadratic equation in two variables.
- Students understand that the points at which the two graphs of the equations intersect correspond to the solutions of the system.

### Lesson Notes

Students review the solution of systems of linear equations, move on to systems of equations that represent a line and a circle and systems that represent a line and a parabola, and make conjectures as to how many points of intersection there can be in a given system of equations. They sketch graphs of a circle and a line to visualize the solution to a system of equations, solve the system algebraically, and note the correspondence between the solution and the intersection. Then they do the same for graphs of a parabola and a line.

The principal standards addressed in this lesson are **A-REI.C.6** (solve systems of linear equations exactly and approximately, e.g., with graphs, focusing on pairs of linear equations in two variables) and **A-REI.C.7** (solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically). The standards MP.5 (use appropriate tools strategically) and MP.8 (look for and express regularity in repeated reasoning) are also addressed.

### Materials

Graph paper, straightedge, compass, and a tool for displaying graphs (e.g., projector, smart board, white board, chalk board, or squared poster paper)

### Classwork

#### Exploratory Challenge 1 (8 minutes)

In this exercise, the students review ideas about systems of linear equations from Module 4 in Grade 8 (**A-REI.C.6**). Consider distributing graph paper for students to use throughout this lesson. Begin by posing the following problem for students to work on individually:

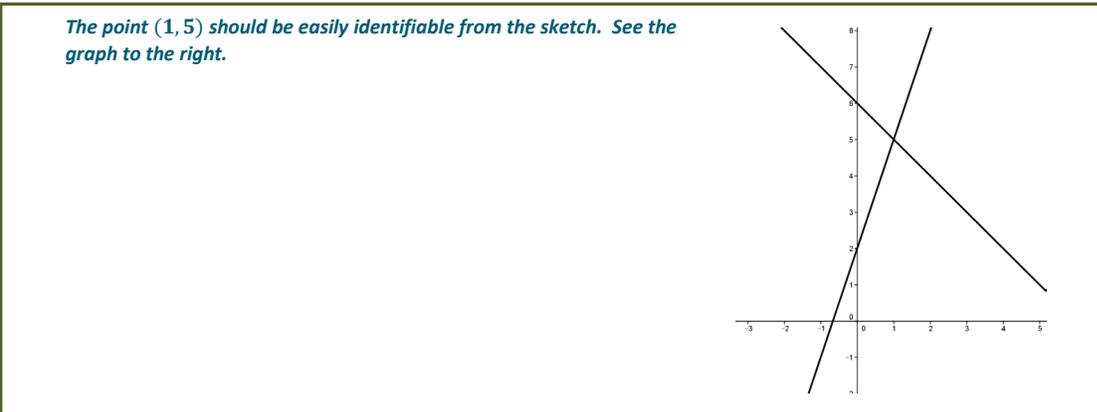
##### Exploratory Challenge 1

- a. Sketch the lines given by  $x + y = 6$  and  $-3x + y = 2$  on the same set of axes, and then solve the pair of equations algebraically to verify your graphical solution.

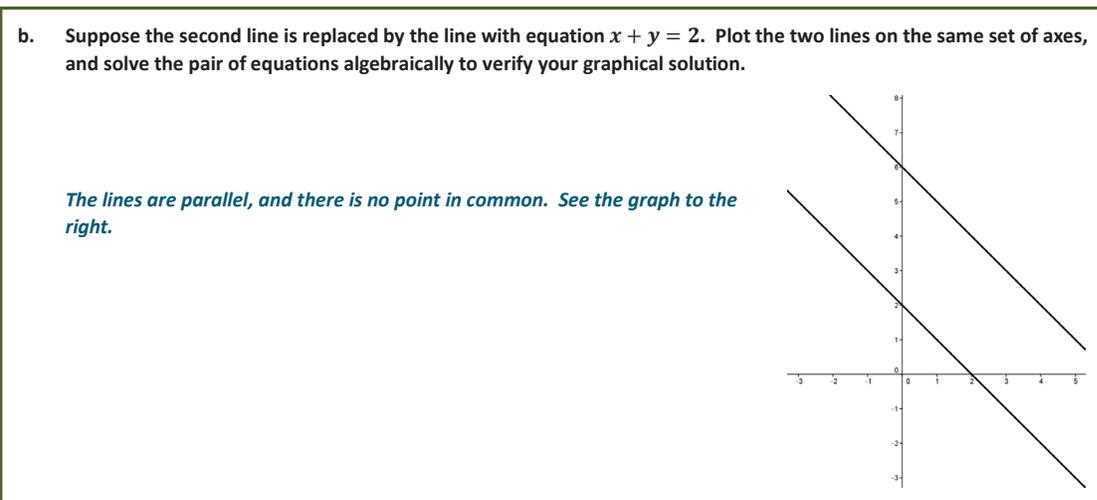
##### Scaffolding:

Circulate to identify students who might be asked to display their sketches and solutions.

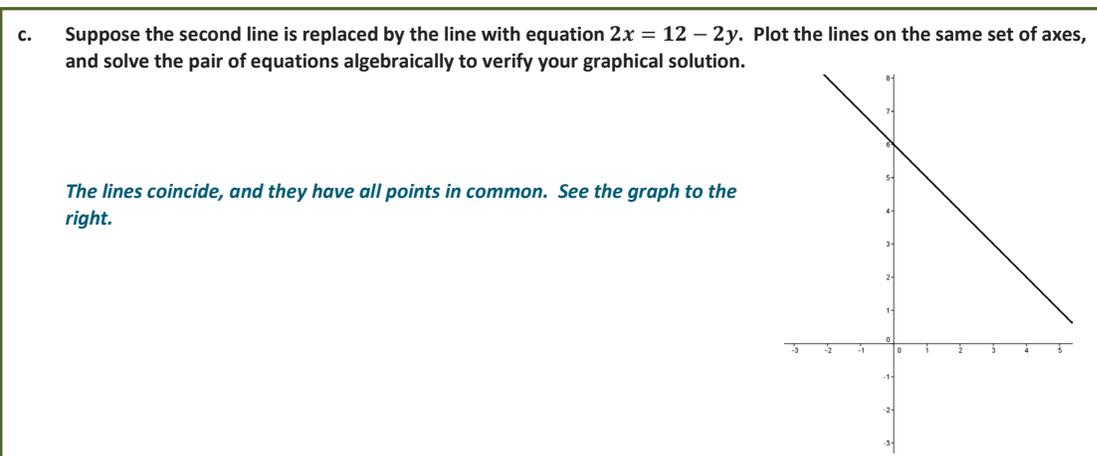
Once the students have made a sketch, ask one of them to use the display tool and draw the two graphs for the rest of the class to see. While the student is doing that, ask the other students how many points are shared (one) and what the coordinates of that point are.



Point out that in this case, there is one solution. Now change the problem as follows. Then discuss the question as a class, and ask one or two students to show their sketches using the display tool.



Point out that in this case, there is no solution. Now change the problem again as follows, and again discuss the question as a class. Then ask one or two students to show their sketches using the display tool.



Point out that in this third case, there is an infinite number of solutions. Discuss the following problem as a class.

- d. We have seen that a pair of lines can intersect in 1, 0, or an infinite number of points. Are there any other possibilities?

*No. Students should convince themselves and each other that these three options exhaust the possibilities for the intersection of two lines.*

### Exploratory Challenge 2 (12 minutes)

In this exercise, students move on to a system of a linear and quadratic equations (A-REI.C.6). Begin by asking students to work in pairs to sketch graphs and develop conjectures about the following item:

#### Exploratory Challenge 2

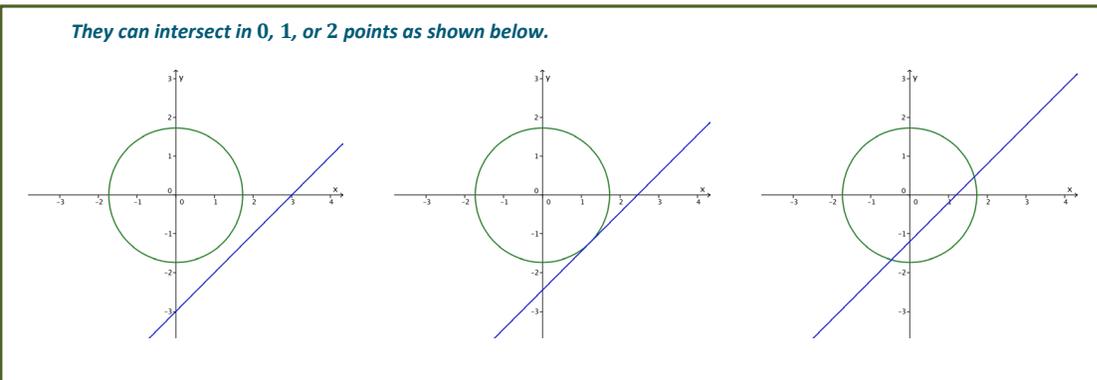
- a. Suppose that instead of equations for a pair of lines, you were given an equation for a circle and an equation for a line. What possibilities are there for the two figures to intersect? Sketch a graph for each possibility.

#### Scaffolding:

- Circulate to assist pairs of students who might be having trouble coming up with all three possibilities.
- For students who are ready, ask them to write equations for the graphs they have sketched.

Once the students have made their sketches, ask one pair to use the display tool and draw the graphs for the rest of the class to see.

*They can intersect in 0, 1, or 2 points as shown below.*



Next, the students should continue to work in pairs to sketch graphs and develop conjectures about the following item (A-REI.C.6):

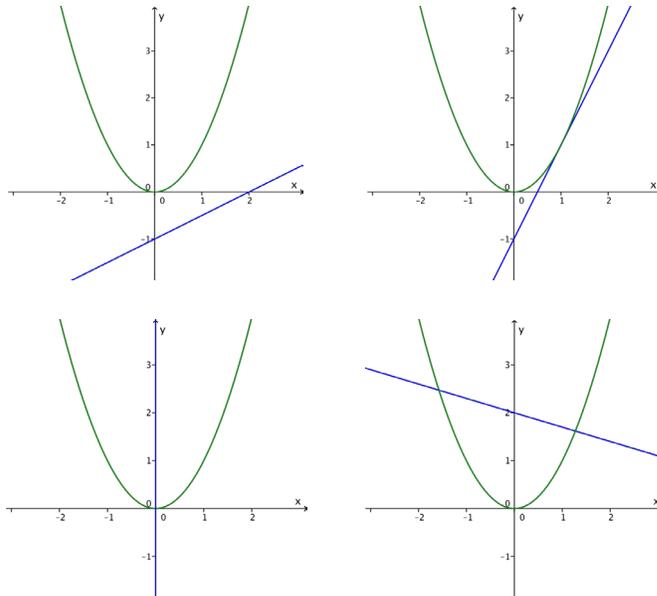
- b. Graph the parabola with equation  $y = x^2$ . What possibilities are there for a line to intersect the parabola? Sketch each possibility.

#### Scaffolding:

- Again circulate to assist pairs of students who might be having trouble coming up with all three possibilities.

Once the students have made their sketches, ask one pair to use the display tool and draw the graphs for the rest of the class to see.

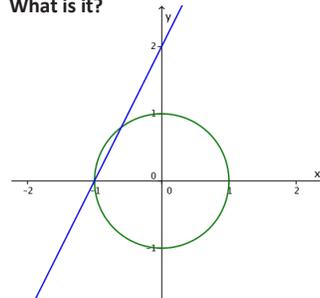
The parabola and line can intersect in 0, 1, or 2 points as shown below. Note that, in contrast to the circle, where all the lines intersecting the circle in one point are tangent to it, lines intersecting the parabola in one point are either tangent to it or are parallel to the parabola's axis of symmetry—in this case, the y-axis.



Next, ask the students to work on the following problem individually (A-REI.C.7):

- c. Sketch the circle given by  $x^2 + y^2 = 1$  and the line given by  $y = 2x + 2$  on the same set of axes. One solution to the pair of equations has a value of  $y$  that is easily identifiable from the sketch. What is it?

The point  $(-1, 0)$  should be easily identifiable from the sketch, but the other point is not.



Once the students have made a sketch, ask one of them to use the display tool and draw the two graphs for the rest of the class to see. While the student is doing that, ask the other students how many points are shared (two) and what the coordinates of those points are.

The students should see that they can substitute the value for  $y$  in the second equation into the first equation. In other words, they need to solve the following quadratic equation (A-REI.B.4).

d. Solve  $x^2 + (2x + 2)^2 = 1$

*Factoring or using the quadratic formula, the students should find that the solutions to the quadratic equation are  $-1$  and  $-\frac{3}{5}$ .*

*If  $x = -1$ , then  $y = 0$ , as the sketch shows, so  $(-1, 0)$  is a solution. If  $x = -\frac{3}{5}$ , then  $y = 2\left(-\frac{3}{5}\right) + 2 = \frac{4}{5}$ , so  $\left(-\frac{3}{5}, \frac{4}{5}\right)$  is another solution.*

Note that the problem above does not explicitly tell students to look for intersection points. Thus, the exercise assesses not only whether they can solve the system but also whether they understand that the intersection points of the graphs correspond to solutions of the system.

Students should understand that to solve the system of equations, we look for points that lie on the line and the circle. The points that lie on the circle are precisely those that satisfy  $x^2 + y^2 = 1$ , and the points that lie on the line are those that satisfy  $y = 2x + 2$ . So points on both are the intersection.

**Exercise 1 (8 minutes)**

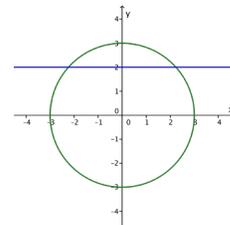
Pose the following three-part problem for students to work on individually and then discuss as a class.

**Exercise 1**

1. Draw a graph of the circle with equation  $x^2 + y^2 = 9$ .

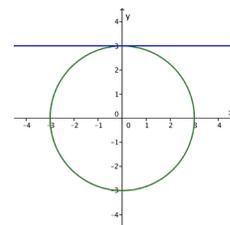
a. What are the solutions to the system of circle and line when the circle is given by  $x^2 + y^2 = 9$  and the line is given by  $y = 2$ ?

*Substituting  $y = 2$  in the equation of the circle yields  $x^2 + 4 = 9$ , so  $x^2 = 5$ , and  $x = \sqrt{5}$  or  $x = -\sqrt{5}$ . The solutions are  $(-\sqrt{5}, 2)$  and  $(\sqrt{5}, 2)$ .*



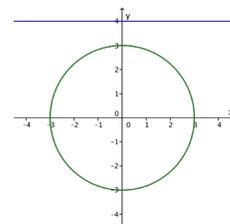
b. What happens when the line is given by  $y = 3$ ?

*Substituting  $y = 3$  in the equation of the circle yields  $x^2 + 9 = 9$ , so  $x^2 = 0$ . The line is tangent to the circle, and the solution is  $(0, 3)$ .*



c. What happens when the line is given by  $y = 4$ ?

*Substituting  $y = 4$  in the equation of the circle yields  $x^2 + 16 = 9$ , so  $x^2 = -7$ . Since there are no real numbers that satisfy  $x^2 = -7$ , there is no solution to this equation. This indicates that the line and circle do not intersect.*



**Exercises 2–6 (8 minutes)**

Students will need graph paper for this portion of the lesson. Complete Exercise 2 in groups so that students can check answers with each other. Then they can do Exercises 3 to 6 individually or in groups as they choose. Assist with the exercises if students have trouble understanding what it means to “verify your results both algebraically and graphically.”

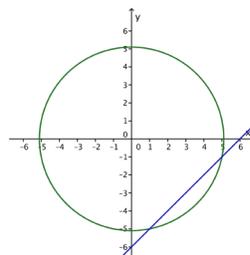
*Scaffolding (for advanced learners):*

- Create two different systems of one linear equation and one quadratic equation that have one solution at  $(0, 2)$ .

**Exercises 2–6**

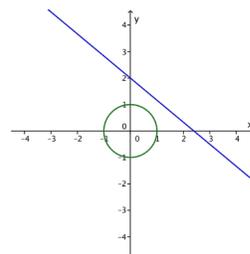
2. By solving the equations as a system, find the points common to the line with equation  $x - y = 6$  and the circle with equation  $x^2 + y^2 = 26$ . Graph the line and the circle to show those points.

*$(5, -1)$  and  $(1, -5)$ . See picture to the right.*



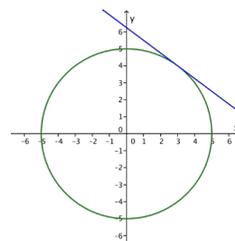
3. Graph the line given by  $5x + 6y = 12$  and the circle given by  $x^2 + y^2 = 1$ . Find all solutions to the system of equations.

*There is no real solution; the line and circle do not intersect. See picture to the right.*



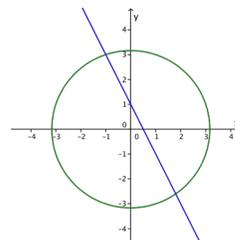
4. Graph the line given by  $3x + 4y = 25$  and the circle given by  $x^2 + y^2 = 25$ . Find all solutions to the system of equations. Verify your result both algebraically and graphically.

*The line is tangent to the circle at  $(3, 4)$ , which is the only solution. See picture to the right.*



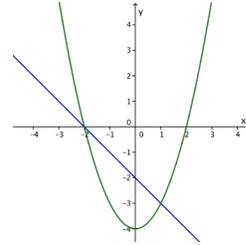
5. Graph the line given by  $2x + y = 1$  and the circle given by  $x^2 + y^2 = 10$ . Find all solutions to the system of equations. Verify your result both algebraically and graphically.

*The line and circle intersect at  $(-1, 3)$  and  $(\frac{9}{5}, -\frac{13}{5})$ , which are the two solutions. See picture to the right.*



6. Graph the line given by  $x + y = -2$  and the quadratic curve given by  $y = x^2 - 4$ . Find all solutions to the system of equations. Verify your result both algebraically and graphically.

*The line and the parabola intersect at  $(1, -3)$  and  $(-2, 0)$ , which are the two solutions. See picture to the right.*



### Closing (4 minutes)

Ask students to respond to these questions with a partner or in writing. Share their responses as a class.

MP.1

- How does graphing a line and a quadratic curve help you solve a system consisting of a linear and a quadratic equation?
- What are the possibilities for the intersection of a line and a quadratic curve, and how are they related to the number of solutions of a system of linear and quadratic equations?

#### Scaffolding:

- Perhaps create a chart with the summary that can serve as a reminder to the students.

Present and discuss the Lesson Summary.

#### Lesson Summary

Here are some steps to consider when solving systems of equations that represent a line and a quadratic curve.

- Solve the linear equation for  $y$  in terms of  $x$ . This is equivalent to rewriting the equation in slope-intercept form. Note that working with the quadratic equation first would likely be more difficult and might cause the loss of a solution.
- Replace  $y$  in the quadratic equation with the expression involving  $x$  from the slope-intercept form of the linear equation. That will yield an equation in one variable.
- Solve the quadratic equation for  $x$ .
- Substitute  $x$  into the linear equation to find the corresponding value of  $y$ .
- Sketch a graph of the system to check your solution.

Be sure to note that in the case of the circle, the reverse process of solving the equation for the circle first—for either  $x$  or  $y$ —and then substituting in the linear equation would have yielded an equation with a complicated radical expression and might have led students to miss part of the solution by considering only the positive square root.

### Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 31: Systems of Equations

### Exit Ticket

Make and explain a prediction about the nature of the solution to the following system of equations and then solve it.

$$x^2 + y^2 = 25$$

$$4x + 3y = 0$$

Illustrate with a graph. Verify your solution and compare it with your initial prediction.

## Exit Ticket Sample Solutions

Make and explain a prediction about the nature of the solution to the following system of equations and then solve it.

$$\begin{aligned}x^2 + y^2 &= 25 \\ 4x + 3y &= 0\end{aligned}$$

Illustrate with a graph. Verify your solution and compare it with your initial prediction.

**Prediction:** By inspection of the equations, students should conclude that the circle is centered at the origin and that the line goes through the origin. So, the solution should consist of two points.

**Solution:** Solve the linear equation for one of the variables:  $y = -\frac{4x}{3}$ .

**Substitute that variable in the quadratic equation:**  $x^2 + \left(-\frac{4x}{3}\right)^2 = 25$ .

**Remove parentheses and combine like terms:**  $25x^2 - 25 \cdot 9 = 0$ ,  
so  $x^2 - 9 = 0$ .

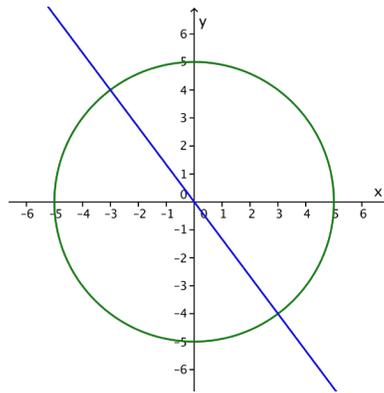
**Solve the quadratic equation in  $x$ :**  $(x + 3)(x - 3) = 0$ , which gives the roots 3 and  $-3$ .

**Substitute into the linear equation:** If  $x = 3$ , then  $y = -4$ ; if  $x = -3$ , then  $y = 4$ .

**As the graph shows, the solution is the two points of intersection of the circle and the line:**  $(3, -4)$  and  $(-3, 4)$ .

**An alternative solution would be to solve the linear equation for  $x$  instead of  $y$ , getting the quadratic equation  $(y + 4)(y - 4) = 0$ , which gives the roots 4 and  $-4$  and the same points of intersection.**

**As noted before, solving the quadratic equation for  $x$  or  $y$  first is not a good procedure. It can lead to a complicated radical expression and loss of part of the solution.**



## Problem Set Sample Solutions

Problem 4 yields a system with no real solution, and the graph shows that the circle and line have no point of intersection in the coordinate plane. In Problems 5 and 6, the curve is a parabola. In Problem 5, the line intersects the parabola in two points, whereas in Problem 6, the line is tangent to the parabola, and there is only one point of intersection. Note that there would also have been only one point of intersection if the line had been the line of symmetry of the parabola.

1. Where do the lines given by  $y = x + b$  and  $y = 2x + 1$  intersect?

*Since we do not know the value of  $b$ , we cannot solve this problem by graphing, and we will have to approach it algebraically. Eliminating  $y$  gives the equation*

$$\begin{aligned}x + b &= 2x + 1 \\ x &= b - 1\end{aligned}$$

*Since  $x = b - 1$ , we have  $y = x + b = (b - 1) + b = 2b - 1$ . Thus, the lines intersect at the point  $(b - 1, 2b - 1)$ .*

2. Find all solutions to the following system of equations.

$$(x - 2)^2 + (y + 3)^2 = 4$$

$$x - y = 3$$

Illustrate with a graph.

Solve the linear equation for one of the variables:  $x = y + 3$ .

Substitute that variable in the quadratic equation:

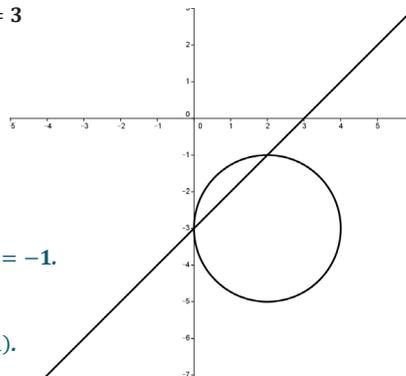
$$(y + 3 - 2)^2 + (y + 3)^2 = 4.$$

Rewrite the equation in standard form:  $2y^2 + 8y + 6 = 0$ .

Solve the quadratic equation:  $2(y + 3)(y + 1) = 0$ , so  $y = -3$  or  $y = -1$ .

If  $y = -3$ , then  $x = 0$ . If  $y = -1$ , then  $x = 2$ .

As the graph shows, the solution is the two points  $(0, -3)$  and  $(2, -1)$ .



3. Find all solutions to the following system of equations.

$$x + 2y = 0$$

$$x^2 - 2x + y^2 - 2y - 3 = 0$$

Illustrate with a graph.

Solve the linear equation for one of the variables:  $x = -2y$ .

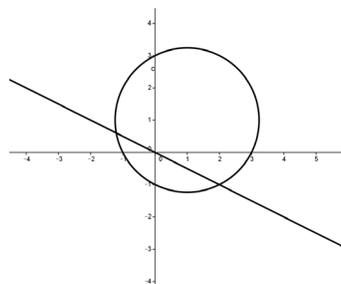
Substitute that variable in the quadratic equation:  $(-2y)^2 - 2(-2y) + y^2 - 2y - 3 = 0$ .

Rewrite the equation in standard form:  $5y^2 + 2y - 3 = 0$ .

Solve the quadratic equation:  $(5y - 3)(y + 1) = 0$ , so  $y = \frac{3}{5}$  or  $y = -1$ .

If  $y = \frac{3}{5}$ , then  $x = -\frac{6}{5}$ . If  $y = -1$ , then  $x = 2$ .

As the graph shows, the solutions are the two points:  $(-\frac{6}{5}, \frac{3}{5})$  and  $(2, -1)$ .



4. Find all solutions to the following system of equations.

$$x + y = 4$$

$$(x + 3)^2 + (y - 2)^2 = 10$$

Illustrate with a graph.

Solve the linear equation for one of the variables:  $x = 4 - y$

Substitute that variable in the quadratic equation:  $(4 - y + 3)^2 + (y - 2)^2 = 10$

Rewrite the equation in standard form:  $2y^2 - 18y + 43 = 0$

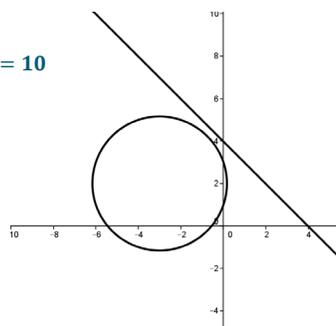
Solve the equation using the quadratic formula:

$$y = \frac{18 + \sqrt{324 - 344}}{4} \text{ or } y = \frac{18 - \sqrt{324 - 344}}{4}$$

So we have  $y = \frac{1}{2}(9 + \sqrt{-5})$  or  $y = \frac{1}{2}(9 - \sqrt{-5})$

Therefore, there is no real solution to the system.

As the graph shows, the line and circle do not intersect.



5. Find all solutions to the following system of equations.

$$y = -2x + 3$$

$$y = x^2 - 6x + 3$$

Illustrate with a graph.

The linear equation is already solved for one of the variables:  $y = -2x + 3$ .

Substitute that variable in the quadratic equation:  $-2x + 3 = x^2 - 6x + 3$ .

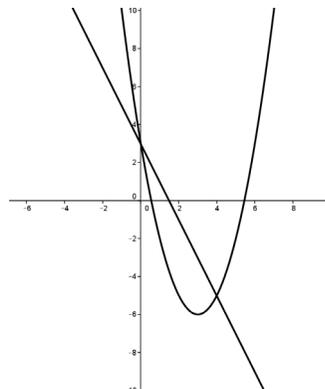
Rewrite the equation in standard form:  $x^2 - 4x = 0$ .

Solve the quadratic equation:  $x(x - 4) = 0$ .

So,  $x = 0$  or  $x = 4$ .

If  $x = 0$ , then  $y = 3$ . If  $x = 4$ , then  $y = -5$ .

As the graph shows, the solutions are the two points  $(0, 3)$  and  $(4, -5)$ .



6. Find all solutions to the following system of equations.

$$-y^2 + 6y + x - 9 = 0$$

$$6y = x + 27$$

Illustrate with a graph.

Solve the second equation for  $x$ :  $x = 6y - 27$ .

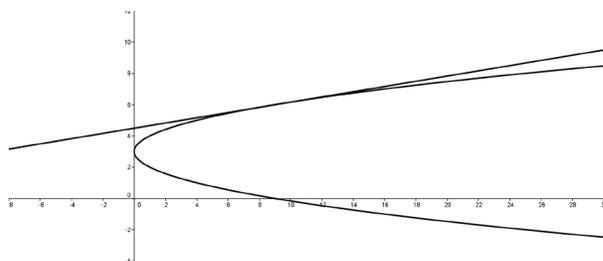
Substitute in the first equation:  $-y^2 + 6y + 6y - 27 - 9 = 0$ .

Combine like terms:  $-y^2 + 12y - 36 = 0$ .

Rewrite the equation in standard form and factor:  $-(y - 6)^2 = 0$ .

Therefore,  $y = 6$ . Then  $x = 6y - 27$ , so  $x = 9$ .

There is only one solution  $(9, 6)$ , and as the graph shows, the line is tangent to the parabola.



An alternative solution would be to solve the linear equation for  $y$  instead of  $x$ , getting the quadratic equation  $(x - 9)(x - 9) = 0$ , which gives the repeated root  $x = 9$  and the same point of tangency  $(9, 6)$ .

Another alternative solution would be to solve the quadratic equation for  $x$ , so that  $x = y^2 - 6y + 9$ . Substituting in the linear equation would yield  $6y = y^2 - 6y + 9 + 27$ . Converting that to standard form would give  $y^2 - 12y + 36 = 0$ , which gives the repeated root  $y = 6$ , as in the first solution. Note that in this case, unlike when the graph of the quadratic equation is a circle, the quadratic equation can be solved for  $x$  in terms of  $y$  without getting a radical expression.

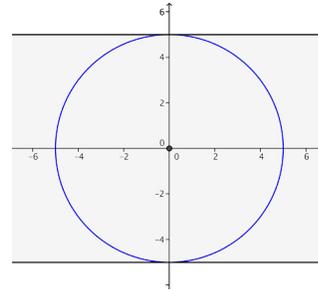
7. If the following system of equations has two solutions, what is the value of  $k$ ?

$$x^2 + y^2 = 25$$

$$y = k$$

Illustrate with a graph.

*The center of the circle is the origin, and the line is parallel to the x-axis. Therefore, as the graph shows, there are two solutions only when  $-5 < k < 5$ .*



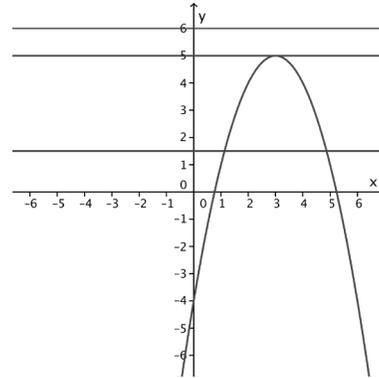
8. If the following system of equations has exactly one solution, what is the value of  $k$ ?

$$y = 5 - (x - 3)^2$$

$$y = k$$

Illustrate with a graph.

*The parabola opens down, and its axis of symmetry is the vertical line  $x = 3$ . The line  $y = k$  is a horizontal line and will intersect the parabola in either two, one, or no points. It intersects the parabola in one point only if it passes through the vertex of the parabola, which is  $k = 5$ .*



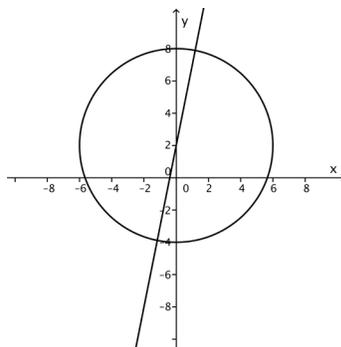
9. If the following system of equations has no solutions, what is the value of  $k$ ?

$$x^2 + (y - k)^2 = 36$$

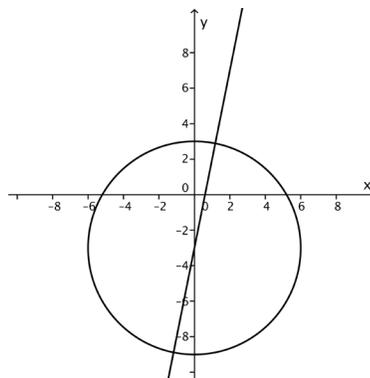
$$y = 5x + k$$

Illustrate with a graph.

*The circle has radius 6 and center  $(0, k)$ . The line has slope 5 and crosses the y-axis at  $(0, k)$ . Since for any value of  $k$  the line passes through the center of the circle, the line intersects the circle twice. (In the figure on the left below  $k = 2$ , and in the one on the right below,  $k = -3$ .) There is no value of  $k$  for which there is no solution.*



$k = 2$



$k = -3$



## Lesson 32: Graphing Systems of Equations

### Student Outcomes

- Students develop facility with graphical interpretations of systems of equations and the meaning of their solutions on those graphs. For example, they can use the distance formula to find the distance between the centers of two circles and thereby determine whether the circles intersect in 0, 1, or 2 points.
- By completing the squares, students can convert the equation of a circle in general form to the center-radius form and, thus, find the radius and center. They can also convert the center-radius form to the general form by removing parentheses and combining like terms.
- Students understand how to solve and graph a system consisting of two quadratic equations in two variables.

### Lesson Notes

This lesson is an extension that goes beyond what is required in the standards. In particular, the standard **A-REI.C.7** (solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically) does not extend to a system of two quadratic equations, which is a natural culmination of the types of systems formed by linear and quadratic equations. The lesson also addresses standard MP.8 (look for and express regularity in repeated reasoning).

The lesson begins with a brief review of the distance formula and its connection both to the Pythagorean Theorem and to the center-radius equation of a circle. The distance formula will be used extensively in the next few lessons, so be sure to review it with students. Students also briefly review how to solve and graph a system of a linear equation and an equation of a circle. They then move to the main focus of the lesson, which is graphing and solving systems of pairs of quadratic equations whose graphs include parabolas as well as circles

### Materials

This lesson requires use of graphing calculators or computer software, such as the Wolfram Alpha engine, the GeoGebra package, or the Geometer's Sketchpad software for graphing geometric figures, plus a tool for displaying graphs, such as a projector, smart board, white board, chalk board, or squared poster paper.

### Classwork

#### Opening (1 minute)

Begin with questions that should remind students of the distance formula and how it is connected to the Pythagorean Theorem.

- Suppose you have a point  $A$  with coordinates  $(1, 3)$ . Find the distance  $AB$  if  $B$  has coordinates:

1.  $(4, 2)$

*Answer:*  $AB = \sqrt{10}$

2.  $(-3, 1)$

*Answer:*  $AB = 2\sqrt{5}$

3.  $(x, y)$

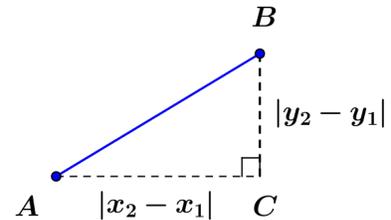
*Answer:*  $AB = \sqrt{(x-1)^2 + (y-3)^2}$

If the students cannot recall the distance formula (in the coordinate plane), they may need to be reminded of it.

**The Distance Formula:** Given two points  $(x_1, y_1)$  and  $(x_2, y_2)$ , the distance  $d$  between these points is given by the formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

If  $A$  and  $B$  are points with coordinates  $(x_1, y_1)$  and  $(x_2, y_2)$ , then the distance between them is the length  $AB$ . Draw horizontal and vertical lines through  $A$  and  $B$  to intersect in point  $C$  and form right triangle  $\triangle ABC$ . The length of the horizontal side is the difference in the  $x$ -coordinates  $|x_2 - x_1|$ , and the length of the vertical side is the difference in the  $y$ -coordinates  $|y_2 - y_1|$ . The Pythagorean Theorem gives the length of the hypotenuse as  $(AB)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$ . Taking the square root gives the distance formula.



**Opening Exercise (3 minutes)**

Make sure the students all have access to, and familiarity with, some technology (calculator or computer software) for graphing lines and circles in the coordinate plane. Have them work individually on the following exercise.

**Opening Exercise**

Given the line  $y = 2x$ , is there a point on the line at a distance 3 from  $(1, 3)$ ? Explain how you know.

*Yes, there are two such points. They are the intersection of the line  $y = 2x$  and the circle  $(x - 1)^2 + (y - 3)^2 = 9$ . (The intersection points are roughly  $(0.07, 0.15)$  and  $(2.73, 5.45)$ .)*

Draw a graph showing where the point is.

*There are actually two such points. See the graph to the right.*

**Scaffolding:**

- Circulate to identify and help students who might have trouble managing the graphing tool.

Students should compare the graph they have drawn with that of a neighbor.

**Exercise 1 (5 minutes)**

This exercise reviews the solution of a simple system consisting of a linear equation and the equation of a circle from the perspective of the defining property of a circle (**A-REI.C.7**).

**Exercise 1**

Solve the system  $(x - 1)^2 + (y - 2)^2 = 2^2$  and  $y = 2x + 2$ .

*Substituting  $2x + 2$  for  $y$  in the quadratic equation allows us to find the  $x$ -coordinates.*

$$\begin{aligned}(x - 1)^2 + ((2x + 2) - 2)^2 &= 4 \\(x^2 - 2x + 1) + 4x^2 &= 4 \\5x^2 - 2x - 3 &= 0 \\(x - 1)(5x + 3) &= 0\end{aligned}$$

So,  $x = 1$  or  $x = -\frac{3}{5}$ , and the intersection points are  $(-\frac{3}{5}, \frac{4}{5})$  and  $(1, 4)$ .

What are the coordinates of the center of the circle?

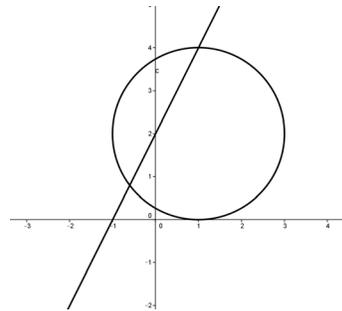
$(1, 2)$

What can you say about the distance from the intersection points to the center of the circle?

*Because they are points on the circle and the radius of the circle is 2, the intersection points are 2 units away from the center. This can be verified by the distance formula.*

Using your graphing tool, graph the line and the circle.

*See the graph at the right.*

**Example 1 (5 minutes)**

It is important to keep in mind that not all quadratic equations in two variables represent circles.

**Example 1**

Rewrite  $x^2 + y^2 - 4x + 2y = -1$  by completing the square in both  $x$  and  $y$ . Describe the circle represented by this equation.

*Rearranging terms gives  $x^2 - 4x + y^2 + 2y = -1$ .*

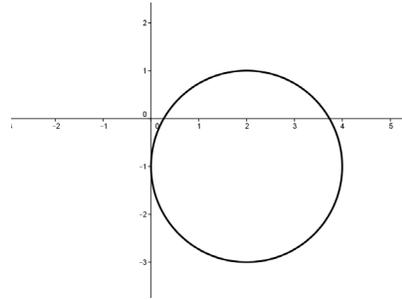
*Then, completing the square in both  $x$  and  $y$ , we have*

$$\begin{aligned}(x^2 - 4x + 4) + (y^2 + 2y + 1) &= -1 + 4 + 1 \\(x - 2)^2 + (y + 1)^2 &= 4.\end{aligned}$$

*This is the equation of a circle with center  $(2, -1)$  and radius 2.*

Using your graphing tool, graph the circle.

See the graph to the right.



In contrast, consider the following equation:  $x^2 + y^2 - 2x - 8y = -19$ .

Rearranging terms gives  $x^2 - 2x + y^2 - 8y = -19$ .

Then, completing the square in both  $x$  and  $y$ , we have

$$\begin{aligned}(x^2 - 2x + 1) + (y^2 - 8y + 16) &= -19 + 1 + 16 \\ (x - 1)^2 + (y - 4)^2 &= -2,\end{aligned}$$

which is not a circle, because then the radius would be  $\sqrt{-2}$ .

What happens when you use your graphing tool with this equation?

The tool cannot draw the graph. There are no points in the plane that satisfy this equation, so the graph is empty.

### Exercise 2 (5 minutes)

Allow students time to think these questions over, draw some pictures, and discuss with a partner before discussing as a class.

#### Exercise 2

Consider a circle with radius 5 and another circle with radius 3. Let  $d$  represent the distance between the two centers. We want to know how many intersections there are of these two circles for different values of  $d$ . Draw figures for each case.

- a. What happens if  $d = 8$ ?

If the distance is 8, then the circles will touch at only one point. We say that the circles are externally tangent.

- b. What happens if  $d = 10$ ?

If the distance is 10, the circles do not intersect, and one circle is outside of the other.

- c. What happens if  $d = 1$ ?

If the distance is 1, the circles do not intersect, but one circle lies inside the other.

- d. What happens if  $d = 2$ ?

If the distance is 2, the circles will touch at only one point, with one circle inside the other. We say that the circles are internally tangent.

e. For which values of  $d$  do the circles intersect in exactly one point? Generalize this result to circles of any radius.  
*If  $d = 8$  or  $d = 2$ , the circles will be tangent. In general, if  $d$  is either the sum or the difference of the radii, then the circles will be tangent.*

f. For which values of  $d$  do the circles intersect in two points? Generalize this result to circles of any radius.  
*If  $2 < d < 8$ , the circles will intersect in two points. In general, if  $d$  is between the sum and the difference of the radii then the circles will be tangent.*

g. For which values of  $d$  do the circles not intersect? Generalize this result to circles of any radius.  
*The circles do not intersect if  $d < 2$  or  $d > 8$ . In general, if  $d$  is smaller than the difference of the radii or larger than the sum of the radii, then the circles will not intersect.*

Example 2 (5 minutes)

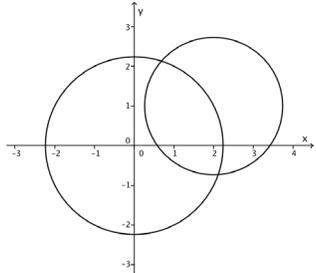
**Example 2**  
 Find the distance between the centers of the two circles with equations below, and use that distance to determine in how many points these circles intersect.

$$x^2 + y^2 = 5$$

$$(x - 2)^2 + (y - 1)^2 = 3$$

*The first circle has center  $(0, 0)$ , and the second circle has center  $(2, 1)$ . Using the distance formula, the distance between the centers of these circles is  $d = \sqrt{(2 - 0)^2 + (1 - 0)^2} = \sqrt{5}$ .*

*Since the distance between the centers is between the sum and the difference of the two radii, that is,  $\sqrt{5} - \sqrt{3} < \sqrt{5} < \sqrt{5} + \sqrt{3}$ , we know that the circles must intersect in two distinct points.*



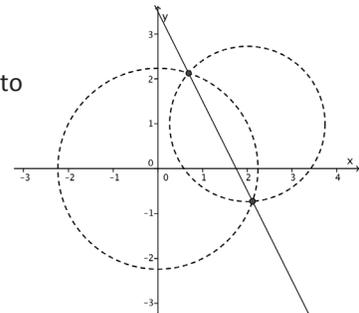
- Find the coordinates of the intersection points of the circles.
  - *Multiplying out the terms in the second equation gives:*

$$x^2 - 4x + 4 + y^2 - 2y + 1 = 3.$$
  - *We subtract the first equation:  $x^2 + y^2 = 5$ .*
- The reason for subtracting is that we are removing repeated information in the two equations.
  - *We get  $-4x - 2y = -7$ , which is the equation of the line through the two intersection points of the circles.*
- To find the intersection points, we find the intersection of the line  $-4x - 2y = -7$  and the circle  $x^2 + y^2 = 5$ .
- As with the other systems of quadratic curves and lines, we solve the linear equation for  $y$  and substitute it into the quadratic equation to find two solutions for  $x$ :  $x = \frac{7}{5} - \frac{\sqrt{51}}{10}$  and  $x = \frac{7}{5} + \frac{\sqrt{51}}{10}$ .

- The corresponding y-values are

$$y = \frac{7}{10} + \frac{\sqrt{51}}{5} \text{ and } y = \frac{7}{10} - \frac{\sqrt{51}}{5}.$$

- The graph of the circles and the line through the intersection points is shown to the right.



**Exercise 3 (4 minutes)**

This exercise concerns a system of equations that represents circles that do not intersect.

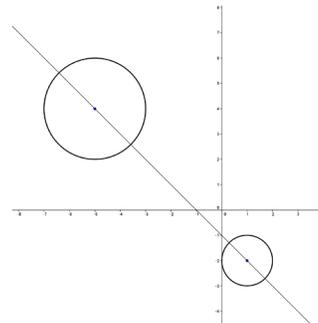
**Exercise 3**

Use the distance formula to show algebraically and graphically that the following two circles do not intersect.

$$(x - 1)^2 + (y + 2)^2 = 1$$

$$(x + 5)^2 + (y - 4)^2 = 4$$

The centers of the two circles are  $(1, -2)$  and  $(-5, 4)$ , and the radii are 1 and 2. The distance between the two centers is  $\sqrt{6^2 + 6^2} = 6\sqrt{2}$ , which is greater than  $1 + 2 = 3$ . The graph to the right also shows that the circles do not intersect.



**Example 3 (10 minutes)**

Work through this example with the whole class, showing students how to find the tangent to a circle at a point and one way to determine how many points of intersection there are for a line and a circle.

**Example 3**

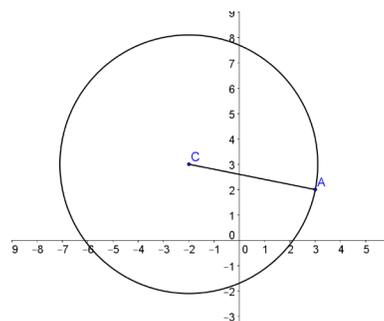
Point  $A(3, 2)$  is on a circle whose center is  $C(-2, 3)$ . What is the radius of the circle?

The distance from  $A$  to  $C$  is given by  $\sqrt{(3 + 2)^2 + (2 - 3)^2} = \sqrt{26}$ , which is the length of the radius.

What is the equation of the circle? Graph it.

Given the center and the radius, we can write the equation of the circle as  $(x + 2)^2 + (y - 3)^2 = 26$ .

The graph is shown at the right.



Use the fact that the tangent at  $A(3, 2)$  is perpendicular to the radius at that point to find the equation of the tangent line. Then graph it.

