Lesson 1: Successive Differences in Polynomials

Classwork

Opening Exercise

John noticed patterns in the arrangement of numbers in the table below.

______	2.4		3.4		4.4		5.4		6.4
______	5.76		11.56		19.36		29.16		40.96
________		5.8		7.8		9.8		11.8	
___________			2		2		2		

Assuming that the pattern would continue, he used it to find the value of 7.4^2. Explain how he used the pattern to find 7.4^2, and then use the pattern to find 8.4^2.

How would you label each row of numbers in the table?

Discussion

Let the sequence $\{a_0, a_1, a_2, a_3, \ldots\}$ be generated by evaluating a polynomial expression at the values $0, 1, 2, 3, \ldots$. The numbers found by evaluating $a_1 - a_0$, $a_2 - a_1$, $a_3 - a_2, \ldots$ form a new sequence which we will call the first differences of the polynomial. The differences between successive terms of the first differences sequence are called the second differences, and so on.

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Example 1

What is the sequence of first differences for the linear polynomial given by $ax + b$, where a and b are constant coefficients?

What is the sequence of second differences for $ax + b$?

Example 2

Find the first, second, and third differences of the polynomial $ax^2 + bx + c$ by filling in the blanks in the following table.

x	$ax^2 + bx + c$	First Differences	Second Differences	Third Differences
0	c			
1	$a + b + c$			
2	$4a + 2b + c$			
3	$9a + 3b + c$			
4	$16a + 4b + c$			
5	$25a + 5b + c$			

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Example 3

Find the second, third, and fourth differences of the polynomial $ax^3 + bx^2 + cx + d$ by filling in the blanks in the following table.

x	$ax^3 + bx^2 + cx + d$	First Differences	Second Differences	Third Differences	Fourth Differences
0	d				
		$a + b + c$			
1	$a + b + c + d$				
		$7a + 3b + c$			
2	$8a + 4b + 2c + d$				
		$19a + 5b + c$			
3	$27a + 9b + 3c + d$				
		$37a + 7b + c$			
4	$64a + 16b + 4c + d$				
		$61a + 9b + c$			
5	$125a + 25b + 5c + d$				

Example 4

What type of relationship does the set of ordered pairs (x, y) satisfy? How do you know? Fill in the blanks in the table below to help you decide. (The first differences have already been computed for you.)

x	y	First Differences	Second Differences	Third Differences
0	2			
		-1		
1	1			
		5		
2	6			
		17		
3	23			
		35		
4	58			
		59		
5	117			

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Find the equation of the form $y = ax^3 + bx^2 + cx + d$ that all ordered pairs (x, y) above satisfy. Give evidence that your equation is correct.

Relevant Vocabulary

Numerical Symbol: A *numerical symbol* is a symbol that represents a specific number. Examples: $1, 2, 3, 4, \pi, -3.2$.

Variable Symbol: A *variable symbol* is a symbol that is a placeholder for a number from a specified set of numbers. The set of numbers is called the *domain of the variable.* Examples: x, y, z.

Algebraic Expression: An *algebraic expression* is either

1. a numerical symbol or a variable symbol or
2. the result of placing previously generated algebraic expressions into the two blanks of one of the four operators ((__)+(__), (__)−(__), (__)×(__), (__)÷(__)) or into the base blank of an exponentiation with an exponent that is a rational number.

Following the definition above, $\left(\left((x) \times (x)\right) \times (x)\right) + \left((3) \times (x)\right)$ is an algebraic expression, but it is generally written more simply as $x^3 + 3x$.

Numerical Expression: A *numerical expression* is an algebraic expression that contains only numerical symbols (no variable symbols) that evaluates to a single number. Example: The numerical expression $\frac{(3 \cdot 2)^2}{12}$ evaluates to 3.

Monomial: A *monomial* is an algebraic expression generated using only the multiplication operator (__×__). The expressions x^3 and $3x$ are both monomials.

Binomial: A *binomial* is the sum of two monomials. The expression $x^3 + 3x$ is a binomial.

Polynomial Expression: A *polynomial expression* is a monomial or sum of two or more monomials.

Sequence: A *sequence* can be thought of as an ordered list of elements. The elements of the list are called the *terms of the sequence.*

Arithmetic Sequence: A sequence is called *arithmetic* if there is a real number d such that each term in the sequence is the sum of the previous term and d.

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Problem Set

1. Create a table to find the second differences for the polynomial $36 - 16t^2$ for integer values of t from 0 to 5.

2. Create a table to find the third differences for the polynomial $s^3 - s^2 + s$ for integer values of s from -3 to 3.

3. Create a table of values for the polynomial x^2, using $n, n+1, n+2, n+3, n+4$ as values of x. Show that the second differences are all equal to 2.

4. Show that the set of ordered pairs (x, y) in the table below satisfies a quadratic relationship. (Hint: Find second differences.) Find the equation of the form $y = ax^2 + bx + c$ that all of the ordered pairs satisfy.

x	0	1	2	3	4	5
y	5	4	-1	-10	-23	-40

5. Show that the set of ordered pairs (x, y) in the table below satisfies a cubic relationship. (Hint: Find third differences.) Find the equation of the form $y = ax^3 + bx^2 + cx + d$ that all of the ordered pairs satisfy.

x	0	1	2	3	4	5
y	20	4	0	20	76	180

6. The distance d ft. required to stop a car traveling at $10v$ mph under dry asphalt conditions is given by the following table.

v	0	1	2	3	4	5
d	0	5	19.5	43.5	77	120

   a. What type of relationship is indicated by the set of ordered pairs?
   b. Assuming that the relationship continues to hold, find the distance required to stop the car when the speed reaches 60 mph, when $v = 6$.
   c. (Challenge) Find an equation that describes the relationship between the speed of the car v and its stopping distance d.

7. Use the polynomial expressions $5x^2 + x + 1$ and $2x + 3$ to answer the questions below.
   a. Create a table of second differences for the polynomial $5x^2 + x + 1$ for the integer values of x from 0 to 5.
   b. Justin claims that for $n \geq 2$, the n^{th} differences of the sum of a degree n polynomial and a linear polynomial are the same as the n^{th} differences of just the degree n polynomial. Find the second differences for the sum $(5x^2 + x + 1) + (2x + 3)$ of a degree 2 and a degree 1 polynomial and use the calculation to explain why Justin might be correct in general.
   c. Jason thinks he can generalize Justin's claim to