The slope of the tangent line is the opposite reciprocal of the slope of  $\overline{AC}$ . The slope of  $\overline{AC}$  is  $\frac{3-2}{-2-3} = -\frac{1}{5}$ , so the slope of the tangent line is 5. Using the point-slope form of the equation of a line with slope 5 and passing through point  $(3, 2)$  gives

$$y - 2 = 5(x - 3)$$

$$y = 5x - 13.$$

The equation of the tangent line is, therefore,  $y = 5x - 13$ .

Find the coordinates of point  $B$ , the second intersection of the line  $\overline{AC}$  and the circle.

The system  $(x + 2)^2 + (y - 3)^2 = 26$  and  $5y = -x + 13$  can be solved by substituting  $x = 13 - 5y$  into the equation of the circle, which yields  $(13 - 5y + 2)^2 + (y - 3)^2 = 26$ . This gives  $26(y - 2)(y - 4) = 0$ . Thus, the  $y$ -coordinate is either 2 or 4. If  $y = 2$ , then  $x = 13 - 5 \cdot 2 = 3$ , and if  $y = 4$ , then  $x = 13 - 5 \cdot 4 = -7$ . Since  $A$  has coordinates  $(3, 2)$ , it follows that  $B$  has coordinates  $(-7, 4)$ .

What is the equation of the tangent to the circle at  $(-7, 4)$ ? Graph it as a check.

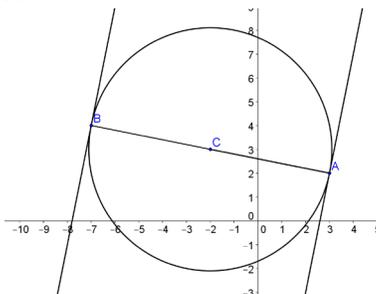
Using the point-slope form of a line with slope 5 and point  $(-7, 4)$ :

$$y - 4 = 5(x + 7)$$

$$y = 5x + 39.$$

The equation of the tangent line is, therefore,  $y = 5x + 39$ .

The graph is shown to the right.



The lines  $y = 5x + b$  are parallel to the tangent lines to the circle at points  $A$  and  $B$ . How is the  $y$ -intercept  $b$  for these lines related to the number of times each line intersects the circle?

When  $b = -13$  or  $b = 39$ , the line is tangent to the circle, intersecting in one point.

When  $-13 < b < 39$ , the line intersects the circle in two points.

When  $b < -13$  or  $b > 39$ , the line and circle do not intersect.

**Closing (2 minutes)**

Ask students to summarize how to convert back and forth between the center-radius equation of a circle and the general quadratic equation of a circle.

Ask students to speculate about what might occur with respect to intersections if one or two of the quadratic equations in the system are not circles.

**Exit Ticket (5 minutes)**

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 32: Graphing Systems of Equations

### Exit Ticket

1. Find the intersection of the two circles

$$x^2 + y^2 - 2x + 4y - 11 = 0$$

and

$$x^2 + y^2 + 4x + 2y - 9 = 0.$$

2. The equations of the two circles in Question 1 can also be written as follows:

$$(x - 1)^2 + (y + 2)^2 = 16$$

and

$$(x + 2)^2 + (y + 1)^2 = 14.$$

Graph the circles and the line joining their points of intersection.

3. Find the distance between the centers of the circles in Questions 1 and 2.

## Exit Ticket Sample Solutions

1. Find the intersection of the two circles

$$x^2 + y^2 - 2x + 4y - 11 = 0$$

and

$$x^2 + y^2 + 4x + 2y - 9 = 0.$$

*Subtract the second equation from the first:  $-6x + 2y - 2 = 0$ .*

*Solve the equation for  $y$ :  $y = 3x + 1$ .*

*Substitute in the first equation:  $x^2 + (3x + 1)^2 - 2x + 4(3x + 1) - 11 = 0$ .*

*Remove parentheses and combine like terms:  $5x^2 + 8x - 3 = 0$ .*

*Substitute in the quadratic equation to find two values:  $x = \frac{-4}{5} - \frac{\sqrt{31}}{5}$  and  $x = \frac{-4}{5} + \frac{\sqrt{31}}{5}$ .*

*The corresponding  $y$ -values are the following:  $y = \frac{-7}{5} - \frac{3\sqrt{31}}{5}$  and  $y = \frac{-7}{5} + \frac{3\sqrt{31}}{5}$ .*

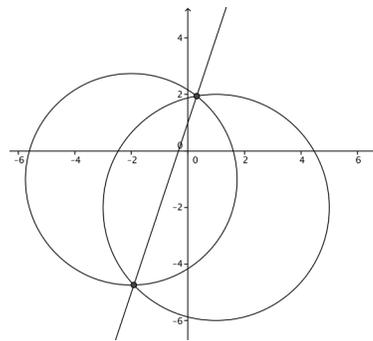
2. The equations of the two circles in Question 1 can also be written as follows:

$$(x - 1)^2 + (y + 2)^2 = 16$$

and

$$(x + 2)^2 + (y + 1)^2 = 14.$$

Graph the circles and the line joining their points of intersection.



*See the graph to the right.*

3. Find the distance between the centers of the circles in Questions 1 and 2.

*The center of the first circle is  $(1, -2)$ , and the center of the second circle is  $(-2, -1)$ . We then have*

$$d = \sqrt{(-2 - 1)^2 + (-1 + 2)^2} = \sqrt{9 + 1} = \sqrt{10}.$$

## Problem Set Sample Solutions

In this Problem Set, after solving some problems dealing with the distance formula, the students continue converting between forms of the equation of a circle and then move on to solving and graphing systems of quadratic equations, some of which represent circles and some of which do not.

1. Use the distance formula to find the distance between the points  $(-1, -13)$  and  $(3, -9)$ .

*Using the formula with  $(-1, -13)$  and  $(3, -9)$ :*

$$d = \sqrt{(3 - (-1))^2 + ((-9) - (-13))^2}$$

$$d = \sqrt{4^2 + 4^2} = \sqrt{16 + 16} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}.$$

*Therefore, the distance is  $4\sqrt{2}$ .*

2. Use the distance formula to find the length of the longer side of the rectangle whose vertices are (1, 1), (3, 1), (3, 7), and (1, 7).

Using the formula with (1, 1) and (1, 7):

$$d = \sqrt{(1-1)^2 + (7-1)^2}$$

$$d = \sqrt{(0)^2 + (6)^2} = \sqrt{36} = 6$$

Therefore, the length of the longer side is 6.

3. Use the distance formula to find the length of the diagonal of the square whose vertices are (0, 0), (0, 5), (5, 5), and (5, 0).

Using the formula with (0, 0) and (5, 5):

$$d = \sqrt{(5-0)^2 + (5-0)^2}$$

$$d = \sqrt{(5-0)^2 + (5-0)^2} = \sqrt{25 + 25} = 5\sqrt{2}$$

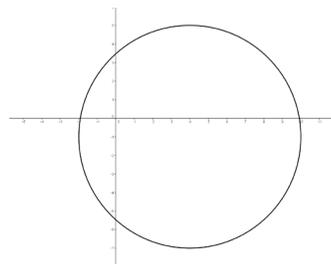
Therefore, the length of the diagonal is  $5\sqrt{2}$ .

Write an equation for the circles in Exercises 4–6 in the form  $(x - h)^2 + (y - k)^2 = r^2$ , where the center is  $(h, k)$  and the radius is  $r$  units. Then write the equation in the standard form  $x^2 + ax + y^2 + by + c = 0$ , and construct the graph of the equation.

4. A circle with center (4, -1) and radius 6 units.

$$(x - 4)^2 + (y + 1)^2 = 36; \text{ standard form: } x^2 - 8x + y^2 + 2y - 19 = 0$$

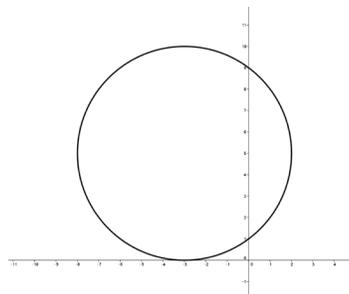
The graph is shown to the right.



5. A circle with center (-3, 5) tangent to the x-axis.

$$(x + 3)^2 + (y - 5)^2 = 25; \text{ standard form: } x^2 + 6x + y^2 - 10y + 9 = 0.$$

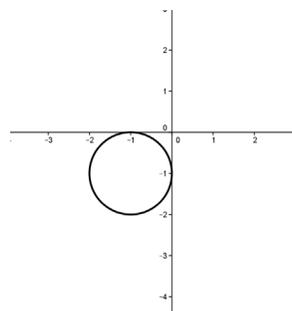
The graph is shown to the right.



6. A circle in the third quadrant, radius 1 unit, tangent to both axes.

$$(x + 1)^2 + (y + 1)^2 = 1; \text{ standard form: } x^2 + 2x + y^2 + 2y + 1 = 0.$$

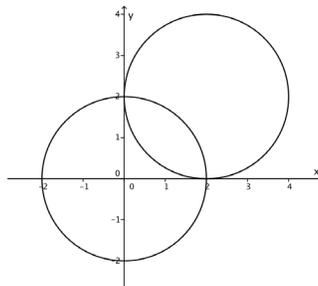
The graph is shown to the right.



7. By finding the radius of each circle and the distance between their centers, show that the circles  $x^2 + y^2 = 4$  and  $x^2 - 4x + y^2 - 4y + 4 = 0$  intersect. Illustrate graphically.

The second circle is  $(x - 2)^2 + (y - 2)^2 = 4$ . Each radius is 2, and the centers are at  $(0, 0)$  and  $(2, 2)$ . The distance between the centers is  $2\sqrt{2}$ , which is less than 4, the sum of the radii.

The graph of the two circles is to the right.



8. Find the points of intersection of the circles  $x^2 + y^2 - 15 = 0$  and  $x^2 - 4x + y^2 + 2y - 5 = 0$ . Check by graphing the equations.

Write the equations as

$$\begin{aligned} x^2 + y^2 &= 15 \\ x^2 + y^2 - 4x + 2y &= 5 \end{aligned}$$

Subtracting the second equation from the first:

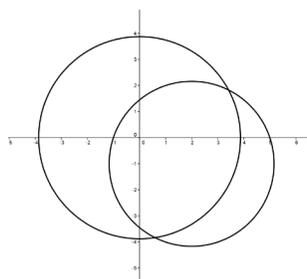
$$4x - 2y = 10,$$

which is equivalent to

$$2x - y = 5.$$

Solving the system  $x^2 + y^2 = 15$  and  $y = 2x - 5$  yields

$(2 + \sqrt{2}, -1 + 2\sqrt{2})$  and  $(2 - \sqrt{2}, -1 - 2\sqrt{2})$ . The graph is to the right.



9. Solve the system  $y = x^2 - 2$  and  $x^2 + y^2 = 4$ . Illustrate graphically.

Substitute  $x^2 = y + 2$  into the second equation:

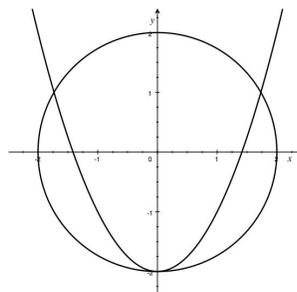
$$\begin{aligned} y + 2 + y^2 &= 4 \\ y^2 + y - 2 &= 0 \\ (y - 1)(y + 2) &= 0 \end{aligned}$$

so  $y = -2$  or  $y = 1$ .

If  $y = -2$ , then  $x^2 = y + 2 = 0$  and thus  $x = 0$ .

If  $y = 1$ , then  $x^2 = y + 2 = 3$ , so  $x = \sqrt{3}$  or  $x = -\sqrt{3}$ .

Thus, there are three solutions  $(0, -2)$ ,  $(\sqrt{3}, 1)$ , and  $(-\sqrt{3}, 1)$ . The graph is to the right.



10. Solve the system  $y = 2x - 13$  and  $y = x^2 - 6x + 3$ . Illustrate graphically.

Substitute  $2x - 13$  for  $y$  in the second equation:  $2x - 13 = x^2 - 6x + 3$ .

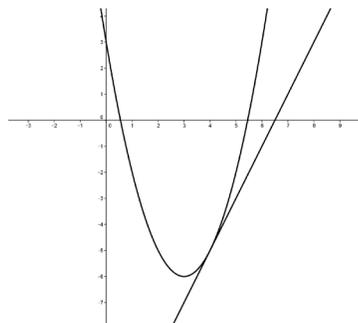
Rewrite the equation in standard form:  $x^2 - 8x + 16 = 0$ .

Solve for  $x$ :  $(x - 4)(x - 4) = 0$ .

The root is repeated, so there is only one solution  $x = 4$ .

The corresponding  $y$ -value is  $y = -5$ , and there is only one solution,  $(4, -5)$ .

As shown to the right, the line is tangent to the parabola.





## Lesson 33: The Definition of a Parabola

### Student Outcomes

- Students model the locus of points at equal distance between a point (focus) and a line (directrix). They construct a parabola and understand this geometric definition of the curve. They use algebraic techniques to derive the analytic equation of the parabola.

### Lesson Notes

A Newtonian reflector telescope uses a parabolic mirror to reflect light to the focus of the parabola, bringing the image of a distant object closer to the eye. This lesson uses the Newtonian telescope to motivate the discussion of parabolas. The precise definitions of a parabola and the axis of symmetry of a parabola are given here. Figure 1 to the right depicts this definition of a parabola. In this diagram,  $FP_1 = P_1Q_1$ ,  $FP_2 = P_2Q_2$ ,  $FP_3 = P_3Q_3$  illustrate that for any point  $P$  on the parabola, the distance between  $P$  and  $F$  is equal to the distance between  $P$  and the line  $L$  along a segment perpendicular to  $L$ .

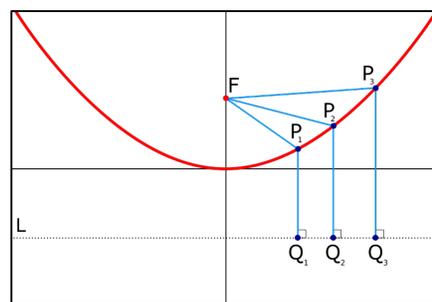


Figure 1

**Parabola:**<sup>G-GPE.A.2</sup> A parabola with directrix  $L$  and focus  $F$  is the set of all points in the plane that are equidistant from the point  $F$  and line  $L$ .

**Axis of Symmetry of a Parabola:**<sup>G-GPE.A.2</sup> The axis of symmetry of a parabola given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus.

**Vertex of a Parabola:**<sup>G-GPE.A.2</sup> The vertex of a parabola is the point where the axis of symmetry intersects the parabola.

This lesson focuses on deriving the analytic equation for a parabola given the focus and directrix (**G-GPE.A.2**) and showing that it is a quadratic equation. In doing so, students are able to tie together many powerful ideas from geometry and algebra, including transformations, coordinate geometry, polynomial equations, the Pythagorean Theorem, and functions.

Parabolas all have the reflective property illustrated in Figure 2. Rays entering the parabola parallel to the axis of symmetry will reflect off the parabola and pass through the focus point  $F$ . A Newtonian telescope uses this property of parabolas.

Parabolas have been studied by mathematicians since at least the 4<sup>th</sup> century B.C. James Gregory's *Optical Promata*, printed in 1663, contains the first known plans for a reflecting telescope using parabolic mirrors, though the idea itself was discussed earlier by many astronomers and mathematicians, including Galileo Galilei, as early as 1616. Isaac Newton, for whom the telescope is now

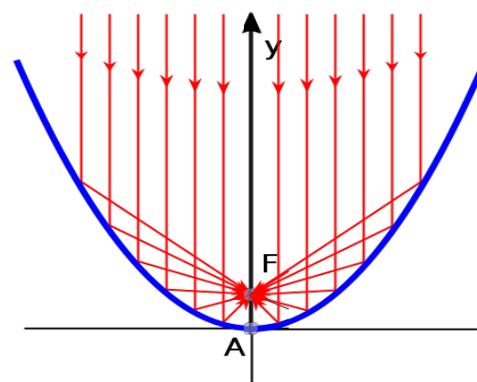
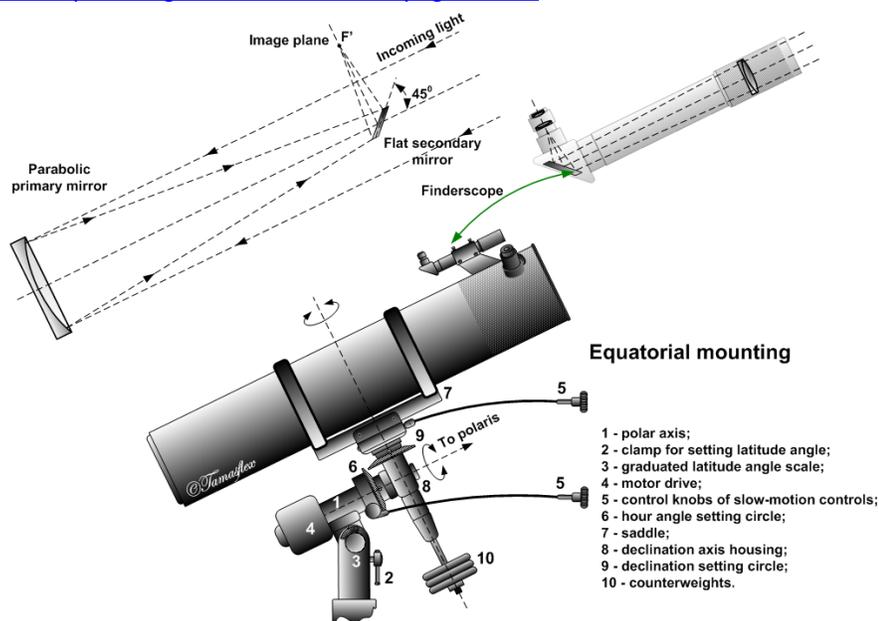


Figure 2

named, needed such a telescope to prove his theory that white light is made up of a spectrum of colors. This theory explained why earlier telescopes that worked by refraction distorted the colors of objects in the sky. However, the technology did not exist at the time to accurately construct a parabolic mirror because of difficulties accurately engineering the curve of the parabola. In 1668, he built a reflecting telescope using a spherical mirror instead of a parabolic mirror, which distorted images but made the construction of the telescope possible. Even with the image distortion caused by the spherical mirror, Newton was able to see the moons of Jupiter without color distortion. Around 1721, John Hadley constructed the first reflecting telescope that used a parabolic mirror.

A Newtonian telescope reflects light back into the tube and requires a second mirror to direct the reflected image to the eyepiece. In a modern Newtonian telescope, the primary mirror is a paraboloid—the surface obtained by rotating a parabola around its axis of symmetry—and a second flat mirror positioned near the focus reflects the image directly to the eyepiece mounted along the side of the tube. A quick image search of the internet will show you simple diagrams of these types of telescopes. This type of telescope remains a popular design today and many amateur astronomers build their own Newtonian telescopes. The diagram shown in the student pages is adapted from this image which can be accessed at <http://en.wikipedia.org/wiki/File:Newton01.png#filelinks>.



## Classwork

### Opening (3 minutes)

The Opening Exercise below gets students thinking about reflections on different shaped lines and curves. From physics, we know that the measure of the angle of reflection of a ray of light is equal to the measure of the angle of incidence when it is bounced off a flat surface. For light reflecting on a curved surface, we measure the angles using the ray of light and the line tangent to the curve where the light ray touches the curve. As described below in the scaffolding box, we mainly want students to understand the shape of the mirror will result in different reflected images. After giving students a few minutes to work on this, ask for their ideas.

- How does each mirror reflect the light?
  - *Mirror 1 bounces light straight back at you.*
  - *Mirror 2 bounces light at a  $90^\circ$  angle across to the other side of the mirror, then it bounces at another  $90^\circ$  angle to away from the mirror.*
  - *Mirror 3 would bounce the light at different angles at different points because the surface is curved.*

Some background information that can help you process the Opening Exercise with students is summarized below.

- Semicircular mirrors do not send all rays of light to a single focus point; this fact can be seen by carefully drawing the path of three rays of light and noticing that they each intersect the other rays in different points after they reflect.
- From physics, the angle of reflection is congruent to the angle of incidence with a line tangent to the curve. On a curved mirror, the slope of the tangent line changes so the rays of light reflect at different angles.
- Remember when working with students to focus on the big ideas: These mirrors will not reflect light back to a single point. Simple student diagrams are acceptable.

After debriefing the opening with your class, introduce the idea of a telescope that uses mirrors to reflect light. We want a curved surface that focuses the incoming light to a single point in order to see reflected images from outer space. The question below sets the stage for this lesson. The rest of the lesson will define a parabola as a curve that meets the requirements of the telescope design.

- Is there a curved shape that will accomplish this goal?

*Scaffolding:*

- Do not get sidetracked if students struggle to accurately draw the reflected light. Work through these as a class if necessary. Emphasize that the light will be reflected differently as the mirror's curvature changes.

**Opening Exercise (2 minutes)**

**Opening Exercise**

Suppose you are viewing the cross-section of a mirror. Where would the incoming light be reflected in each type of design? Sketch your ideas below.

Mirror 1
Mirror 2
Mirror 3

*In Mirror 1, the light would reflect back onto the light rays. In Mirror 2, the light would reflect from one side of the mirror horizontally to the other side, and then reflect back upwards vertically. Incoming and outgoing rays would be parallel. In Mirror 3, the light would reflect back at different angles because the mirror is curved.*

Mirror 1
Mirror 2
Mirror 3

MP.1 & MP.4

To transition from the Opening Exercise to the next discussion, tell students that telescopes work by reflecting light. If we want to see an image without distortion using a telescope, then we need the reflected light to focus on one point. Mirror 3 comes the closest to having this property but does not reflect the rays of light back to a single point. You can model this by showing a sample of a student solution or by providing a sketch that you created.

**Discussion (15 minutes): Telescope Design**

Lead a whole class discussion that ties together the definition of a parabola and its reflective property with Newton’s telescope design requirements. A Newtonian telescope needs a mirror that will focus all the light on a single point to prevent a distorted image. A parabola by definition meets this requirement. During this discussion, you will share the definition of a parabola and how the focus and directrix give the graph its shape. If the distance between the focus and the directrix changes, the parabola’s curvature changes. If we rotate a parabola 180° around the axis of symmetry, we get a curved surface called a *paraboloid*; this is the shape of a parabolic mirror. A Newtonian telescope requires a fairly flat mirror in order to see images of objects that are astronomically far away, so he built his mirror based on a parabola with a relatively large distance between its focus and directrix.

**Discussion**

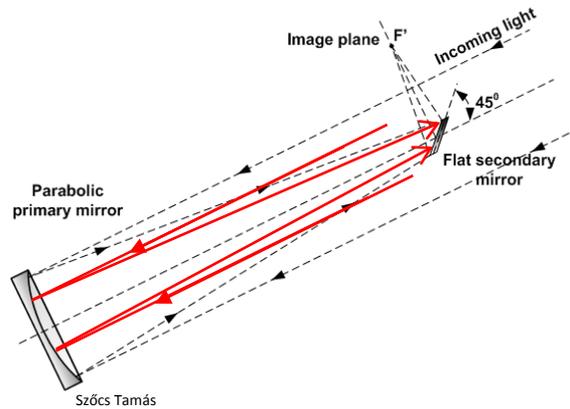
When Newton designed his reflector telescope he understood two important ideas. Figure 1 shows a diagram of this type of telescope.

- The curved mirror needs to focus all the light to a single point that we will call the focus. An angled flat mirror is placed near this point and reflects the light to the eyepiece of the telescope.
- The reflected light needs to arrive at the focus at the same time, otherwise the image is distorted.

Figure 1

The diagram, labeled Figure 1, illustrates the light path in a Newtonian telescope. It shows a large, concave **Parabolic primary mirror** at the bottom left. Dashed lines represent **Incoming light** rays entering from the top right. These rays reflect off the primary mirror and converge towards a point labeled **F'**, which is the **Image plane**. A smaller, flat **Flat secondary mirror** is positioned at a 45-degree angle to intercept the light before it reaches the focus. The light is then reflected out through the side of the telescope. The name **Szócs Tamás** is written at the bottom of the diagram.

In the diagram below, the dotted and solid lines show the incoming light. Model how to add these additional lines to the diagram. Make sure students annotate this on their student pages.



Next, discuss the definition of parabola that appears in the student pages and the reflective property of parabolic curves. Take time to explain what the term equidistant means and how we define distance between a given point and a given line as the shortest distance, which will always be the length of the segment that lies on a line perpendicular to the given line whose endpoints are the given point and the intersection point of the given line and the perpendicular line. Ask students to recall the definition of a circle from Lessons 30–31 and use this to explore the definition of a parabola. Before reading through the definition, give students a ruler and ask them to measure the segments  $\overline{FP_1}$ ,  $\overline{Q_1P_1}$ ,  $\overline{FP_2}$ ,  $\overline{Q_2P_2}$ , etc., in Figure 2. Then have them locate a few more points on the curve and measure the distance from the curve to point  $F$  and from the curve to the horizontal line  $L$ .

**Definition:** A *parabola with directrix  $L$  and focus point  $F$*  is the set of all points in the plane that are equidistant from the point  $F$  and line  $L$ .

Figure 2 to the right illustrates this definition of a parabola. In this diagram,  $FP_1 = P_1Q_1$ ,  $FP_2 = P_2Q_2$ ,  $FP_3 = P_3Q_3$  showing that for any point  $P$  on the parabola, the distance between  $P$  and  $F$  is equal to the distance between  $P$  and the line  $L$ .

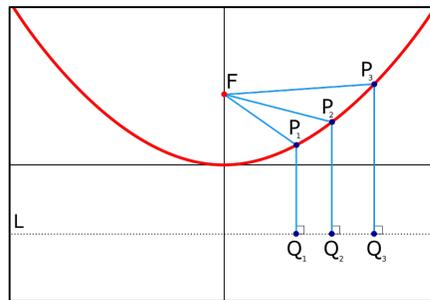


Figure 2

All parabolas have the reflective property illustrated in Figure 3. Rays parallel to the axis will reflect off the parabola and through the focus point,  $F$ .

Thus, a mirror shaped like a rotated parabola would satisfy Newton’s requirements for his telescope design.

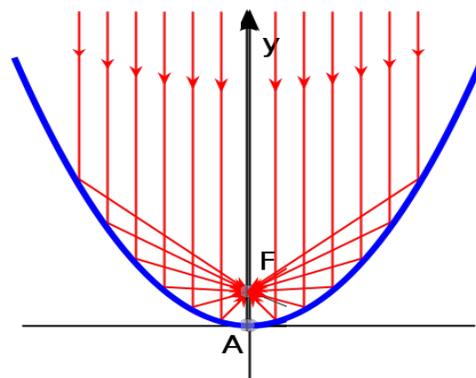


Figure 3

Then, move on to Figures 4 and 5. Here we transition back to thinking about the telescope and show how a mirror in the shape of a parabola (as opposed to say a semi-circle or other curve) will reflect light to the focus point. You could talk about fun house mirrors (modeled by some smartphone apps) that distort images as an example of how other curved surfaces reflect light differently.

- If we want the light to be reflected to the focus at exactly the same time, then what must be true about the distances between the focus and any point on the mirror and the distances between the directrix and any point on the mirror?
  - *Those distances must be equal.*

*Scaffolding:*

- To help students master the new vocabulary associated with parabolas, make a poster using the diagram shown below and label the parts. Adjust as needed for your classes but make sure to include the focus point and directrix in your poster along with marked congruent segments illustrating the definition.

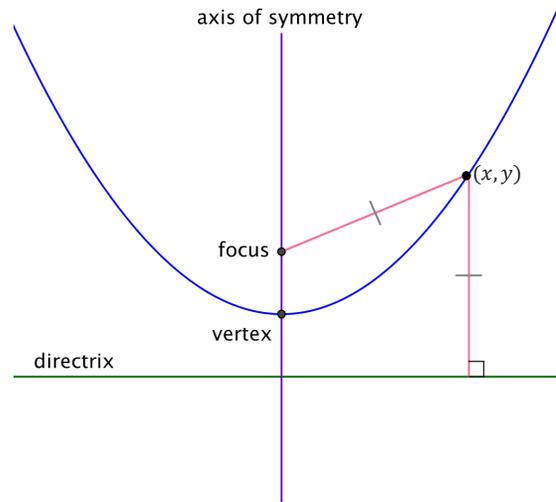


Figure 4 below shows several different line segments representing the reflected light with one endpoint on the curved mirror that is a parabola and the other endpoint at the focus. Anywhere the light hits this type of curved surface, it always reflects to the focus,  $F$ , at exactly the same time.

Figure 5 shows the same image with a directrix. Imagine for a minute that the mirror was not there. Then, the light would arrive at the directrix all at the same time. Since the distance from each point on the parabolic mirror to the directrix is the same as the distance from the point on the mirror to the focus, and the speed of light is constant, it takes the light the same amount of time to travel to the focus as it would have taken it to travel to the directrix. In the diagram, this means that  $AF = AF_A$ ,  $BF = BF_B$ , and so on. Thus, the light rays arrive at the focus at the same time, and the image is not distorted.

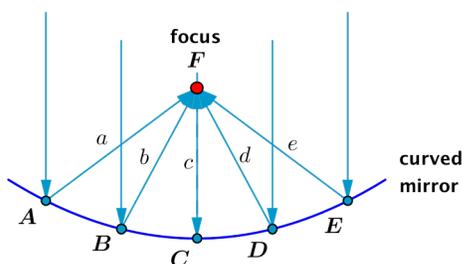


Figure 4

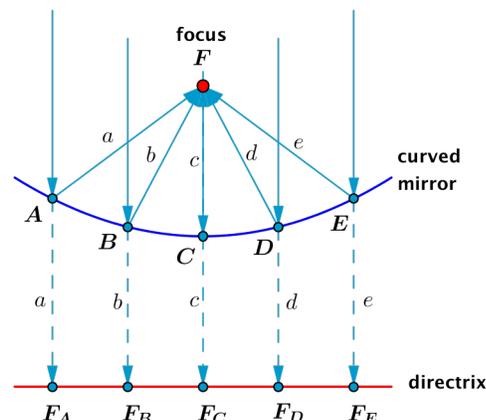


Figure 5

To further illustrate the definition of a parabola, ask students to mark on Figure 5 how the lengths  $AF$ ,  $BF$ ,  $CF$ ,  $DF$ , and  $EF$  are equal to the lengths  $AF_A$ ,  $BF_B$ ,  $CF_C$ ,  $DF_D$ , and  $EF_E$ , respectively.

- How does this definition fit the requirements for a Newtonian telescope?
  - *The definition states exactly what we need to make the incoming light hit the focus at the exact same time since the distance between any point on the curve to the directrix is equal to the distance between any point on the curve and the focus.*
- A parabola looks like the graph of what type of function?
  - *It looks like the graph of an even degree polynomial function.*

Transition to Example 1 by announcing that the prediction that an equation for a parabola would be a quadratic equation will be confirmed using a specific example.

**Example 1 (13 minutes): Finding an Analytic Equation for a Parabola**

This example derives an equation for a parabola given the focus and directrix. As you work through this example, give students time to record the steps along with you. Refer students back to the way the distance formula was used in the definition of a circle and explain we can use it here as well to find an analytic equation for this type of curve.

*Scaffolding:*

- Provide some additional practice with using distance formula to find the length of a line segment for struggling learners.
- Post the distance formula on the wall in your classroom.
- Draw a diagram with the points  $F(0,2)$ ,  $F'(4,0)$ , and  $A(4,5)$  labeled. Have them find the lengths of  $FA$  and  $F'A$ .

**Example 1**  
Given a focus and a directrix, create an equation for a parabola.

**Focus:**  $F = (0, 2)$   
**Directrix:**  $x$ -axis  
**Parabola:**  $P = \{(x, y) \mid (x, y) \text{ is equidistant to } F \text{ and to the } x\text{-axis.}\}$

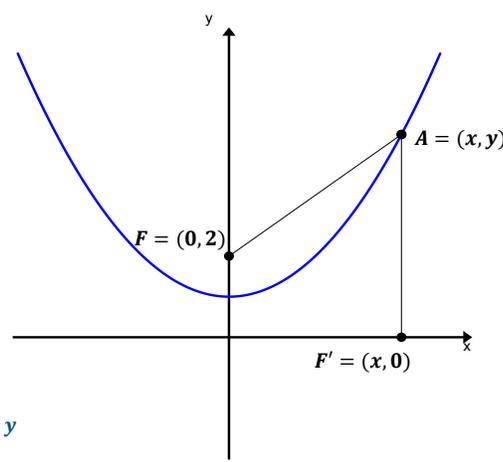
Let  $A$  be any point  $(x, y)$  on the parabola  $P$ . Let  $F'$  be a point on the directrix with the same  $x$ -coordinate as point  $A$ .

What is the length of  $AF'$ ?

$AF' = y$

Use the distance formula to create an expression that represents the length of  $AF$ .

$AF = \sqrt{(x - 0)^2 + (y - 2)^2}$



Create an equation that relates the two lengths and solve it for  $y$ .

Therefore,

$$P = \{(x, y) \mid \sqrt{(x-0)^2 + (y-2)^2} = y\}.$$

The two segments have equal lengths.

$$AF' = AF$$

The length of each segment.

$$y = \sqrt{(x-0)^2 + (y-2)^2}$$

Square both sides of the equation.

$$y^2 = x^2 + (y-2)^2$$

Expand the binomial.

$$y^2 = x^2 + y^2 - 4y + 4$$

Solve for  $y$ .

$$4y = x^2 + 4$$

$$y = \frac{1}{4}x^2 + 1$$

Replacing this equation in the definition of  $P = \{(x, y) \mid (x, y) \text{ is equidistant to } F \text{ and to the } x\text{-axis}\}$  gives the statement  $P = \{(x, y) \mid y = \frac{1}{4}x^2 + 1\}$ .

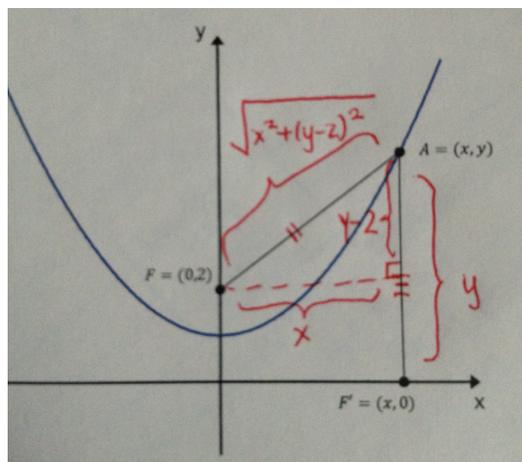
Thus, the parabola  $P$  is the graph of the equation  $y = \frac{1}{4}x^2 + 1$ .

Verify that this equation appears to match the graph shown.

Consider the point where the  $y$ -axis intersects the parabola; let this point have coordinates  $(0, b)$ . From the graph, we see that  $0 < b < 2$ . The distance from the focus  $(0, 2)$  to  $(0, b)$  is  $2 - b$  units, and the distance from the directrix to  $(0, b)$  is  $b$  units. Since  $(0, b)$  is on the parabola, we have  $2 - b = b$ , so that  $b = 1$ . From this perspective, we see that the point  $(0, 1)$  must be on the parabola. Does this point satisfy the equation we found? Let  $x = 0$ . Then our equation gives  $y = \frac{1}{4}x^2 + 1 = \frac{1}{4}(0)^2 + 1$ , so  $(0, 1)$  satisfies the equation. This is the only point that we have determined to be on the parabola at this point, but it provides evidence that the equation matches the graph.

Use these questions below as you work through Example 1. Have students mark the congruent segments on their diagram and record the derivation of the equation as you work it out in front of the class. As you are working through this example, you can remind students that when we work with the distance formula we are applying the Pythagorean Theorem in the coordinate plane. This refers back to their work in both Grade 8 and high school Geometry.

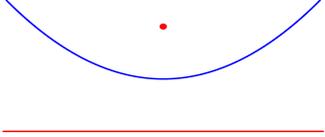
- According to the definition of a parabola, which two line segments in the diagram must have equal measure? Mark them congruent on your diagram.
  - $AF$  must be equal to  $AF'$ .
- How long is  $AF'$ ? How do you know?
  - It is  $y$  units long. The  $y$ -coordinate of point  $A$  is  $y$ .
- Recall the distance formula, and use it to create an expression equal to the length of  $\overline{AF}$ .
  - The distance formula is  $D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ , where  $(x_1, y_1)$  and  $(x_2, y_2)$  are two points in the Cartesian plane.
- How can you tell if this equation represents a quadratic function?
  - The degree of  $x$  will be 2 and the degree of  $y$  will be 1 and each  $x$  will correspond to exactly one  $y$ .



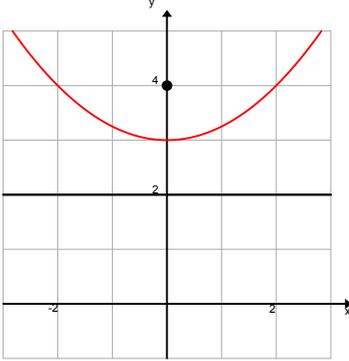
A marked up diagram is shown to the right.

Exercises 1–2 (4 minutes)

**Exercises 1–2**

- Demonstrate your understanding of the definition of a parabola by drawing several pairs of congruent segments given the parabola, its focus, and directrix. Measure the segments that you drew to confirm the accuracy of your sketches in either centimeters or inches.
 
- Derive the analytic equation of a parabola given the focus of  $(0, 4)$  and the directrix  $y = 2$ . Use the diagram to help you work this problem.
  - Label a point  $(x, y)$  anywhere on the parabola.
  - Write an expression for the distance from the point  $(x, y)$  to the directrix.
 
$$y - 2$$
  - Write an expression for the distance from the point  $(x, y)$  to the focus.
 
$$\sqrt{(x - 0)^2 + (y - 4)^2}$$
  - Apply the definition of a parabola to create an equation in terms of  $x$  and  $y$ . Solve this equation for  $y$ .
 
$$y - 2 = \sqrt{(x - 0)^2 + (y - 4)^2}$$

*Solved for  $y$ , we find an equivalent equation is  $y = \frac{1}{4}x^2 + 3$ .*
  - What is the translation that takes the graph of this parabola to the graph of the equation derived in Example 1?
 

*A translation down two units will take this graph of this parabola to the one derived in Example 1.*

Closing (3 minutes)

In this lesson, we limit the discussion to parabolas with a horizontal directrix. We will show in later lessons that all parabolas are similar and that the equations are quadratic regardless of the orientation of the parabola in the plane. Have students answer these questions individually in writing, then discuss their responses as a whole class.

- What is a parabola?
  - *A parabola is a geometric figure that represents the set of all points equidistant from a point called the focus and a line called the directrix.*
- Why are parabolic mirrors used in telescope designs?
  - *Parabolic mirrors are used in telescope designs because they focus reflected light to a single point.*
- What type of analytic equation can be used to model parabolas?
  - *The analytic equation of a parabola whose directrix is a horizontal line is quadratic in  $x$ .*

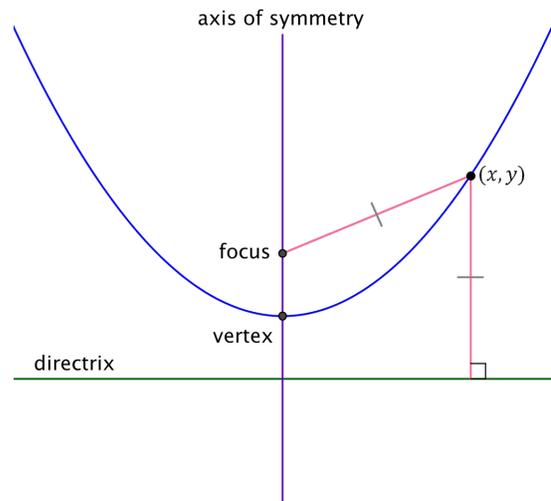
## Lesson Summary

**Parabola:** A parabola with *directrix line*  $L$  and *focus point*  $F$  is the set of all points in the plane that are equidistant from the point  $F$  and line  $L$ .

**Axis of symmetry:** The *axis of symmetry of a parabola* given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus.

**Vertex of a parabola:** The *vertex of a parabola* is the point where the axis of symmetry intersects the parabola.

In the Cartesian plane, the distance formula can help us to derive an analytic equation for the parabola.



## Exit Ticket (5 minutes)



## Exit Ticket Sample Solutions

1. Derive an analytic equation for a parabola whose focus is  $(0, 4)$  and directrix is the  $x$ -axis. Explain how you got your answer.

Let  $(x, y)$  be a point on the parabola. Then, the distance between this point and the focus is given by  $\sqrt{(x - 0)^2 + (y - 4)^2}$ . The distance between the point  $(x, y)$  and the directrix is  $y$ . Then,

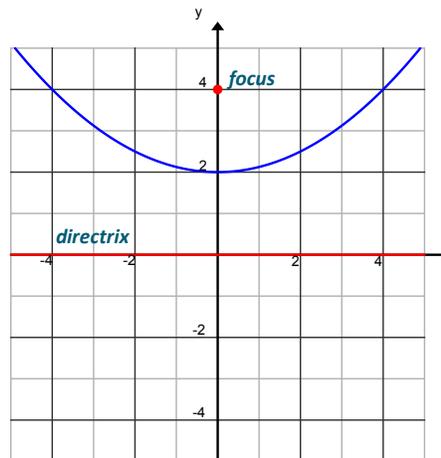
$$y = \sqrt{(x - 0)^2 + (y - 4)^2}$$

$$y^2 = x^2 + y^2 - 8y + 16$$

$$8y = x^2 + 16$$

$$y = \frac{1}{8}x^2 + 2$$

2. Sketch the parabola from Question 1. Label the focus and directrix.

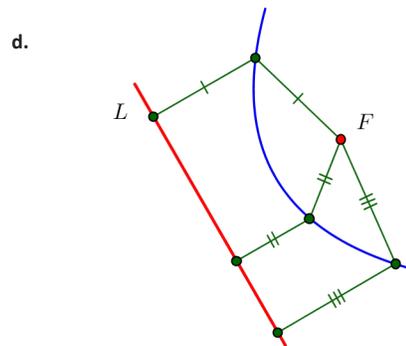
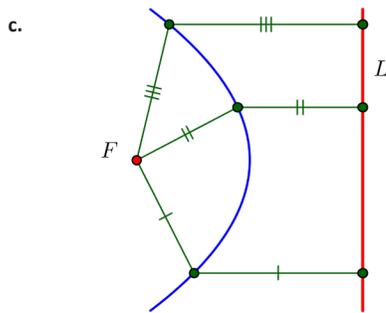
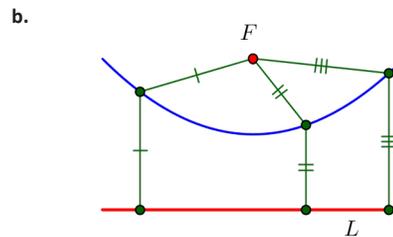
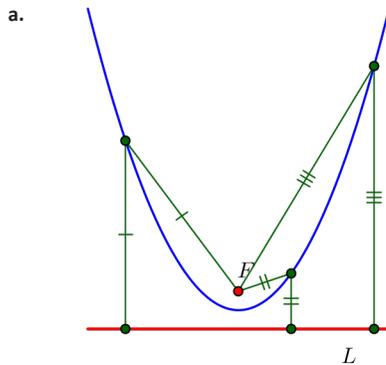


**Problem Set Sample Solutions**

These questions are designed to reinforce the ideas presented in this session. The first few questions focus on applying the definition of a parabola to sketch parabolas. Then the questions scaffold to creating an analytic equation for a parabola given its focus and directrix. Finally, questions near the end of the Problem Set help students to recall transformations of graphs of functions to prepare them for work in future lessons on proving when parabolas are congruent and that all parabolas are similar.

1. Demonstrate your understanding of the definition of a parabola by drawing several pairs of congruent segments given each parabola, its focus, and directrix. Measure the segments that you drew in either inches or centimeters to confirm the accuracy of your sketches.

*Measurements will depend on the location of the segments and the size of the printed document. Segments that should be congruent should be close to the same length.*



2. Find the distance from the point  $(4, 2)$  to the point  $(0, 1)$ .  
*The distance is  $\sqrt{17}$  units.*
3. Find the distance from the point  $(4, 2)$  to the line  $y = -2$ .  
*The distance is 4 units.*
4. Find the distance from the point  $(-1, 3)$  to the point  $(3, -4)$ .  
*The distance is  $\sqrt{65}$  units.*

5. Find the distance from the point  $(-1, 3)$  to the line  $y = 5$ .

*The distance is 2 units.*

6. Find the distance from the point  $(x, 4)$  to the line  $y = -1$ .

*The distance is 5 units.*

7. Find the distance from the point  $(x, -3)$  to the line  $y = 2$ .

*The distance is 5 units.*

8. Find the values of  $x$  for which the point  $(x, 4)$  is equidistant from  $(0, 1)$  and the line  $y = -1$ .

*If  $\sqrt{(x-0)^2 + (4-1)^2} = 5$ , then  $x = 4$  or  $x = -4$ .*

9. Find the values of  $x$  for which the point  $(x, -3)$  is equidistant from  $(1, -2)$  and the line  $y = 2$ .

*If  $\sqrt{(x-1)^2 + (-3-(-2))^2} = 5$ , then  $x = 1 + 2\sqrt{6}$  or  $x = 1 - 2\sqrt{6}$ .*

10. Consider the equation  $y = x^2$ .

- a. Find the coordinates of the three points on the graph of  $y = x^2$  whose  $x$ -values are 1, 2, and 3.

*The coordinates are  $(1, 1)$ ,  $(2, 4)$ , and  $(3, 9)$ .*

- b. Show that each of the three points in part (a) is equidistant from the point  $(0, \frac{1}{4})$  and the line  $y = -\frac{1}{4}$ .

*For  $(1, 1)$ , show that*

$$\sqrt{(1-0)^2 + \left(1-\frac{1}{4}\right)^2} = \sqrt{1 + \frac{9}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$$

*and*

$$1 - \left(-\frac{1}{4}\right) = \frac{5}{4}.$$

*For  $(2, 4)$ , show that*

$$\sqrt{(2-0)^2 + \left(4-\frac{1}{4}\right)^2} = \sqrt{4 + \frac{225}{16}} = \sqrt{\frac{289}{16}} = \frac{17}{4}$$

*and*

$$4 - \left(-\frac{1}{4}\right) = \frac{17}{4}.$$

*For  $(3, 9)$ , show that*

$$\sqrt{(3-0)^2 + \left(9-\frac{1}{4}\right)^2} = \sqrt{9 + \left(\frac{35}{4}\right)^2} = \sqrt{\frac{1369}{16}} = \frac{37}{4}$$

*and*

$$9 - \left(-\frac{1}{4}\right) = \frac{37}{4}.$$

- c. Show that if the point with coordinates  $(x, y)$  is equidistant from the point  $(0, \frac{1}{4})$ , and the line  $y = -\frac{1}{4}$ , then  $y = x^2$ .

The distance from  $(x, y)$  to  $(0, \frac{1}{4})$  is  $\sqrt{(x-0)^2 + (y-\frac{1}{4})^2}$ , and the distance from  $(x, y)$  to the line  $y = -\frac{1}{4}$  is  $y - (-\frac{1}{4}) = y + \frac{1}{4}$ . Setting these distances equal gives

$$\sqrt{(x-0)^2 + (y-\frac{1}{4})^2} = y + \frac{1}{4}$$

$$\sqrt{x^2 + y^2 - \frac{1}{2}y + \frac{1}{16}} = y + \frac{1}{4}$$

$$x^2 + y^2 - \frac{1}{2}y + \frac{1}{16} = (y + \frac{1}{4})^2$$

$$x^2 - \frac{1}{2}y + \frac{1}{16} = \frac{1}{2}y + \frac{1}{16}$$

$$x^2 = y$$

Thus, if a point  $(x, y)$  is the same distance from the point  $(0, \frac{1}{4})$  and the line  $y = -\frac{1}{4}$ , then  $(x, y)$  lies on the parabola  $y = x^2$ .

11. Given the equation  $y = \frac{1}{2}x^2 - 2x$ ,

- a. Find the coordinates of the three points on the graph of  $y = \frac{1}{2}x^2 - 2x$  whose  $x$ -values are  $-2, 0$ , and  $4$ .

The coordinates are  $(-2, 6), (0, 0), (4, 0)$ .

- b. Show that each of the three points in part (a) is equidistant from the point  $(2, -\frac{3}{2})$ , and the line  $y = -\frac{5}{2}$ .

For  $(-2, 6)$ , show that

$$\sqrt{(-2-2)^2 + (6-(-\frac{3}{2}))^2} = \sqrt{16 + \frac{225}{4}} = \sqrt{\frac{289}{4}} = \frac{17}{2}$$

and

$$6 - (-\frac{5}{2}) = \frac{17}{2}.$$

For  $(0, 0)$ , show that

$$\sqrt{(0-2)^2 + (0-(-\frac{3}{2}))^2} = \sqrt{4 + \frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

and

$$0 - (-\frac{5}{2}) = \frac{5}{2}.$$

For  $(4, 0)$ , show that

$$\sqrt{(4-2)^2 + (0-(-\frac{3}{2}))^2} = \sqrt{4 + \frac{9}{4}} = \sqrt{\frac{25}{4}} = \frac{5}{2}$$

and

$$0 - (-\frac{5}{2}) = \frac{5}{2}.$$

- c. Show that if the point with coordinates  $(x, y)$  is equidistant from the point  $(2, -\frac{3}{2})$  and the line  $y = -\frac{5}{2}$  then  $y = \frac{1}{2}x^2 - 2x$ .

The distance from  $(x, y)$  to  $(2, -\frac{3}{2})$  is  $\sqrt{(x-2)^2 + (y + \frac{3}{2})^2}$ , and the distance from  $(x, y)$  to the line  $y = -\frac{5}{2}$  is  $y - (-\frac{5}{2}) = y + \frac{5}{2}$ . Setting these distances equal gives

$$\begin{aligned} \sqrt{(x-2)^2 + (y + \frac{3}{2})^2} &= y + \frac{5}{2} \\ \sqrt{x^2 - 4x + y^2 + 3y + \frac{25}{4}} &= y + \frac{5}{2} \\ x^2 - 4x + y^2 + 3y + \frac{25}{4} &= y^2 + 5y + \frac{25}{4} \\ x^2 - 4x &= 2y \\ \frac{1}{2}(x^2 - 2x) &= y \end{aligned}$$

Thus, if a point  $(x, y)$  is the same distance from the point  $(2, -\frac{3}{2})$ , and the line  $y = -\frac{5}{2}$ , then  $(x, y)$  lies on the parabola  $y = \frac{1}{2}(x^2 - 2x)$ .

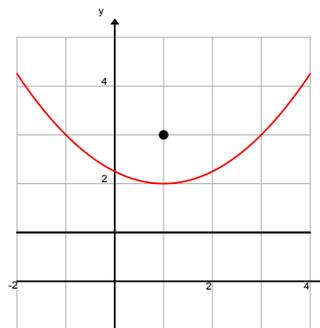
12. Derive the analytic equation of a parabola with focus  $(1, 3)$  and directrix  $y = 1$ . Use the diagram to help you work this problem.

- a. Label a point  $(x, y)$  anywhere on the parabola.
- b. Write an expression for the distance from the point  $(x, y)$  to the directrix.

$$y - 1$$

- c. Write an expression for the distance from the point  $(x, y)$  to the focus  $(1, 3)$ .

$$\sqrt{(x-1)^2 + (y-3)^2}$$



- d. Apply the definition of a parabola to create an equation in terms of  $x$  and  $y$ . Solve this equation for  $y$ .

$$\begin{aligned} y - 1 &= \sqrt{(x-1)^2 + (y-3)^2} \\ (y-1)^2 &= (x-1)^2 + (y-3)^2 \\ y^2 - 2y + 1 &= (x-1)^2 + y^2 - 6y + 9 \\ 4y &= (x-1)^2 + 8 \\ y &= \frac{1}{4}(x-1)^2 + 2 \\ y &= \frac{1}{4}x^2 - \frac{1}{2}x + \frac{9}{4} \end{aligned}$$

- e. Describe a sequence of transformations that would take this parabola to the parabola with equation  $y = \frac{1}{4}x^2 + 1$  derived in Example 1.

A translation 1 unit to the left and 1 unit downward will take this parabola to the one derived in Example 1.

13. Consider a parabola with focus  $(0, -2)$  and directrix on the  $x$ -axis.

a. Derive the analytic equation for this parabola.

$$y = -\frac{1}{4}x^2 - 1$$

- b. Describe a sequence of transformations that would take the parabola with equation  $y = \frac{1}{4}x^2 + 1$  derived in Example 1 to the graph of the parabola in part (a).

*Reflect the graph in Example 1 across the  $x$ -axis to obtain this parabola.*

14. Derive the analytic equation of a parabola with focus  $(0, 10)$  and directrix on the  $x$ -axis.

$$y = \frac{1}{20}x^2 + 5$$



## Lesson 34: Are All Parabolas Congruent?

### Student Outcomes

- Students learn the vertex form of the equation of a parabola and how it arises from the definition of a parabola.
- Students perform geometric operations, such as rotations, reflections, and translations, on arbitrary parabolas to discover standard representations for their congruence classes. In doing so, they learn that all parabolas with the same distance  $p$  between the focus and the directrix are congruent to the graph of  $y = \frac{1}{2p}x^2$ .

### Lesson Notes

This lesson builds upon the previous lesson and applies transformations to show that all parabolas with the same distance between their focus and directrix are congruent. Recall that two figures in the plane are congruent if there exists a finite sequence of rigid motions that maps one onto the other, so it makes sense for us to discuss congruency of parabolas. The lesson closes with a theorem and proof detailing the answer to the question posed in the lesson title. By using transformations in this lesson to determine the conditions under which two parabolas are congruent, this lesson builds coherence with the work students did in Geometry. This lesson specifically asks students to consider how we can use transformations to prove two figures are congruent. Additionally, the lesson reinforces the connections between geometric transformations and transformations of the graphs of functions.

There are many opportunities to provide scaffolding in this lesson for students who are not ready to move quickly to abstract representation. Use technology, patty paper or transparencies, and simple hand-drawn graphs as appropriate to support student learning throughout this lesson. Use the anchor poster created in Lesson 33, and keep key vocabulary words and formulas (e.g., the distance formula) displayed for student reference.

You might choose to break this lesson up into two days; on the first day, explore the definition of congruent parabolas, sketch parabolas given their focus and directrix, and explore the consequences of changing the distance between the focus and directrix,  $p$ . On the second day, derive the analytic equation for a parabola with a given focus and directrix and vertex at the origin, and prove the theorem on parabola congruence.

### Classwork

#### Opening Exercise (7 minutes)

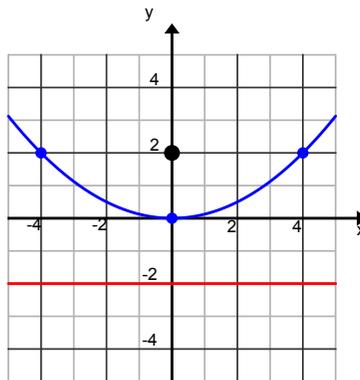
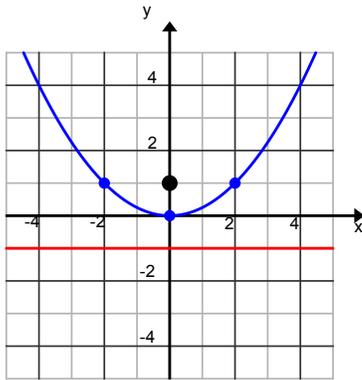
Allow students to discuss their approaches to this exercise with a partner or in small groups. Keep encouraging students to consider the definition of a parabola as they try to sketch the parabolas. Encourage students who draw a haphazard curve to consider how they could make sure their graph is the set of points equidistant from the focus and directrix. Throughout this lesson, provide students with access to graphing calculators or other computer graphing software that they can use to test and confirm conjectures. You can further model this and provide additional scaffolding by using an online applet located at the website <http://www.intmath.com/plane-analytic-geometry/parabola-interactive.php>. Within the applet, students can move either the focus or the directrix to change the value of  $p$ . They can also slide a point along the parabola noting the equal distances between that point and the focus and that point and the directrix. Depending on the level of your students, you can begin these exercises by moving directly to the applet or by having them start the exercises and use the applet later in this section.

Opening Exercise

Are all parabolas congruent? Use the following questions to support your answer.

- a. Draw the parabola for each focus and directrix given below.

*The solution is shown below.*



- b. What do we mean by congruent parabolas?

*Two parabolas would be congruent if we could find a sequence of rigid motions that takes one parabola onto the other. We could translate the vertex of the first one onto the vertex of the second one, then rotate the image of the first one so that the directrices are parallel and both parabolas open in the same direction. If the transformed first parabola coincides with the original second parabola, then the two original parabolas are congruent.*

- c. Are the two parabolas from part (a) congruent? Explain how you know.

*These two parabolas are not congruent. They have the same vertex but different y-values for each x in the domain, except for the point (0, 0). If we translate the first one somewhere else, then the vertices will not align. If we reflect or rotate, then both parabolas will not open upward. There is no rigid transformation or set of transformations that takes the graph of one parabola onto the other.*

- d. Are all parabolas congruent?

*No. We just found two that are not congruent.*

- e. Under what conditions might two parabolas be congruent? Explain your reasoning.

*Once we align the vertices and get the directrices parallel and parabolas opening in the same direction through rotation or reflection, the parabolas will have the same shape if the focus and directrix are aligned. Thus, two parabolas will be congruent if they have the same distance between the focus and directrix.*

Scaffolding:

- If students struggle with vocabulary, refer them back to the anchor poster.
- As an extension, ask students to derive the equation for the parabola as was shown in Example 1 of the previous lesson.

MP.3

Debrief this exercise using the following questions, which can also be used as scaffolds if students are struggling to begin this problem. During the debrief, record student thinking on chart paper to be used for reference at the end of this lesson as a means to confirm or refute their conjectures.

At some point in this discussion, students should recognize that it would be nice if we had a name for the distance between the focus and directrix of the parabola. When appropriate, let them know that this distance will be denoted by  $p$  in this lesson and lessons that follow. If they do not mention it here, it will be brought into the discussion in Exercise 5.

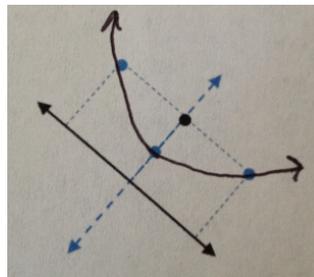
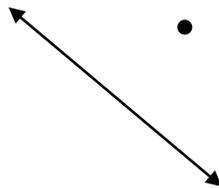
- How can you use the definition of a parabola to quickly locate at least three points on the graph of the parabola with a focus  $(0,1)$  and directrix  $y = -1$ ?
  - *Along the  $y$ -axis, the distance between the focus and the directrix is 2, so one point, the vertex, will be halfway between them at  $(0,0)$ . Since the distance between the focus and the directrix is  $p = 2$  units, if we go 2 units to the left and right of the focus, we will be able to locate two more points on the parabola at  $(2,1)$  and  $(-2,1)$ .*
- How can you use the definition of a parabola to quickly locate at least three points on the graph of the parabola with a focus  $(0,2)$  and directrix  $y = -2$ ?
  - *The distance between the focus and the directrix is  $p = 4$  units, so the vertex is halfway between the focus and directrix at  $(0,0)$ . Moving right and left 4 units from the focus gives points  $(4,2)$  and  $(-4,2)$ .*
- Generalize this process of finding three points on a parabola with given focus and directrix.
  - *Find the vertex halfway between the focus and directrix, and let  $p$  be the distance between the focus and directrix. Then, sketch a line through the focus parallel to the directrix. Locate the two points  $p$  units along that line in either direction from the focus. The vertex and these two points are all on the parabola.*
- How does this process help us determine if two parabolas are congruent?
  - *Two parabolas will be congruent if the two points found through this process are the same distance away from the focus. That is, two parabolas will be congruent if they have the same value of  $p$ , or the same distance between the focus and directrix.*

### Exercises 1–5 (5 minutes)

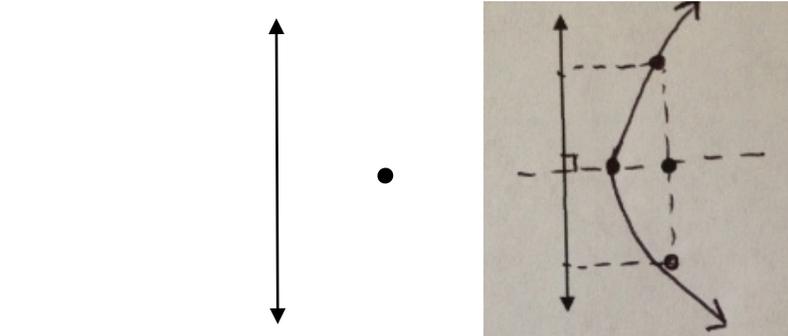
Students practice drawing parabolas given a focus and a directrix. These parabolas are NOT all oriented with a horizontal directrix. Let students struggle with how to construct a sketch of the parabola. Remind them of their work in the Opening Exercise. You may choose to do one problem together to model the process. Draw a line perpendicular to the directrix through the focus. Locate the midpoint of the segment connecting the focus and directrix. Then, create a square on either side of this line with side length equal to the distance between the focus and the directrix. One vertex of each square that is not on the directrix or axis of symmetry is another point on the parabola. If you have dynamic geometry software, students could also model this construction using technology.

#### Exercises 1–5

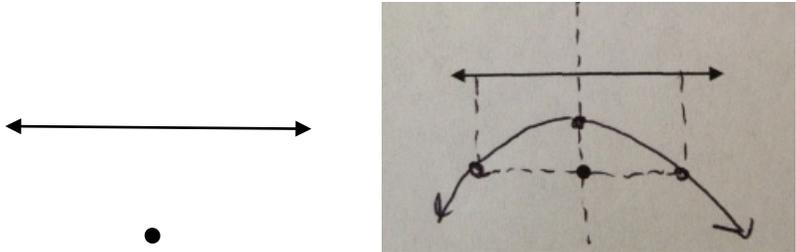
1. Draw the parabola with the given focus and directrix.



2. Draw the parabola with the given focus and directrix.



3. Draw the parabola with the given focus and directrix.



*Scaffolding:*  
 Use sentence frames for this exercise to support English Language Learners.  
 For example, in Exercise 5, use “As the value of  $p$  grows, the parabola \_\_\_\_\_.”  
 Later in the lesson, use “All parabolas with the same distance between the \_\_\_\_ and \_\_\_\_ of the parabola are congruent.”

Give students time to work through these exercises alone or in small groups. Then, have a few students present their approaches on the board. Be sure to emphasize that by applying the definition, we can produce a fairly accurate sketch of a parabola.

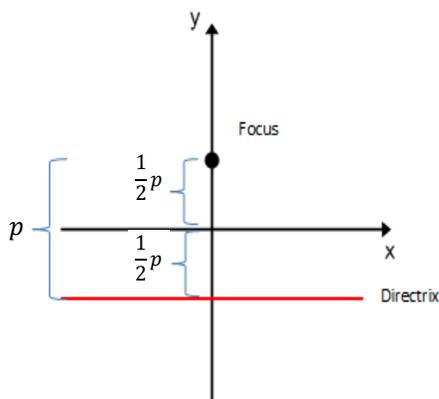
- What two geometric objects determine the set of points that forms a parabola?
  - A point called the focus and a line called the directrix.

Direct students to compare and contrast Exercises 1–3 and discuss the implications within their small groups.

4. What can you conclude about the relationship between the parabolas in Exercises 1–3?  
*The parabolas are all the same size and shape because the distance between the focus and the directrix stayed the same. These parabolas should be congruent.*

Direct students attention to the diagram in Exercise 5. Have students respond individually and then discuss in small groups.

5. Let  $p$  be the number of units between the focus and the directrix, as shown. As the value of  $p$  increases, what happens to the shape of the resulting parabola?



*As the value of  $p$  increases, the graph is dilated and shrinks vertically compared to graphs with a smaller value of  $p$ . It appears to get flatter.*

**Example 1 (12 minutes): Derive an Equation for a Parabola**

In this example, lead students through the process of creating an equation that represents a parabola with horizontal directrix, vertex at the origin, and the distance between the focus and directrix  $p > 0$ . This process will be similar to the work done in yesterday’s lesson, except we are working with a general case instead of a specified value for  $p$ . Scaffolding may be necessary for students who are not ready to move to the abstract level. For those students, continued modeling with selected  $p$  values and use of the applet mentioned earlier will help bridge the gap between concrete and abstract. Also, recall that several of the exercises in the previous lesson worked through this process with specific points. Remind students for whom this work is tedious that by deriving a general formula, we can simplify our work going forward, which is the heart of MP.7 and MP.8.

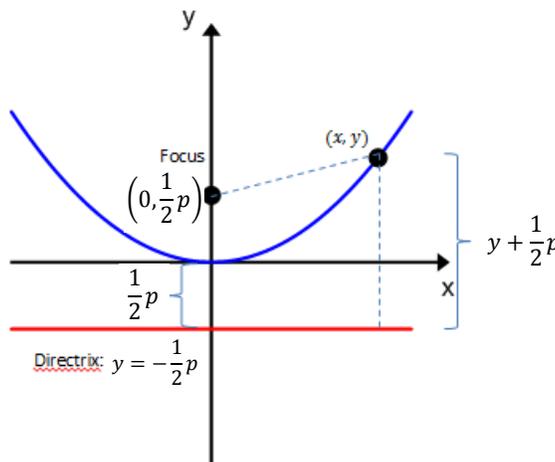
*Scaffolding:*

For more advanced students, derive the general vertex form of a parabola with vertex at  $(h, k)$ , distance  $p$  between the focus and directrix, and horizontal directrix:

$$y = \pm \frac{1}{2p}(x - h)^2 + k.$$

**Example 1**

Consider a parabola  $P$  with distance  $p > 0$  between the focus with coordinates  $(0, \frac{1}{2}p)$ , and directrix  $y = -\frac{1}{2}p$ . What is the equation that represents this parabola?



- What are the coordinates of the vertex?
  - *The coordinates of the vertex are (0,0).*
- Find a formula for the distance between the focus and the point  $(x, y)$ .
  - $\sqrt{(x - 0)^2 + \left(y - \frac{1}{2}p\right)^2}$
- Find a formula for the distance between  $(x, y)$  and the directrix.
  - $y + \frac{1}{2}p$
- By the definition of a parabola, these distances are equal. Create an equation.
  - $y + \frac{1}{2}p = \sqrt{(x - 0)^2 + \left(y - \frac{1}{2}p\right)^2}$

- Solve the equation for  $y$ .
  - *Start by squaring both sides. When you expand the squared binomials, the  $y^2$  and  $\frac{1}{4}p^2$  terms drop out of the equation. You may need to provide some additional scaffolding for students who are still not fluent with expanding binomial expressions.*

$$\begin{aligned} \left(y + \frac{1}{2}p\right)^2 &= x^2 + \left(y - \frac{1}{2}p\right)^2 \\ \left(y + \frac{1}{2}p\right)^2 - \left(y - \frac{1}{2}p\right)^2 &= x^2 \\ y^2 + py + \frac{1}{4}p^2 - y^2 + py - \frac{1}{4}p^2 &= x^2 \\ 2py &= x^2 \\ y &= \frac{1}{2p}x^2 \end{aligned}$$

- *Therefore,  $P = \left\{(x, y) \mid y = \frac{1}{2p}x^2\right\}$ .*

- How does this result verify the conjecture that as  $p$  increases, the parabola gets flatter? (Note to teacher: Now is a good time to demonstrate with the applet <http://www.intmath.com/plane-analytic-geometry/parabola-interactive.php>.)
  - *We can confirm the conjecture that the graph of a parabola vertically shrinks as  $p$  increases because the expression  $\frac{1}{2p}$  will get smaller as  $p$  gets larger; thus, the parabola appears flatter.*

*Scaffolding:*

As an alternative to Example 1, use graphing software to explore the relationship between the distance between the focus and directrix, and the coefficient of  $x^2$ . Organize the results in a table like the one shown below, and ask students to generalize the patterns they are seeing.

Equation	Focus	Directrix	Distance from Focus to Directrix
$y = \frac{1}{2}x^2$	$\left(0, \frac{1}{2}\right)$	$y = -\frac{1}{2}$	1
$y = \frac{1}{4}x^2$	(0,1)	$y = -1$	2
$y = \frac{1}{6}x^2$	$\left(0, \frac{3}{2}\right)$	$y = -\frac{3}{2}$	3
$y = \frac{1}{8}x^2$	(0,2)	$y = -2$	4
$y = \frac{1}{2p}x^2$	$\left(0, \frac{1}{2}p\right)$	$y = -\frac{1}{2}p$	$p$

MP.7

**Discussion (3 minutes)**

The goal of this brief discussion is to introduce the vertex form of the equation of a parabola without going through the entire derivation of the formula and to make the connection between what we are doing now with parabolas and what was done in Algebra I, Module 2, Topic B with quadratic functions. Completing the square on any quadratic function  $y = f(x)$  will give us the equation of a parabola in vertex form, which we could then quickly graph since we would know the coordinates of the vertex and distance  $p$  from focus to directrix.

**Discussion**

We have shown that any parabola with a distance  $p > 0$  between the focus  $(0, \frac{1}{2}p)$  and directrix  $y = -\frac{1}{2}p$  has a vertex at the origin and is represented by a quadratic equation of the form  $y = \frac{1}{2p}x^2$ .

Suppose that the vertex of a parabola with a horizontal directrix that opens upward is  $(h, k)$ , and the distance from the focus to directrix is  $p > 0$ . Then, the focus has coordinates  $(h, k + \frac{1}{2}p)$ , and the directrix has equation  $y = k - \frac{1}{2}p$ . If we go through the above derivation with focus  $(h, k + \frac{1}{2}p)$  and directrix  $y = k - \frac{1}{2}p$ , we should not be surprised to get a quadratic equation. In fact, if we complete the square on that equation, we can write it in the form  $y = \frac{1}{2p}(x - h)^2 + k$ .

In Algebra I, Module 4, Topic B, we saw that any quadratic function can be put into vertex form:  $f(x) = a(x - h)^2 + k$ . Now we see that any parabola that opens upward can be described by a quadratic function in vertex form, where  $a = \frac{1}{2p}$ .

If the parabola opens downward, then the equation is  $y = -\frac{1}{2p}(x - h)^2 + k$ , and the graph of any quadratic equation of this form is a parabola with vertex at  $(h, k)$ , distance  $p$  between focus and directrix, and opening downward. Likewise, we can derive analogous equations for parabolas that open to the left and right. This discussion is summarized in the box below.

**Vertex Form of a Parabola**

Given a parabola  $P$  with vertex  $(h, k)$ , horizontal directrix, and distance  $p > 0$  between focus and directrix, the analytic equation that describes the parabola  $P$  is

- $y = \frac{1}{2p}(x - h)^2 + k$  if the parabola opens upward, and
- $y = -\frac{1}{2p}(x - h)^2 + k$  if the parabola opens downward.

Conversely, if  $p > 0$ , then

- The graph of the quadratic equation  $y = \frac{1}{2p}(x - h)^2 + k$  is a parabola that opens upward with vertex at  $(h, k)$  and distance  $p$  from focus to directrix, and
- The graph of the quadratic equation  $y = -\frac{1}{2p}(x - h)^2 + k$  is a parabola that opens downward with vertex at  $(h, k)$  and distance  $p$  from focus to directrix.

Given a parabola  $P$  with vertex  $(h, k)$ , vertical directrix, and distance  $p > 0$  between focus and directrix, the analytic equation that describes the parabola  $P$  is:

- $x = \frac{1}{2p}(y - k)^2 + h$  if the parabola opens to the right, and
- $x = -\frac{1}{2p}(y - k)^2 + h$  if the parabola opens to the left.

Conversely, if  $p > 0$ , then

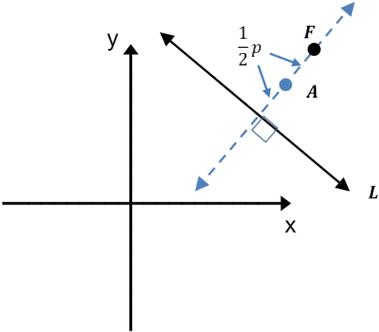
- The graph of the quadratic equation  $x = \frac{1}{2p}(y - k)^2 + h$  is a parabola that opens to the right with vertex at  $(h, k)$  and distance  $p$  from focus to directrix, and
- The graph of the quadratic equation  $x = -\frac{1}{2p}(y - k)^2 + h$  is a parabola that opens to the left with vertex at  $(h, k)$  and distance  $p$  from focus to directrix.

**Example 2 (8 minutes)**

The goal of this section is to present and prove the theorem that all parabolas that have the same distance from the focus to the directrix (that is, the same value of  $p$ ) are congruent. Start by having students sketch the parabola defined by the focus and directrix on the diagram shown on the student pages.

**Example 2**

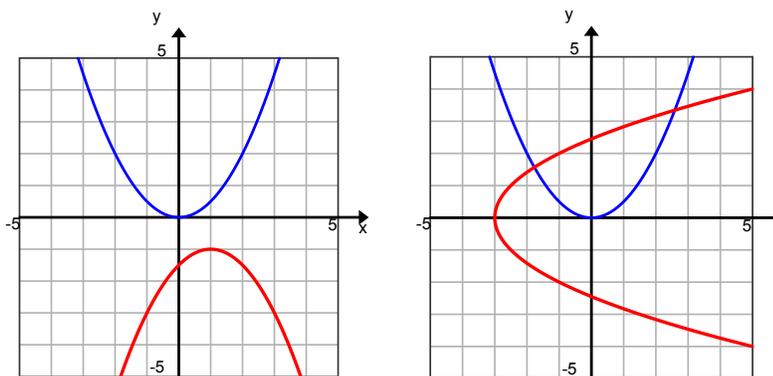
**Theorem:** Given a parabola  $P$  given by a directrix  $L$  and a focus  $F$  in the Cartesian plane, then  $P$  is congruent to the graph of  $y = \frac{1}{2p}x^2$ , where  $p$  is the distance from  $F$  to  $L$ .



A more concrete approach also appears below and could be used as an alternative approach to the formal proof or as a precursor to the formal proof. If you provide this additional scaffolding, this lesson may need to extend to an additional day to provide time to prove the theorem below rather than just demonstrate it by the examples provided in the scaffolds.

The following examples provide concrete evidence to support the proof provided on the next pages. Use this before or after the proof based on the needs of your students. Print graphs of these parabolas on paper. Then, print the graph of  $y = \frac{1}{2}x^2$  on a transparency. Use this to help students understand the rigid transformations that will map the given parabola onto  $y = \frac{1}{2}x^2$ .

- The graphs of  $y = \frac{1}{2}x^2$  and another parabola are shown in each coordinate plane. Describe a series of rigid transformations that will map the given parabola onto the graph of  $y = \frac{1}{2}x^2$ .



- *There are many sequences of transformations that can take the red parabola to the blue one. For the graphs on the left, translate the red graph one unit vertically up and one unit to the left, then reflect the resulting graph about the  $x$ -axis. For the graphs on the right, rotate the graph  $90^\circ$  counter-clockwise about the point  $(-3,0)$ , then translate the graph 3 units to the right.*

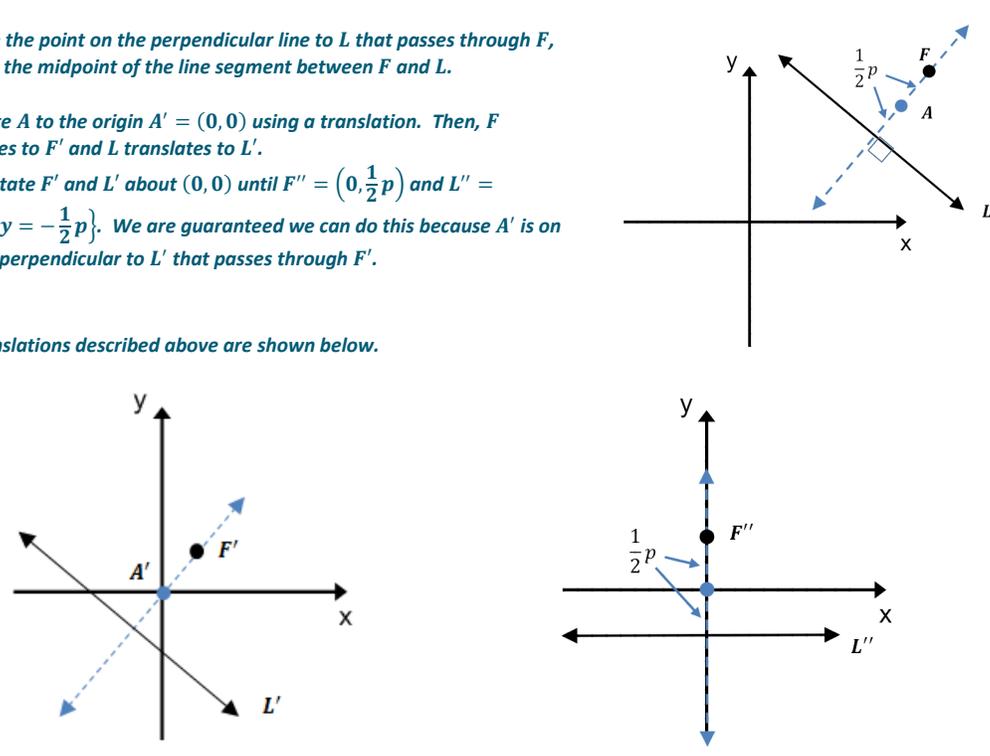
- Suppose we changed the distance between the focus and the directrix to 2 units instead of 1, and then mapped the given parabolas onto the graph of  $y = \frac{1}{4}x^2$ . Would the resulting graphs be the same? Why?
  - *The resulting graphs would be the same since the distance between the focus and the directrix for the parabola  $y = \frac{1}{4}x^2$  is 2 units. The parabolas would map exactly onto the parabola whose vertex is  $(0,0)$ .*
- Under what conditions would two parabolas be congruent? How could you verify this using transformations that map one parabola onto the other?
  - *If the distance between the focus and the directrix is the same, then the two parabolas will be congruent. You could describe a series of rotations, reflections, and/or translations that will map one parabola onto the other. These rigid transformations preserve the size and shape of the graph and show that the two figures are congruent.*

**Proof**

Let  $A$  be the point on the perpendicular line to  $L$  that passes through  $F$ , which is the midpoint of the line segment between  $F$  and  $L$ .

Translate  $A$  to the origin  $A' = (0, 0)$  using a translation. Then,  $F$  translates to  $F'$  and  $L$  translates to  $L'$ .

Next, rotate  $F'$  and  $L'$  about  $(0, 0)$  until  $F'' = (0, \frac{1}{2}p)$  and  $L'' = \{(x, y) \mid y = -\frac{1}{2}p\}$ . We are guaranteed we can do this because  $A'$  is on the line perpendicular to  $L'$  that passes through  $F'$ .



The translations described above are shown below.

Since  $P$  is determined by  $F$  and  $L$ , the first translation takes  $P$  to a parabola  $P'$  such that  $P \cong P'$ . The rotation takes  $P'$  to a parabola  $P''$  such that  $P' \cong P''$ . Therefore,  $P \cong P''$  by transitivity.

Now, by Example 1 above,

$$P'' = \{(x, y) \mid y = \frac{1}{2p}x^2\};$$

that is,  $P''$  is the graph of the equation  $y = \frac{1}{2p}x^2$ , which is what we wanted to prove.

Depending on the level of your students, you can begin with the scaffold examples using  $y = \frac{1}{2}x^2$ ; you can begin with the proof of this theorem can be presented directly by you; or you can have students try to work through the proof with some scaffolding and support.

Consider these discussion questions to help students get started with thinking about a proof if you choose to have them work collaboratively in small groups to create a proof.

- How can we apply transformations to show that every parabola is congruent to  $y = \frac{1}{2p}x^2$ , where  $p$  is the distance between the focus and the directrix?
  - *The parabola's vertex is (0,0). We could translate any parabola so that its vertex is also (0,0).*
  - *Then we would need to rotate the parabolas so that the directrix is a horizontal line and the focus is a point along the y-axis.*

You may need to remind students that translations and rotations are rigid transformations and, therefore, guarantee that the parabolas determined by the focus and directrix as they are translated and rotated will remain the same shape and size. For more information, see Module 1 from high school Geometry.

### Exercises 6–9 (4 minutes): Reflecting on the Theorem

Have students respond individually and then share within their groups. Post a few responses on the board for a whole class debrief and correct any misconceptions at that point.

#### Exercises 6–9

6. Restate the results of the theorem from Example 2 in your own words.

*All parabolas that have the same distance between the focus point and the directrix are congruent.*

7. Create the equation for a parabola that is congruent to  $y = 2x^2$ . Explain how you determined your answer.

*$y = 2x^2 + 1$ . As long as the coefficient of the  $x^2$  term is the same, the parabolas will be congruent.*

8. Create an equation for a parabola that IS NOT congruent to  $y = 2x^2$ . Explain how you determined your answer.

*$y = x^2$ . As long as the coefficient of the  $x^2$  term is different, the parabolas will not be congruent.*

9. Write the equation for two different parabolas that are congruent to the parabola with focus point (0,3) and directrix line  $y = -3$ .

*The distance between the focus and the directrix is 6 units. Therefore, any parabola with a coefficient of  $\frac{1}{2p} = \frac{1}{2(6)} = \frac{1}{12}$  will be congruent to this parabola. Here are two options:  $y = \frac{1}{12}x^2 + 1$  and  $x = \frac{1}{12}y^2$ .*

### Closing (2 minutes)

Revisit the conjecture from the beginning of this lesson: Under what conditions will two parabolas be congruent? Give students time to reflect on this question in writing before reviewing the points listed below. Summarize the following key points as you wrap up this lesson. In the next lesson, we will consider whether or not all parabolas are similar. This lesson has established that given a distance  $p$  between the directrix and focus, all parabolas with equal values of  $p$  are congruent to the parabola that is the graph of the equation  $y = \frac{1}{2p}x^2$ .

- The points of a parabola are determined by the directrix and a focus.
- Every parabola is congruent to a parabola defined by a focus on the  $y$ -axis and a directrix that is parallel to the  $x$ -axis.
- All parabolas that have the same distance between the focus and the directrix are congruent.
- When the focus is at  $(0, \frac{1}{2}p)$  and the directrix is given by the equation  $y = -\frac{1}{2}p$ , then the parabola is the graph of the equation  $y = \frac{1}{2p}x^2$ .
- When the vertex is at  $(h, k)$ , and the distance from the focus to directrix is  $p > 0$ , then:
  - If it opens upward, the parabola is the graph of the equation  $y = \frac{1}{2p}(x - h)^2 + k$ ;
  - If it opens downward, the parabola is the graph of the equation  $y = -\frac{1}{2p}(x - h)^2 + k$ ;
  - If it opens to the right, the parabola is the graph of the equation  $x = \frac{1}{2p}(y - k)^2 + h$ ;
  - If it opens to the left, the parabola is the graph of the equation  $x = -\frac{1}{2p}(y - k)^2 + h$ .

**Exit Ticket (4 minutes)**

Name \_\_\_\_\_

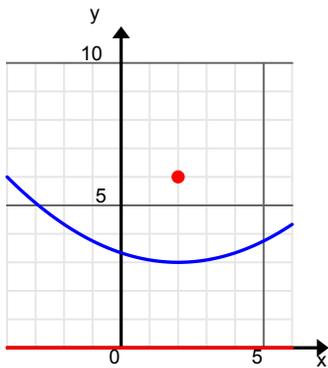
Date \_\_\_\_\_

## Lesson 34: Are All Parabolas Congruent?

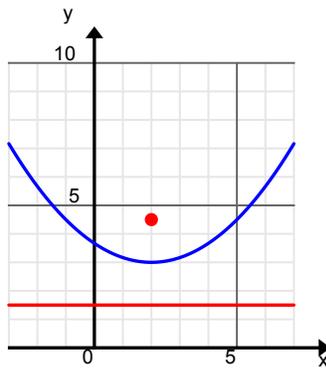
### Exit Ticket

Which parabolas shown below are congruent to the parabola that is the graph of the equation  $y = \frac{1}{12}x^2$ ? Explain how you know.

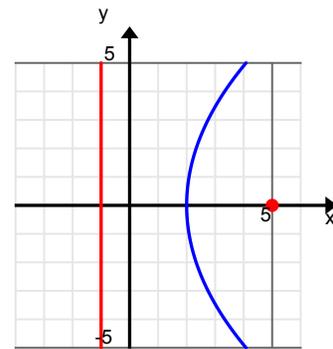
a.



b.



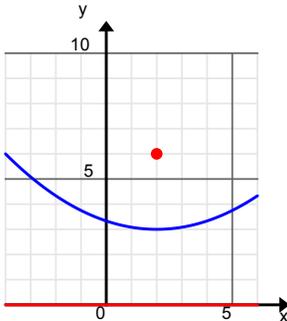
c.

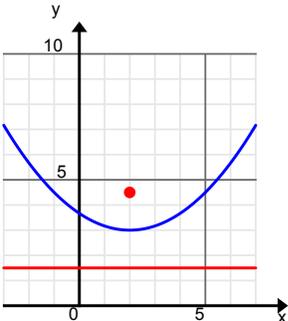


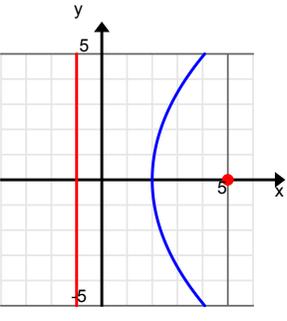
Exit Ticket Sample Solutions

Which parabolas shown below are congruent to the parabola that is the graph of the equation  $y = \frac{1}{12}x^2$ ? Explain how you know.

*The p-value is 6. So, any parabola where the distance between the focus and the directrix is equal to 6 units will be congruent to the parabola that is the graph of the equation  $y = \frac{1}{12}x^2$ . Of the parabolas shown below, (a) and (c) meet this condition, but (b) does not.*

a. 

b. 

c. 

Problem Set Sample Solutions

Problems 1–9 in this Problem Set review how to create the analytic equation of a parabola. Students may use the process from the previous lesson, or use the vertex form of the equation of a parabola included in this lesson. Starting with Problem 10, the focus of the Problem Set shifts to recognizing when parabolas are congruent.

1. Show that if the point with coordinates  $(x, y)$  is equidistant from  $(4, 3)$  and the line  $y = 5$ , then  $y = -\frac{1}{4}x^2 + 2x$ .

*Students might start with the equation  $\sqrt{(x - 4)^2 + (y - 3)^2} = 5 - y$  and solve for  $y$  as follows:*

$$\begin{aligned} \sqrt{(x - 4)^2 + y^2 - 6y + 9} &= 5 - y \\ (x - 4)^2 + y^2 - 6y + 9 &= 25 - 10y + y^2 \\ (x - 4)^2 &= -4y + 16 \\ 4y &= -(x - 4)^2 + 16 \\ y &= -\frac{1}{4}(x^2 - 8x + 16) + 4 \\ y &= -\frac{1}{4}x^2 + 2x. \end{aligned}$$

*Or, they might apply what we have learned about the vertex form of the equation of a parabola. Since the directrix is above the focus, we know the parabola opens downward, so  $p$  will be negative. Since the distance from the point  $(4, 3)$  to the line  $y = 5$  is 2 units, we know that  $p = -2$ . The vertex is halfway between the focus and directrix, so the coordinates of the vertex are  $(4, 4)$ . Then, the vertex form of the equation that represents the parabola is*

$$\begin{aligned} y &= -\frac{1}{4}(x - 4)^2 + 4 \\ y &= -\frac{1}{4}x^2 + 2x. \end{aligned}$$

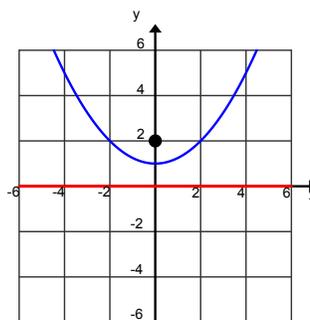
2. Show that if the point with coordinates  $(x, y)$  is equidistant from the point  $(2, 0)$  and the line  $y = -4$ , then  $y = \frac{1}{8}(x - 2)^2 - 2$ .

*Students might start with the equation  $\sqrt{(x - 2)^2 + (y - 0)^2} = y + 4$ , and then solve it for  $y$ , or they might apply the vertex form of the equation of a parabola. Since the vertex is above the directrix, we know that the parabola opens upward, and  $p > 0$ . Since the distance from the point  $(2, 0)$  to the line  $y = -4$  is 4 units, we know that  $p = 4$ . The vertex is halfway between the focus and directrix, so the vertex is  $(2, -2)$ . Thus, the equation that represents the parabola is  $y = \frac{1}{8}(x - 2)^2 - 2$ .*

3. Find the equation of the set of points which are equidistant from  $(0, 2)$  and the  $x$ -axis. Sketch this set of points.

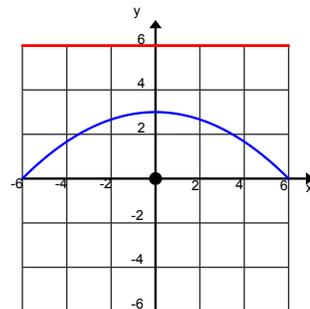
*The focus is  $(0, 2)$ , and the directrix is the  $x$ -axis. Thus, the vertex is the point  $(0, 1)$ , which is halfway between the vertex and directrix. Since the parabola opens upward,  $p > 0$ , so  $p = 2$ . Then, the vertex form of the equation of the parabola is*

$$y = \frac{1}{4}x^2 + 1.$$



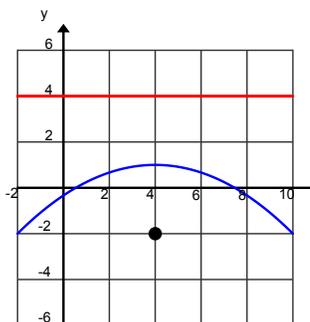
4. Find the equation of the set of points which are equidistant from the origin and the line  $y = 6$ . Sketch this set of points.

$$y = -\frac{1}{12}x^2 + 3$$



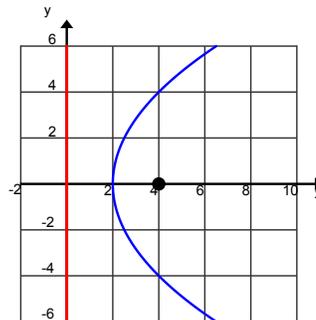
5. Find the equation of the set of points which are equidistant from  $(4, -2)$  and the line  $y = 4$ . Sketch this set of points.

$$y = -\frac{1}{12}(x - 4)^2 + 1$$



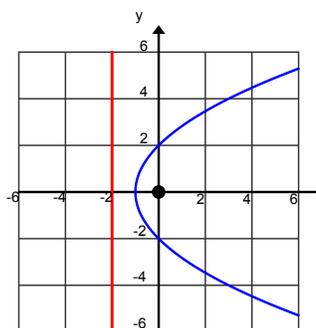
6. Find the equation of the set of points which are equidistant from  $(4, 0)$  and the  $y$ -axis. Sketch this set of points.

$$x = \frac{1}{8}y^2 + 2$$



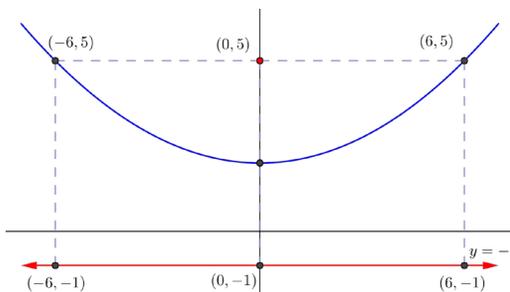
7. Find the equation of the set of points which are equidistant from the origin and the line  $x = -2$ . Sketch this set of points.

$$x = \frac{1}{4}y^2 - 1$$

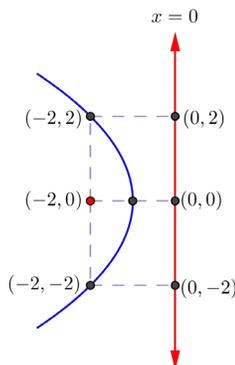


8. Use the definition of a parabola to sketch the parabola defined by the given focus and directrix.

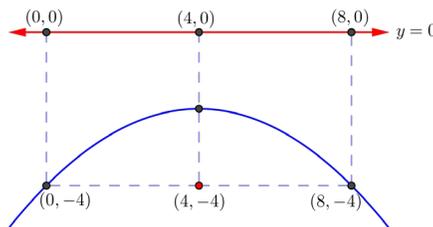
- a. Focus:  $(0, 5)$  Directrix:  $y = -1$



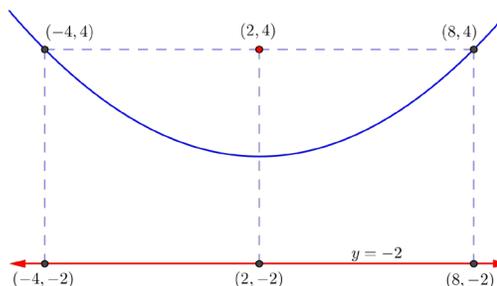
- b. Focus:  $(-2, 0)$  Directrix:  $y$ -axis



c. Focus:  $(4, -4)$  Directrix:  $x$ -axis



d. Focus:  $(2, 4)$  Directrix:  $y = -2$



9. Find an analytic equation for each parabola described in Problem 8.

a.  $P = \{(x, y) \mid y = \frac{1}{12}x^2 + 2\}$ ; thus,  $P$  is the graph of the equation  $y = \frac{1}{12}x^2 + 2$ .

b.  $P = \{(x, y) \mid x = -\frac{1}{4}y^2 - 1\}$ ; thus,  $P$  is the graph of the equation  $x = -\frac{1}{4}y^2 - 1$ .

c.  $P = \{(x, y) \mid y = -\frac{1}{8}(x - 4)^2 - 2\}$ ; thus,  $P$  is the graph of the equation  $y = -\frac{1}{8}(x - 4)^2 - 2$ .

d.  $P = \{(x, y) \mid y = \frac{1}{12}(x - 2)^2 + 1\}$ ; thus,  $P$  is the graph of the equation  $y = \frac{1}{12}(x - 2)^2 + 1$ .

10. Are any of the parabolas described in Problem 9 congruent? Explain your reasoning.

*(a)  $p = 6$ , (b)  $p = 2$ , (c)  $p = 4$ , and (d)  $p = 6$ ; therefore, the parabolas in parts (a) and (d) are congruent because they have the same distance between the focus and directrix.*

11. Sketch each parabola, labeling its focus and directrix.

*Each sketch should have the appropriate vertex, focus, and directrix and be fairly accurate. Sketches for (a) and (c) are shown.*

a.  $y = \frac{1}{2}x^2 + 2$

*Distance between focus and directrix is 1 unit, vertex  $(0, 2)$ , focus  $(0, 2.5)$ , directrix  $y = 1.5$*

b.  $y = -\frac{1}{4}x^2 + 1$

*Distance between focus and directrix is 2 units, vertex  $(0, 1)$ , focus  $(0, 0)$ , directrix  $y = 2$*

c.  $x = \frac{1}{8}y^2$

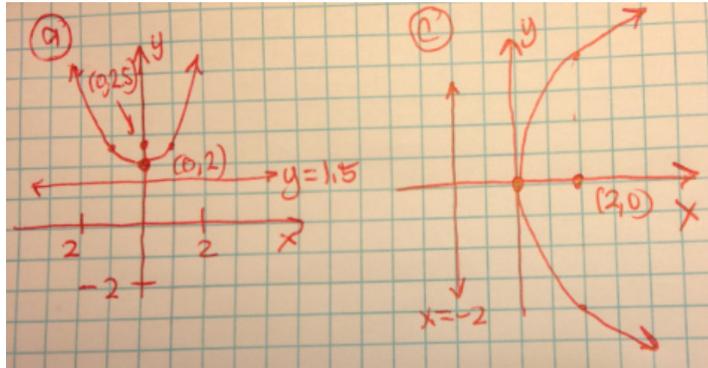
*Distance between focus and directrix is 4 units, vertex  $(0, 0)$ , focus  $(2, 0)$ , directrix  $x = -2$*

d.  $x = \frac{1}{2}y^2 + 2$

Distance between focus and directrix is 1 unit, vertex (2, 0), focus (2.5, 0), directrix  $x = 1.5$

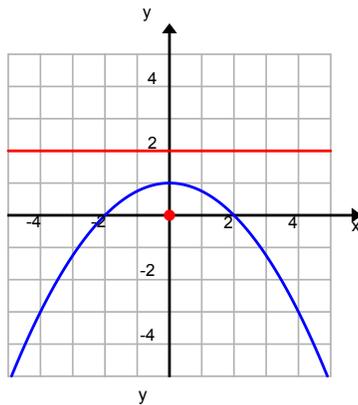
e.  $y = \frac{1}{10}(x - 1)^2 - 2$

Distance between focus and directrix is 5 units, vertex (1, -2), focus (1, 0.5), directrix  $y = -4.5$

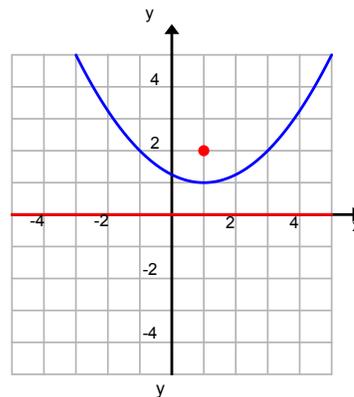


12. Determine which parabolas are congruent to the parabola that is the graph of the equation  $y = -\frac{1}{4}x^2$ .

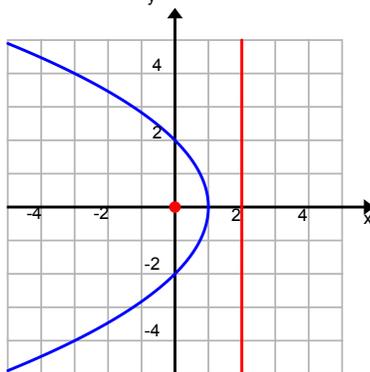
a.



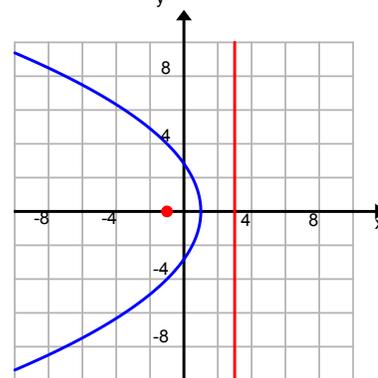
c.



b.



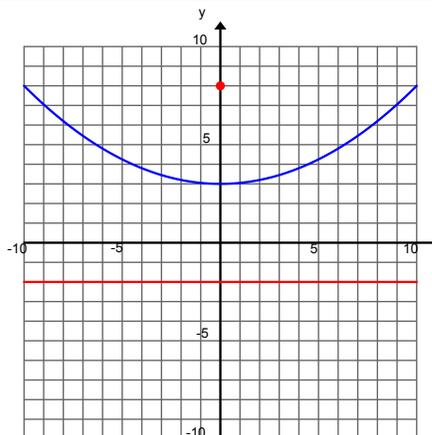
d.



Parabolas (a), (b), and (c) are congruent because all have  $p = 2$ . Parabola (d) has  $p = 1$ , so it is not congruent to the others.

13. Determine which equations represent the graph of a parabola that is congruent to the parabola shown to the right.

- a.  $y = \frac{1}{20}x^2$
- b.  $y = \frac{1}{10}x^2 + 3$
- c.  $y = -\frac{1}{20}x^2 + 8$
- d.  $y = \frac{1}{5}x^2 + 5$
- e.  $x = \frac{1}{10}y^2$
- f.  $x = \frac{1}{5}(y - 3)^2$
- g.  $x = \frac{1}{20}y^2 + 1$



The parabolas in parts (a), (c), and (g) are congruent and are congruent to the parabola shown. They all have the same distance of 10 units between the focus and the directrix like the parabola shown.

14. Jemma thinks that the parabola whose graph is the equation  $y = \frac{1}{3}x^2$  is NOT congruent to the parabola whose graph is the equation  $y = -\frac{1}{3}x^2 + 1$ . Do you agree or disagree? Create a convincing argument to support your reasoning.

*Jemma is wrong. These two parabolas are congruent. If you translate the graph of  $y = -\frac{1}{3}x^2 + 1$  down one unit and then reflect the resulting graph about the x-axis, the resulting graph will have the equation  $y = \frac{1}{3}x^2$ . Alternately, the focus and directrix of each parabola are the same distance apart, 1.5 units.*

15. Let  $P$  be the parabola with focus  $(2, 6)$  and directrix  $y = -2$ .

a. Write an equation whose graph is a parabola congruent to  $P$  with a focus  $(0, 4)$ .

*$y = \frac{1}{16}x^2$  is one option. The directrix for this parabola is  $y = -4$ . Another possible solution would be the parabola with focus  $(0, 4)$  and directrix  $y = 12$ . The equation would be  $y = -\frac{1}{16}x^2 + 8$ .*

b. Write an equation whose graph is a parabola congruent to  $P$  with a focus  $(0, 0)$ .

$$y = \frac{1}{16}x^2 - 4$$

c. Write an equation whose graph is a parabola congruent to  $P$  with the same directrix, but different focus.

*The focus would be a reflection of the original focus across the directrix, or  $(2, -10)$ . The equation would be  $y = -\frac{1}{16}(x - 2)^2 - 6$ .*

d. Write an equation whose graph is a parabola congruent to  $P$  with the same focus, but with a vertical directrix.

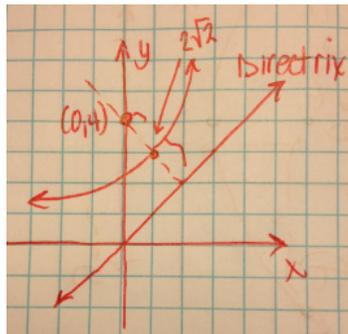
$$x = \frac{1}{16}(y - 6)^2 - 2 \text{ or } x = -\frac{1}{16}(y - 6)^2 + 8$$

MP.3

16. Let  $P$  be the parabola with focus  $(0, 4)$  and directrix  $y = x$ .

a. Sketch this parabola.

*Sketch is shown to the right.*



b. By how many degrees would you have to rotate  $P$  about the focus to make the directrix line horizontal?

*One possible answer is a clockwise rotation of  $45^\circ$ .*

c. Write an equation in the form  $y = \frac{1}{2a}x^2$  whose graph is a parabola that is congruent to  $P$ .

*The distance between the focus and the directrix is  $2\sqrt{2}$ . The equation is  $y = \frac{1}{4\sqrt{2}}x^2$ .*

d. Write an equation whose graph is a parabola with a vertical directrix that is congruent to  $P$ .

*Since the exact focus and directrix are not specified, there are infinitely many possible parabolas. A vertical directrix does require that the  $y$ -term be squared. Thus,  $x = \frac{1}{4\sqrt{2}}y^2$  satisfies the conditions specified in the problem.*

e. Write an equation whose graph is  $P'$ , the parabola congruent to  $P$  that results after  $P$  is rotated clockwise  $45^\circ$  about the focus.

*The directrix will be  $y = 4 - 2\sqrt{2}$ . The equation is  $y = \frac{1}{4\sqrt{2}}x^2 + 4 - \sqrt{2}$ .*

f. Write an equation whose graph is  $P''$ , the parabola congruent to  $P$  that results after the directrix of  $P$  is rotated  $45^\circ$  about the origin.

*The focus will be  $(2\sqrt{2}, 2\sqrt{2})$ , and the directrix will be the  $x$ -axis. The equation is  $y = \frac{1}{4\sqrt{2}}(x - 2\sqrt{2})^2 + \sqrt{2}$ .*

EXTENSION

17. Consider the function  $f(x) = \frac{2x^2 - 8x + 9}{-x^2 + 4x - 5}$ , where  $x$  is a real number.

a. Use polynomial division to rewrite  $f$  in the form  $f(x) = q + \frac{r}{-x^2 + 4x - 5}$  for some real numbers  $q$  and  $r$ .

*Using polynomial division,  $f(x) = -2 + \frac{-1}{-x^2 + 4x - 5}$ .*

b. Find the  $x$ -value where the maximum occurs for the function  $f$ , without using graphing technology. Explain how you know.

*We can rewrite  $f$  as  $f(x) = -2 + \frac{1}{x^2 - 4x + 5}$ . Since  $x^2 - 4x + 5 = (x - 2)^2 + 1$ , the graph of  $y = x^2 - 4x + 5$  is a parabola with vertex  $(2, 1)$  that opens upward. Thus, the lowest point on the graph is  $(2, 1)$ . The function  $f$  will take on its maximum value when  $\frac{1}{x^2 - 4x + 5}$  is maximized; this happens when the value of  $x^2 - 4x + 5$  is minimized. Since we have already seen that  $x^2 - 4x + 5$  is minimized at  $x = 2$ , the expression  $\frac{1}{x^2 - 4x + 5}$  takes on its maximum value when  $x = 2$ , and, thus, the original function  $f$  takes on its maximum value when  $x = 2$ .*

MP.2



## Lesson 35: Are All Parabolas Similar?

### Student Outcomes

- Students apply the geometric transformation of dilation to show that all parabolas are similar.

### Lesson Notes

In the previous lesson, students used transformations to prove that all parabolas with the same distance between the focus and directrix are congruent. In the process, they made a connection between geometry, coordinate geometry, transformations, equations, and functions. In this lesson, students explore how dilation can be applied to prove that all parabolas are similar.

Students may express disagreement with or confusion about the claim that all parabolas are similar because the various graphical representations of parabolas they have seen do not appear to have the “same shape.” Because a parabola is an open figure as opposed to a closed figure, like a triangle or quadrilateral, it is not easy to see similarity among parabolas. Students must understand that we are strictly defining *similar* via similarity transformations; in other words, two parabolas are similar because there is a sequence of translations, rotations, reflections, and dilations that takes one parabola to the other. In the last lesson, we showed that every parabola is congruent to the graph of the equation  $y = \frac{1}{2p}x^2$  for some  $p > 0$ ; in this lesson, we need only consider dilations.

When students claim that two parabolas are not similar, they should be reminded that the parts of the parabolas they are looking at may well appear to be different in size or magnification, but the parabolas themselves are not different in shape. Remind students that similarity is established by dilation; in other words, by magnifying a figure in both the horizontal and vertical directions. By analogy, although circles with different radii have different curvature, every student should agree that any circle can be dilated to be the same size and shape as any other circle; thus, all circles are similar.

Quadratic curves such as parabolas belong to a family of curves known as *conic sections*. The technical term in mathematics for how much a conic section deviates from being circular is *eccentricity*, and two conic sections with the same eccentricity are similar. Circles have eccentricity 0, and parabolas have eccentricity 1. After this lesson, you might ask students to research and write a report on eccentricity.

### Classwork

Provide graph paper for students as they work the first five exercises. They will first examine three congruent parabolas and then make a conjecture about whether or not all parabolas are similar. Finally, they will explore this conjecture by graphing parabolas of the form  $y = ax^2$  that have different  $a$ -values.

#### Scaffolding:

- Allow students access to graphing calculators or software to focus on conceptual understanding if they are having difficulty sketching the graphs.
- Consider providing students with transparencies with a variety of parabolas drawn on them (as in prior lesson), such as  $y = x^2$ ,  $y = \frac{1}{2}x^2$ , and  $y = \frac{1}{4}x^2$  to help them illustrate these principles.

Exercises 1–5 (4 minutes)

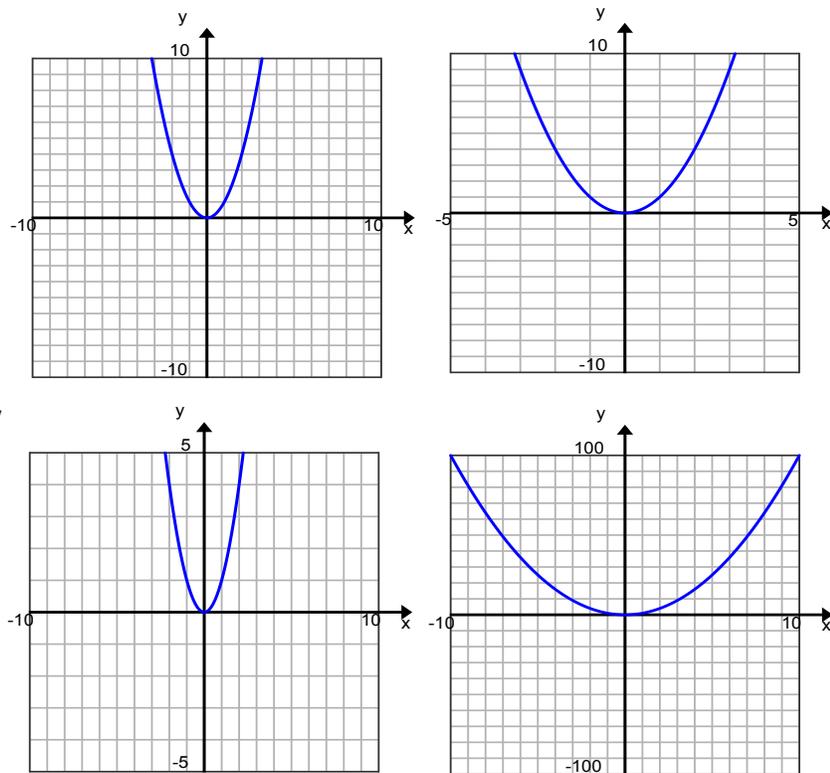
Exercises 1–6

1. Write the equation of two parabolas that are congruent to  $y = x^2$  and explain how you determined your equations.
2. Sketch the graph of  $y = x^2$  and the two parabolas you created on the same coordinate axes.
3. Write the equation of two parabolas that are NOT congruent to  $y = x^2$ . Explain how you determined your equations.
4. Sketch the graph of  $y = x^2$  and the two non-congruent parabolas you created on the same coordinate axes.
5. Use your work on Exercises 1–4 to answer the question posed in the lesson title: Are all parabolas similar? Explain your reasoning.

MP.3

Discussion

After students have examined the fact that when we change the  $a$ -value in the parabola equation, the resulting graph is basically the same shape, you can further emphasize this point by exploring the graph of  $y = x^2$  on a graphing calculator or graphing program on your computer. Use the same equation but different viewing windows so students can see that we can create an image of what appears to be a different parabola by transforming the dimensions of the viewing window. However, the images are just a dilation of the original that is created when we change the scale. See the images to the right. Each figure is a graph of the equation  $y = x^2$  with different scales on the horizontal and vertical axes.



Exercise 6 (5 minutes)

In this exercise, students derive the analytic equation for a parabola given its graph, focus, and directrix. Students have worked briefly with parabolas with a vertical directrix in previous lessons, so this exercise will be an opportunity for the teacher to assess whether or not students are able to transfer and extend their thinking to a slightly different situation.

6. The parabola at right is the graph of what equation?

a. Label a point  $(x, y)$  on the graph of  $P$ .

b. What does the definition of a parabola tell us about the distance between the point  $(x, y)$  and the directrix  $L$ , and the distance between the point  $(x, y)$  and the focus  $F$ ?

*Let  $(x, y)$  be any point on the graph of  $P$ . Then, these distances are equal because  $P = \{(x, y) | (x, y) \text{ is equidistant from } F \text{ and } L\}$ .*

c. Create an equation that relates these two distances.

*Distance from  $(x, y)$  to  $F$ :  $\sqrt{(x - 2)^2 + (y - 0)^2}$*

*Distance from  $(x, y)$  to  $L$ :  $x + 2$*

*Therefore, any point on the parabola has coordinates  $(x, y)$  that satisfy  $\sqrt{(x - 2)^2 + (y - 0)^2} = x + 2$ .*

d. Solve this equation for  $x$ .

*The equation can be solved as follows.*

$$\begin{aligned} \sqrt{(x - 2)^2 + (y - 0)^2} &= x + 2 \\ (x - 2)^2 + y^2 &= (x + 2)^2 \\ x^2 - 4x + 4 + y^2 &= x^2 + 4x + 4 \\ y^2 &= 8x \\ x &= \frac{1}{8}y^2 \end{aligned}$$

*Thus,*

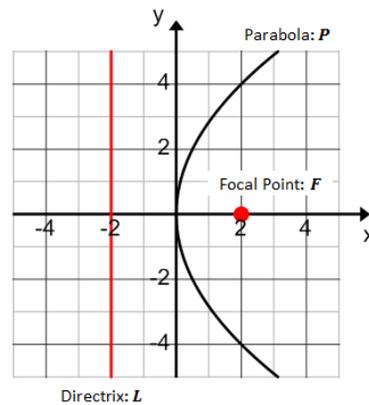
$$P = \{(x, y) | x = \frac{1}{8}y^2\}.$$

e. Find two points on the parabola  $P$ , and show that they satisfy the equation found in part (d).

*By observation,  $(2, 4)$  and  $(2, -4)$  are points on the graph of  $P$ . Both points satisfy the equation that defines  $P$ .*

$$(2, 4): \frac{1}{8}(4)^2 = \frac{16}{8} = 2$$

$$(2, -4): \frac{1}{8}(-4)^2 = \frac{16}{8} = 2$$



**Discussion (8 minutes)**

After giving students time to work through the Exercises 1–5, ask the following questions to revisit concepts from Algebra I, Module 3.

- In the previous exercise, is  $P$  a function of  $x$ ?
  - No, because the  $x$ -value 2 corresponds to two  $y$ -values.
- Is  $P$  a function of  $y$ ?
  - Yes. If you rotate the Cartesian plane by  $90^\circ$ , you can see that it would be a function. Alternately, if we take  $y$  to be in the domain and  $x$  to be in the range, then each  $y$ -value on  $P$  corresponds to exactly one  $x$ -value, which is the definition of a function.

These two questions remind students that just because we typically use the variable  $x$  to represent the domain element of an algebraic function, this does not mean that it must always represent the domain element.

Next, transition to summarizing what was learned in the last two lessons. We have defined a parabola and determined the conditions required for two parabolas to be congruent. Use the following questions to summarize these ideas.

- What have we learned about the definition of a parabola?
  - *The points on a parabola are equidistant from the directrix and the focus.*
- What transformations can be applied to a parabola to create a parabola congruent to the original one?
  - *If the directrix and the focus are transformed by a rigid motion (e.g., translation, rotation, or reflection), then the new parabola defined by the transformed directrix and focus will be congruent to the original.*

Essentially, every parabola that has a distance of  $p$  units between its focus and directrix is congruent to a parabola with focus  $(0, \frac{1}{2}p)$  and directrix  $y = -\frac{1}{2}p$ . What is the equation of this parabola?

$$P = \left\{ (x, y) \mid y = \frac{1}{2p}x^2 \right\}$$

Thus, all parabolas that have the same distance between the focus and the directrix are congruent.

The family of graphs given by the equation  $y = \frac{1}{2p}x^2$  for  $p > 0$  describes the set of non-congruent parabolas, one for each value of  $p$ .

Ask students to consider the question from the lesson title. Chart responses to revisit at the end of this lesson to confirm or refute their claims.

#### Discussion

How many of you think that all parabolas are similar? Explain why you think so.

*In geometry, two figures were similar if a sequence of transformations would take one figure onto the other one.*

What could we do to show that two parabolas are similar? How might you show this?

*Since every parabola can be transformed into a congruent parabola by applying one or more rigid transformations, perhaps similar parabolas can be created by applying a dilation which is a non-rigid transformation.*

MP.3

To check to see if all parabolas are similar, we only need to show that any parabola that is the graph of  $y = \frac{1}{2p}x^2$  for  $p > 0$  is similar to the graph of  $y = x^2$ . This is done through a dilation by some scale factor  $k > 0$  at the origin  $(0,0)$ . Note that a dilation of the graph of a function is the same as performing a horizontal scaling followed by a vertical scaling that students studied in Algebra I, Module 3.

### Exercises 7–10 (8 minutes)

The following exercises review the function transformations studied in Algebra I that are required to define dilation at the origin. These exercises provide students with an opportunity to recall what they learned in a previous course so that they can apply it here. Students must read points on the graphs to determine that the vertical scaling is by a factor of 2 for the graphs on the left and by a factor of  $\frac{1}{2}$  for the graphs on the right. In Algebra I, Module 3, we saw that the graph of a function can be transformed with a non-rigid transformation in two ways: *vertical* and *horizontal scaling*.

A *vertical scaling* of a graph by a scale factor  $k > 0$  takes every  $y$ -value of points  $(x, y)$  on the graph of  $y = f(x)$  to  $ky$ . The result of the transformation is given by the graph of  $y = kf(x)$ .

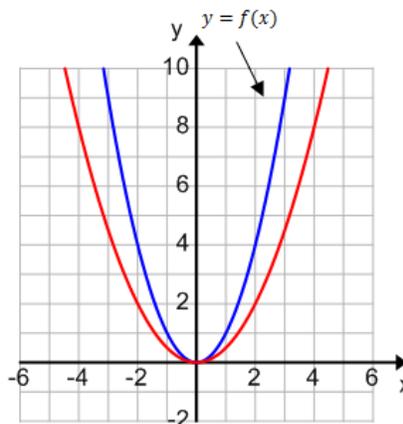
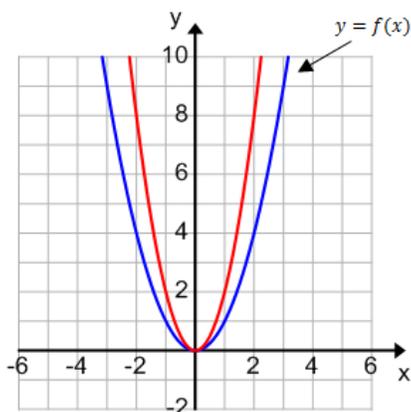
A *horizontal scaling* of a graph by a scale factor  $k > 0$  takes every  $x$ -value of points  $(x, y)$  on the graph of  $y = f(x)$  to  $kx$ . The result of the transformation is given by the graph of  $y = f\left(\frac{1}{k}x\right)$ .

*Scaffolding:*

- Allow students access to graphing calculators or software to focus on conceptual understanding if they are having difficulty sketching the graphs.
- The graphs shown in Exercises 7 and 8 are  $f(x) = x^2$ ,  $g(x) = 2x^2$ , and  $h(x) = \frac{1}{2}x^2$ . The graphs shown in Exercises 9 and 10 are  $f(x) = x^2$ ,  $g(x) = \left(\frac{1}{2}x\right)^2$ , and  $h(x) = (2x)^2$ .

**Exercises 7–10**

Use the graphs below to answer Exercises 7 and 8.



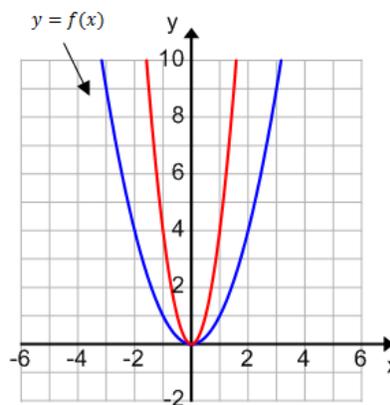
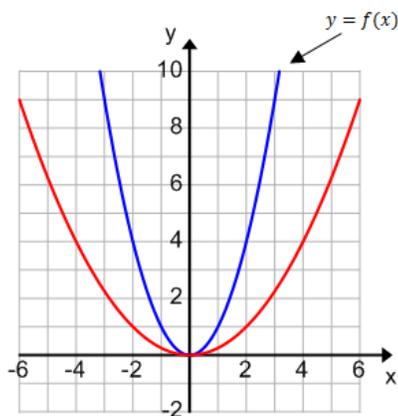
7. Suppose the unnamed red graph on the left coordinate plane is the graph of the function  $g$ . Describe  $g$  as a vertical scaling of the graph of  $y = f(x)$ ; that is, find a value of  $k$  so that  $g(x) = kf(x)$ . What is the value of  $k$ ? Explain how you determined your answer.

*The graph of  $g$  is a vertical scaling of the graph of  $f$  by a factor of 2. Thus,  $g(x) = 2f(x)$ . By comparing points on the graph of  $f$  to points on the graph of  $g$ , you can see that the  $y$ -values on  $g$  are all twice the  $y$ -values on  $f$ .*

8. Suppose the unnamed red graph on the right coordinate plane is the graph of the function  $h$ . Describe  $h$  as a vertical scaling of the graph of  $y = f(x)$ ; that is, find a value of  $k$  so that  $h(x) = kf(x)$ . Explain how you determined your answer.

*The graph of  $h$  is a vertical scaling of the graph of  $f$  by a factor of  $\frac{1}{2}$ . Thus,  $h(x) = \frac{1}{2}f(x)$ . By comparing points on the graph of  $f$  to points on the graph of  $h$ , you can see that the  $y$ -values on  $h$  are all half of the  $y$ -values on  $f$ .*

Use the graphs below to answer Exercises 9–10.



9. Suppose the unnamed function graphed in red on the left coordinate plane is  $g$ . Describe  $g$  as a horizontal scaling of the graph of  $y = f(x)$ . What is the value of  $k$ ? Explain how you determined your answer.

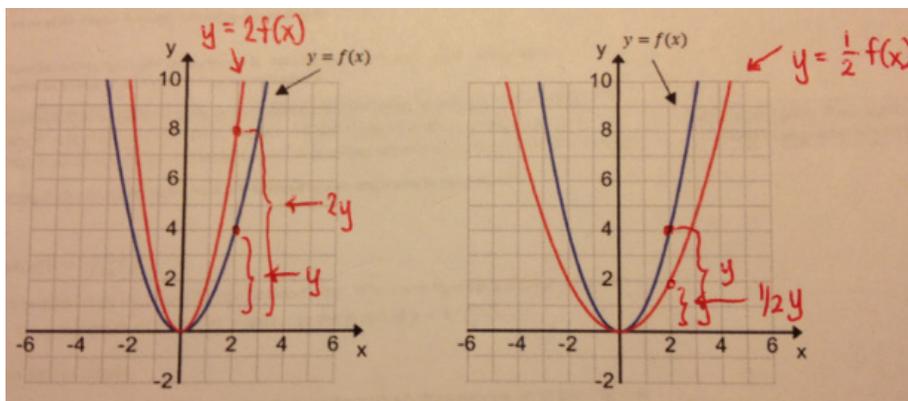
*The graph of  $g$  is a horizontal scaling of the graph of  $f$  by a factor of 2. Thus,  $g(x) = f\left(\frac{1}{2}x\right)$ . By comparing points on the graph of  $f$  to points on the graph of  $g$ , you can see that for the same  $y$ -values, the  $x$ -values on  $g$  are all twice the  $x$ -values on  $f$ .*

10. Suppose the unnamed function graphed in red on the right coordinate plane is  $h$ . Describe  $h$  as a horizontal scaling of the graph of  $y = f(x)$ . What is the value of  $k$ ? Explain how you determined your answer.

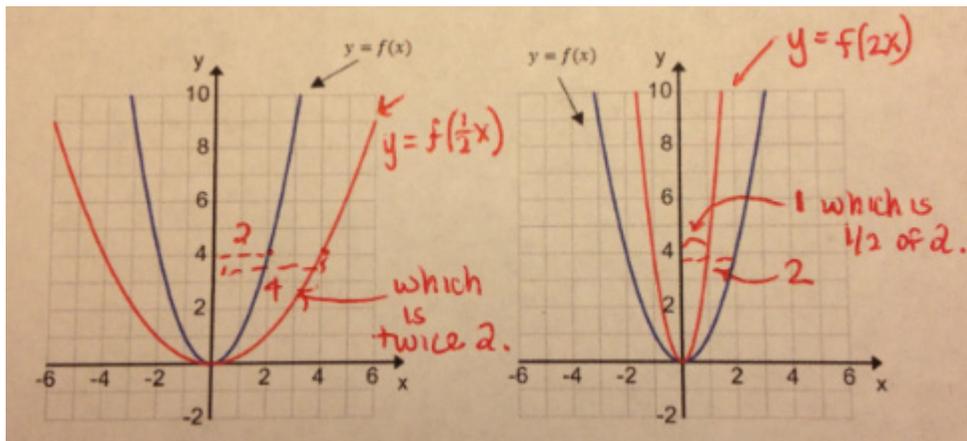
*The graph of  $h$  is a horizontal scaling of the graph of  $f$  by a factor of  $\frac{1}{2}$ . Thus,  $h(x) = f(2x)$ . By comparing points on the graph of  $f$  to points on the graph of  $h$ , you can see that for the same  $y$ -values, the  $x$ -values on  $h$  are all half of the  $x$ -values on  $f$ .*

When you debrief these exercises, model marking up the diagrams to illustrate the vertical and horizontal scaling. A sample is provided below.

Marked up diagrams for vertical scaling in Exercises 7 and 8:



Marked up diagrams for horizontal scaling in Exercises 9 and 10:



After working through Exercises 7–10, pose the following discussion question.

- If a dilation by scale factor  $k$  involves both horizontal and vertical scaling by a factor of  $k$ , how could we express the dilation of the graph of  $y = f(x)$ ?
  - You could combine both types of scaling. Thus,  $y = kf\left(\frac{1}{k}x\right)$ .

Explain the definition of dilation at the origin as a combination of a horizontal and then vertical scaling by the same factor. Exercises 1–3 in the Problem Set will address this idea further.

**Definition:** A dilation at the origin  $D_k$  is a horizontal scaling by  $k > 0$  followed by a vertical scaling by the same factor  $k$ . In other words, this dilation of the graph of  $y = f(x)$  is the graph of the equation  $y = kf\left(\frac{1}{k}x\right)$ .

It will be important for students to clearly understand that this dilation of the graph of  $y = f(x)$  is the graph of the equation  $y = kf\left(\frac{1}{k}x\right)$ . Remind students of the following two facts that they studied in high school Geometry:

1. When one figure is a dilation of another figure, the two figures are similar.
2. A dilation at the origin is just a particular type of dilation transformation.

Thus, the graph of  $y = f(x)$  is similar to the graph of  $y = kf\left(\frac{1}{k}x\right)$ . Students may realize here that their thinking about “stretching” the graph creating a similar parabola is not quite enough to prove that all parabolas are similar because we must consider both a horizontal and vertical dilation in order to connect back to the geometric definition of similar figures.

**Example 1 (5 minutes): Dilation at the Origin**

This example helps students gain a level of comfort with the notation and mathematics before moving on to proving that all parabolas are similar.

**Example 1**

Let  $f(x) = x^2$  and let  $k = 2$ . Write a formula for the function  $g$  that results from dilating  $f$  at the origin by a factor of  $\frac{1}{2}$ .

*The new function will have equation  $g(x) = 2f\left(\frac{1}{2}x\right)$ . Since  $f(x) = x^2$ , the new function will have equation  $g(x) = 2\left(\frac{1}{2}x\right)^2$ . That is,  $g(x) = \frac{1}{2}x^2$ .*

What would the results be for  $k = 3, 4$ , or  $5$ ? What about  $k = \frac{1}{2}$ ?

*For  $k = 3$ ,  $g(x) = \frac{1}{3}x^2$ .*

*For  $k = 4$ ,  $g(x) = \frac{1}{4}x^2$ .*

*For  $k = 5$ ,  $g(x) = \frac{1}{5}x^2$ .*

*For  $k = \frac{1}{2}$ ,  $g(x) = 2x^2$ .*

After working through this example, the following questions will help prepare students for the upcoming proof using a general parabola from the earlier discussion.

- Based on this example, what can you conclude about these parabolas?
  - *They are all similar to one another because they represent dilations of the graph at the origin of the original function.*
- Based on this example, what can you conclude about these parabolas?
- Is this enough information to prove ALL parabolas are similar?
  - *No, we have only proven that these specific parabolas are similar.*
- How could we prove that all parabolas are similar?
  - *We would have to use the patterns we observed here to make a generalization and algebraically show that it works in the same way.*

**Scaffolding:**

Some students might find this derivation easier if you use the parabola  $y = ax^2$ . Then, the proof would be as follows:

If  $f(x) = ax^2$ , then the graph of  $f$  is similar to the graph of the equation

$$y = k \left( a \left( \frac{1}{k} x \right) \right)^2.$$

Simplifying the right side gives:

$$y = \frac{a}{k} x^2.$$

We want this new parabola to be similar to  $y = x^2$ , which it will be if  $\frac{a}{k} = 1$ .

Therefore, let  $a = k$ . Thus, dilating the graph of  $y = ax^2$  about the origin by a factor of  $a$ , we have shown this parabola is similar to  $y = x^2$ .

To further support students, supply written reasons, such as those provided, as you work through these steps on the board.

**Discussion (8 minutes): Prove All Parabolas Are Similar**

In this discussion, you will work through a dilation at the origin on a general parabola with equation  $y = \frac{1}{2p}x^2$  to transform it to our basic parabola with equation  $y = x^2$  by selecting the appropriate value of  $k$ . At that point, we can argue that all parabolas are similar. Walk through the outline below slowly, and ask the class for input at each step, but expect that much of this discussion will be teacher-centered. For students not ready to show this result at an abstract level, have them work in small groups to show that a few parabolas, such as

$$y = \frac{1}{2}x^2, y = 4x^2, \text{ and } y = \frac{1}{8}x^2,$$

are similar to  $y = x^2$  by finding an appropriate dilation about the origin. Then, generalize from these examples in the following discussion.

- Recall from Lesson 34 that any parabola is congruent to an “upright” parabola of the form  $y = \frac{1}{2p}x^2$ , where  $p$  is the distance between the vertex and directrix. That is, given any parabola we can rotate, reflect and translate it so that it has its vertex at the origin and axis of symmetry along the  $y$ -axis. We now want to show that all parabolas of the form  $y = \frac{1}{2p}x^2$  are similar to the parabola  $y = x^2$ . To do this, we will apply a dilation at the origin to the parabola  $y = \frac{1}{2p}x^2$ . We just need to find the right value of  $k$  for the dilation.
- Recall that the graph of  $y = f(x)$  is similar to the graph of  $y = kf\left(\frac{1}{k}x\right)$ .

If  $f(x) = \frac{1}{2p}x^2$ , then the graph of  $f$  is similar to the graph of the equation  $y = kf\left(\frac{1}{k}x\right) = k\left(\frac{1}{2p}\left(\frac{1}{k}x\right)^2\right)$ , which simplifies to  $y = \frac{1}{2pk}x^2$ .

We want to find the value of  $k$  that dilates the graph of  $f(x) = \frac{1}{2p}x^2$  into  $y = x^2$ . That is, we need to choose the dilation factor  $k$  so that  $y = \frac{1}{2p}x^2$  becomes  $y = x^2$ ; therefore, we want  $\frac{1}{2pk} = 1$ . Solving this equation for  $k$  gives  $k = \frac{1}{2p}$ .

- Therefore, if we dilate the parabola  $y = \frac{1}{2p}x^2$  about the origin by a factor of  $\frac{1}{2p}$ , we have

$$\begin{aligned} y &= kf\left(\frac{1}{k}x\right) \\ &= k\left(\frac{1}{2p}\left(\frac{1}{k}x\right)^2\right) \\ &= \frac{1}{2pk}x^2 \\ &= x^2. \end{aligned}$$

Thus, we have shown that the original parabola is similar to  $y = x^2$ .

- In the previous lesson, we showed that any parabola is congruent to a parabola  $y = \frac{1}{2p}x^2$  for some value of  $p$ . Now, we have shown that every parabola of the form  $y = \frac{1}{2p}x^2$  is similar to our basic parabola  $y = x^2$ . Then, any parabola in the plane is similar to the basic parabola  $y = x^2$ .
- Further, all parabolas are similar to each other because we have just shown that they are all similar to the same parabola.

**Closing (3 minutes)**

Revisit the title of this lesson by asking students to summarize what they learned about the reason why all parabolas are similar. Then, take time to bring closure to this cycle of three lessons. The work students have engaged in has drawn together three different domains: geometry, algebra, and functions. In working through these examples and exercises and engaging in the discussions presented here, students can gain an appreciation for how mathematics can model real-world scenarios. The past three lessons show the power of using algebra and functions to solve problems in geometry.

Solving geometric problems using algebra and functions is one of the most powerful techniques we have to solve science, engineering, and technology problems.

**Lesson Summary**

- We started with a geometric figure of a parabola defined by geometric requirements and recognized that it involved the graph of an equation we studied in algebra.
- We used algebra to prove that all parabolas with the same distance between the focus and directrix are congruent to each other, and in particular, they are congruent to a parabola with vertex at the origin, axis of symmetry along the  $y$ -axis, and equation of the form  $y = \frac{1}{2p}x^2$ .
- Noting that the equation for a parabola with axis of symmetry along the  $y$ -axis is of the form  $y = f(x)$  for a quadratic function  $f$ , we proved that all parabolas are similar using transformations of functions.

**Exit Ticket (4 minutes)**

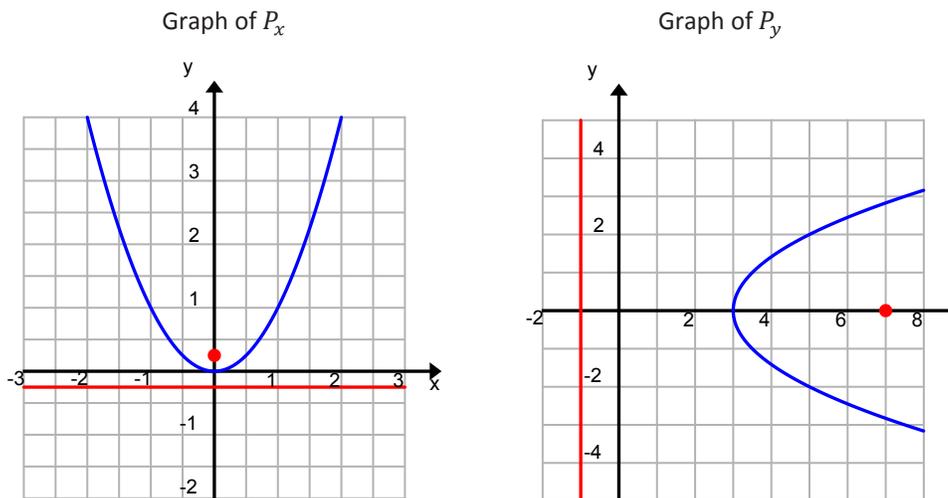
Name \_\_\_\_\_

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## Lesson 35: Are All Parabolas Similar?

### Exit Ticket

1. Describe the sequence of transformations that will transform the parabola  $P_x$  into the similar parabola  $P_y$ .



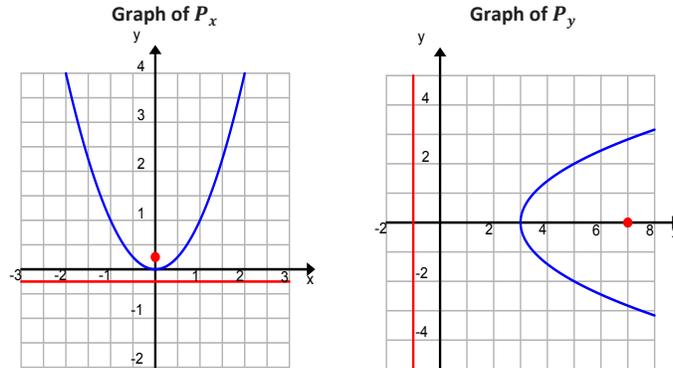
2. Are the two parabolas defined below similar or congruent or both? Justify your reasoning.

Parabola 1: The parabola with a focus of  $(0,2)$  and a directrix line of  $y = -4$

Parabola 2: The parabola that is the graph of the equation  $y = \frac{1}{6}x^2$

Exit Ticket Sample Solutions

1. Describe the sequence of transformations that would transform the parabola  $P_x$  into the similar parabola  $P_y$ .



Vertical scaling by a factor of  $\frac{1}{2}$ , vertical translation up 3 units, and a  $90^\circ$  rotation clockwise about the origin.

2. Are the two parabolas defined below similar or congruent or both?

Parabola 1: The parabola with a focus of  $(0, 2)$  and a directrix line of  $y = -4$

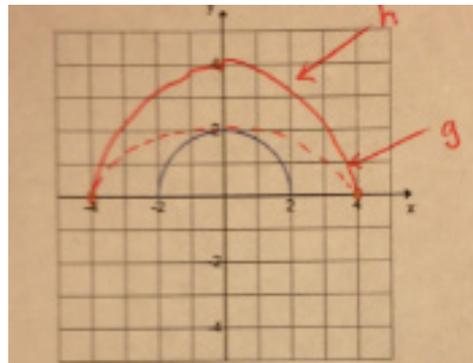
Parabola 2: The parabola that is the graph of the equation  $y = \frac{1}{6}x^2$

They are similar but not congruent because the distance between the focus and the directrix on Parabola 1 is 6 units, but on Parabola 2, it is only 3 units. Alternatively, students may describe that you cannot apply a series of rigid transformations that will map Parabola 1 onto Parabola 2. However, by using a dilation and a series of rigid transformations, the two parabolas can be shown to be similar since ALL parabolas are similar.

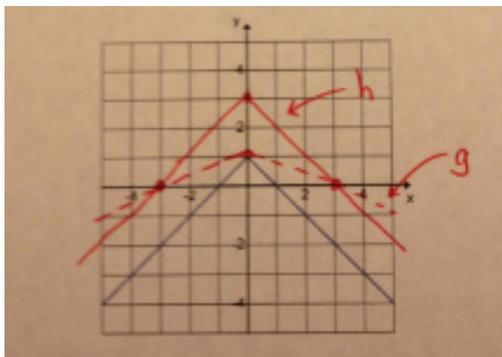
MP.3

Problem Set Sample Solutions

1. Let  $f(x) = \sqrt{4 - x^2}$ . The graph of  $f$  is shown below. On the same axes, graph the function  $g$ , where  $g(x) = f(\frac{1}{2}x)$ . Then, graph the function  $h$ , where  $h(x) = 2g(x)$ .



2. Let  $f(x) = -|x| + 1$ . The graph of  $f$  is shown below. On the same axes, graph the function  $g$ , where  $g(x) = f\left(\frac{1}{3}x\right)$ . Then, graph the function  $h$ , where  $h(x) = 3g(x)$ .



3. Based on your work in Problems 1 and 2, describe the resulting function when the original function is transformed with a horizontal and then a vertical scaling by the same factor,  $k$ .

*The resulting function is scaled by a factor of  $k$  in both directions. It is a dilation about the origin of the original figure and is similar to it.*

4. Let  $f(x) = x^2$ .

- a. What are the focus and directrix of the parabola that is the graph of the function  $f(x) = x^2$ ?

*Since  $\frac{1}{2p} = 1$ , we know  $p = \frac{1}{2}$ , and that is the distance between the focus and the directrix. The point  $(0, 0)$  is the vertex of the parabola and the midpoint of the segment connecting the focus and the directrix. Since the distance between the focus and vertex is  $\frac{1}{2}p = \frac{1}{4}$ , which is the same as the distance between the vertex and directrix; therefore, the focus has coordinates  $(0, \frac{1}{4})$ , and the directrix is  $y = -\frac{1}{4}$ .*

- b. Describe the sequence of transformations that would take the graph of  $f$  to each parabola described below.

- i. Focus:  $(0, -\frac{1}{4})$ , directrix:  $y = \frac{1}{4}$

*This parabola is a reflection of the graph of  $f$  across the  $x$ -axis.*

- ii. Focus:  $(\frac{1}{4}, 0)$ , directrix:  $x = -\frac{1}{4}$

*This parabola is a  $90^\circ$  clockwise rotation of the graph of  $f$ .*

- iii. Focus:  $(0, 0)$ , directrix:  $y = -\frac{1}{2}$

*This parabola is a vertical translation of the graph of  $f$  down  $\frac{1}{4}$  unit.*

- iv. Focus:  $(0, \frac{1}{4})$ , directrix:  $y = -\frac{3}{4}$

*This parabola is a vertical scaling of the graph of  $f$  by a factor of  $\frac{1}{2}$  and a vertical translation of the resulting image down  $\frac{1}{4}$  unit.*

- v. Focus:  $(0, 3)$ , directrix:  $y = -1$

*This parabola is a vertical scaling of the graph of  $f$  by a factor of  $\frac{1}{8}$  and a vertical translation of the resulting image up 1 unit.*

- c. Which parabolas are similar to the parabola that is the graph of  $f$ ? Which are congruent to the parabola that is the graph of  $f$ ?

*All of the parabolas are similar. We have proven that all parabolas are similar. The congruent parabolas are (i), (ii), and (iii). These parabolas are the result of a rigid transformation of the original parabola that is the graph of  $f$ . They have the same distance between the focus and directrix line as the original parabola.*

5. Derive the analytic equation for each parabola described in Problem 4(b) by applying your knowledge of transformations.

i.  $y = -x^2$

ii.  $x = y^2$

iii.  $y = x^2 - \frac{1}{4}$

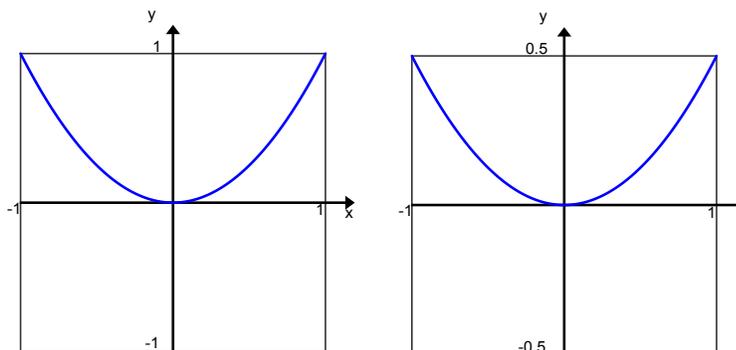
iv.  $y = \frac{1}{2}x^2 - \frac{1}{4}$

v.  $y = \frac{1}{8}x^2 + 1$

6. Are all parabolas the graph of a function of  $x$  in the  $xy$ -plane? If so, explain why, and if not, provide an example (by giving a directrix and focus) of a parabola that is not.

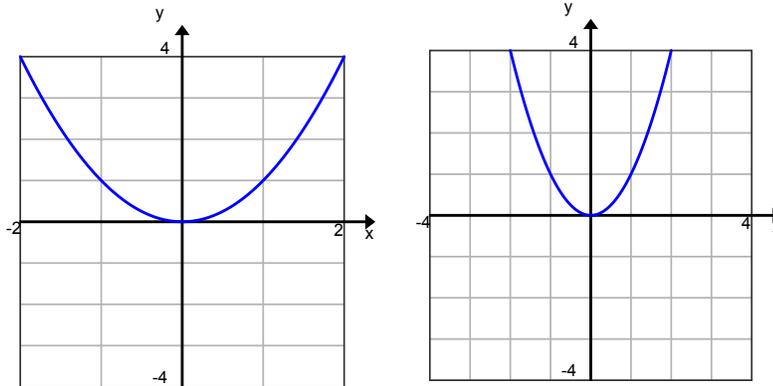
*No, they are not. Examples include the graph of the equation  $x = y^2$ , or a list stating a directrix and focus. For example, students may give the example of a directrix given by  $x = -2$  and focus  $(2, 0)$ , or an even more interesting example, such as a directrix given by  $y = x$  with focus  $(-1, -1)$ . Any line and any point not on that line define a parabola.*

7. Are the following parabolas congruent? Explain your reasoning.



*They are not congruent but they are similar. I can see that the parabola on the left appears to contain the point  $(1, 1)$ , while the parabola on the right appears to contain the point  $(1, \frac{1}{2})$ . This implies that the graph of the parabola on the right is a dilation of the graph of the parabola on the left, so they are not congruent.*

8. Are the following parabolas congruent? Explain your reasoning.



*They are congruent. Both graphs contain the points (0,0), (1,1), and (2,4) that satisfy the equation  $y = x^2$ . The scales are different on these graphs, making them appear non-congruent.*

9. Write the equation of a parabola congruent to  $y = 2x^2$  that contains the point (1, -2). Describe the transformations that would take this parabola to your new parabola.

*There are many solutions. Two possible solutions:*

*Reflect the graph about the x-axis to get  $y = -2x^2$ .*

*OR*

*Translate the graph down four units to get  $y = 2x^2 - 4$ .*

10. Write the equation of a parabola similar to  $y = 2x^2$  that does NOT contain the point (0, 0), but does contain the point (1, 1).

*There are many solutions. One solution is  $y = (x - 1)^2 + 1$ . This parabola is congruent to  $y = x^2$  and, therefore, similar to the original parabola, but the graph has been translated horizontally and vertically to contain the point (1, 1), but not the point (0, 0).*



Topic D:

## A Surprise from Geometry—Complex Numbers Overcome All Obstacles

N-CN.A.1, N-CN.A.2, N-CN.C.7, A-REI.A.2, A-REI.B.4b, A-REI.C.7

<b>Focus Standard:</b>	N-CN.A.1	Know there is a complex number $i$ such that $i^2 = -1$ , and every complex number has the form $a + bi$ with $a$ and $b$ real.
	N-CN.A.2	Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
	N-CN.C.7	Solve quadratic equations with real coefficients that have complex solutions.
	A-REI.A.2	Solve simple rational and radical equations in one variable, and give examples showing how extraneous solutions may arise.
	A-REI.B.4	Solve quadratic equations in one variable. <ul style="list-style-type: none"> <li>b. Solve quadratic equations by inspection (e.g., for <math>x^2 = 49</math>), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as <math>a \pm bi</math> for real numbers <math>a</math> and <math>b</math>.</li> </ul>
	A-REI.C.7	Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically. <i>For example, find the points of intersection between the line <math>y = -3x</math> and the circle <math>x^2 + y^2 = 3</math>.</i>
	<b>Instructional Days:</b>	5
<b>Lesson 36:</b>	Overcoming a Third Obstacle to Factoring—What If There Are No Real Number Solutions? (P) <sup>1</sup>	
<b>Lesson 37:</b>	A Surprising Boost from Geometry (P)	
<b>Lesson 38:</b>	Complex Numbers as Solutions to Equations (P)	
<b>Lesson 39:</b>	Factoring Extended to the Complex Realm (P)	
<b>Lesson 40:</b>	Obstacles Resolved—A Surprising Result (S)	

<sup>1</sup> Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson

In Topic D, students extend their facility with finding zeros of polynomials to include complex zeros. Lesson 36 presents a third obstacle to using factors of polynomials to solve polynomial equations. Students begin by solving systems of linear and non-linear equations to which no real solutions exist, and then relate this to the possibility of quadratic equations with no real solutions. Lesson 37 introduces complex numbers through their relationship to geometric transformations. That is, students observe that scaling all numbers on a number line by a factor of  $-1$  turns the number line out of its one-dimensionality and rotates it  $180^\circ$  through the plane. They then answer the question, “What scale factor could be used to create a rotation of  $90^\circ$ ?” In Lesson 38, students discover that complex numbers have real uses; in fact, they can be used in finding real solutions of polynomial equations. In Lesson 39, students develop facility with properties and operations of complex numbers and then apply that facility to factor polynomials with complex zeros. Lesson 40 brings the module to a close with the result that every polynomial can be rewritten as the product of linear factors, which is not possible without complex numbers. Even though standards **N-CN.C.8** and **N-CN.C.9** are not assessed at the Algebra II level, they are included instructionally to develop further conceptual understanding.



## Lesson 36: Overcoming a Third Obstacle to Factoring— What If There Are No Real Number Solutions?

### Student Outcomes

- Students understand the possibility that an equation—or a system of equations—has no real solutions. Students identify these situations and make the appropriate geometric connections.

### Lesson Notes

Lessons 36–40 provide students with the necessary tools to find solutions to polynomial equations outside the realm of the real numbers. This lesson illustrates how to both analytically and graphically identify a system of equations that has no real number solution. In the next lesson, the imaginary unit  $i$  is defined, and students begin to work with complex numbers through the familiar geometric context of rotation. Students will realize that the set of complex numbers inherits the arithmetic and algebraic properties from the real numbers. The work with complex solutions to polynomial equations in these lessons culminates with the Fundamental Theorem of Algebra in Lesson 40, the last lesson in this module.

### Classwork

#### Opening (1 minutes)

This lesson illustrates how to identify a system of equations that has no real number solution, both graphically and analytically. In this lesson, we explore systems of equations that have no real number solutions.

#### Opening Exercise 1 (5 minutes)

Instruct students to complete the following exercise individually and then to pair up with a partner after a few minutes to compare their answers. Allow students to search for solutions analytically or graphically as they choose. After a few minutes, ask students to share their answers and solution methods. Both an analytic and a graphical solution should be presented for each system, either by a student or by the teacher if all students used the same approach. Circulate while students are working, and take note of which students are approaching the question analytically and which are approaching the question graphically.

**Opening Exercise**

Find all solutions to each of the systems of equations below using any method.

$$2x - 4y = -1$$

$$3x - 6y = 4$$

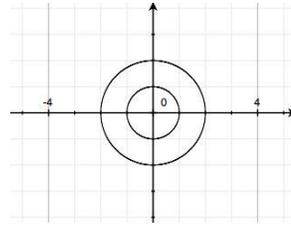
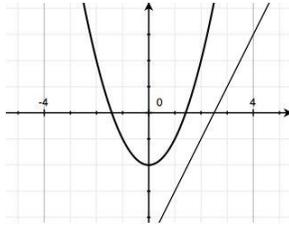
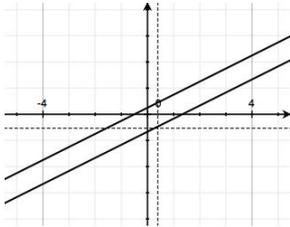
$$y = x^2 - 2$$

$$y = 2x - 5$$

$$x^2 + y^2 = 1$$

$$x^2 + y^2 = 4$$

All three systems have no real number solutions, which is evident from the non-intersecting graphs in each. Instead of graphing the systems, students may have used an analytic approach such as the approach outlined in the Discussion below.



**Discussion (10 minutes)**

Ask students to explain their reasoning for each of the three systems in the Opening Exercise with both approaches shown for each part. This means that six students should have the opportunity to present their solutions to the class. It is important that you go through both the analytical and graphical approaches for each system so that students draw the connection between graphs that do not intersect and systems that have no analytic solution. Be sure that you have some way to display the graph of each system of equations as you lead students through this discussion.

Part (a):

- Looking at the graph of the first system  $\begin{cases} 2x - 4y = -1 \\ 3x - 6y = 4 \end{cases}$ , how can we tell that there is no solution?
  - The two lines never intersect.
  - The two lines are parallel.
- Using an algebraic approach, how can we tell that there is no solution?
  - If we multiply both sides of the top equation by 3 and the bottom equation by 2, we see that an equivalent system can be written.

$$6x - 12y = -3$$

$$6x - 12y = 8$$

Subtracting the first equation from the second results in the false number sentence

$$0 = 11.$$

Thus, there are no real numbers  $x$  and  $y$  that satisfy both equations.

- The graphs of these equations are lines. What happens if we put them in slope-intercept form?
  - *Rewriting both linear equations in slope-intercept form, the system from part (a) can be written as*

$$y = \frac{1}{2}x + \frac{1}{4}$$

$$y = \frac{1}{2}x - \frac{2}{3}$$

*From what we know about graphing lines, the lines associated to these equations have the same slope and different  $y$ -intercepts, so they will be parallel. Since parallel lines do not intersect, the lines have no points in common and, therefore, this system has no solution.*

Part (b):

- Looking at the graph of the second system  $\begin{cases} y = x^2 - 2 \\ y = 2x - 5 \end{cases}$ , how can we tell that there is no solution?
  - *The line and the parabola never intersect.*
- Can we confirm, algebraically, that the system in part (b) has no real solution?
  - *Yes. Since  $y = x^2 - 2$  and  $y = 2x - 5$ , we must have  $x^2 - 2 = 2x - 5$ , which is equivalent to the quadratic equation  $x^2 - 2x + 3 = 0$ . Solving for  $x$  using the quadratic formula, we get*

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)} = 1 \pm \frac{\sqrt{-8}}{2}$$

*Since the square root of a negative real number is not a real number, there is no real number  $x$  that satisfies the equation  $x^2 - 2x + 3 = 0$ ; therefore, there is no point in the plane with coordinates  $(x, y)$  that satisfies both equations in the original system.*

Part (c):

- Looking at the graph of the final system  $\begin{cases} x^2 + y^2 = 1 \\ x^2 + y^2 = 4 \end{cases}$ , how can we tell that there is no solution?
  - *The circles are concentric, meaning that they have the same center and different radii. Thus, they never intersect, and there are no points that lie on both circles.*
- Can we algebraically confirm that the system in part (c) has no solution?
  - *Yes. If we try to solve this system, we could subtract the first equation from the second, giving the false number sentence  $0 = 3$ . Since this statement is false, we know that there are no values of  $x$  and  $y$  that satisfy both equations simultaneously; thus, the system has no solution.*

At this point, ask students to summarize in writing or with a partner what they have learned so far. Use this brief exercise as an opportunity to check for understanding.

### Exercise 1 (4 minutes)

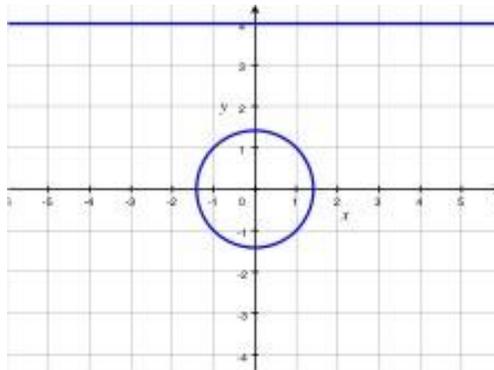
Have students work individually, and then check their answers with a partner. Make sure they write out their steps as done in the sample solutions. After a few minutes, invite students to share one or two solutions on the board.

Exercises 1–4

1. Are there any real number solutions to the system  $\begin{cases} y = 4 \\ x^2 + y^2 = 2 \end{cases}$ ? Support your findings both analytically and graphically.

$$\begin{aligned} x^2 + (4)^2 &= 2 \\ x^2 + 16 &= 2 \\ x^2 &= -14 \end{aligned}$$

Since  $x^2$  is non-negative for all real numbers  $x$ , there are no real numbers  $x$  so that  $x^2 = -14$ . Then, there is no pair of real numbers  $(x, y)$  that solves the system consisting of the line  $y = 4$  and the circle  $x^2 + y^2 = 2$ . Thus, the line  $y = 4$  does not intersect the circle  $x^2 + y^2 = 2$  in the real plane. This is confirmed graphically as follows.



Discussion (7 minutes)

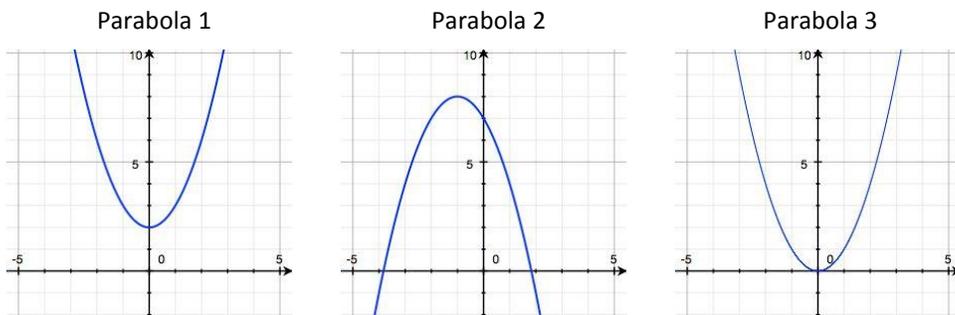
We are still withholding any mention of complex numbers or complex solutions; those will be introduced in the next lesson. Make sure your students understand that analytical findings can be confirmed graphically and vice-versa. We turn our focus to quadratic equations in one variable  $x$  without real solutions and to how the absence of any real solution  $x$  can be confirmed by graphing a system of equations with two variables  $x$  and  $y$ .

Scaffolding:

Feel free to assign an optional extension exercise, such as: “Which of these equations will have no solution? Explain how you know in terms of a graph.”

$$\begin{aligned} x^2 + 5 &= 0 \\ x^2 - 4 &= 0 \\ x^2 + 1 &= 0 \\ x^2 - 10 &= 0 \end{aligned}$$

Solution:  $x^2 + 5 = 0$  and  $x^2 + 1 = 0$  will not have real solutions because the graphs of the equations  $y = x^2 + 5$  and  $y = x^2 + 1$  do not intersect the  $x$ -axis, the line given by  $y = 0$ .



Present students with the following graphs of parabolas:

- Remember that a parabola with a vertical axis of symmetry is the graph of an equation of the form  $y = ax^2 + bx + c$  for some real number coefficients  $a, b$ , and  $c$  with  $a \neq 0$ . We can consider the solutions of the quadratic equation  $ax^2 + bx + c = 0$  to be the  $x$ -coordinates of solutions to the system of equations  $y = ax^2 + bx + c$  and  $y = 0$ . Thus, when we are investigating whether a quadratic equation  $ax^2 + bx + c = 0$  has a solution, we can think of this as finding the  $x$ -intercepts of the graph of  $y = ax^2 + bx + c$ .
- Which of these three parabolas are represented by a quadratic equation  $y = ax^2 + bx + c$  that has no solution to  $ax^2 + bx + c = 0$ ? Explain how you know.
  - *Parabola 1 because its graph does not intersect the  $x$ -axis. No  $x$ -intercepts of the parabola means there are no solutions to the associated equation  $ax^2 + bx + c = 0$ .*

- Now, consider Parabola 2, which is the graph of the equation  $y = 8 - (x + 1)^2$ . How many solutions are there to the equation  $8 - (x + 1)^2 = 0$ ? Explain how you know.
  - *Because Parabola 2 intersects the  $x$ -axis twice, the system consisting of  $y = 8 - (x + 1)^2$  and  $y = 0$  has two real solutions. The graph suggests that the system will have one positive solution and one negative solution.*
- Now, consider Parabola 3, which is the graph of the equation  $y = x^2$ . How does the graph tell us how many solutions there are to the equation  $x^2 = 0$ ? Explain how you know.
  - *Parabola 3 touches the  $x$ -axis only at  $(0, 0)$  so the parabola and the line with equation  $y = 0$  intersect at only one point. Accordingly, the system has exactly one solution, and there is exactly one solution to the equation  $x^2 = 0$ .*

Pause, and ask students to again summarize what they have learned, either in writing or orally to a neighbor. Students should be making connections between the graph of the quadratic equation  $y = ax^2 + bx + c$  (which is a parabola), the number of  $x$ -intercepts of the graph, and the number of solutions to the system consisting of  $y = 0$  and  $y = ax^2 + bx + c$ .

**Exercises 2–4 (12 minutes)**

Students should work individually or in pairs on these exercises. To solve these problems analytically, they need to understand that they can determine the  $x$ -coordinates of the intersection points of the graphs of these geometric figures by solving an equation. Make sure your students are giving their answers to these questions as coordinate pairs. Encourage students to solve the problems analytically and verify the solutions graphically.

2. Does the line  $y = x$  intersect the parabola  $y = -x^2$ ? If so, how many times, and where? Draw graphs on the same set of axes.

$$x = -x^2$$

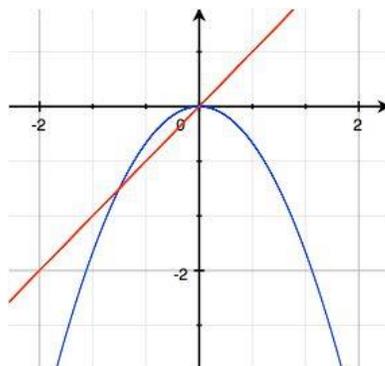
$$x + x^2 = 0$$

$$x(1 + x) = 0$$

$$x = 0 \text{ or } x = -1$$

If  $x = 0$ , then  $y = -x^2 = 0$ , and if  $x = -1$ , then  $y = -x^2 = -(-1)^2 = -1$ .

The line  $y = x$  intersects the parabola  $y = -x^2$  at two distinct points:  $(0, 0)$  and  $(-1, -1)$ .



**Scaffolding:**

- Consider having students follow the instructor along using a graphing calculator to show that the graph of  $y = x$  intersects the graph of  $y = -x^2$  twice, at the points indicated.
- Consider tasking advanced students to generate a system that meets certain criteria. For example, ask them to write the equations of a circle and a parabola that intersect once at  $(0, 1)$ . One appropriate answer is  $x^2 + y^2 = 1$  and  $y = x^2 + 1$ .

3. Does the line  $y = -x$  intersect the circle  $x^2 + y^2 = 1$ ? If so, how many times, and where? Draw graphs on the same set of axes.

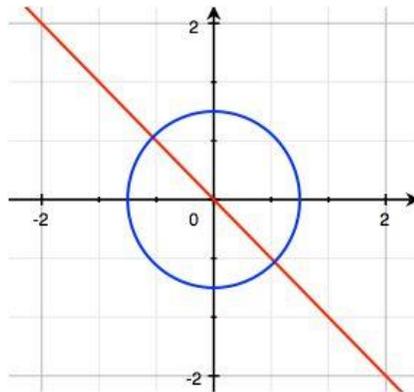
$$x^2 + (-x)^2 = 1$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$x = -\frac{\sqrt{2}}{2} \text{ or } x = \frac{\sqrt{2}}{2}$$

The line  $y = -x$  intersects the circle  $x^2 + y^2 = 1$  at two distinct points:  $(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$  and  $(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$ .



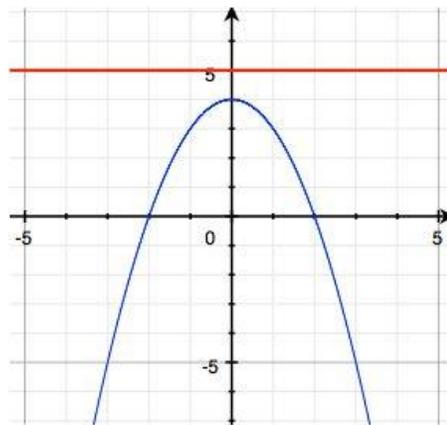
4. Does the line  $y = 5$  intersect the parabola  $y = 4 - x^2$ ? Why or why not? Draw the graphs on the same set of axes.

$$5 = 4 - x^2$$

$$1 = -x^2$$

$$x^2 = -1$$

A squared real number cannot be negative, so the line  $y = 5$  does not intersect the parabola  $y = 4 - x^2$ .



Before moving on, discuss these results as a whole class. Have students put both graphical and analytical solutions to each exercise on the board. Start to reinforce the connection that when the graphs intersect, the related system of equations has real solutions, and when the graphs do not intersect, there are no real solutions to the related system of equations.

**Closing (2 minutes)**

Have students discuss with their neighbors the key points from today's lesson. Encourage them to discuss the relationship between the solution(s) to a quadratic equation of the form  $ax^2 + bx + c = 0$  and the system

$$y = ax^2 + bx + c$$
$$y = 0.$$

They should discuss an understanding of the relationship between any solution(s) to a system of two equations and the  $x$ -coordinate of any point(s) of intersection of the graphs of the equations in the system.

The Lesson Summary below contains key findings from today's lesson.

**Lesson Summary**

An equation or a system of equations may have one or more solutions in the real numbers, or it may have no real number solution.

Two graphs that do not intersect in the real plane describe a system of two equations without a real solution. If a system of two equations does not have a real solution, the graphs of the two equations do not intersect in the real plane.

A quadratic equation in the form  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ , that has no real solution indicates that the graph of  $y = ax^2 + bx + c$  does not intersect the  $x$ -axis.

**Exit Ticket (4 minutes)**

In this Exit Ticket, students will show that a particular system of two equations has no real solutions. They will demonstrate this both analytically and graphically.

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 36: Overcoming a Third Obstacle—What If There Are No Real Number Solutions?

### Exit Ticket

Solve the following system of equations or show that it does not have a real solution. Support your answer analytically and graphically.

$$y = x^2 - 4$$

$$y = -(x + 5)$$

## Exit Ticket Sample Solutions

Solve the following system of equations or show that it does not have a real solution. Support your answer analytically and graphically.

$$y = x^2 - 4$$

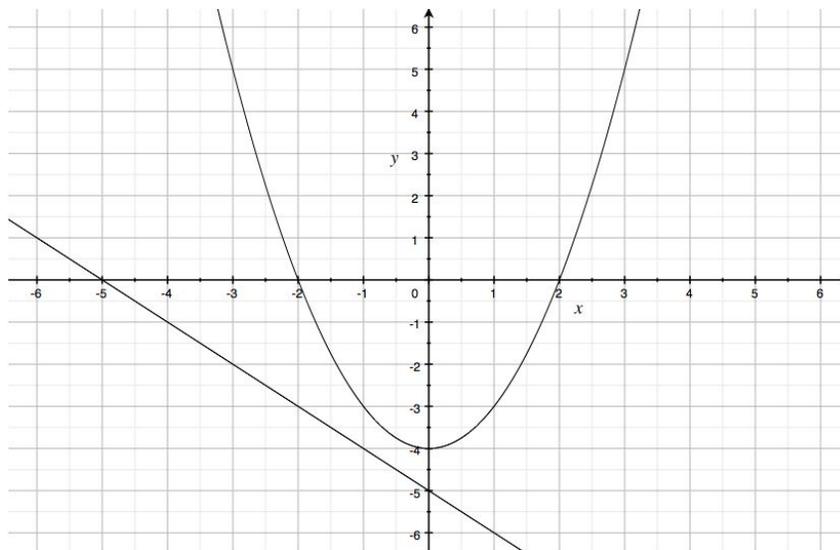
$$y = -(x + 5)$$

We distribute over the set of parentheses in the second equation and rewrite the system.

$$y = x^2 - 4$$

$$y = -(x + 5)$$

The graph of the system shows a parabola and a line that do not intersect. As such, we know that the system does not have a real solution.



Algebraically,

$$x^2 - 4 = -x - 5$$

$$x^2 + x + 1 = 0.$$

Using the quadratic formula with  $a = 1$ ,  $b = 1$ , and  $c = 1$ ,

$$x = \frac{-1 + \sqrt{1^2 - 4(1)(1)}}{2(1)} \text{ or } x = \frac{-1 - \sqrt{1^2 - 4(1)(1)}}{2(1)},$$

which indicates that the solutions would be  $\frac{-1 + \sqrt{-3}}{2}$  and  $\frac{-1 - \sqrt{-3}}{2}$ .

Since the square root of a negative number is not a real number, there is no real number  $x$  that solves this equation. Thus, the system has no solution  $(x, y)$  where  $x$  and  $y$  are real numbers.

## Problem Set Sample Solutions

1. For each part, solve the system of linear equations, or show that no real solution exists. Graphically support your answer.

a. 
$$\begin{aligned} 4x + 2y &= 9 \\ x + y &= 3 \end{aligned}$$

*Multiply the bottom equation by 4.*

$$\begin{aligned} 4x + 2y &= 9 \\ 4x + 4y &= 12 \end{aligned}$$

*Subtract top from bottom.*

$$2y = 3$$

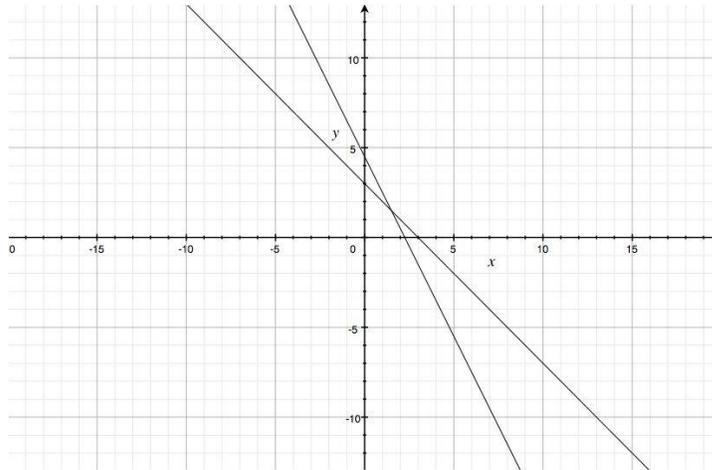
*We get  $y = \frac{3}{2}$ .*

*Substitute  $\frac{3}{2}$  for  $y$  in the original bottom equation.*

$$x + \frac{3}{2} = 3$$

*We get  $x = \frac{3}{2}$ .*

*The lines from the system intersect at  $(\frac{3}{2}, \frac{3}{2})$ .*



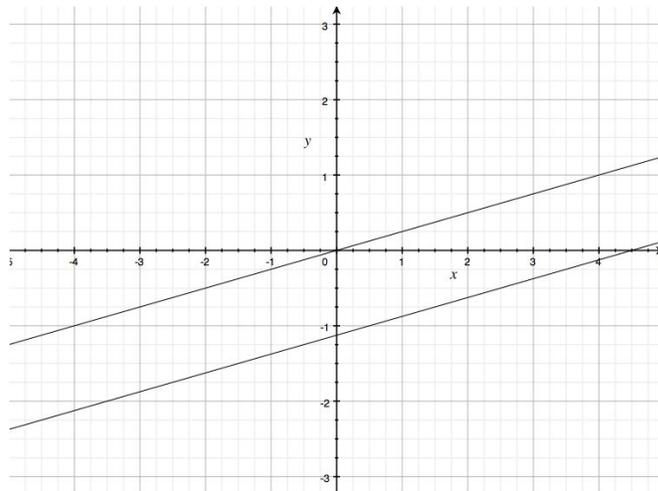
b.  $2x - 8y = 9$   
 $3x - 12y = 0$

Multiply the top equation by 3 and the bottom equation by 2 on both sides.

$$6x - 24y = 27$$

$$6x - 24y = 0$$

Subtracting the bottom equation from the top equation gives  $27 = 0$ , but  $27 = 0$  is a false number sentence. Thus, there is no solution to the system. The graph of the system appropriately shows two parallel lines.



2. Solve the following system of equations, or show that no real solution exists. Graphically confirm your answer.

$$3x^2 + 3y^2 = 6$$

$$x - y = 3$$

We can factor out 3 from the top equation and isolate  $y$  in the bottom equation to give us a better idea of what the graphs of the equations in the system look like. The first equation represents a circle centered at the origin with radius  $\sqrt{2}$ , and the second equation represent the line  $y = x - 3$ , which is simply the  $45^\circ$  line through the origin,  $y = x$ , shifted down by 3 units.

Algebraically,

$$3x^2 + 3(x - 3)^2 = 6$$

$$x^2 + (x - 3)^2 = 2$$

$$x^2 + (x^2 - 6x + 9) = 2$$

$$2x^2 - 6x + 7 = 0$$

We solve for  $x$  using the quadratic formula:

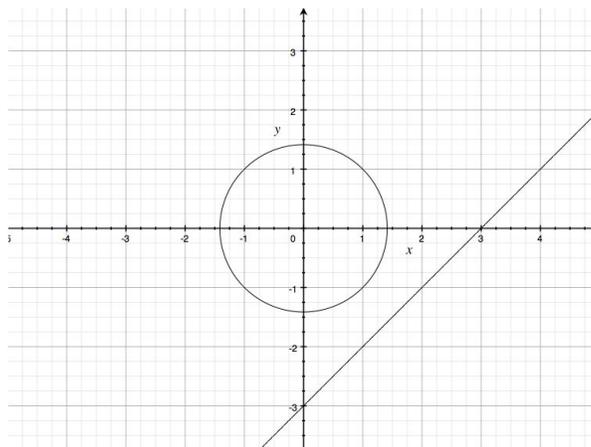
$$a = 2, b = -6, c = 7$$

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(2 \cdot 7)}}{2 \cdot 2}$$

$$x = \frac{6 \pm \sqrt{36 - 56}}{4}$$

The solutions would be  $\frac{6 + \sqrt{-20}}{4}$  and  $\frac{6 - \sqrt{-20}}{4}$ .

Since both solutions for  $x$  contain a square root of a negative number, no real solution  $x$  exists; so, the system has no solution  $(x, y)$  where  $x$  and  $y$  are real numbers.



3. Find the value of  $k$  so that the graph of the following system of equations has no solution.

$$\begin{aligned} 3x - 2y - 12 &= 0 \\ kx + 6y - 10 &= 0 \end{aligned}$$

First, we rewrite the linear equations in the system in slope-intercept form.

$$\begin{aligned} y &= \frac{3}{2}x - 6 \\ y &= -\frac{k}{6}x + \frac{10}{6} \end{aligned}$$

There is no solution to this system when the lines are parallel. Two lines are parallel when they share the same slope and have different  $y$ -intercepts. Here, the first line has slope  $\frac{3}{2}$  and  $y$ -intercept  $-6$ , and the second line has slope  $-\frac{k}{6}$  and  $y$ -intercept  $\frac{10}{6}$ . The lines have different  $y$ -intercepts and will be parallel when  $-\frac{k}{6} = \frac{3}{2}$ .

$$\begin{aligned} \frac{3}{2} &= -\frac{k}{6} \\ 2k &= -18 \\ k &= -9 \end{aligned}$$

Thus, there is no solution only when  $k = -9$ .

4. Offer a geometric explanation to why the equation  $x^2 - 6x + 10 = 0$  has no real solutions.

The graph of  $y = x^2 - 6x + 10$  opens upward (since the leading coefficient is positive) and takes on its lowest value at the vertex  $(3, 1)$ . Hence, it does not intersect the  $x$ -axis, and, therefore, the equation has no real solutions.

5. Without his pencil or calculator, Joey knows that  $2x^3 + 3x^2 - 1 = 0$  has at least one real solution. How does he know?

The graph of every cubic polynomial function intersects the  $x$ -axis at least once because the end behaviors are opposite: one end goes up and the other goes down. This means that the graph of any cubic equation  $y = ax^3 + bx^2 + cx + d$  must have at least one  $x$ -intercept. Thus, every cubic equation must have at least one real number solution.

6. The graph of the quadratic equation  $y = x^2 + 1$  has no  $x$ -intercepts. However, Gia claims that when the graph of  $y = x^2 + 1$  is translated by a distance of 1 in a certain direction, the new (translated) graph would have exactly one  $x$ -intercept. Further, if  $y = x^2 + 1$  is translated by a distance greater than 1 in the same direction, the new (translated) graph would have exactly two  $x$ -intercepts. Support or refute Gia's claim. If you agree with her, in which direction did she translate the original graph? Draw graphs to illustrate.

By translating the graph of  $y = x^2 + 1$  DOWN by 1 unit, the new graph has equation  $y = x^2$ , which has one  $x$ -intercept at  $x = 0$ . When translating the original graph DOWN by more than 1 unit, the new graph will cross the  $x$ -axis exactly twice.

7. In the previous problem, we mentioned that the graph of  $y = x^2 + 1$  has no  $x$ -intercepts. Suppose that  $y = x^2 + 1$  is one of two equations in a system of equations and that the other equation is a line. Give an example of a linear equation such that this system has exactly one solution.

The line with equation  $y = 1$  is tangent to  $y = x^2 + 1$  only at  $(0, 1)$ ; so, there would be exactly one real solution to the system.

$$\begin{aligned} y &= x^2 + 1 \\ y &= 1 \end{aligned}$$

Another possibility is an equation of any vertical line, such as  $x = -3$  or  $x = 4$ , or  $x = a$  for any real number  $a$ .

MP.3

8. In prior problems, we mentioned that the graph of  $y = x^2 + 1$  has no  $x$ -intercepts. Does the graph of  $y = x^2 + 1$  intersect the graph of  $y = x^3 + 1$ ?

*Setting these equations together, we can rearrange terms to get  $x^3 - x^2 = 0$ , which is an equation we can solve by factoring. We have  $x^2(x - 1) = 0$ , which has solutions at 0 and 1. Thus, the graphs of these equations intersect when  $x = 0$  and when  $x = 1$ . When  $x = 0$ ,  $y = 1$ , and when  $x = 1$ ,  $y = 2$ . Thus, the two graphs intersect at the points  $(0, 1)$  and  $(1, 2)$ .*

*The quick answer: The highest term in both equations has degree 3. The third degree term does not cancel when setting the two equations (in terms of  $x$ ) equal to each other. All cubic equations have at least one real solution, so the two graphs intersect at least at one point.*



## Lesson 37: A Surprising Boost from Geometry

### Student Outcomes

- Students define a complex number in the form  $a + bi$ , where  $a$  and  $b$  are real numbers and the imaginary unit  $i$  satisfies  $i^2 = -1$ . Students geometrically identify  $i$  as a multiplicand effecting a  $90^\circ$  counterclockwise rotation of the real number line. Students locate points corresponding to complex numbers in the complex plane.
- Students understand complex numbers as a superset of the real numbers; i.e., a complex number  $a + bi$  is real when  $b = 0$ . Students learn that complex numbers share many similar properties of the real numbers: associative, commutative, distributive, addition/subtraction, multiplication, etc.

### Lesson Notes

Students first receive an introduction to the imaginary unit  $i$  and develop an algebraic and geometric understanding of the complex numbers (**N-CN.A.1**). The lesson then underscores that complex numbers also satisfy the properties of operations that real numbers do (**N-CN.A.2**). Finally, students perform exercises to reinforce their understanding of and facility with complex numbers in an algebraic arena. This lesson ties into the work in the next lesson, which involves complex solutions to quadratic equations (**N-CN.C.7**).

Complex numbers are neither *imaginary*, as in make believe, nor *complex*, as in complicated. Students first encounter them when they classify equations such as  $x^2 + 1 = 0$  as having no real number solutions. At that point, we do not introduce the possibility that a solution exists within a superset of the real numbers called the complex numbers. At the end of this module, we briefly introduce the idea that every polynomial  $P$  of degree  $n$  has  $n$  values  $r_i$  for which  $P(r_i) = 0$ , where  $n$  is a whole number and  $r_i$  is a real or complex number. Further, in preparation for students' work in Precalculus, we state (but do not expect students to know) that  $P$  can be written as the product of  $n$  linear factors, a result known as the Fundamental Theorem of Algebra. The usefulness of complex numbers as solutions to polynomial equations comes with a cost: While real numbers can be ordered (put in order from smallest to greatest), complex numbers cannot be compared; for example, the complex number  $\frac{1}{2} + i\frac{\sqrt{3}}{2}$  is not larger or smaller than  $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ . However, this is a small price to pay. Students will begin to see just how important complex numbers are to geometry and computer science in Modules 1 and 2 in Precalculus. In college level science and engineering courses, complex numbers are used in conjunction with differential equations to model circular motion and periodic phenomena in two dimensions.

### Classwork

#### Opening (1 minute)

We introduce a geometric context for complex numbers by demonstrating the analogous relationship between rotations in the plane and multiplication. The intention is for students to develop a deep understanding of  $i$  through geometry.

- Today, we define a new number system that allows us to identify solutions to some equations that have no real number solutions. The complex numbers, as you will see, in fact share many properties with the real numbers with which you are familiar. We will be taking a geometric approach to introducing complex numbers.

**Opening Exercise (5 minutes)**

Have students work alone on this motivating Opening Exercise. This exercise provides the context and invites the necessity for defining an alternative number system, namely the complex numbers. Go over parts (a), (b), and (c) with the class; then, suggest that part (d) may be solvable using an alternative number system. Have students table this thought while beginning a geometrically-oriented discussion.

<b>Opening Exercise</b>	
Solve each equation for $x$ .	
a. $x - 1 = 0$	1
b. $x + 1 = 0$	-1
c. $x^2 - 1 = 0$	1, -1
d. $x^2 + 1 = 0$	No real solution

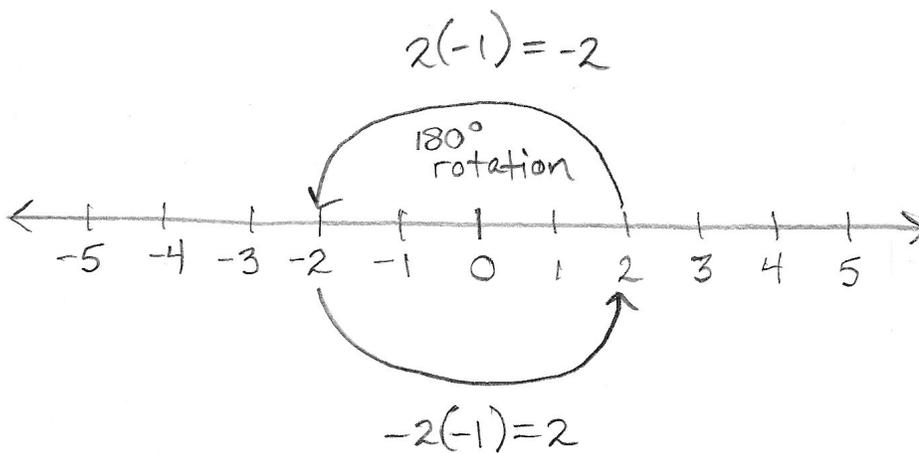
*Scaffolding:*

- There were times in the past when people would have said that an equation such as  $x^2 = 2$  also had no solution.

**Discussion (20 minutes)**

Before beginning, allow students to prepare graph paper for drawing images as the discussion unfolds. At the close of this discussion, have students work with partners to summarize at least one thing they learned; then, provide time for some teacher-guided note-taking to capture the definition of the imaginary unit and its connection to geometric rotation.

Recall that multiplying by  $-1$  rotates the number line in the plane by  $180^\circ$  about the point 0.

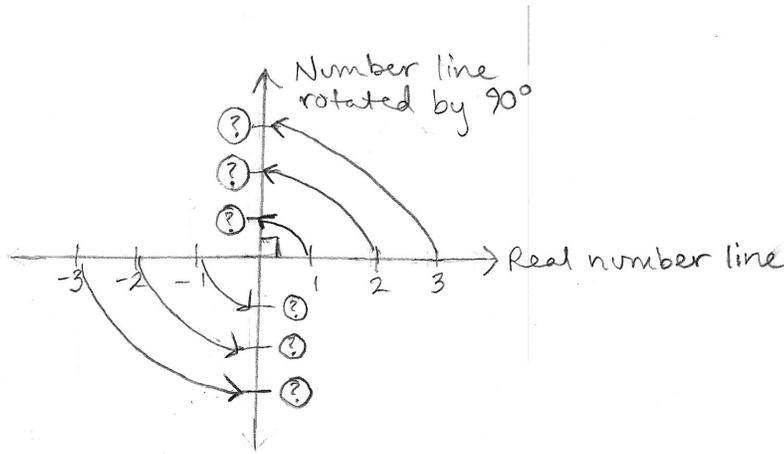


*Scaffolding:*

- You can demonstrate the rotation concept by drawing the number line carefully on a piece of white paper, drawing an identical number line on a transparency, putting a pin at zero, and rotating the transparency to show that the number line is rotating. For example, go from 2 to  $-2$ . This, of course, is the same as multiplying by  $-1$ .

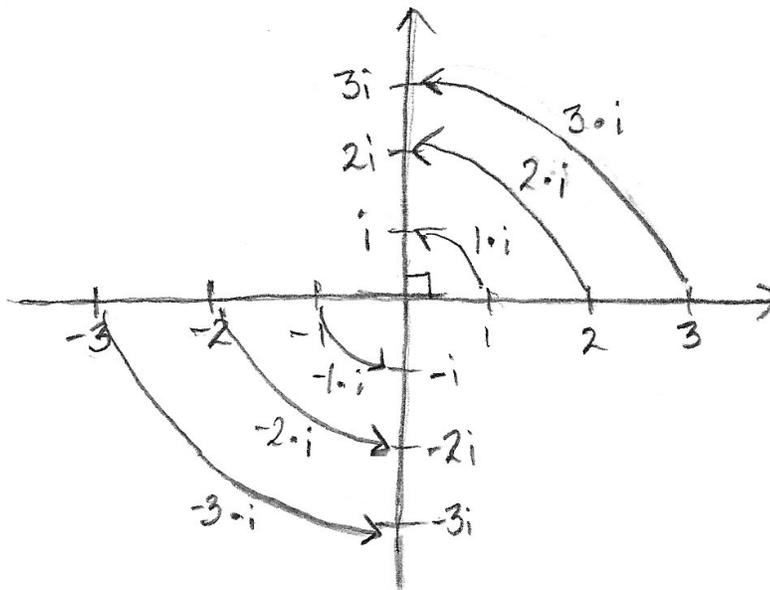
Pose this interesting thought question to students: Is there a number we can multiply by that corresponds to a  $90^\circ$  rotation?

Students may find that this is a strange question. First, such a number *does not* take the number line to itself, so we have to *imagine* another number line that is a 90° rotation of the original:

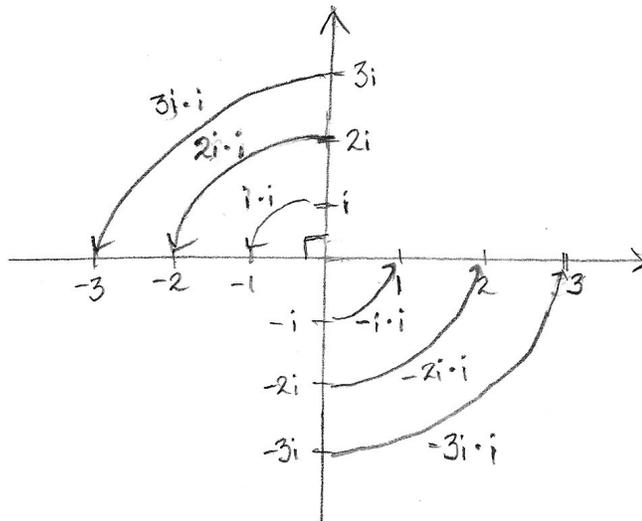


This is like the coordinate plane. However, how should we label the points on the vertical axis?

Well, since we *imagined* such a number existed, let's call it the imaginary axis and subdivide it into units of something called *i*. Then, the point 1 on the number line rotates to  $1 \cdot i$  on the rotated number line and so on, as follows:



- What happens if we multiply a point on the vertical number line by  $i$ ?
  - We rotate that point by  $90^\circ$  counterclockwise:



When we perform two  $90^\circ$  rotations, it is the same as performing a  $180^\circ$  rotation, so multiplying by  $i$  twice results in the same rotation as multiplying by  $-1$ . Since two rotations by  $90^\circ$  is the same as a single rotation by  $180^\circ$ , two rotations by  $90^\circ$  is equivalent to multiplication by  $i$  twice, and one rotation by  $180^\circ$  is equivalent to multiplication by  $-1$ , we have

MP.2

$$i^2 \cdot x = -1 \cdot x$$

for any real number  $x$ ; thus,

$$i^2 = -1.$$

- Why might this new number  $i$  be useful?
  - Recall from the Opening Exercise that there are no real solutions to the equation  $x^2 + 1 = 0$ .  
However, this new number  $i$  is a solution.

$$(i)^2 + 1 = -1 + 1 = 0$$

In fact, “solving” the equation  $x^2 + 1 = 0$ , we get

$$\begin{aligned} x^2 &= -1 \\ \sqrt{x^2} &= \sqrt{-1} \\ x &= \sqrt{-1} \text{ or } x = -\sqrt{-1}. \end{aligned}$$

However, because we know from above that  $i^2 = -1$ , and  $(-i)^2 = (-1)^2(i)^2 = -1$ , we have two solutions to the quadratic equation  $x^2 = -1$ , which are  $i$  and  $-i$ .

These result suggests that “ $i = \sqrt{-1}$ .” That seems a little weird, but this new imagined number  $i$  already appears to solve problems we could not solve before.

For example, in Algebra I, when we applied the quadratic formula to  $x^2 + 2x + 5 = 0$ , we found that

$$x = \frac{-2 + \sqrt{2^2 - 4(1)(5)}}{2(1)} \text{ or } x = \frac{-2 - \sqrt{2^2 - 4(1)(5)}}{2(1)}$$

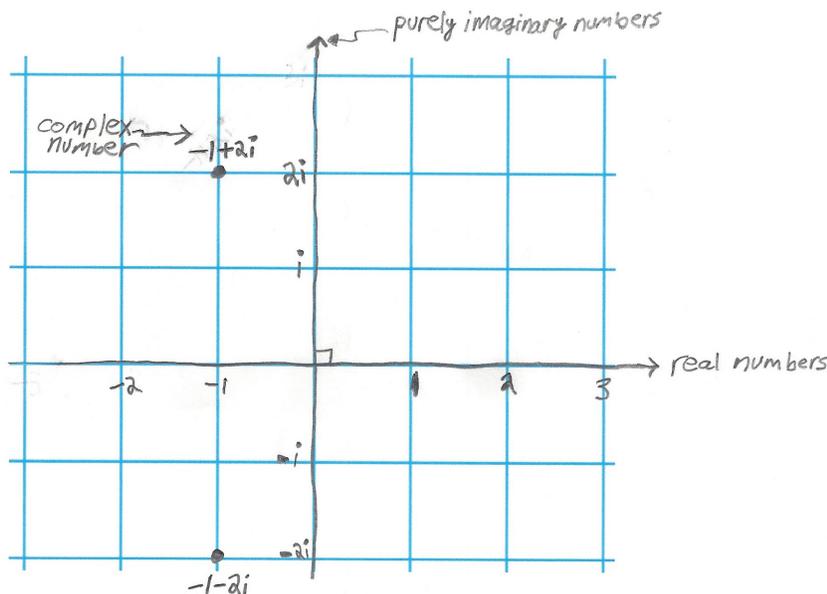
$$x = \frac{-2 + \sqrt{-16}}{2} \text{ or } x = \frac{-2 - \sqrt{-16}}{2}.$$

Recognizing the negative number under the square root, we reported that the equation  $x^2 + 2x + 5 = 0$  has no real solutions. Now, however, we can write

$$\sqrt{-16} = \sqrt{16 \cdot -1} = \sqrt{16} \cdot \sqrt{-1} = 4i.$$

Therefore,  $x = -1 + 2i$  or  $x = -1 - 2i$ , which means  $-1 + 2i$  and  $-1 - 2i$  are the solutions to  $x^2 + 2x + 5 = 0$ .

The solutions  $-1 + 2i$  and  $-1 - 2i$  are numbers called complex numbers, which we can locate in the complex plane.



**Scaffolding:**

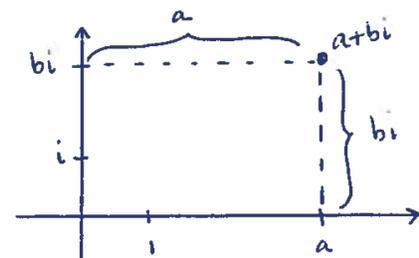
- Name a few complex numbers for students to plot on their graph paper. This will build an understanding of their locations in this coordinate system. For example, consider  $-2i - 3$ ,  $-i$ ,  $i$ ,  $i - 1$ , and  $\frac{3}{2}i + 2$ . Make sure students are also cognizant of the fact that real numbers are also complex numbers, e.g.,  $-\frac{3}{2}$ ,  $0$ ,  $1$ ,  $\pi$ .

In fact, all complex numbers can be written in the form  $a + bi$ ,

where  $a$  and  $b$  are real numbers. Just as we can represent real numbers on the number line, we can represent complex numbers in the complex plane. Each complex number  $a + bi$  can be located in the complex plane in the same way we locate the point  $(a, b)$  in the Cartesian plane. From the origin, translate  $a$  units horizontally along the real axis and  $b$  units vertically along the imaginary axis.

Since complex numbers are built from real numbers, we should be able to add, subtract, multiply, and divide them. They should also satisfy the commutative, associative, and distributive properties, just as real numbers do.

Let's check how some of these operations work for complex numbers.



**Examples 1–2 (4 minutes): Addition and Subtraction with Complex Numbers**

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Addition of variable expressions is a matter of re-arranging terms according to the properties of operations. Often, we call this “combining like terms.” These properties of operations apply to complex numbers.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

**Example 1: Addition with Complex Numbers**

Compute  $(3 + 4i) + (7 - 20i)$ .

$$(3 + 4i) + (7 - 20i) = 3 + 4i + 7 - 20i = (3 + 7) + (4 - 20)i = 10 - 16i$$

**Example 2: Subtraction with Complex Numbers**

Compute  $(3 + 4i) - (7 - 20i)$ .

$$(3 + 4i) - (7 - 20i) = 3 + 4i - 7 + 20i = (3 - 7) + (4 + 20)i = -4 + 20i$$

*Scaffolding:*

If necessary, further examples of addition and multiplication with complex numbers are as follows:

- $(6 - i) + (3 - 2i) = 9 - 3i$
- $(3 + 2i)(-3 + 2i) = -13$
- $(5 + 4i)(2 - i) = 14 + 3i$
- $(2 + \sqrt{3}i)(-2 + \sqrt{3}i) = -7$
- $(1 - 6i)^2 = 37 - 12i$
- $(-3 - i)((2 - 4i) + (1 + 3i)) = -10$

**Examples 3-4 (6 minutes): Multiplication with Complex Numbers**

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Multiplication uses the properties of operations and the fact that  $i^2 = -1$ . It is analogous to polynomial multiplication.

$$(a + bi) \cdot (c + di) = ac + bci + adi + bdi^2 = (ac - bd) + (bc + ad)i$$

**Example 3: Multiplication with Complex Numbers**

Compute  $(1 + 2i)(1 - 2i)$ .

$$\begin{aligned} (1 + 2i)(1 - 2i) &= 1 + 2i - 2i - 4i^2 \\ &= 1 + 0 - 4(-1) \\ &= 1 + 4 \\ &= 5 \end{aligned}$$

**Example 4: Multiplication with Complex Numbers**

Verify that  $-1 + 2i$  and  $-1 - 2i$  are solutions to  $x^2 + 2x + 5 = 0$ .

$-1 + 2i$ :

$$\begin{aligned} (-1 + 2i)^2 + 2(-1 + 2i) + 5 &= 1 - 4i + 4i^2 - 2 + 4i + 5 \\ &= 4i^2 - 4i + 4i + 1 - 2 + 5 \\ &= -4 + 0 + 4 \\ &= 0 \end{aligned}$$

$-1 - 2i$ :

$$\begin{aligned} (-1 - 2i)^2 + 2(-1 - 2i) + 5 &= 1 + 4i + 4i^2 - 2 - 4i + 5 \\ &= 4i^2 + 4i - 4i + 1 - 2 + 5 \\ &= -4 + 0 + 4 \\ &= 0 \end{aligned}$$

So, both complex numbers  $-1 - 2i$  and  $-1 + 2i$  are solutions to the quadratic equation  $x^2 + 2x + 5 = 0$ .

**Closing (4 minutes)**

Close by asking students to write or discuss with a neighbor some reasons for defining the set of complex numbers in the first place. Have them explain the importance of complex numbers satisfying the arithmetic properties of real numbers. How does geometry help explain  $i$ ?

The Lesson Summary box presents key findings from today's lesson.

**Lesson Summary**

Multiplying by  $i$  rotates every complex number in the complex plane by  $90^\circ$  about the origin.

Every complex number is in the form  $a + bi$ , where  $a$  is the real part and  $b$  is the imaginary part of the number. Real numbers are also complex numbers; the real number  $a$  can be written as the complex number  $a + 0i$ .

Adding two complex numbers is analogous to combining like terms in a polynomial expression.

Multiplying two complex numbers is like multiplying two binomials, except one can use  $i^2 = -1$  to further write the expression in simpler form.

Complex numbers satisfy the associative, commutative, and distributive properties.

Complex numbers can now allow us to find solutions to equations that previously had no real number solutions.

**Exit Ticket (5 minutes)**

In this Exit Ticket, students reduce a complex expression into its  $a + bi$  form and then locate the corresponding point on the complex plane.

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## Lesson 37: A Surprising Boost from Geometry

### Exit Ticket

Express the quantities below in  $a + bi$  form, and graph the corresponding points on the complex plane. If you use one set of axes, be sure to label each point appropriately.

$$(1 + i) - (1 - i)$$

$$(1 + i)(1 - i)$$

$$i(2 - i)(1 + 2i)$$

## Exit Ticket Sample Solutions

Express the quantities below in  $a + bi$  form, and graph the corresponding points on the complex plane. If you use one set of axes, be sure to label each point appropriately.

$$(1 + i) - (1 - i)$$

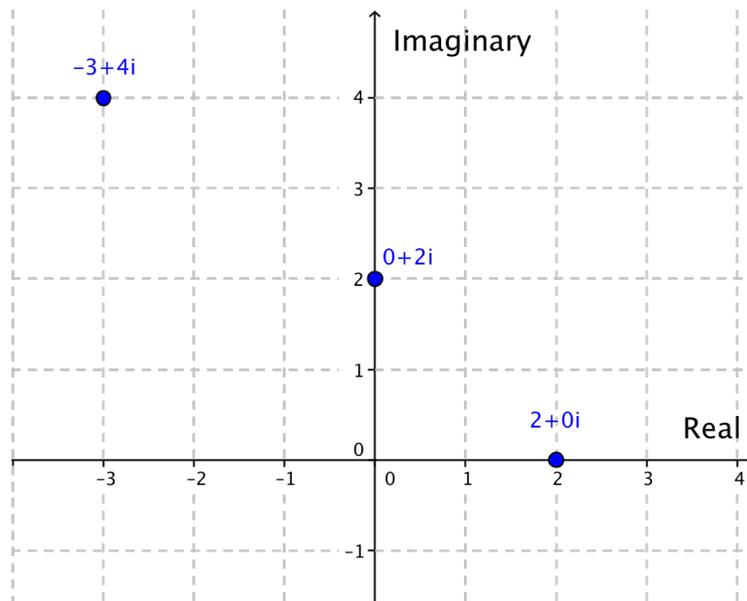
$$(1 + i)(1 - i)$$

$$i(2 - i)(1 + 2i)$$

$$\begin{aligned}(1 + i) - (1 - i) &= 0 + 2i \\ &= 2i\end{aligned}$$

$$\begin{aligned}(1 + i)(1 - i) &= 1 + i - i - i^2 \\ &= 1 - i^2 \\ &= 1 + 1 \\ &= 2 + 0i \\ &= 2\end{aligned}$$

$$\begin{aligned}i(2 - i)(1 + 2i) &= i(2 + 4i - i - 2i^2) \\ &= i(2 + 3i - 2(-1)) \\ &= i(2 + 3i + 2) \\ &= i(4 + 3i) \\ &= 4i + 3i^2 \\ &= -3 + 4i\end{aligned}$$



**Problem Set Sample Solutions**

This problem set offers your students an opportunity to practice and gain facility with complex numbers and complex number arithmetic.

1. Locate the point on the complex plane corresponding to the complex number given in parts (a)–(h). On one set of axes, label each point by its identifying letter. For example, the point corresponding to  $5 + 2i$  should be labeled “a.”

a.  $5 + 2i$

b.  $3 - 2i$

c.  $-2 - 4i$

d.  $-i$

e.  $\frac{1}{2} + i$

f.  $\sqrt{2} - 3i$

g.  $0$

h.  $-\frac{3}{2} + \frac{\sqrt{3}}{2}i$

2. Express each of the following in  $a + bi$  form.

a.  $(13 + 4i) + (7 + 5i)$   
 $(13 + 7) + (4 + 5)i = 20 + 9i$

b.  $(5 - i) - 2(1 - 3i)$   
 $5 - i - 2 + 6i = 3 + 5i$

c.  $((5 - i) - 2(1 - 3i))^2$   
 $(3 + 5i)^2 = 9 + 30i + 25i^2$   
 $= 9 + 30i + (-25)$   
 $= -16 + 30i$

d.  $(3 - i)(4 + 7i)$   
 $12 - 4i + 21i - 7i^2 = 12 + 17i - (-7)$   
 $= 19 + 17i$

e.  $(3 - i)(4 + 7i) - ((5 - i) - 2(1 - 3i))$   
 $(19 + 17i) - (3 + 5i) = (19 - 3) + (17 - 5)i$   
 $= 16 + 12i$

3. Express each of the following in  $a + bi$  form.

a.  $(2 + 5i) + (4 + 3i)$

$$\begin{aligned}(2 + 5i) + (4 + 3i) &= (2 + 4) + (5 + 3)i \\ &= 6 + 8i\end{aligned}$$

b.  $(-1 + 2i) - (4 - 3i)$

$$\begin{aligned}(-1 + 2i) - (4 - 3i) &= -1 + 2i - 4 + 3i \\ &= -5 + 5i\end{aligned}$$

c.  $(4 + i) + (2 - i) - (1 - i)$

$$\begin{aligned}(4 + i) + (2 - i) - (1 - i) &= 4 + i + 2 - i - 1 + i \\ &= 5 + i\end{aligned}$$

d.  $(5 + 3i)(3 + 5i)$

$$\begin{aligned}(5 + 3i)(3 + 5i) &= 5 \cdot 3 + 3 \cdot 3i + 5 \cdot 5i + 3i \cdot 5i \\ &= 15 + 9i + 25i + 15i^2 \\ &= 15 + 34i - 15 \\ &= 0 + 34i \\ &= 34i\end{aligned}$$

e.  $-i(2 - i)(5 + 6i)$

$$\begin{aligned}-i(2 - i)(5 + 6i) &= -i(10 - 5i + 12i - 6i^2) \\ &= -i(10 + 7i + 6) \\ &= -i(16 + 7i) \\ &= -16i - 7i^2 \\ &= -16i + 7 \\ &= 7 - 16i\end{aligned}$$

f.  $(1 + i)(2 - 3i) + 3i(1 - i) - i$

$$\begin{aligned}(1 + i)(2 - 3i) + 3i(1 - i) - i &= (2 + 2i - 3i - 3i^2) + 3i - 3i^2 - i \\ &= 2 + 2i - 3i + 3 + 3i + 3 - i \\ &= 8 + i\end{aligned}$$

4. Find the real values of  $x$  and  $y$  in each of the following equations using the fact that if  $a + bi = c + di$ , then  $a = c$  and  $b = d$ .

a.  $5x + 3yi = 20 + 9i$

$$5x = 20$$

$$x = 4$$

$$3yi = 9i$$

$$y = 3$$

MP.7

MP.7

b.  $2(5x + 9) = (10 - 3y)i$

$$2(5x + 9) + 0i = 0 + (10 - 3y)i$$

$$2(5x + 9) = 0$$

$$x = -\frac{9}{5}$$

$$0i = (10 - 3y)i$$

$$10 - 3y = 0$$

$$y = \frac{10}{3}$$

c.  $3(7 - 2x) - 5(4y - 3)i = x - 2(1 + y)i$

$$3(7 - 2x) = x$$

$$21 - 6x = x$$

$$21 = 7x$$

$$x = 3$$

$$-5(4y - 3)i = -2(1 + y)i$$

$$-5(4y - 3) = -2(1 + y)$$

$$-20y + 15 = -2 - 2y$$

$$17 = 18y$$

$$y = \frac{17}{18}$$

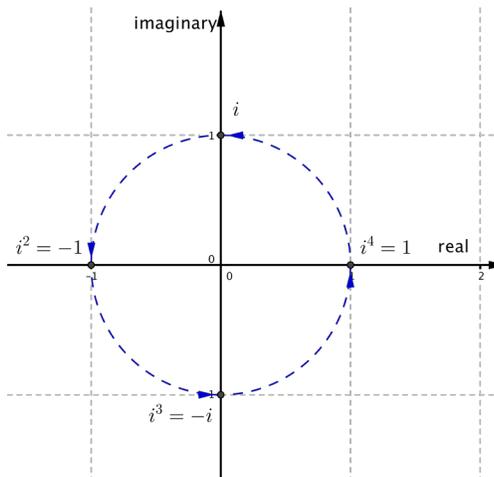
5. Since  $i^2 = -1$ , we see that

$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$

$$i^4 = i^2 \cdot i^2 = -1 \cdot -1 = 1.$$

Plot  $i$ ,  $i^2$ ,  $i^3$ , and  $i^4$  on the complex plane and describe how multiplication by each rotates points in the complex plane.

*Multiplying by  $i$  rotates points by  $90^\circ$  counterclockwise around  $(0, 0)$ . Multiplying by  $i^2 = -1$  rotates points by  $180^\circ$  about  $(0, 0)$ . Multiplying by  $i^3 = -i$  rotates points counterclockwise by  $270^\circ$  about the origin, which is equivalent to rotation by  $90^\circ$  clockwise about the origin. Multiplying by  $i^4$  rotates points counterclockwise by  $360^\circ$ , which is equivalent to not rotating at all. The points  $i$ ,  $i^2$ ,  $i^3$ , and  $i^4$  are plotted below on the complex plane.*



6. Express each of the following in  $a + bi$  form.

a.  $i^5$   $0 + i$

b.  $i^6$   $-1 + 0i$

c.  $i^7$   $0 - i$

d.  $i^8$   $1 + 0i$

e.  $i^{102}$   $-1 + 0i$

*A simple approach is to notice that every 4 multiplications by  $i$  result in four  $90^\circ$  rotations, which takes  $i^4$  back to 1. Therefore, divide 102 by 4, which is 25 with a remainder 2. So, 102  $90^\circ$  rotations will take  $i^{102}$  onto  $-1$ .*

MP.8

7. Express each of the following in  $a + bi$  form.

$(1 + i)^2$

$$\begin{aligned} (1 + i)(1 + i) &= 1 + i + i + i^2 \\ &= 1 + 2i - 1 \\ &= 2i \end{aligned}$$

$(1 + i)^4$

$$\begin{aligned} (1 + i)^4 &= ((1 + i)^2)^2 \\ &= (2i)^2 \\ &= 4i^2 \\ &= -4 \end{aligned}$$

$(1 + i)^6$

$$\begin{aligned} (1 + i)^6 &= (1 + i)^2(1 + i)^4 \\ &= (2i)(-4) \\ &= -8i \end{aligned}$$

8. Evaluate  $x^2 - 6x$  when  $x = 3 - i$ .

$-10$

9. Evaluate  $4x^2 - 12x$  when  $x = \frac{3}{2} - \frac{i}{2}$ .

$-10$

10. Show by substitution that  $\frac{5-i\sqrt{5}}{5}$  is a solution to  $5x^2 - 10x + 6 = 0$ .

$$\begin{aligned} 5\left(\frac{5-i\sqrt{5}}{5}\right)^2 - 10\left(\frac{5-i\sqrt{5}}{5}\right) + 6 &= \frac{1}{5}(5-i\sqrt{5})(5-i\sqrt{5}) - 2(5-i\sqrt{5}) + 6 \\ &= \frac{1}{5}(25 - 10i\sqrt{5} + 5i^2) - 2(5-i\sqrt{5}) + 6 \\ &= \frac{1}{5}(25 - 10i\sqrt{5} - 5) - 2(5-i\sqrt{5}) + 6 \\ &= 5 - 2i\sqrt{5} - 1 - 10 + 2i\sqrt{5} + 6 \\ &= 0 \end{aligned}$$

11. a. Evaluate the four products below.

Evaluate  $\sqrt{9} \cdot \sqrt{4}$ .  $3 \cdot 2 = 6$

Evaluate  $\sqrt{9} \cdot \sqrt{-4}$ .  $3 \cdot 2i = 6i$

Evaluate  $\sqrt{-9} \cdot \sqrt{4}$ .  $3i \cdot 2 = 6i$

Evaluate  $\sqrt{-9} \cdot \sqrt{-4}$ .  $3i \cdot 2i = 6i^2 = -6$

b. Suppose  $a$  and  $b$  are positive real numbers. Determine whether the following quantities are equal or not equal.

$\sqrt{a} \cdot \sqrt{b}$  and  $\sqrt{-a} \cdot \sqrt{-b}$  *not equal*

$\sqrt{-a} \cdot \sqrt{b}$  and  $\sqrt{a} \cdot \sqrt{-b}$  *equal*

MP.7



## Lesson 38: Complex Numbers as Solutions to Equations

### Student Outcomes

- Students solve quadratic equations with real coefficients that have complex solutions (**N-CN.C.7**). They recognize when the quadratic formula gives complex solutions and write them as  $a + bi$  for real numbers  $a$  and  $b$ . (**A-REI.B.4b**)

### Lesson Notes

This lesson models how to solve quadratic equations over the set of complex numbers. Students relate the sign of the discriminant to the nature of the solution set for a quadratic equation. Continue to encourage students to make connections between the graphs of a quadratic equation,  $y = ax^2 + bx + c$ , and the number and type of solutions to the equation  $ax^2 + bx + c = 0$ .

### Classwork

#### Opening (2 minutes)

In Algebra I, students learned that when the quadratic formula resulted in an expression that contained a negative number in the radicand, the equation would have no real solution. Now, we have defined the imaginary unit as  $i = \sqrt{-1}$ . That allows us to solve quadratic equations over the set of complex numbers and see that every quadratic equation has at least one solution.

#### Opening Exercises (5 minutes)

Have students work on this opening exercise alone or in pairs. In this exercise, students apply the quadratic formula to three different relatively simple quadratic equations: one with two real roots, one with one real repeated root, and one with two imaginary roots. Students are then asked to explain the results in terms of the discriminant. Afterward, go over the answers with the class.

Review the quadratic formula  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$  before beginning this exercise, and define the *discriminant* as the number under the radical; that is, the discriminant is the quantity  $b^2 - 4ac$ .

#### Scaffolding:

Advanced students may be able to handle a more abstract framing of—in essence—the same exercise. The exercise below offers the advanced student an opportunity to discover the discriminant and its significance on his or her own.

“Recall that a quadratic equation can have exactly two distinct real solutions, exactly one distinct real solution, or exactly two distinct complex solutions. What is the quadratic formula that we can use to solve an equation in the form  $ax^2 + bx + c = 0$ , where  $a$ ,  $b$ , and  $c$  are real numbers and  $a \neq 0$ . Analyze this formula to decide when the equation will have two, one, or no real solutions.”

Solution:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The “type” of solutions to a quadratic equation hinges on the expression under the radical in the quadratic formula, namely,  $b^2 - 4ac$ . When  $b^2 - 4ac < 0$ , both solutions will have imaginary parts. When  $b^2 - 4ac > 0$ , the quadratic equation has two distinct real solutions. When  $b^2 - 4ac = 0$ , the quadratic formula simplifies as  $x = -\frac{b}{2a}$ . In this case, there is only one real solution, which we call a zero of multiplicity two.

Opening Exercises

1. Use the quadratic formula to solve the following quadratic equations. Calculate the discriminant for each equation.

a.  $x^2 - 9 = 0$

*The equation  $x^2 - 9 = 0$  has two real solutions:  $x = 3$  and  $x = -3$ . The discriminant of  $x^2 - 9 = 0$  is 36.*

b.  $x^2 - 6x + 9 = 0$

*The equation  $x^2 - 6x + 9 = 0$  has one real solution:  $x = 3$ . The discriminant of  $x^2 - 6x + 9 = 0$  is 0.*

c.  $x^2 + 9 = 0$

*The equation  $x^2 + 9 = 0$  has two complex solutions:  $x = 3i$  and  $x = -3i$ . The discriminant of  $x^2 + 9 = 0$  is  $-36$ .*

2. How does the value of the discriminant for each equation relate the number of solutions you found?

*If the discriminant is negative, we get complex solutions. If the discriminant is zero, we get one real solution. If the discriminant is positive, we get two real solutions.*

Discussion (8 minutes)

The expression under the radical in the quadratic formula  $b^2 - 4ac$  is called the discriminant.

- Why do you think we call it the discriminant?
  - *In English, a discriminant is a characteristic that allows “something” (e.g., an object, a person, a function) among a group of other “somethings” to be distinguished.*
  - *In this case, the discriminant distinguishes a quadratic equation by its number and type of solutions: one real solution (repeated), two real solutions, or two complex solutions.*
- Let’s examine the situation when the discriminant is zero. Why does a quadratic equation with discriminant zero have only one real solution?

Scaffolding:

- English Language Learners may benefit from a Frayer diagram or other vocabulary exercise for the word “discriminant.”

- *When the discriminant is zero, the quadratic formula gives the single solution  $-\frac{b \pm 0}{2a} = -\frac{b}{2a}$ .*

- Why is the solution when  $b^2 - 4ac = 0$  a repeated zero?

- *If  $b^2 - 4ac = 0$ , then  $c = \frac{b^2}{4a}$ , and we can factor the quadratic expression  $ax^2 + bx + c$  as follows:*

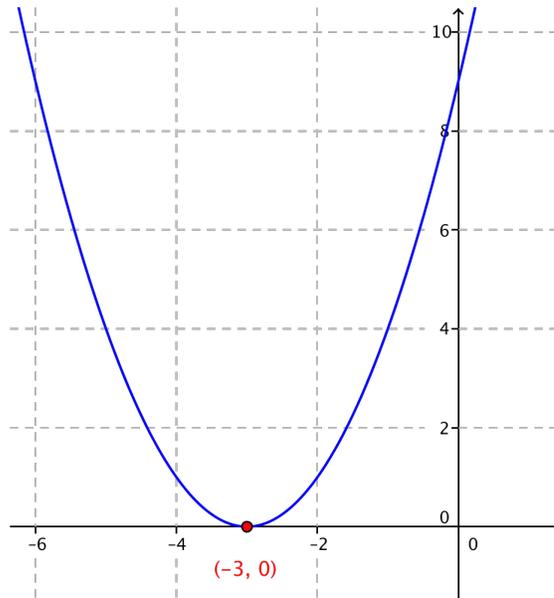
$$ax^2 + bx + c = ax^2 + bx + \frac{b^2}{4a} = a \left( x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \right) = a \left( x + \frac{b}{2a} \right)^2.$$

*From what we know of factoring quadratic expressions from Lesson 11,  $-\frac{b}{2a}$  is a repeated zero.*

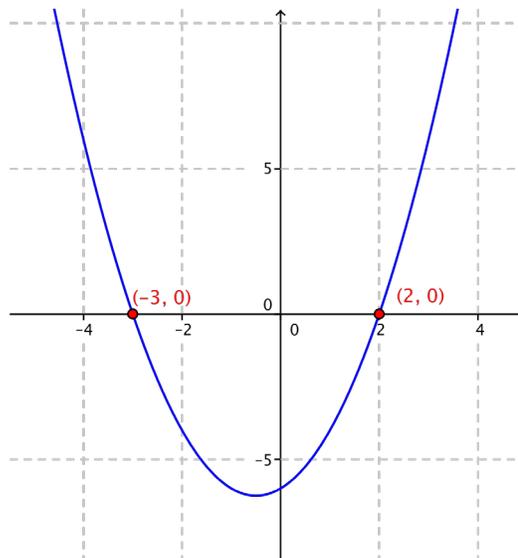
- *Analytically, the solutions can be thought of as  $-\frac{b+0}{2a}$  and  $-\frac{b-0}{2a}$ , which are both  $-\frac{b}{2a}$ . So, there are two solutions that are the same number.*

- *Geometrically, we can write the equation of the parabola as  $y = a \left( x + \frac{b}{2a} \right)^2$ , so the vertex of this parabola is  $\left( -\frac{b}{2a}, 0 \right)$ , meaning the vertex of the parabola lies on the  $x$ -axis. Thus, the parabola is tangent to the  $x$ -axis and intersects the  $x$ -axis only at the point  $\left( -\frac{b}{2a}, 0 \right)$ .*

- For example, the graph of  $y = x^2 + 6x + 9$  intersects the  $x$ -axis only at  $(-3, 0)$ , as follows.

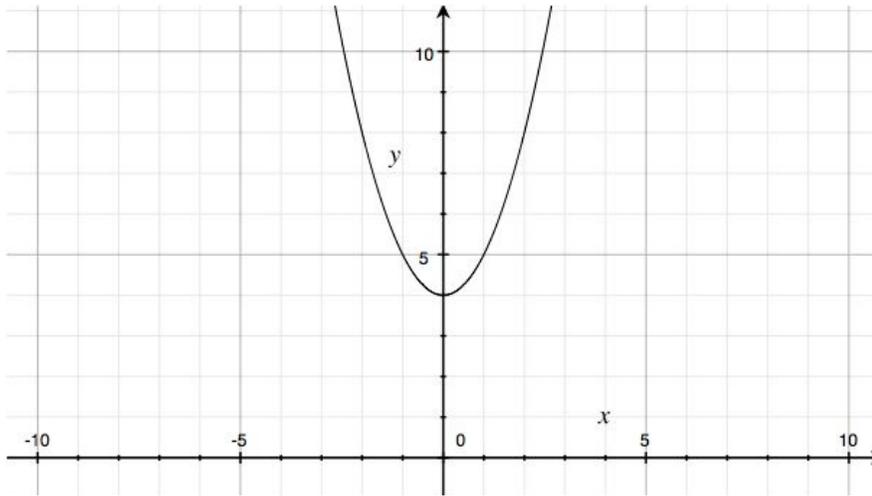


- Describe the graph of a quadratic equation with positive discriminant.
  - If the discriminant is positive, then the quadratic formula gives two different real solutions.
  - Two real solutions mean the graph intersects the  $x$ -axis at two distinct real points.
- For example, the graph of  $y = x^2 + x - 6$  intersects the  $x$ -axis at  $(-3, 0)$  and  $(2, 0)$ , as follows.



- Describe the graph of a quadratic equation with negative discriminant.
  - Since the discriminant is negative, the quadratic formula will give two different complex solutions.
  - Since there are no real solutions, the graph does not cross or touch the  $x$ -axis in the real plane.

- For example, the graph of  $y = x^2 + 4$ , shown below, does not intersect the  $x$ -axis.



### Example 1 (5 minutes)

Consider the equation  $3x + x^2 = -7$ .

- What does the value of the discriminant tell us about number of solutions to this equation?
  - The equation in standard form is  $x^2 + 3x + 7 = 0$ .
  - $a = 1, b = 3, c = 7$
  - The discriminant is  $3^2 - 4(1)(7) = -19$ . The negative discriminant indicates that no real solutions exist. There are two complex solutions.
- Solve the equation. Does the number of solutions match the information provided by the discriminant? Explain.
  - Using the quadratic formula,  

$$x = \frac{-3 + \sqrt{-19}}{2} \text{ or } x = \frac{-3 - \sqrt{-19}}{2}.$$
  - The solutions, in  $a + bi$  form, are  $-\frac{3}{2} + \frac{\sqrt{19}}{2}i$  and  $-\frac{3}{2} - \frac{\sqrt{19}}{2}i$ .
  - The two complex solutions are consistent with the rule for a negative discriminant.

### Exercise (15 minutes)

Have students work individually on this exercise; then, have them work with a partner or in a small group to check their solutions. You could also conduct this exercise using personal white boards and have your students show their answers to each question after a few minutes. When many students are stuck, invite them to exchange papers with a partner to check for errors. Having students identify errors in their work or the work of others will help them to build fluency when working with these complicated expressions. Debrief this exercise by showing the related graph of the equation in the coordinate plane and verify that the number of solutions corresponds to the number of  $x$ -intercepts.

MP.3

## Exercise

Compute the value of the discriminant of the quadratic equation in each part. Use the value of the discriminant to predict the number and type of solutions. Find all real and complex solutions.

a.  $x^2 + 2x + 1 = 0$

We have  $a = 1$ ,  $b = 2$ , and  $c = 1$ . Then

$$b^2 - 4ac = 2^2 - 4(1)(1) = 0.$$

Note that the discriminant is zero, so this equation has exactly one real solution.

$$x = \frac{-(2) \pm \sqrt{0}}{2(1)} = -1$$

Thus, the only solution is  $-1$ .

b.  $x^2 + 4 = 0$

We have  $a = 1$ ,  $b = 0$ , and  $c = 4$ . Then

$$b^2 - 4ac = -16.$$

Note that the discriminant is negative, so this equation has two complex solutions.

$$x = \frac{-0 \pm \sqrt{-16}}{2(1)}$$

Thus, the two complex solutions are  $2i$  and  $-2i$ .

c.  $9x^2 - 4x - 14 = 0$

We have  $a = 9$ ,  $b = -4$ , and  $c = -14$ . Then

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4(9)(-14) \\ &= 16 + 504 \\ &= 520. \end{aligned}$$

Note that the discriminant is positive, so this equation has two distinct real solutions.

Using the quadratic formula,

$$x = \frac{-(-4) \pm 2\sqrt{130}}{2(9)}.$$

So, the two real solutions are  $\frac{2+\sqrt{130}}{9}$  and  $\frac{2-\sqrt{130}}{9}$ .

d.  $3x^2 + 4x + 2 = 0$

We have  $a = 3$ ,  $b = 4$ , and  $c = 2$ . Then

$$\begin{aligned} b^2 - 4ac &= 4^2 - 4(3)(2) \\ &= 16 - 24 \\ &= -8. \end{aligned}$$

The discriminant is negative, so there will be two complex solutions. Using the quadratic formula,

$$x = \frac{-4 \pm \sqrt{-8}}{2(3)}.$$

So, the two complex solutions are  $-\frac{2}{3} + \frac{\sqrt{2}}{3}i$  and  $-\frac{2}{3} - \frac{\sqrt{2}}{3}i$ .

e.  $x = 2x^2 + 5$

We can rewrite this equation in standard form with  $a = 2$ ,  $b = -1$ , and  $c = 5$ :

$$2x^2 - x + 5 = 0.$$

Then

$$\begin{aligned} b^2 - 4ac &= (-1)^2 - 4(2)(5) \\ &= 1 - 40 \\ &= -39. \end{aligned}$$

The discriminant is negative, so there will be two complex solutions. Using the quadratic formula,

$$\begin{aligned} x &= \frac{-(-1) \pm \sqrt{-39}}{2(2)} \\ x &= \frac{1 \pm i\sqrt{39}}{4}. \end{aligned}$$

The two solutions are  $\frac{1}{4} + \frac{\sqrt{39}}{4}i$  and  $\frac{1}{4} - \frac{\sqrt{39}}{4}i$ .

f.  $8x^2 + 4x + 32 = 0$

We can factor 4 from the left side of this equation to obtain  $4(2x^2 + x + 8) = 0$ , and we know that a product is zero when one of the factors are zero. Since  $4 \neq 0$ , we must have  $2x^2 + x + 8 = 0$ . This is a quadratic equation with  $a = 2$ ,  $b = 1$ , and  $c = 8$ . Then

$$b^2 - 4ac = 1^2 - 4(2)(8) = -63.$$

The discriminant is negative, so there will be two complex solutions. Using the quadratic formula,

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{-63}}{2(2)} \\ x &= \frac{-1 \pm 3i\sqrt{7}}{4}. \end{aligned}$$

The complex solutions are  $-\frac{1}{4} + \frac{3\sqrt{7}}{4}i$  and  $-\frac{1}{4} - \frac{3\sqrt{7}}{4}i$ .

**Scaffolding:**

You may assign advanced students to create quadratic equations that have specific solutions. For example, request a quadratic equation that has only the solution  $-5$ .

Answer:  $x^2 - 10x + 25 = 0$ . This follows from the expansion of the left side of  $(x + 5)^2 = 0$ . You may also request a quadratic equation with solution set  $3 + i$  and  $3 - i$ . The answer is  $x^2 - 6x + 10 = 0$ .

**Closing (5 minutes)**

As you summarize this lesson, ask your students to create a graphic organizer that allows them to compare and contrast the nature of the discriminant, the number and types of solutions to  $ax^2 + bx + c = 0$ , and the graphs of the equation  $y = ax^2 + bx + c$ . Have them record a problem of each type from the previous exercise as an example in their graphic organizer.

**Lesson Summary**

- A quadratic equation with real coefficients and a real constant may have real or complex solutions.
- Given a quadratic equation  $ax^2 + bx + c = 0$ , the discriminant  $b^2 - 4ac$  indicates whether the equation has two distinct real solutions, one real solution, or two complex solutions.
  - If  $b^2 - 4ac > 0$ , there are two real solutions to  $ax^2 + bx + c = 0$ .
  - If  $b^2 - 4ac = 0$ , there is one real solution to  $ax^2 + bx + c = 0$ .
  - If  $b^2 - 4ac < 0$ , there are two complex solutions to  $ax^2 + bx + c = 0$ .

**Exit Ticket (5 minutes)**

The Exit Ticket gives students the opportunity to demonstrate their mastery of this lesson's content.

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 38: Complex Numbers as Solutions to Equations

### Exit Ticket

Use the discriminant to predict the nature of the solutions to the equation  $4x - 3x^2 = 10$ . Then, solve the equation.

## Exit Ticket Sample Solutions

Use the discriminant to predict the nature of the solutions to the equation  $4x - 3x^2 = 10$ . Then, solve the equation.

$$3x^2 - 4x + 10 = 0$$

We have  $a = 3$ ,  $b = -4$ , and  $c = 10$ . Then

$$\begin{aligned} b^2 - 4ac &= (-4)^2 - 4(3)(10) \\ &= 16 - 120 \\ &= -104. \end{aligned}$$

The value of the discriminant is negative, indicating that there are two complex solutions.

$$\begin{aligned} x &= \frac{-(-4) \pm \sqrt{-104}}{2(3)} \\ x &= \frac{4 \pm 2i\sqrt{26}}{6} \end{aligned}$$

Thus, the two solutions are  $\frac{2}{3} + \frac{\sqrt{26}}{3}i$  and  $\frac{2}{3} - \frac{\sqrt{26}}{3}i$ .

## Problem Set Sample Solutions

The Problem Set offers students more practice solving quadratic equations with complex solutions.

1. Give an example of a quadratic equation in standard form that has...

a. Exactly two distinct real solutions.

Since  $(x + 1)(x - 1) = x^2 - 1$ , the equation  $x^2 - 1 = 0$  has two distinct real solutions, 1 and  $-1$ .

b. Exactly one distinct real solution.

Since  $(x + 1)^2 = x^2 + 2x + 1$ , the equation  $x^2 + 2x + 1 = 0$  has only one real solution,  $-1$ .

c. Exactly two complex (non-real) solutions.

Since  $x^2 + 1 = 0$  has no solutions in the real numbers, this equation must have two complex solutions. They are  $i$  and  $-i$ .

2. Suppose we have a quadratic equation  $ax^2 + bx + c = 0$  so that  $a + c = 0$ . Does the quadratic equation have one solution or two distinct solutions? Are they real or complex? Explain how you know.

If  $a + c = 0$ , then either  $a = c = 0$ ,  $a > 0$  and  $c < 0$ , or  $a < 0$  and  $c > 0$ .

The definition of a quadratic polynomial requires that  $a \neq 0$ , so either  $a > 0$  and  $c < 0$  or  $a < 0$  and  $c > 0$ .

In either case,  $ac < 0$ . Because  $b^2$  is positive and  $4ac$  is positive, we know  $b^2 - 4ac > 0$ .

Therefore, a quadratic equation  $ax^2 + bx + c = 0$  always has two distinct real solutions when  $a + c = 0$ .

3. Solve the equation  $5x^2 - 4x + 3 = 0$ .

We have a quadratic equation with  $a = 5$ ,  $b = -4$ , and  $c = 3$ .

$$x = \frac{-(-4) \pm 2\sqrt{-11}}{2(5)}$$

So, the solutions are  $\frac{2+i\sqrt{11}}{5}$  and  $\frac{2-i\sqrt{11}}{5}$ .

4. Solve the equation  $2x^2 + 8x = -9$ .

In standard form, this is the quadratic equation  $2x^2 + 8x + 9 = 0$  with  $a = 2$ ,  $b = 8$ , and  $c = 9$ .

$$x = \frac{-8 \pm 2\sqrt{-2}}{2(2)} = \frac{-4 \pm i\sqrt{2}}{2}$$

Thus, the solutions are  $2 + \frac{i\sqrt{2}}{2}$  and  $2 - \frac{i\sqrt{2}}{2}$ .

5. Solve the equation  $9x - 9x^2 = 3 + x + x^2$ .

In standard form, this is the quadratic equation  $10x^2 - 8x + 3 = 0$  with  $a = 10$ ,  $b = -8$ , and  $c = 3$ .

$$x = -\frac{-(-8) \pm 2\sqrt{-14}}{2(10)} = \frac{8 \pm 2i\sqrt{14}}{20}$$

Thus, the solutions are  $\frac{4+i\sqrt{14}}{10}$  and  $\frac{4-i\sqrt{14}}{10}$ .

6. Solve the equation  $3x^2 - x + 1 = 0$ .

This is a quadratic equation with  $a = 3$ ,  $b = -1$ , and  $c = 1$ .

$$x = -\frac{-(-1) \pm \sqrt{-11}}{2(3)} = \frac{1 \pm i\sqrt{11}}{6}$$

Thus, the solutions are  $\frac{1+i\sqrt{11}}{6}$  and  $\frac{1-i\sqrt{11}}{6}$ .

7. Solve the equation  $6x^4 + 4x^2 - 3x + 2 = 2x^2(3x^2 - 1)$ .

When expanded, this is a quadratic equation with  $a = 6$ ,  $b = -3$ , and  $c = 2$ .

$$6x^4 + 4x^2 - 3x + 2 = 6x^4 - 2x^2$$

$$6x^2 - 3x + 2 = 0$$

$$x = \frac{-(-3) \pm \sqrt{\{-39\}}}{2(6)}$$

So, the solutions are  $x = \frac{3+i\sqrt{39}}{12}$  and  $x = \frac{3-i\sqrt{39}}{12}$ .

8. Solve the equation  $25x^2 + 100x + 200 = 0$ .

We can factor 25 from the left side of this equation to obtain  $25(x^2 + 4x + 8) = 0$ , and we know that a product is zero when one of the factors is zero. Since  $25 \neq 0$ , we must have  $x^2 + 4x + 8 = 0$ . This is a quadratic equation with  $a = 1$ ,  $b = 4$ , and  $c = 8$ . Then

$$x = \frac{-4 \pm 4\sqrt{-1}}{2},$$

and the solutions are  $-2 + 2i$  and  $-2 - 2i$ .

9. Write a quadratic equation in standard form such that  $-5$  is its only solution.

$$(x + 5)^2 = 0$$

$$x^2 + 10x + 25 = 0$$

10. Is it possible that the quadratic equation  $ax^2 + bx + c = 0$  has a positive real solution if  $a$ ,  $b$ , and  $c$  are all positive real numbers?

No. The solutions are  $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$  and  $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$ . If  $b$  is positive, the second one of these will be negative. So, we need to think about whether or not the first one can be positive. If  $-b + \sqrt{b^2 - 4ac} > 0$ , then  $\sqrt{b^2 - 4ac} > b$ ; so,  $b^2 - 4ac > b^2$ , and  $-4ac > 0$ . This means that either  $a$  or  $c$  must be negative. So, if all three coefficients are positive, then there cannot be a positive solution to  $ax^2 + bx + c = 0$ .

11. Is it possible that the quadratic equation  $ax^2 + bx + c = 0$  has a positive real solution if  $a$ ,  $b$ , and  $c$  are all negative real numbers?

No. If  $a$ ,  $b$ , and  $c$  are all negative, then  $-a$ ,  $-b$ , and  $-c$  are all positive. The solutions of  $ax^2 + bx + c = 0$  are the same as the solutions to  $-ax^2 - bx - c = 0$ , and by Problem 10, this equation has no positive real solution since it has all positive coefficients.

Extension:

12. Show that if  $k > 3.2$ , the solutions of  $5x^2 - 8x + k = 0$  are not real numbers.

We have  $a = 5$ ,  $b = -8$ , and  $c = k$ , then

$$b^2 - 4ac = (-8)^2 - 4 \cdot 5 \cdot k$$

$$= 64 - 20k.$$

When the discriminant is negative, the solutions of the quadratic function are not real numbers.

$$0 > 64 - 20k$$

$$20k > 64$$

$$k > \frac{64}{20}$$

$$k > 3.2$$

Thus, if  $k > 3.2$ , then the discriminant is negative and the solutions of  $5x^2 - 8x + k = 0$  are not real numbers.

MP.2

13. Let  $k$  be a real number, and consider the quadratic equation  $(k + 1)x^2 + 4kx + 2 = 0$ .

a. Show that the discriminant of  $(k + 1)x^2 + 4kx + 2 = 0$  defines a quadratic function of  $k$ .

*The discriminant of a quadratic equation written in the form  $ax^2 + bx + c = 0$  is  $b^2 - 4ac$ .*

*Here,  $a = k + 1$ ,  $b = 4k$ , and  $c = 2$ . We get*

$$\begin{aligned} b^2 - 4ac &= (4k)^2 - 4 \cdot (k + 1) \cdot 2 \\ &= 16k^2 - 8(k + 1) \\ &= 16k^2 - 8k - 8. \end{aligned}$$

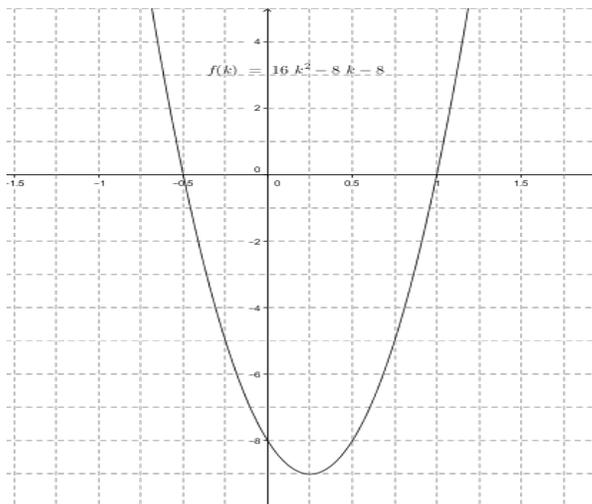
*With  $k$  unknown, we can write  $f(k) = 16k^2 - 8k - 8$ , which is a quadratic function of  $k$ .*

b. Find the zeros of the function in part (a) and make a sketch of its graph.

*If  $f(k) = 0$ , then we have*

$$\begin{aligned} 0 &= 16k^2 - 8k - 8 \\ &= 2k^2 - k - 1 \\ &= 2k^2 - 2k + k - 1 \\ &= 2k(k - 1) + 1(k - 1) \\ &= (k - 1)(2k + 1). \end{aligned}$$

*Then,  $k - 1 = 0$  or  $2k + 1 = 0$ . So,  $k = 1$  or  $k = -\frac{1}{2}$ .*



c. For what value of  $k$  are there two distinct real solutions to the given quadratic equation?

*The original quadratic equation has two distinct real solutions when the discriminant given by  $f(k)$  is positive. This occurs for all real numbers  $k$  such that  $k < -\frac{1}{2}$  or  $k > 1$ .*

d. For what value of  $k$  are there two complex solutions to the given quadratic equation?

*There are two complex solutions when  $f(k) < 0$ . This occurs for all real numbers  $k$  such that  $-\frac{1}{2} < k < 1$ .*

e. For what value of  $k$  is there one solution to the given quadratic equation?

*There is one solution when  $f(k) = 0$ . This occurs at  $k = -\frac{1}{2}$  and  $k = 1$ .*

14. We can develop two formulas that can help us find errors in calculated solutions of quadratic equations.

- a. Find a formula for the sum  $S$  of the solutions of the quadratic equation  $ax^2 + bx + c = 0$ .

The zeros of the quadratic equation are given by  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ . Then

$$\begin{aligned} S &= \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b + \sqrt{b^2 - 4ac} + -b - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-b + -b + \sqrt{b^2 - 4ac} - \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-2b}{2a} \\ &= -\frac{b}{a} \end{aligned}$$

Thus,  $S = -\frac{b}{a}$ .

- b. Find a formula for the product  $R$  of the solutions of the quadratic equation  $ax^2 + bx + c = 0$ .

$$R = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Note that the numerators differ only in that one is a sum, and one is a difference. The formula  $(m + n) \cdot (m - n) = m^2 - n^2$  applies where  $m = -b$  and  $n = \sqrt{b^2 - 4ac}$ . We get

$$\begin{aligned} R &= \frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{2a \cdot 2a} \\ &= \frac{b^2 - b^2 + 4ac}{4a^2} \\ &= \frac{4ac}{4a^2} \\ &= \frac{c}{a} \end{aligned}$$

So, the product is  $R = \frac{c}{a}$ .

- c. June calculated the solutions 7 and  $-1$  to the quadratic equation  $x^2 - 6x + 7 = 0$ . Do the formulas from parts (a) and (b) detect an error in her solutions? If not, determine if her solution is correct.

The sum formula agrees with June's calculations. From June's zeros,

$$7 + -1 = 6,$$

and from the formula,

$$S = \frac{6}{1} = 6.$$

However, the product formula does not agree with her calculations. From June's zeros,

$$7 \cdot -1 = -7,$$

and from the formula,

$$R = \frac{7}{1} = 7.$$

June's solutions are not correct:  $(7)^2 - 6(7) + 7 = 49 - 42 + 7 = 14$ ; so, 7 is not a solution to this quadratic equation. Likewise,  $1 - 6 + 7 = 2$ , so 1 is also not a solution to this equation. Thus, the formulas caught her error.

- d. Paul calculated the solutions  $3 - i\sqrt{2}$  and  $3 + i\sqrt{2}$  to the quadratic equation  $x^2 - 6x + 7 = 0$ . Do the formulas from parts (a) and (b) detect an error in his solutions? If not, determine if his solutions are correct.

*In part (c), we calculated that  $R = 7$  and  $S = 6$ . From Paul's zeros,*

$$S = 3 + i\sqrt{2} + 3 - i\sqrt{2} = 6,$$

*and for the product,*

$$\begin{aligned} R &= (3 + i\sqrt{2}) \cdot (3 - i\sqrt{2}) \\ &= 3^2 - (i\sqrt{2})^2 \\ &= 9 - 1 \cdot 2 \\ &= 11. \end{aligned}$$

*This disagrees with the calculated version of  $R$ . So, the formulas do find that he made an error.*

- e. Joy calculated the solutions  $3 - \sqrt{2}$  and  $3 + \sqrt{2}$  to the quadratic equation  $x^2 - 6x + 7 = 0$ . Do the formulas from parts (a) and (b) detect an error in her solutions? If not, determine if her solutions are correct.

*Joy's zeros will have the same sum as Paul's, so  $S = 6$ , which agrees with the sum from the formula. For the product of her zeros we get*

$$\begin{aligned} R &= (3 - \sqrt{2})(3 + \sqrt{2}) \\ &= 9 - 2 \\ &= 7, \end{aligned}$$

*which agrees with the formulas.*

*Checking her solutions in the original equation, we find*

$$\begin{aligned} (3 - \sqrt{2})^2 - 6(3 - \sqrt{2}) + 7 &= (9 - 6\sqrt{2} + 2) - 18 + 6\sqrt{2} + 7 \\ &= 0, \\ (3 + \sqrt{2})^2 - 6(3 + \sqrt{2}) + 7 &= (9 + 6\sqrt{2} + 2) - 18 - 6\sqrt{2} + 7 \\ &= 0. \end{aligned}$$

*Thus, Joy has correctly found the solutions of this quadratic equation.*

- f. If you find solutions to a quadratic equations that match the results from parts (a) and (b), does that mean your solutions are correct?

*Not necessarily. We only know that if the sum and product of the solutions do not match  $S$  and  $R$ , then we have not found a solution. Evidence suggests that if the sum and product of the solutions do match  $S$  and  $R$ , then we have found the correct solutions, but we do not know for sure until we check.*

- g. Summarize the results of this exercise.

*For a quadratic equation of the form  $ax^2 + bx + c = 0$ , the sum of the solutions is given by  $S = -\frac{b}{a}$  and the product of the solutions is given by  $R = \frac{c}{a}$ . So, multiplying and adding the calculated solutions will identify if we have made an error. Passing these checks, however, does not guarantee that the numbers we found are the correct solutions.*

MP.2



## Lesson 39: Factoring Extended to the Complex Realm

### Student Outcomes

- Students solve quadratic equations with real coefficients that have complex solutions. Students extend polynomial identities to the complex numbers.
- Students note the difference between solutions to the equation and the  $x$ -intercepts of the graph of said equation.

### Lesson Notes

This lesson extends the factoring concepts and techniques covered in Topic B of this module to the complex number system and specifically addresses **N-CN.C.7**. Students will learn how to solve and express the solutions to any quadratic equation. Students observe that complex solutions to polynomial equations with real coefficients occur in conjugate pairs, and that only real solutions to polynomial equations are also the  $x$ -intercepts of the graph of the related polynomial function. In essence, this is the transition lesson to the next lesson on factoring all polynomials into linear factors.

### Classwork

#### Opening (1 minute)

Since your introduction to the complex number system, you have hopefully recognized its theme of sharing arithmetic and algebraic properties with the real numbers. Today, we extend factoring polynomial expressions and finding solutions to polynomial equations to the complex realm.

#### Opening Exercise (8 minutes)

Have students individually complete this opening exercise. Students will eventually identify the expressions in this exercise as cases of the identity  $(x + ai)(x - ai) = x^2 + a^2$ . But, for now, allow them the experience of the algebra and to confirm for themselves that the imaginary terms combine to 0 in each example, resulting in polynomials in standard form with real coefficients. Invite students to the board to write their solutions, and let the class have the first opportunity to correct any mistakes, should it be necessary.

#### Opening Exercise

Rewrite each expression as a polynomial in standard form.

a.  $(x + i)(x - i)$

$$\begin{aligned}(x + i)(x - i) &= x^2 + ix - ix - i^2 \\ &= x^2 - i^2 \\ &= x^2 - (-1) \\ &= x^2 + 1\end{aligned}$$

b.  $(x + 5i)(x - 5i)$

$$\begin{aligned}(x + 5i)(x - 5i) &= x^2 + 5ix - 5ix - 25i^2 \\ &= x^2 - 25i^2 \\ &= x^2 - 25(-1) \\ &= x^2 + 25\end{aligned}$$

c.  $(x - (2 + i))(x - (2 - i))$

$$\begin{aligned}(x - (2 + i))(x - (2 - i)) &= x^2 - (2 + i)x - (2 - i)x + [(2 + i)(2 - i)] \\ &= x^2 - 2x - ix - 2x + ix + [4 - i^2] \\ &= x^2 - 4x + [4 - (-1)] \\ &= x^2 - 4x + 5\end{aligned}$$

### Discussion (5 minutes)

Here we begin a dialogue that discusses patterns and regularity observed in the Opening Exercise. As you pose each question, give students time to discuss them with a partner or in their small groups. To encourage students to be accountable for responding to questions during discussion, you can have them write answers on personal whiteboards, show a thumbs-up when they have an idea, whisper their idea to a partner before asking for a response with the whole group, or show their agreement or disagreement to a question by showing a thumbs up/down.

- Do you observe any patterns among parts (a)–(c) in the opening exercise?
  - *After each expression is expanded and like terms are collected, we have quadratic polynomials with real coefficients. The imaginary terms were opposites and combined to 0.*
- How could you generalize the patterns into a rule (or identity)?
  - *Parts (a) and (b) are instances of the identity*  
 $(x + ai)(x - ai) = x^2 + a^2$ .
- What about part (c)? Do you notice an instance of the same identity?
  - *Yes.*  
 $(x - (2 + i))(x - (2 - i)) = ((x - 2) - i)((x - 2) + i)$
- Where have we seen a similar identity to  $(x + ai)(x - ai) = x^2 + a^2$ ?
  - *Recall the polynomial identity  $(x + a)(x - a) = x^2 - a^2$  from Lesson 6.*
- Recall the quick mental arithmetic we learned in Lesson 7. Can you compute  $(3 + 2i)(3 - 2i)$  really quickly?
  - *Yes.  $(3 + 2i)(3 - 2i) = 3^2 + 2^2 = 9 + 4 = 13$*
- How about  $(9 + 4i)(9 - 4i)$ ?
  - $(9 + 4i)(9 - 4i) = 9^2 + 4^2 = 81 + 16 = 97$

### Exercises 1–2 (5 minutes)

Students understand that the expansion of  $(x + ai)(x - ai)$  is a polynomial with real coefficients; the imaginary terms disappear when working through the algebra. Now, students are expected to understand this process in reverse; in other words, they factor polynomials with real coefficients but complex factors.

MP.7

## Exercises 1–4

Completely factor the following polynomial expressions.

1.  $x^2 + 9$

$$x^2 + 9 = (x + 3i)(x - 3i)$$

2.  $x^2 + 5$

$$x^2 + 5 = (x + i\sqrt{5})(x - i\sqrt{5})$$

## Discussion (6 minutes)

This discussion is the introduction to conjugate pairs in the context of complex numbers. Relate this idea back to the idea of conjugate pairs for radical expressions from Lesson 29.

- In Lesson 29, we saw that the conjugate of an expression such as  $x + \sqrt{5}$  is the expression  $x - \sqrt{5}$ , and if we multiply a radical expression by its conjugate, the result is a rational expression—the radical part disappears.

$$(x + \sqrt{5})(x - \sqrt{5}) = x^2 - x\sqrt{5} + x\sqrt{5} - 5 = x^2 - 5$$

- Analogously, complex numbers  $a + bi$  have conjugates. The conjugate of  $a + bi$  is  $a - bi$ . Then, we see that

$$(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2.$$

- We have observed, for real values of  $x$  and  $a$ , that the expression  $(x + ai)(x - ai)$  is a real number. The factors  $(x + ai)$  and  $(x - ai)$  form a conjugate pair.

Quadratic expressions with real coefficients, as we have seen, can be decomposed into real factors or non-real complex factors. However, non-real factors must be members of a conjugate pair; hence, a quadratic expression with real coefficients cannot have exactly one complex factor.

- Similarly, quadratic equations can have real or non-real complex solutions. If there are complex solutions, they will be conjugates of each other.
- Can a polynomial equation with real number coefficients have just one complex solution?
  - *No. If there is a complex solution, then the conjugate is also a solution. Complex solutions come in pairs.*
- Now, can a polynomial equation have real and non-real solutions?
  - *Yes, as long as all non-real complex solutions occur in conjugate pairs.*
  - *For example, the polynomial equation  $(x^2 + 1)(x^2 - 1) = 0$  has two real solutions, 1 and  $-1$ , and two complex solutions,  $i$  and  $-i$ , form a conjugate pair.*
- If you know that  $2i$  is a solution to the polynomial equation  $P(x) = 0$ , can you tell me another solution?
  - *Complex solutions come in conjugate pairs, so if  $2i$  is a solution to the equation, then its conjugate,  $-2i$ , is also a solution.*

## Scaffolding:

- Show conjugate pairs graphically by graphing (as in the previous lessons) a parabola with 0, 1, and 2 solutions and cubic curves with a various number of solutions. Let student determine visually what is possible.
- Additionally, consider having students complete a Frayer diagram for the term “conjugate.”
- As an extension, ask students to generate conjugate pairs.

At this point, have students write down or discuss with their neighbors what they have learned so far. The teacher should walk around the room and check for understanding.

**Exercise 3 (6 minutes)**

Students should work in groups of 2–4 on this exercise. Invite students to the board to present their solutions.

3. Consider the polynomial  $P(x) = x^4 - 3x^2 - 4$ .

a. What are the solutions to  $x^4 - 3x^2 - 4 = 0$ ?

$$\begin{aligned} x^4 - 3x^2 - 4 &= 0 \\ (x^2)^2 - 3x^2 - 4 &= 0 \\ (x^2 + 1)(x^2 - 4) &= 0 \\ (x + i)(x - i)(x + 2)(x - 2) &= 0 \end{aligned}$$

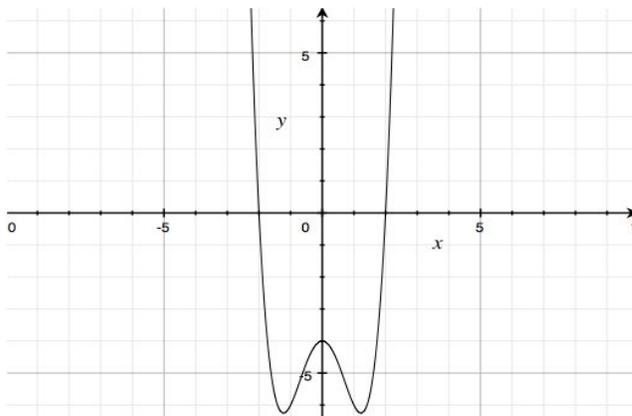
The solutions are  $-i, i, -2,$  and  $2$ .

b. How many  $x$ -intercepts does the graph of the equation  $y = x^4 - 3x^2 - 4$  have? What are the coordinates of the  $x$ -intercepts?

The graph of  $y = x^4 - 3x^2 - 4$  has two  $x$ -intercepts:  $(-2, 0)$  and  $(2, 0)$ .

c. Are solutions to the polynomial equation  $P(x) = 0$  the same as the  $x$ -intercepts of the graph of  $y = P(x)$ ? Justify your reasoning.

No. Only the real solutions to the equation are  $x$ -intercepts of the graph. By comparing the graph of the polynomial in part (b) to the equation's solutions from part (c), you can see that only the real number solutions to the equation correspond to the  $x$ -intercepts in the Cartesian plane.



*Scaffolding:*

- Consider having groups work Exercise 3 with different polynomials:

$$\begin{aligned} &4x^3 - x \\ &x^3 - 4x^2 + 29x \\ &x^3 - 6x^2 + 25x \\ &x^4 - 1 \\ &x^4 + 3x^2 - 4 \\ &x^4 - 4x^3 + 5x^2 \\ &x^4 - 4x^3 + 13x^2 \\ &x^4 - 2x^3 - 10x^2 \\ &x^4 + 13x^2 + 36 \end{aligned}$$

MP.3

**Exercise 4 (5 minutes)**

Transition students to the next exercise by announcing that we now want to reverse our thinking. In the previous problem, we solved an equation to find the solutions. Now, pose the question: Can we construct an equation if we know its solutions? You may want to remind students that when a polynomial equation is written in factored form  $(x - r_1)(x - r_2) \dots (x - r_n) = 0$ , the solutions to the equation are  $r_1, r_2, \dots, r_n$ . Students will apply what they learned in the previous exercise to create a polynomial equation given its solutions. The problems scaffold from easier to more difficult. Students are encouraged to rewrite the factored form to show the polynomial in standard form for additional practice with complex numbers, but you may choose to have them leave the polynomial in factored form if time is a concern. Have students work with a partner on this exercise.

4. Write a polynomial  $P$  with the lowest possible degree that has the given solutions. Explain how you generated each answer.

a.  $-2, 3, -4i, 4i$

*The polynomial  $P$  has two real zeroes and two complex zeroes. Since the two complex zeroes are members of a conjugate pair,  $P$  may have as few as four total factors. Therefore,  $P$  has degree at least 4.*

$$\begin{aligned} P(x) &= (x + 2)(x - 3)(x + 4i)(x - 4i) \\ &= (x^2 - x - 6)(x^2 - 16i^2) \\ &= (x^2 - x - 6)(x^2 + 16) \\ &= x^4 - x^3 - 6x^2 + 16x^2 - 16x - 96 \\ &= x^4 - x^3 + 10x^2 - 16x - 96 \end{aligned}$$

b.  $-1, 3i$

*The polynomial  $P$  has one real zero and two complex zeroes because complex zeroes come in pairs. Since  $3i$  and  $-3i$  form a conjugate pair,  $P$  has at least three total factors. Therefore,  $P$  has degree at least 3.*

$$\begin{aligned} P(x) &= (x + 1)(x - 3i)(x + 3i) \\ &= (x + 1)(x^2 - 9i^2) \\ &= (x + 1)(x^2 + 9) \\ &= x^3 + x^2 + 9x + 9 \end{aligned}$$

c.  $0, 2, 1 + i, 1 - i$

*Since  $1 + i$  and  $1 - i$  are complex conjugates,  $P$  is at least a 4<sup>th</sup> degree polynomial.*

$$\begin{aligned} P(x) &= x(x - 2)(x - (1 + i))(x - (1 - i)) \\ &= x(x - 2)[(x - 1) - i][(x - 1) + i] \\ &= x(x - 2)[(x - 1)^2 - i^2] \\ &= x(x - 2)[(x^2 - 2x + 1) + 1] \\ &= x(x - 2)(x^2 - 2x + 2) \\ &= x(x^3 - 2x^2 + 2x - 2x^2 + 4x - 4) \\ &= x(x^3 - 4x^2 + 6x - 4) \\ &= x^4 - 4x^3 + 6x^2 - 4x \end{aligned}$$

d.  $\sqrt{2}, -\sqrt{2}, 3, 1 + 2i$

*Since  $1 + 2i$  is a complex solution to  $P(x) = 0$ , its conjugate,  $1 - 2i$ , must also be a complex solution. Thus,  $P$  is at least a fifth-degree polynomial.*

$$\begin{aligned} P(x) &= (x - \sqrt{2})(x + \sqrt{2})(x - 3)(x - (1 + 2i))(x - (1 - 2i)) \\ &= (x^2 - 2)(x - 3)[(x - 1) - 2i][(x - 1) + 2i] \\ &= (x^2 - 2)(x - 3)[(x - 1)^2 - 4i^2] \\ &= (x^2 - 2)(x - 3)[(x^2 - 2x + 1) + 4] \\ &= (x^2 - 2)(x - 3)(x^2 - 2x + 5) \\ &= (x^3 - 3x^2 - 2x + 6)(x^2 - 2x + 5) \\ &= x^5 - 5x^4 + 9x^3 - 5x^2 - 22x + 30 \end{aligned}$$

e.  $2i, 3 - i$

*The complex conjugates of  $2i$  and  $3 - i$  are  $-2i$  and  $3 + i$ , respectively. So,  $P$  is at least a fourth-degree polynomial.*

$$\begin{aligned} P(x) &= (x - 2i)(x + 2i)(x - (3 - i))(x - (3 + i)) \\ &= (x^2 - 4i^2)[(x - 3) + i][(x - 3) - i] \\ &= (x^2 + 4)[(x - 3)^2 - i^2] \\ &= (x^2 + 4)[(x^2 - 6x + 9) + 1] \\ &= (x^2 + 4)(x^2 - 6x + 10) \\ &= x^4 - 6x^3 + 14x^2 - 24x + 40 \end{aligned}$$

### Closing (3 minutes)

Have students break into small groups to discuss what they learned today. Today's lesson is summarized in the box below.

#### Lesson Summary

- Polynomial equations with real coefficients can have real or complex solutions or they can have both.
- Complex solutions to polynomial equations with real coefficients are always members of a conjugate pair.
- Real solutions to polynomial equations are the same as  $x$ -intercepts of the associated graph, but complex solutions are not.

### Exit Ticket (6 minutes)

In this Exit Ticket, students solve quadratic equations with real and complex solutions.



Exit Ticket Sample Solutions

1. Solve the quadratic equation  $x^2 + 9 = 0$ . What are the  $x$ -intercepts of the graph of the function  $f(x) = x^2 + 9$ ?

$$\begin{aligned} x^2 + 9 &= 0 \\ x^2 &= -9 \\ x &= \sqrt{-9} \text{ or } x = -\sqrt{-9} \\ x &= 3\sqrt{-1} \text{ or } x = -3\sqrt{-1} \\ x &= 3i \text{ or } x = -3i \end{aligned}$$

The  $x$ -intercepts of the graph of the function  $f(x) = x^2 + 9$  are any real solutions to the equation  $x^2 + 9 = 0$ . However, since both solutions to  $x^2 + 9 = 0$  are not real, the function  $f(x) = x^2 + 9$  does not have any  $x$ -intercepts.

2. Find the solutions to  $2x^5 - 5x^3 - 3x = 0$ . What are the  $x$ -intercepts of the graph of the function  $f(x) = 2x^5 - 5x^3 - 3x$ ?

$$\begin{aligned} (2x^4 - 5x^2 - 3) &= 0 \\ x(x^2 - 3)(2x^2 + 1) &= 0 \\ x(x + \sqrt{3})(x - \sqrt{3})(2x^2 + 1) &= 0 \\ x(x + \sqrt{3})(x - \sqrt{3})\left(x + \frac{i\sqrt{2}}{2}\right)\left(x - \frac{i\sqrt{2}}{2}\right) &= 0 \end{aligned}$$

Thus,  $x = 0$ ,  $x = -\sqrt{3}$ ,  $x = \sqrt{3}$ ,  $x = -\frac{i\sqrt{2}}{2}$ , or  $x = \frac{i\sqrt{2}}{2}$ .

The solutions are  $0$ ,  $\sqrt{3}$ ,  $-\sqrt{3}$ ,  $\frac{i\sqrt{2}}{2}$ , and  $-\frac{i\sqrt{2}}{2}$ .

The  $x$ -intercepts of the graph of the function  $f(x) = 2x^5 - 5x^3 - 3x$  are the real solutions to the equation  $2x^5 - 5x^3 - 3x = 0$ , so the  $x$ -intercepts are  $0$ ,  $\sqrt{3}$ , and  $-\sqrt{3}$ .

Problem Set Sample Solutions

1. Rewrite each expression in standard form.

a.  $(x + 3i)(x - 3i)$   
 $x^2 + 3^2 = x^2 + 9$

b.  $(x - a + bi)(x - (a + bi))$   
 $(x - a + bi)(x - (a + bi)) = ((x - a) + bi)((x - a) - bi)$   
 $= (x - a)^2 + b^2$   
 $= x^2 - 2ax + a^2 + b^2$

c.  $(x + 2i)(x - i)(x + i)(x - 2i)$   
 $(x + 2i)(x - 2i)(x + i)(x - i) = (x^2 + 2^2)(x^2 + 1^2)$   
 $= (x^2 + 4)(x^2 + 1)$   
 $= x^4 + 5x^2 + 4$

d.  $(x + i)^2 \cdot (x - i)^2$

$$(x + i)(x - i) \cdot (x + i)(x - i) = (x^2 + 1)(x^2 + 1) \\ = x^4 + 2x^2 + 1$$

2. Suppose in Problem 1 that you had no access to paper, writing utensils, or technology. How do you know that the expressions in parts (a)–(d) are polynomials with real coefficients?

*In part (a), the identity  $(x + ai)(x - ai) = x^2 + a^2$  can be applied. Since the number  $a$  is real, the resulting polynomial will have real coefficients. The remaining three expressions can all be rearranged to take advantage of the conjugate pairs identity. In parts (c) and (d), regrouping terms will produce products of polynomial expressions with real coefficients, which will again have real coefficients.*

3. Write a polynomial equation of degree 4 in standard form that has the solutions  $i, -i, 1, -1$ .

*The first step is writing the equation in factored form:*

$$(x + i)(x - i)(x + 1)(x - 1) = 0.$$

*Then, use the commutative property to rearrange terms and apply the difference of squares formula:*

$$(x + i)(x - i)(x + 1)(x - 1) = (x^2 + 1)(x^2 - 1) \\ = x^4 - 1.$$

*So, the standard form of the equation is*

$$x^4 - 1 = 0.$$

4. Explain the difference between  $x$ -intercepts and solutions to an equation. Give an example of a polynomial with real coefficients that has twice as many solutions as  $x$ -intercepts. Write it in standard form.

*The  $x$ -intercepts are the real solutions to a polynomial equation with real coefficients. The solutions to an equation can be real or not real. The previous problem is an example of a polynomial with twice as many solutions than  $x$ -intercepts. Or, we could consider the equation  $x^4 - 6x^3 + 13x^2 - 12x + 4 = 0$ , which has zeros of multiplicity 2 at both 1 and 2.*

5. Find the solutions to  $x^4 - 5x^2 - 36 = 0$  and the  $x$ -intercepts of the graph of  $y = x^4 - 5x^2 - 36$ .

$$(x^2 + 4)(x^2 - 9) = 0 \\ (x + 2i)(x - 2i)(x + 3)(x - 3) = 0$$

*Since the solutions are  $2i, -2i, 3,$  and  $-3$ , and only real solutions to the equation are  $x$ -intercepts of the graph, the  $x$ -intercepts are 3 and  $-3$ .*

6. Find the solutions to  $2x^4 - 24x^2 + 40 = 0$  and the  $x$ -intercepts of the graph of  $y = 2x^4 - 24x^2 + 40$ .

$$2(x^4 - 12x^2 + 20) = 0 \\ 2(x^2 - 10)(x^2 - 2) = 0$$

*Since all of the solutions  $\sqrt{10}, -\sqrt{10}, \sqrt{2}$  and  $-\sqrt{2}$  are real numbers, the  $x$ -intercepts of the graph are  $\sqrt{10}, -\sqrt{10}, \sqrt{2}$  and  $-\sqrt{2}$ .*

7. Find the solutions to  $x^4 - 64 = 0$  and the  $x$ -intercepts of the graph of  $y = x^4 - 64$ .

$$(x^2 + 8)(x^2 - 8) = 0 \\ (x + \sqrt{8}i)(x - \sqrt{8}i)(x + \sqrt{8})(x - \sqrt{8}) = 0$$

*The  $x$ -intercepts are  $2\sqrt{2}$  and  $2 - \sqrt{2}$ .*

8. Use the fact that  $x^4 + 64 = (x^2 - 4x + 8)(x^2 + 4x + 8)$  to explain how you know that the graph of  $y = x^4 + 64$  has no  $x$ -intercepts. You need not find the solutions.

*The  $x$ -intercepts of  $y = x^4 + 64$  are solutions to  $(x^2 - 4x + 8)(x^2 + 4x + 8) = 0$ . Both  $x^2 - 4x + 8 = 0$  and  $x^2 + 4x + 8 = 0$  have negative discriminant values of  $-16$ , so the equations  $x^2 - 4x + 8 = 0$  and  $x^2 + 4x + 8 = 0$  have no real solutions. Thus, the equation  $x^4 + 64 = 0$  has no real solutions, and the graph of  $y = x^4 + 64$  has no  $x$ -intercepts.*

*Since  $x^4 + 64 = 0$  has no real solutions, the graph of  $y = x^4 + 64$  has no  $x$ -intercepts.*



## Lesson 40: Obstacles Resolved—A Surprising Result

### Student Outcomes

- Students understand the Fundamental Theorem of Algebra; that all polynomial expressions factor into linear terms in the realm of complex numbers. Consequences, in particular, for quadratic and cubic equations are understood.

### Lesson Notes

There is no real consensus in the literature about what exactly constitutes the Fundamental Theorem of Algebra; it is stated differently in different texts. The two-part theorem stated in this lesson encapsulates the main ideas of the theorem and its corollaries while remaining accessible to students. The first part of what is stated here as the Fundamental Theorem of Algebra is the one that we are not mathematically equipped to prove or justify at this level; this part states that every polynomial equation has at least one solution in the complex numbers and will need to be accepted without proof. The consequence of this first part is what we find really interesting—that every polynomial expression factors into the same number of linear factors as its degree. Justification for this second part of the Fundamental Theorem of Algebra is accessible for students as long as we can accept the first part without needing proof. Since every polynomial of degree  $n \geq 1$  will factor into  $n$  linear factors, we know that any polynomial function of degree  $n$  will have  $n$  zeros (including repeated zeros).

### Classwork

#### Opening Exercise (5 minutes)

At the beginning of the lesson, we focus on the most familiar of polynomial expressions, the quadratic equations. Ensure that students understand the link provided by the Remainder Theorem between solutions of polynomial equations and factors of the associated polynomial expression. Allow students time to work on the Opening Exercise, and then debrief.

#### Opening Exercise

Write each of the following quadratic expressions as a product of linear factors. Verify that the factored form is equivalent.

- |                     |                          |
|---------------------|--------------------------|
| 1. $x^2 + 12x + 27$ | $(x + 3)(x + 9)$         |
| 2. $x^2 - 16$       | $(x + 4)(x - 4)$         |
| 3. $x^2 + 16$       | $(x + 4i)(x - 4i)$       |
| 4. $x^2 + 4x + 5$   | $(x + 2 + i)(x + 2 - i)$ |

**Discussion (7 minutes)**

Remind students about the Remainder Theorem, studied earlier in the module. The Remainder Theorem states that if  $P$  is a polynomial function, and  $P(a) = 0$  for some value of  $a$ , then  $x - a$  is a factor of  $P$ . The Remainder Theorem plays an important role in the development of this lesson, linking the solutions of a polynomial equation to the factors of the associated polynomial expression.

- With a partner, describe any patterns you see in the Opening Exercise.
- Can every quadratic polynomial be written in terms of linear factors? If so, how many linear factors?
  - Yes, two.
- How do you know?
  - Every quadratic equation has two solutions that can be found using the quadratic formula. These solutions of the equation lead to linear factors of the quadratic polynomial.
- What types of solutions can a quadratic equation have? What does this mean about the graph of the corresponding function?
  - The equation has either two real solutions, one real solution, or two complex solutions. These situations correspond to the graph having two  $x$ -intercepts, one  $x$ -intercept, or no  $x$ -intercepts.

Be sure that students realize that real numbers are also complex numbers; if  $a$  is real, then we can write it as  $a + 0i$ .

**Example 1 (8 minutes)**

The purpose of this example is to help students move fluently between the concepts of  $x$ -intercepts of the graph of a polynomial equation  $y = P(x)$ , the solutions of the polynomial equation  $P(x) = 0$ , and the factors in the factored form of the associated polynomial  $P$ . Talk the students through parts (a)–(e), and then allow them time to work alone or in pairs on part (f) before completing the discussion.

**Example 1**

Consider the polynomial  $P(x) = x^3 + 3x^2 + x - 5$  whose graph is shown to the right.

- a. Looking at the graph, how do we know that there is only one real solution?

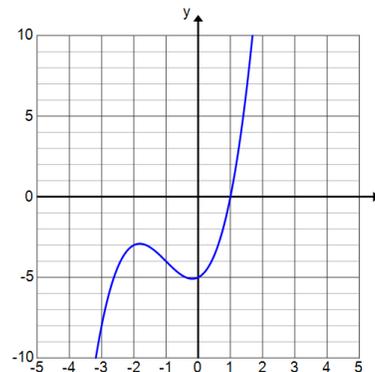
*The graph has only one  $x$ -intercept.*

- b. Is it possible for a cubic polynomial function to have no zeros?

*No. Since the opposite ends of the graph of a cubic function go in opposite directions, the graph must cross the  $x$ -axis at some point. Since the graph must have an  $x$ -intercept, the function must have a zero.*

- c. From the graph, what appears to be one solution to the equation  $x^3 + 3x^2 + x - 5 = 0$ ?

*The only real solution appears to be 1.*



d. How can we verify that this is a solution?

Evaluate the function at 1; that is, verify that  $P(1) = 0$ .

$$P(1) = (1)^3 + 3(1)^2 + 1 - 5 = 1 + 2 + 1 - 5 = 0$$

e. According to the Remainder Theorem, what is one factor of the cubic expression  $x^3 + 3x^2 + x - 5$ ?

$$(x - 1)$$

f. Factor out the expression you found in part (e) from  $x^3 + 3x^2 + x - 5$ .

Using polynomial division, we see that  $x^3 + 3x^2 + x - 5 = (x - 1)(x^2 + 4x + 5)$ .

g. What are all of the solutions to  $x^3 + 3x^2 + x - 5 = 0$ ?

The quadratic equation  $x^2 + 4x + 5 = 0$  has solutions  $2 - i$  and  $2 + i$  by the quadratic formula, so the original equation has solutions  $1, 2 - i$ , and  $2 + i$ .

h. Write the expression  $x^3 + 3x^2 + x - 5$  in terms of linear factors.

The factored form of the cubic expression is  $x^3 + 3x^2 + x - 5 = (x - 1)(x - (2 - i))(x - (2 + i))$ .

Scaffolding:

- For students who are struggling with part (g), point out that the remaining quadratic polynomial is the same as one of the problems from the Opening Exercise.
- As an extension, ask students to create a polynomial equation that has three real and two complex solutions.

- We established earlier in the lesson that all quadratic expressions can be written in terms of two linear factors. How many factors did our cubic expression have?
  - Three.
- Is it true that every cubic expression can be factored into three linear factors?
  - Yes, because a cubic equation will always have at least one real solution that corresponds to a linear factor of the expression. What is left over will be a quadratic expression, which can be written in terms of two linear factors.

If students don't seem ready to answer the last question or are unsure of the answer, let them work through Exercise 1 and then re-address it.

Exercises 1–2 (6 minutes)

Give students time to work through the two exercises and then lead the discussion that follows.

Exercises 1–2

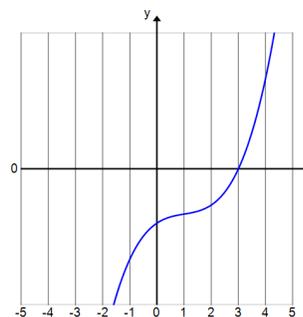
Write each polynomial in terms of linear factors. The graph of  $y = x^3 - 3x^2 + 4x - 12$  is provided for Exercise 2.

1.  $f(x) = x^3 + 5x$

$$f(x) = x(x + i\sqrt{5})(x - i\sqrt{5})$$

2.  $g(x) = x^3 - 3x^2 + 4x - 12$

$$g(x) = (x - 3)(x + 2i)(x - 2i)$$



Discussion (3 minutes)

MP.3

- Do your results from Exercises 1 and 2 agree with our conclusions from Example 1?
  - Yes, each cubic function could be written as a product of three linear factors.
- Make a conjecture about what might happen if we factored a degree 4 polynomial. What about a degree 5 polynomial? Explain your reasoning.
  - A degree 4 polynomial should have 4 linear factors. Based on the previous examples, it seems that a polynomial has as many linear factors as its degree. Similarly, a degree 5 polynomial should be able to be written as a product of 5 linear factors.
- Our major conclusion in this lesson is a two-part theorem known as the Fundamental Theorem of Algebra (FTA). Part 1 of the Fundamental Theorem of Algebra says that every polynomial equation has at least one solution in the complex numbers. Does that agree with our experience?
  - Yes.
- Think about how we factor a polynomial expression  $P$ : We find one solution  $a$  to  $P(x) = 0$ , then we factor out the term  $(x - a)$ . We are left with a new polynomial expression of one degree lower than  $P$ , so we look for another solution, and repeat until we have factored everything into linear parts.
- Consider the polynomial  $P(x) = x^4 - 3x^3 + 6x^2 - 12x + 8$  in the next example.

Scaffolding:

To meet a variety of student needs, ask students to:

- Restate the FTA in their own words (either in writing or verbally).
- Illustrate the FTA with an example. For instance, show that the FTA is true for some polynomial function.
- Apply the FTA to some examples. For instance, how many linear factors would be in the factored form of  $x^5 - 3x + 1$ ?

Example 2 (8 minutes)

While we do not have the mathematical tools or experience needed to prove the Fundamental Theorem of Algebra (either part), this example illustrates how the logic of the second part of the FTA works. Lead students through this example, allowing time for factoring and discussion at each step.

**Example 2**

Consider the polynomial function  $P(x) = x^4 - 3x^3 + 6x^2 - 12x + 8$ , whose corresponding graph  $y = x^4 - 3x^3 + 6x^2 - 12x + 8$  is shown to the right. How many zeros does  $P$  have?

a. Part 1 of the Fundamental Theorem of Algebra says that this equation will have at least one solution in the complex numbers. How does this align with what we can see in the graph to the right?

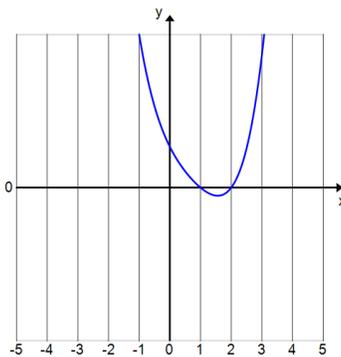
*Since the graph has 2 x-intercepts, there appear to be 2 zeros to the function. We were guaranteed one zero, but we know there are at least two.*

b. Identify one zero from the graph.

*One zero is 1. (The other is 2.)*

c. Use polynomial division to factor out one linear term from the expression  $x^4 - 3x^3 + 6x^2 - 12x + 8$ .

*$x^4 - 3x^3 + 6x^2 - 12x + 8 = (x - 1)(x^3 - 2x^2 + 4x - 8)$*



- d. Now we have a cubic polynomial to factor. We know by part 1 of the Fundamental Theorem of Algebra that a polynomial function will have at least one real zero. What is that zero in this case?

*The original polynomial function had real zeros at 1 and 2, so the cubic function  $P(x) = x^3 - 2x^2 + 4x - 8$  has a zero at 2.*

- e. Use polynomial division to factor out another linear term of  $x^4 - 3x^3 + 6x^2 - 12x + 8$ .

$$x^4 - 3x^3 + 6x^2 - 12x + 8 = (x - 1)(x^3 - 2x^2 + 4x - 8) = (x - 1)(x - 2)(x^2 + 4)$$

- f. Are we done? Can we factor this polynomial any further?

*At this point, we can see that  $x^2 + 4 = (x + 2i)(x - 2i)$ , so*

$$x^4 - 3x^3 + 6x^2 - 12x + 8 = (x - 1)(x - 2)(x + 2i)(x - 2i).$$

- g. Now that the polynomial is in factored form, we can quickly see how many solutions there are to the original equation  $x^4 - 3x^3 + 6x^2 - 12x + 8 = 0$ .

*If  $x^4 - 3x^3 + 6x^2 - 12x + 8 = 0$ , then  $(x - 1)(x - 2)(x + 2i)(x - 2i) = 0$ , so the solutions are 1, 2,  $2i$  and  $-2i$ . So, the polynomial  $P$  has 4 zeros; 2 are real numbers, and 2 are complex numbers.*

- h. What if we had started with a polynomial function of degree 8?

*We would find the first zero, and factor out a linear term, leaving a polynomial of degree 7. We would then find another zero, factor out a linear term, leaving a polynomial of degree 6. We would repeat this process until we had a quadratic polynomial remaining; then, we would factor that with the help of the quadratic formula. We would have 8 linear factors at the end of the process that correspond to the 8 zeros of the original function.*

MP.8

The logic we just followed leads to part 2 of the Fundamental Theorem of Algebra, which is the result that we have already conjectured: A polynomial of degree  $N \geq 1$  will factor into  $N$  linear factors with complex coefficients. Collectively, these two results are often just referred to as the Fundamental Theorem of Algebra. Although we have only worked with polynomials with real coefficients, the FTA applies to polynomial functions with real coefficients, such as  $P(x) = x^3 + 2x^2 - 4$  as well as to polynomial functions with non-real coefficients, such as  $P(x) = x^3 + 3ix^2 + 4 - 2i$ . We have not attempted to justify the first part, but the students should be able to justify the second part of the theorem.

#### Fundamental Theorem of Algebra

1. Every polynomial function of degree  $N \geq 1$  with real or complex coefficients has at least one real or complex zero.
2. Every polynomial of degree  $N \geq 1$  with real or complex coefficients factors into  $N$  linear terms with real or complex coefficients.

- Why is the Fundamental Theorem of Algebra so “fundamental” to mathematics?
  - *The Fundamental Theorem says that the complex number system contains every zero of every polynomial function. We do not need to look anywhere else to find zeros to these types of functions.*

- Notice that the Fundamental Theorem just tells us that the factorization of the polynomial exists; it does not help us actually find it. If we had been given a polynomial function that did not have any real zeros, it would have been very hard to start the factorization process.

### Closing (3 minutes)

- With a partner, summarize the key points of this lesson.
- What does the Fundamental Theorem of Algebra guarantee?
  - A polynomial of degree  $N \geq 1$  will factor into  $N$  linear factors, and the associated function will have  $N$  zeros, some of which may be repeated.*
- Why is this important?
  - The Fundamental Theorem of Algebra ensures that there are as many zeros as we'd expect for a polynomial function, and that factoring will always (in theory) work to find solutions to polynomial equations.*
- Illustrate the Fundamental Theorem of Algebra with an example.

#### Lesson Summary

Every polynomial function of degree  $n$ , for  $n \geq 1$ , has  $n$  roots over the complex numbers, counted with multiplicity. Therefore, such polynomials can always be factored into  $n$  linear factors, and the obstacles to factoring we saw before have all disappeared in the larger context of allowing solutions to be complex numbers.

The Fundamental Theorem of Algebra:

- If  $P$  is a polynomial function of degree  $n \geq 1$ , with real or complex coefficients, then there exists at least one number  $r$  (real or complex) such that  $P(r) = 0$ .
- If  $P$  is a polynomial function of degree  $n \geq 1$ , given by  $P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  with real or complex coefficients  $a_i$ , then  $P$  has exactly  $n$  zeros  $r_1, r_2, \dots, r_n$  (not all necessarily distinct), such that  $P(x) = a(x - r_1)(x - r_2) \dots (x - r_n)$ .

### Exit Ticket (5 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 40: Obstacles Resolved—A Surprising Result

### Exit Ticket

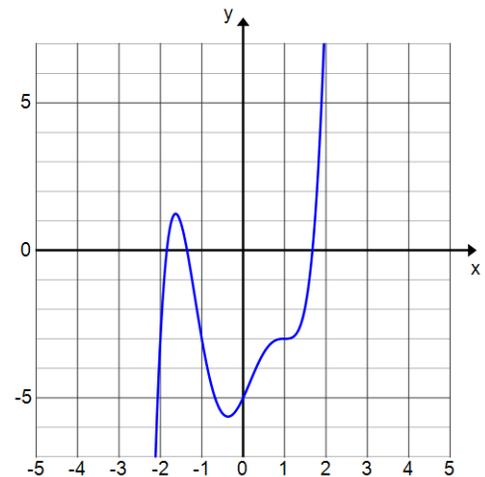
Consider the degree 5 polynomial function  $P(x) = x^5 - 4x^3 + 2x^2 + 3x - 5$ , whose graph is shown below. You do not need to factor this polynomial to answer the questions below.

1. How many linear factors is  $P$  guaranteed to have? Explain.

2. How many zeros does  $P$  have? Explain.

3. How many real zeros does  $P$  have? Explain.

4. How many complex zeros does  $P$  have? Explain.



Exit Ticket Sample Solutions

Consider the degree 5 polynomial function  $P(x) = x^5 - 4x^3 + 2x^2 + 3x - 5$  whose graph is shown below. You do not need to factor this polynomial to answer the questions below.

1. How many linear factors is  $P$  guaranteed to have? Explain.

*The polynomial expression must have 5 linear factors. The Fundamental Theorem of Algebra guarantees that a polynomial function can be written in terms of linear factors and must have the same number of linear factors as its degree.*

2. How many zeros does  $P$  have? Explain.

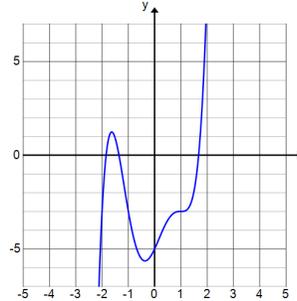
*Since  $P$  can be written in terms of 5 linear factors, the equation  $P$  must have 5 zeros (counted with multiplicity).*

3. How many real zeros does  $P$  have? Explain.

*The graph crosses the  $x$ -axis 3 times, which means that three of the zeros are real numbers.*

4. How many complex zeros does  $P$  have? Explain.

*Since  $P$  must have 5 total zeros and only 3 of them are real, there must be 2 complex zeros.*



Problem Set Sample Solutions

1. Write each quadratic function below in terms of linear factors.

a.  $f(x) = x^2 - 25$

$f(x) = (x + 5)(x - 5)$

b.  $f(x) = x^2 + 25$

$f(x) = (x + 5i)(x - 5i)$

c.  $f(x) = 4x^2 + 25$

$f(x) = (2x + 5i)(2x - 5i)$

d.  $f(x) = x^2 - 2x + 1$

$f(x) = (x - 1)(x - 1)$

e.  $f(x) = x^2 - 2x + 4$

$f(x) = (x - 1 + i\sqrt{3})(x - 1 - i\sqrt{3})$

2. Consider the polynomial function  $P(x) = (x^2 + 4)(x^2 + 1)(2x + 3)(3x - 4)$ .

- a. Express  $P$  in terms of linear factors.

$P(x) = (x + 2i)(x - 2i)(x + i)(x - i)(2x + 3)(3x - 4)$

- b. Fill in the blanks of the following sentence.

The polynomial  $P$  has degree \_\_\_\_\_ and can, therefore, be written in terms of \_\_\_\_\_ linear factors.

The function  $P$  has \_\_\_\_\_ zeros. There are \_\_\_\_\_ real zeros and \_\_\_\_\_ complex zeros. The graph of  $y = P(x)$  has \_\_\_\_\_  $x$ -intercepts.

*The polynomial  $P$  has degree 6 and can, therefore, be written in terms of 6 linear factors. The function  $P$  has 6 solutions. There are 2 real zeros and 4 complex zeros. The graph of  $y = P(x)$  has 2  $x$ -intercepts.*

3. Express each cubic function below in terms of linear factors.

a.  $f(x) = x^3 - 6x^2 - 27x$

$$f(x) = x(x - 9)(x + 3)$$

b.  $f(x) = x^3 - 16x^2$

$$f(x) = x^2(x - 16)$$

c.  $f(x) = x^3 + 16x$

$$f(x) = x(x + 4i)(x - 4i)$$

4. For each cubic function below, one of the zeros is given. Express each cubic function in terms of linear factors.

a.  $f(x) = 2x^3 - 9x^2 - 53x - 24; f(8) = 0$

$$f(x) = (x - 8)(2x + 1)(x + 3)$$

b.  $f(x) = x^3 + x^2 + 6x + 6; f(-1) = 0$

$$f(x) = (x + 1)(x + i\sqrt{6})(x - i\sqrt{6})$$

5. Determine if each statement is always true or sometimes false. If it is sometimes false, explain why it is not always true.

a. A degree 2 polynomial function will have two linear factors.

*Always true.*

b. The graph of a degree 2 polynomial function will have two  $x$ -intercepts.

*False. It is possible for the solutions to a degree 2 polynomial to be complex, in which case the graph would not cross the  $x$ -axis. It is also possible for the graph to have only one  $x$ -intercept if the vertex lies on the  $x$ -axis.*

c. The graph of a degree 3 polynomial function might not cross the  $x$ -axis.

*False. A degree 3 polynomial must cross the  $x$ -axis at least one time.*

d. A polynomial function of degree  $n$  can be written in terms of  $n$  linear factors.

*Always true.*

6. Consider the polynomial function  $f(x) = x^6 - 9x^3 + 8$ .

a. How many linear factors does  $x^6 - 9x^3 + 8$  have? Explain.

*Since the degree is 6, the polynomial must have 6 linear factors.*

b. How is this information useful for finding the zeros of  $f$ ?

*We know that the function has 6 zeros since there are 6 linear factors. Each factor corresponds to a zero of the function.*

c. Find the zeros of  $f$ . (Hint: Let  $Q = x^3$ . Rewrite the equation in terms of  $Q$  to factor.)

$$1, 2, -1 + i\sqrt{3}, -1 - i\sqrt{3}, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$$

7. Consider the polynomial function  $P(x) = x^4 - 6x^3 + 11x^2 - 18$ .

a. Use the graph to find the real zeros of  $P$

*The real zeros are  $-1$  and  $3$ .*

b. Confirm that the zeros are correct by evaluating the function  $P$  at those values.

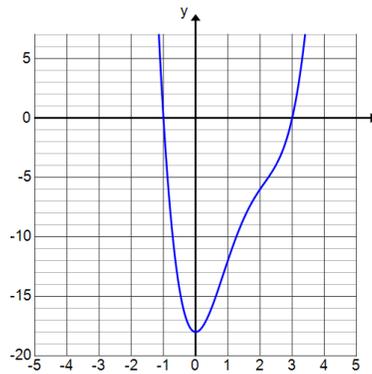
*$P(-1) = 0$  and  $P(3) = 0$*

c. Express  $P$  in terms of linear factors.

*$P(x) = (x + 1)(x - 3)(x - (2 + i\sqrt{2}))(x - (2 - i\sqrt{2}))$*

d. Find all zeros of  $P$ .

*$-1, 3, 2 - i\sqrt{2}, 2 + i\sqrt{2}$*



8. Penny says that the equation  $x^3 - 8 = 0$  has only one solution,  $x = 2$ . Use the Fundamental Theorem of Algebra to explain to her why she is incorrect.

*Because  $x^3 - 8$  is a degree 3 polynomial, the Fundamental Theorem of Algebra guarantees that  $x^3 - 8$  can be written as the product of three linear factors; therefore, the corresponding equation has 3 solutions. One of the 3 solutions is 2. We know that 2 cannot be the only solution because  $(x - 2)(x - 2)(x - 2) \neq x^3 - 8$ .*

9. Roger says that the equation  $x^2 - 12x + 36 = 0$  has only one solution, 6. Regina says Roger is wrong and that the Fundamental Theorem of Algebra guarantees that a quadratic equation must have two solutions. Who is correct and why?

*Roger is correct. While the Fundamental Theorem of Algebra guarantees that a quadratic polynomial can be written in terms of two linear factors, the factors are not necessarily distinct. We know that  $x^2 - 12x + 36 = (x - 6)(x - 6)$ , so the equation  $x^2 - 12x + 36 = 0$  has only one solution, which is 6.*

MP.3

Name \_\_\_\_\_

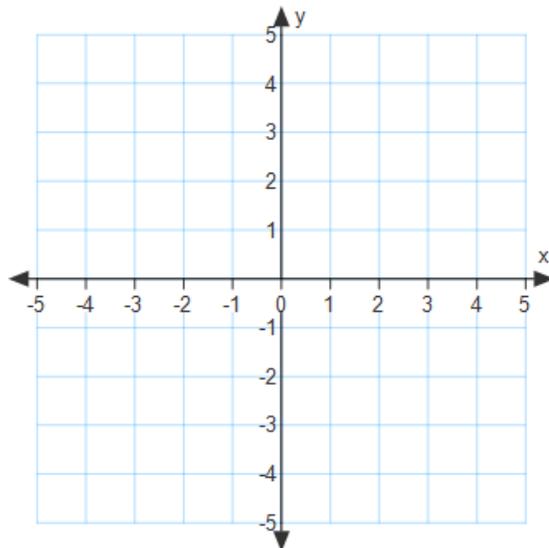
Date \_\_\_\_\_

1. A *parabola* is defined as the set of points in the plane that are equidistant from a fixed point (called the *focus* of the parabola) and a fixed line (called the *directrix* of the parabola).

Consider the parabola with focus point  $(1, 1)$  and directrix the horizontal line  $y = -3$ .

- a. What are the coordinates of the vertex of the parabola?

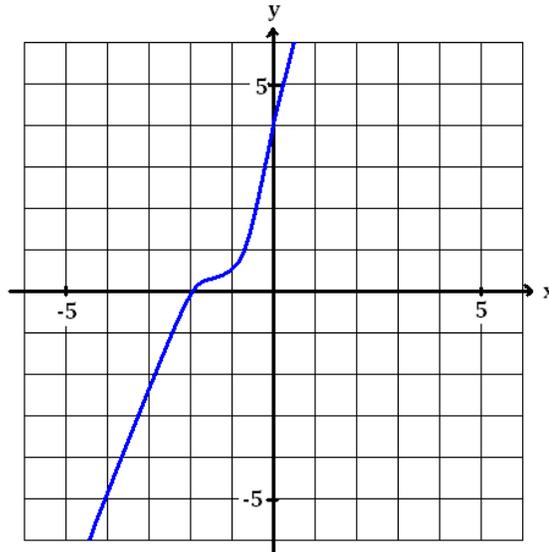
- b. Plot the focus and draw the directrix on the graph below. Then draw a rough sketch of the parabola.



- c. Find the equation of the parabola with this focus and directrix.
- d. What is the  $y$ -intercept of this parabola?
- e. Demonstrate that your answer from part (d) is correct by showing that the  $y$ -intercept you identified is indeed equidistant from the focus and the directrix.

- f. Is the parabola in this question (with focus point  $(1, 1)$  and directrix  $y = -3$ ) congruent to a parabola with focus  $(2, 3)$  and directrix  $y = -1$ ? Explain.
- g. Is the parabola in this question (with focus point  $(1, 1)$  and directrix  $y = -3$ ) congruent to the parabola with equation given by  $y = x^2$ ? Explain.
- h. Are the two parabolas from part (g) similar? Why or why not?

2. The graph of the polynomial function  $f(x) = x^3 + 4x^2 + 6x + 4$  is shown below.

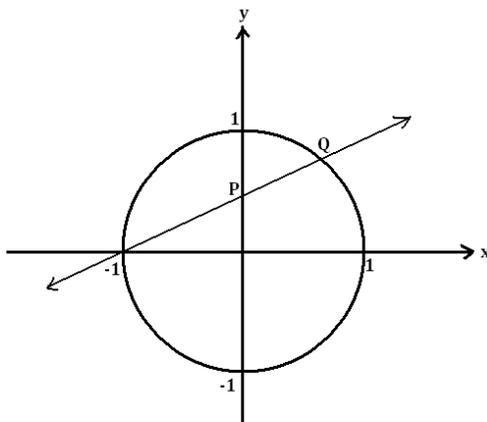


a. Based on the appearance of the graph, what does the real solution to the equation  $x^3 + 4x^2 + 6x + 4 = 0$  appear to be? Jiji does not trust the accuracy of the graph. Prove to her algebraically that your answer is in fact a zero of  $y = f(x)$ .

b. Write  $f$  as a product of a linear factor and a quadratic factor, each with real-number coefficients.

- c. What is the value of  $f(10)$ ? Explain how knowing the linear factor of  $f$  establishes that  $f(10)$  is a multiple of 12.
- d. Find the two complex-number zeros of  $y = f(x)$ .
- e. Write  $f$  as a product of three linear factors.

3. A line passes through the points  $(-1, 0)$  and  $P = (0, t)$  for some real number  $t$  and intersects the circle  $x^2 + y^2 = 1$  at a point  $Q$  different from  $(-1, 0)$ .



- a. If  $t = \frac{1}{2}$ , so that the point  $P$  has coordinates  $(0, \frac{1}{2})$ , find the coordinates of the point  $Q$ .

A *Pythagorean triple* is a set of three positive integers  $a$ ,  $b$ , and  $c$  satisfying  $a^2 + b^2 = c^2$ . For example, setting  $a = 3$ ,  $b = 4$ , and  $c = 5$  gives a Pythagorean triple.

- b. Suppose that  $\left(\frac{a}{c}, \frac{b}{c}\right)$  is a point with rational-number coordinates lying on the circle  $x^2 + y^2 = 1$ . Explain why then  $a$ ,  $b$ , and  $c$  form a Pythagorean triple.

- c. Which Pythagorean triple is associated with the point  $Q = \left(\frac{5}{13}, \frac{12}{13}\right)$  on the circle?

- d. If  $Q = \left(\frac{5}{13}, \frac{12}{13}\right)$ , what is the value of  $t$  so that the point  $P$  has coordinates  $(0, t)$ ?

e. Suppose we set  $x = \frac{1-t^2}{1+t^2}$  and  $y = \frac{2t}{1+t^2}$ , for a real number  $t$ . Show that  $(x, y)$  is then a point on the circle  $x^2 + y^2 = 1$ .

f. Set  $t = \frac{3}{4}$  in the formulas  $x = \frac{1-t^2}{1+t^2}$  and  $y = \frac{2t}{1+t^2}$ . Which point on the circle  $x^2 + y^2 = 1$  does this give? What is the associated Pythagorean triple?

- g. Suppose  $t$  is a value greater than 1,  $P = (0, t)$ , and  $Q$  is the point in the second quadrant (different from  $(-1, 0)$ ) at which the line through  $(-1, 0)$  and  $P$  intersects the circle  $x^2 + y^2 = 1$ . Find the coordinates of the point  $Q$  in terms of  $t$ .

- 4.
- a. Write a system of two equations in two variables where one equation is quadratic and the other is linear such that the system has no solution. Explain, using graphs, algebra, and/or words, why the system has no solution.
- b. Prove that  $x = \sqrt{-5x - 6}$  has no solution.

- c. Does the following system of equations have a solution? If so, find one. If not, explain why not.

$$\begin{aligned}2x + y + z &= 4 \\x - y + 3z &= -2 \\-x + y + z &= -2\end{aligned}$$

A Progression Toward Mastery

Assessment Task Item		STEP 1 Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem.	STEP 2 Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem.	STEP 3 A correct answer with some evidence of reasoning or application of mathematics to solve the problem <u>or</u> an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem.	STEP 4 A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem.
1	a–c  N-Q.A.2 F-IF.C.7c G-GPE.A.2	(a) Student provides incorrect vertex coordinates. (b) Student sketches a parabola that does not open up or a parabola that is horizontal. (c) Student provides an equation that is not in the form of a vertical parabola.	(a) Student provides either an incorrect $x$ - or $y$ -coordinate. (b) Student provides a sketch of a parabola that opens up but with little or no scale or labels. (c) Student provides an incorrect equation using the vertex from part (a); $a$ -value is incorrect due to conceptual errors.	(a) Student provides the correct vertex. (b) Student provides a sketch of a parabola that opens up with correct vertex. The sketch may be incomplete or lack sufficient labels or scale. (c) Student provides a parabola equation with correct vertex. Work showing $a$ -value calculation may contain minor errors.	(a) Student provides the correct vertex. (b) Student provides a well-labeled and accurate sketch of a parabola that opens up and includes the focus, directrix, and vertex. (c) Student provides the correct parabola equation in vertex or standard form with or without work showing how $a = \frac{1}{8}$ .
	d–e  N-Q.A.2 F-IF.C.7c G-GPE.A.2	(d) Student provides incorrect $y$ -intercept. No work is shown or a conceptual error is made. (e) Student makes no attempt or provides two incorrect distances.	(d) Student provides incorrect $y$ -intercept. No work is shown or a conceptual error is made (e.g., $y = 0$ instead of $x = 0$ ). (e) Student provides one correct distance but not both, using student’s incorrect $y$ -intercept and the given focus and directrix. <u>OR</u> Student provides the correct $y$ -intercept in part (d), but the student is unable to compute one or both distances	(d) Student substitutes $x = 0$ to determine the $y$ -intercept, but may make a minor calculation error. (e) Student provides the correct distance to the directrix using student’s $y$ -intercept. Student provides the correct distance between focus and $y$ -intercept using student’s $y$ -intercept. Note that if these are not equal, the student solution should indicate that they should be based on the definition	(d) Student correctly identifies the $y$ -intercept. (e) Student correctly identifies the distance to directrix and applies the distance formula to calculate the distance from focus and $y$ -intercept. Both are equal to $\frac{17}{8}$ .

			between the y-intercept and the given focus and directrix.	of a parabola.	
	<b>f–h</b> <b>N-Q.A.2</b> <b>F-IF.C.7c</b> <b>G-GPE.A.2</b>	Student incorrectly answers two or more parts with no justification in all three parts. <u>OR</u> Student incorrectly answers all three parts with faulty or no justification.	Student incorrectly answers two or more parts. Minimal justification is provided that includes a reference to the $a$ -value.	Student correctly answers all three parts with no justification. <u>OR</u> Student correctly answers two out of three parts with correct justification.	Student correctly answers all three parts. Justification states that parabolas with equal $a$ -values are congruent, but all parabolas are similar.
<b>2</b>	<b>a–b</b> <b>A-SSE.A.2</b> <b>A-APR.A.1</b> <b>A-APR.B.2</b> <b>A-APR.B.3</b> <b>A-REI.A.1</b> <b>A-REI.B.4b</b> <b>F-IF.C.7c</b>	(a) Student concludes that $x = -2$ is NOT a zero due to conceptual or major calculation errors (e.g., incorrect application of division algorithm) <u>OR</u> Student shows no work at all. (b) Student does not provide factored form or it is incorrect.	(a) Student concludes that $x = -2$ is NOT a zero due to minor calculation errors and provides limited justification for the solution. (b) Student does not provide factored form or it is incorrect.	(a) Student concludes that $x = -2$ is a zero, but may not support this mathematically or verbally. (b) Student correctly identifies both factors, but the polynomial may not be written in factored form $((x + 2)(x^2 + 2x + 2))$ .	(a) Student concludes that $x = -2$ is a zero by showing $f(-2) = 0$ or using division and getting a remainder equal to 0. Work or written explanation supports conclusion. (b) Student writes $f$ in correct factored form and provides work to support the solution. Note that the work may be done in part (a).
	<b>c</b> <b>A-SSE.A.2</b> <b>A-APR.B.2</b> <b>A-APR.B.3</b> <b>A-REI.A.1</b> <b>A-REI.B.4b</b> <b>F-IF.C.7c</b>	Student provides an incorrect value of $f(10)$ and a conclusion regarding 12 being a factor is missing or unsupported by any mathematical work or explanation.	Student provides an incorrect value of $f(10)$ and an incorrect factored form of $f$ . The student does not attempt to find the numerical factors of $f(10)$ or divide $f(10)$ by 12 to see if the remainder is 0.	Student provides a complete solution, but the solution may contain minor calculation errors on the value of $f(10)$ . <u>OR</u> Student concludes that 12 is NOT a factor of $f(10)$ because the student used an incorrectly factored form of $f$ in the first place or an incorrect value for $f(10)$ .	Student correctly identifies the solution as $f(10) = 1464$ . The explanation clearly communicates that 12 is a factor of $f(10)$ , i.e., when $x = 10$ , $(x + 2)$ is 12.
	<b>d–e</b> <b>A-SSE.A.2</b> <b>A-APR.A.1</b> <b>A-APR.B.2</b> <b>A-APR.B.3</b> <b>A-REI.A.1</b>	(d) Student does not use quadratic formula or uses incorrect formula. (e) Student provides incorrect complex roots and the solution is not a cubic equivalent to	(d) Student makes minor errors in the quadratic formula and provides incorrect roots. (e) Student uses the incorrect roots from (d), but the solution is a cubic equivalent to $(x + 2)(x - r_1)(x - r_2)$ , where	(d) Student provides the correct complex roots using the quadratic formula (does not have to be in simplest form). (e) Student provides a cubic polynomial using $-2$ and complex roots from (d). May contain	(d) Student provides the correct complex roots expressed as $(-1 \pm i)$ . (e) Student provides a cubic polynomial equivalent to $(x + 2)(x - (1 + i))(x - (1 - i))$ . It is acceptable to leave it in factored

	<b>A-REI.B.4b</b> <b>F-IF.C.7c</b>	$(x + 2)(x - r_1)(x - r_2)$ , where $r_1$ and $r_2$ are complex conjugates.	$r_1$ and $r_2$ are the student solutions to (d).	minor errors (e.g., leaving out parentheses on $(x - (1 + i))$ ) or a multiplication error when writing the polynomial in standard form.	form.
<b>3</b>	<b>a</b> <b>A-APR.C.4</b> <b>A-APR.D.6</b> <b>A-REI.A.2</b> <b>A-REI.C.6</b> <b>A-REI.C.7</b>	Student provides an incorrect equation of the line and makes major mathematical errors in attempting to solve a system of a linear and non-linear equation.	Student provides an incorrect equation of the line, but the solution shows substitution of the student's linear equation into the circle equation. The solution to the system may also contain minor calculation errors. <u>OR</u> Student provides a correct equation of the line, but the student is unable to solve the system due to major mathematical errors.	Student provides the correct equation of the line. The solution to the system may contain minor calculation errors. The correct solution is not expressed as an ordered pair or the solution only includes a correct $x$ - or $y$ -value for point $Q$ .	Student provides the correct equation of the line and the correct solution to the system of equations. The solution is expressed as an ordered pair, $Q\left(\frac{3}{5}, \frac{4}{5}\right)$ .
	<b>b–c</b> <b>A-APR.C.4</b> <b>A-APR.D.6</b> <b>A-REI.A.2</b> <b>A-REI.C.6</b> <b>A-REI.C.7</b>	(b) Student does not provide an answer or the answer is incorrect showing limited understanding of they were asked to do. (c) Student does not provide a triple or it is incorrect.	(b) Student substitutes $\left(\frac{a}{c}, \frac{b}{c}\right)$ into the equation of the circle, but fails to show that this equation is equivalent to $a^2 + b^2 = c^2$ . (c) Student provides an incorrect triple.	(b) Student provides an almost-complete solution (i.e., student substitutes $\left(\frac{a}{c}, \frac{b}{c}\right)$ into the equation of the circle and states that the point satisfies the Pythagorean Triple condition but doesn't show why). The solution may contain minor algebra mistakes. (c) Student identifies 5, 12, 13 as the triple.	(b) Student provides a correct solution showing substitution of $\left(\frac{a}{c}, \frac{b}{c}\right)$ into the equation of a circle. The work clearly demonstrates this equation is equivalent to $a^2 + b^2 = c^2$ . (c) Student identifies 5, 12, 13 as the triple.
	<b>d–f</b> <b>A-APR.C.4</b> <b>A-APR.D.6</b> <b>A-REI.A.2</b> <b>A-REI.C.6</b> <b>A-REI.C.7</b>	Student does not provide a solution or provides an incomplete solution to (d), (e), and (f) with major mathematical errors.	(d) Student provides correct slope of the line but fails to identify correct value of $t$ . (e) Student substitutes coordinates into $x^2 + y^2 = 1$ , but makes major errors in attempt to show they satisfy the equation. (f) Student substitutes $\frac{3}{4}$ for $t$ , but the solution is incorrect.	(d) Student provides the correct slope and equation of the line but fails to identify the correct value of $t$ . (e) Student substitutes coordinates into $x^2 + y^2 = 1$ and simplifies to show they satisfy the equation. (f) Student identifies point $Q$ and the triple correctly.	(d) Student provides the correct slope and equation of the line and correctly identifies the $t$ -value. (e) Student substitutes coordinates into $x^2 + y^2 = 1$ and simplifies to show they satisfy the equation. (f) Student identifies point $Q$ and the triple correctly.

				Note that one or more parts may contain minor calculation errors.	Note that all solutions use proper mathematical notation and clearly demonstrate student understanding.
	<p><b>g</b></p> <p><b>A-APR.C.4</b>  <b>A-APR.D.6</b>  <b>A-REI.A.2</b>  <b>A-REI.C.6</b>  <b>A-REI.C.7</b></p>	Student does not provide a solution or provides an incomplete solution with major mathematical errors.	The solution may include an accurate sketch and the equation of the line $y = tx + t$ but little additional work.	Student attempts to solve the system by substituting $y = tx + t$ into the circle equation and recognizes the need to apply the quadratic equation to solve for $x$ . May contain algebraic errors.	Student provides a complete and correct solution, showing sufficient work and calculation of both the $x$ - and $y$ -coordinate of the point.
4	<p><b>a</b></p> <p><b>A-REI.A.2</b>  <b>A-REI.B.4b</b>  <b>A-REI.C.6</b>  <b>A-REI.C.7</b></p>	Student does not provide work or it is incomplete. System does not include a linear and quadratic equation.	Student provides a system that has a solution, but student work indicates understanding that the graphs of the equations should not intersect or that algebraically the system has no real number solutions.	Student provides a system that does not have a solution, but the justification may reveal minor errors in student's thought process. If a graphical justification is the only one provided the graph must be scaled sufficiently to provide a convincing argument that the two equations do not intersect.	Student provides a system that does not have a solution. Justification includes a graphical, verbal explanation, or algebraic explanation that clearly demonstrates student thinking.
	<p><b>b–c</b></p> <p><b>A-REI.A.2</b>  <b>A-REI.B.4b</b>  <b>A-REI.C.6</b>  <b>A-REI.C.7</b></p>	Student provides incorrect solutions with little or no supporting work shown.	Student provides incorrect solutions to parts (b) and (c). Solutions are limited and reveal major mathematical errors in the solution process.	Student provides incorrect solutions to part (b) or part (c). Solutions show considerable understanding of the processes, but may contain minor errors.	Student provides correct solutions with sufficient work shown. <u>AND</u> Mathematical work or verbal explanation show why $-2$ and $-3$ are NOT solutions to part (b).

Name \_\_\_\_\_

Date \_\_\_\_\_

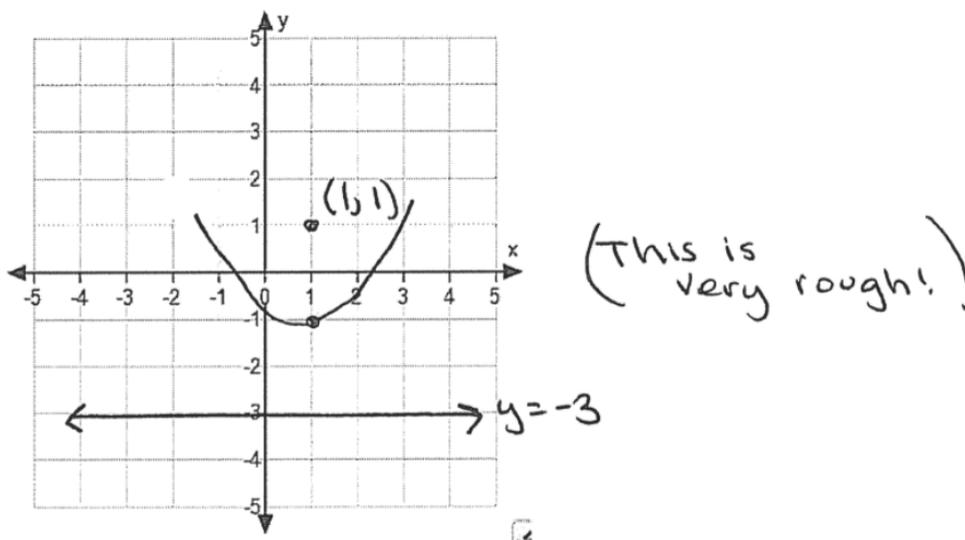
1. A *parabola* is defined as the set of points in the plane that are equidistant from a fixed point (called the *focus* of the parabola) and a fixed line (called the *directrix* of the parabola).

Consider the parabola with focus point  $(1, 1)$  and directrix the horizontal line  $y = -3$ .

- a. What are the coordinates of the vertex of the parabola?

$(1, -1)$

- b. Plot the focus and draw the directrix on the graph below. Then draw a rough sketch of the parabola.



- c. Find the equation of the parabola with this focus and directrix.

A point  $(x, y)$  on the parabola is equidistant from the directrix and the focus:

$$y+3 = \sqrt{(x-1)^2 + (y-1)^2}$$

$$(y+3)^2 = (x-1)^2 + (y-1)^2$$

$$y^2 + 6y + 9 = (x-1)^2 + y^2 - 2y + 1$$

$$8y = (x-1)^2 - 8$$

$$y = \frac{1}{8}(x-1)^2 - 1$$

- d. What is the y-intercept of this parabola?

At the y-intercept,  $x=0$ , so  $y = \frac{1}{8}(-1)^2 - 1 = -\frac{7}{8}$

$-\frac{7}{8}$  is the y intercept.

- e. Demonstrate that your answer from part (d) is correct by showing that the y-intercept you identified is indeed equidistant from the focus and the directrix.

The distance of  $(0, -\frac{7}{8})$  from the focus is:

$$\sqrt{(0-1)^2 + (-\frac{7}{8}-1)^2} = \sqrt{1 + (\frac{15}{8})^2} = \sqrt{\frac{289}{64}} = \frac{17}{8}$$

The distance of  $(0, -\frac{7}{8})$  from the line  $y=3$  is:

$$|(-\frac{7}{8}) - (-3)| = 2\frac{1}{8}$$

These are the same!

- f. Is the parabola in this question (with focus point  $(1, 1)$  and directrix  $y = -3$ ) congruent to a parabola with focus  $(2, 3)$  and directrix  $y = -1$ ? Explain.

The parabola with focus  $(2, 3)$  and directrix  $y = -1$  is  $y + 1 = \sqrt{(x - 2)^2 + (y - 3)^2}$ . Solving for  $y$ ,  $(y + 1)^2 = (x - 2)^2 + (y - 3)^2$ ,

$$y + 2y + 1 = (x - 2)^2 + y^2 - 6y + 9$$

$$8y = (x - 2)^2 + 8$$

$$y = \frac{1}{8}(x - 2)^2 + 1$$

This parabola is congruent to the parabola with focus point  $(1, 1)$  and directrix  $y = -3$  because the leading coefficients are the same.

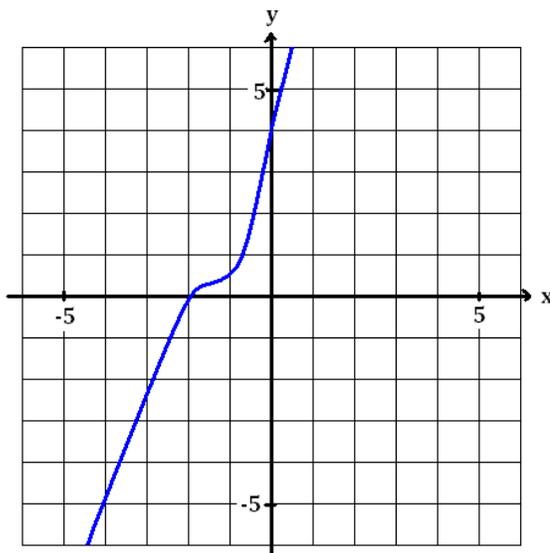
- g. Is the parabola in this question (with focus point  $(1, 1)$  and directrix  $y = -3$ ) congruent to the parabola with equation given by  $y = x^2$ ? Explain.

No,  $y = x^2$  and  $y = \frac{1}{8}(x - 1)^2 + 1$  do not have the same leading coefficient so they are not congruent. ( $y = x^2$  has a leading coefficient 1, and  $y = \frac{1}{8}(x - 1)^2 + 1$  has a leading coefficient  $\frac{1}{8}$ .)

- h. Are the two parabolas from part (g) similar? Why or why not?

Yes! Because all parabolas are similar.

2. The graph of the polynomial function  $f(x) = x^3 + 4x^2 + 6x + 4$  is shown below.



- a. Based on the appearance of the graph, what does the real solution to the equation  $x^3 + 4x^2 + 6x + 4 = 0$  appear to be? Jiju does not trust the accuracy of the graph. Prove to her algebraically that your answer is in fact a zero of  $y = f(x)$ .

The real zero appears to be  $x = -2$ .

$$f(-2) = (-2)^3 + 4(-2)^2 + 6(-2) + 4$$

$$= -8 + 16 - 12 + 4$$

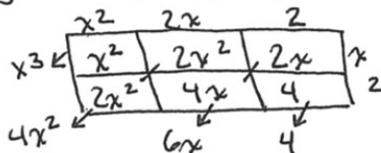
$$= 0$$

- b. Write  $f$  as a product of a linear factor and a quadratic factor, each with real-number coefficients.

Since  $x = -2$  is a zero,  $x + 2$  must be a factor.  
 Dividing  $f(x)$  by  $(x + 2)$  gives  $(x^2 + 2x + 2)$

So,

$$f(x) = (x + 2)(x^2 + 2x + 2)$$



- c. What is the value of  $f(10)$ ? Explain how knowing the linear factor of  $f$  establishes that  $f(10)$  is a multiple of 12.

$$\begin{aligned} f(10) &= (10+2)(100+20+2) \\ &= 12(122) \\ &= 1220+244 \\ &= 1464 \end{aligned}$$

$f(10)$  is a multiple of 12 because  $f(x)$  has a linear factor of  $x+2$ , and  $x+2=12$ , when  $x=10$ .

- d. Find the two complex-number zeros of  $y = f(x)$ .

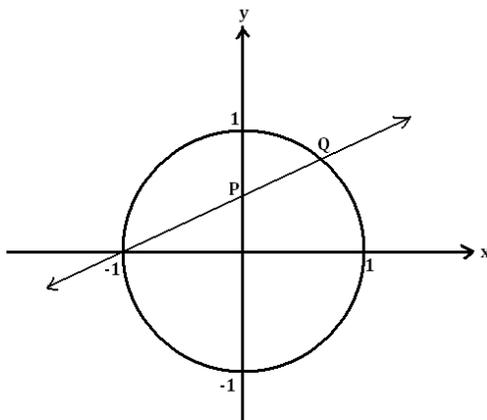
We need to solve  $x^2 + 2x + 2 = 0$ .

$$\begin{aligned} x^2 + 2x + 1 &= -1 \\ (x+1)^2 &= -1 \\ x+1 &= \pm i \\ x &= -1 \pm i \end{aligned}$$

- e. Write  $f$  as a product of three linear factors.

$$f(x) = (x+2)(x - (-1+i))(x - (-1-i))$$

3. A line passes through the points  $(-1, 0)$  and  $P = (0, t)$  for some real number  $t$  and intersects the circle  $x^2 + y^2 = 1$  at a point  $Q$  different from  $(-1, 0)$ .



- a. If  $t = \frac{1}{2}$ , so that the point  $P$  has coordinates  $(0, \frac{1}{2})$ , find the coordinates of the point  $Q$ .

The slope of  $\overleftrightarrow{PQ} = t = \frac{1}{2}$

Line  $\overleftrightarrow{PQ}$  has equation  $y = \frac{1}{2}(x+1)$

Point  $Q$  lies on the line  $y = \frac{1}{2}(x+1)$  and the circle  $x^2 + y^2 = 1$ .

$$\text{So, } x^2 + (\frac{1}{2}(x+1))^2 = 1$$

$$x^2 + \frac{1}{4}(x^2 + 2x + 1) = 1$$

$$4x^2 + x^2 + 2x + 1 = 4$$

$$5x^2 + 2x + 1 = 4$$

$$25x^2 + 10x + 5 = 20$$

$$25x^2 + 10x + 1 = 16$$

$$(5x+1)^2 = 16$$

$$5x+1 = 4 \text{ or } 5x+1 = -4$$

$$5x = 3 \text{ or } -5$$

$$x = \frac{3}{5} \text{ or } -1$$

Since  $Q$  is in the first quadrant, choose  $x = \frac{3}{5}$ .

$$\text{Then } y = \frac{1}{2}(\frac{3}{5} + 1) = \frac{4}{5}.$$

The point  $Q$  is  $(\frac{3}{5}, \frac{4}{5})$ .

A Pythagorean triple is a set of three positive integers  $a$ ,  $b$ , and  $c$  satisfying  $a^2 + b^2 = c^2$ . For example, setting  $a = 3$ ,  $b = 4$ , and  $c = 5$  gives a Pythagorean triple.

- b. Suppose that  $(\frac{a}{c}, \frac{b}{c})$  is a point with rational-number coordinates lying on the circle  $x^2 + y^2 = 1$ . Explain why then  $a$ ,  $b$ , and  $c$  form a Pythagorean triple.

We have  $(\frac{a}{c})^2 + (\frac{b}{c})^2 = 1$ .  
 That is  $\frac{a^2 + b^2}{c^2} = 1$ . Thus  $a^2 + b^2 = c^2$ .  
 If  $a$ ,  $b$ , and  $c$  are integers, this is a Pythagorean triple.

- c. Which Pythagorean triple is associated with the point  $Q = (\frac{5}{13}, \frac{12}{13})$  on the circle?

5, 12, 13

- d. If  $Q = (\frac{5}{13}, \frac{12}{13})$ , what is the value of  $t$  so that the point  $P$  has coordinates  $(0, t)$ ?

Slope  $\overline{PQ} = \frac{t-0}{0-(-1)} = t$ , using points  $(-1, 0)$  and  $(0, t)$ .

Using points  $(\frac{5}{13}, \frac{12}{13})$  and  $(-1, 0)$ , slope  $\overline{PQ} = \frac{\frac{12}{13} - 0}{\frac{5}{13} - (-1)}$   
 $= \frac{\frac{12}{13}}{\frac{18}{13}} = \frac{2}{3}$ .

Thus,  $t = \frac{2}{3}$ .

- e. Suppose we set  $x = \frac{1-t^2}{1+t^2}$  and  $y = \frac{2t}{1+t^2}$ , for a real number  $t$ . Show that  $(x, y)$  is then a point on the circle  $x^2 + y^2 = 1$ .

We need to show that  $\left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2$  equals 1.

$$\begin{aligned} \left(\frac{1-t^2}{1+t^2}\right)^2 + \left(\frac{2t}{1+t^2}\right)^2 &= \frac{1-2t^2+t^2+4t^2}{(1+t^2)^2} \\ &= \frac{t^4+2t^2+1}{(1+t^2)^2} \\ &= \frac{(t^2+1)^2}{(t^2+1)^2} = 1 \end{aligned}$$

We're good!

- f. Set  $t = \frac{3}{4}$  in the formulas  $x = \frac{1-t^2}{1+t^2}$  and  $y = \frac{2t}{1+t^2}$ . Which point on the circle  $x^2 + y^2 = 1$  does this give? What is the associated Pythagorean triple?

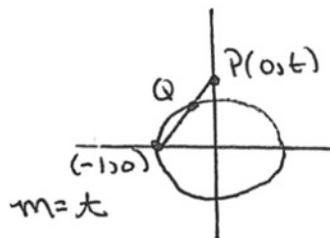
$$\text{For } t = \frac{3}{4}, x = \frac{1 - \frac{9}{16}}{1 + \frac{9}{16}} = \frac{\frac{7}{16}}{\frac{25}{16}} = \frac{7}{25}$$

$$\text{and } y = \frac{2\left(\frac{3}{4}\right)}{1 + \frac{9}{16}} = \frac{\frac{6}{4}}{\frac{25}{16}} = \frac{24}{25}$$

So  $(x, y)$  is  $\left(\frac{7}{25}, \frac{24}{25}\right)$

and the Pythagorean triple is 7, 24, 25.

- g. Suppose  $t$  is a value greater than 1,  $P = (0, t)$ , and  $Q$  is the point in the second quadrant (different from  $(-1, 0)$ ) at which the line through  $(-1, 0)$  and  $P$  intersects the circle  $x^2 + y^2 = 1$ . Find the coordinates of the point  $Q$  in terms of  $t$ .



Line  $\overline{PQ}$  has equation  $y = t(x+1)$ .

Point  $Q$  lies on the line  $y = t(x+1)$

and the circle  $x^2 + y^2 = 1$ .

So,  $x^2 + t^2(x+1)^2 = 1$ . Solving for  $x$ :

$$x^2 + t^2(x^2 + 2x + 1) = 1$$

$$(1+t^2)x^2 + 2t^2x + t^2 - 1 = 0$$

$$x = \frac{-2t^2 \pm \sqrt{4t^4 - 4(1+t^2)(t^2-1)}}{2(1+t^2)} = \frac{-t^2 \pm \sqrt{t^4 - (t^2-1)}}{1+t^2} = \frac{-t^2 \pm 1}{1+t^2}$$

$x = \frac{1-t^2}{1+t^2}$  or  $x = -1$ . Since we are looking for a point different than  $P$ , we choose  $x = \frac{1-t^2}{1+t^2}$

Substituting back into the equation of line  $\overline{PQ}$ ,

$$y = t \left( \frac{1-t^2}{1+t^2} + 1 \right)$$

$$= t \left( \frac{1-t^2+1+t^2}{1+t^2} \right)$$

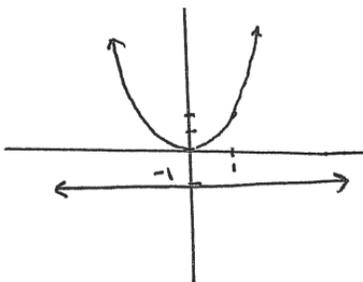
$$= \frac{2t}{1+t^2}$$

The point  $Q$  is  $\left( \frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2} \right)$ .

- 4.
- a. Write a system of two equations in two variables where one equation is quadratic and the other is linear such that the system has no solution. Explain, using graphs, algebra, and/or words, why the system has no solution.

$$y = x^2$$

$$y = -1$$



From the graph, these two curves do not intersect, and so there is no solution to this system of equations.

- b. Prove that  $x = \sqrt{-5x - 6}$  has no solution.

If  $x = \sqrt{-5x - 6}$  holds for some number  $x$ , then  $x^2 = -5x - 6$  would hold for that number, too.

$$\text{That is, } x^2 + 5x + 6 = 0$$

$$(x+3)(x+2) = 0$$

$$x = -3 \text{ or } x = -2$$

But,  $x = -3$  does not work:  $-3 \neq \sqrt{15-6}$   
 and  $x = -2$  does not work:  $-2 \neq \sqrt{10-6}$ .  
 So there is no solution after all.

- c. Does the following system of equations have a solution? If so, find one. If not, explain why not.

$$\begin{array}{l} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{array} \quad \begin{array}{l} 2x + y + z = 4 \\ x - y + 3z = -2 \\ -x + y + z = -2 \end{array}$$

$$\textcircled{1} - \textcircled{3} \Rightarrow 3x = 6 \\ x = 2$$

$$\textcircled{1} \quad 2(2) + y + z = 4 \Rightarrow y + z = 0$$

$$\textcircled{2} \quad 2 - y + 3z = -2 \Rightarrow -y + 3z = -4$$

$$\textcircled{3} \quad -2 + y + z = -2 \Rightarrow y + z = 0$$

$$\textcircled{1} + \textcircled{2} \Rightarrow 4z = -4 \\ z = -1 \Rightarrow y = 1$$

$$\text{check: } 4 + 1 - 1 = 4 \checkmark$$

$$2 - 1 - 3 = -2 \checkmark$$

$$-2 + 1 - 1 = -2 \checkmark$$

The solution is  $(2, 1, -1)$ .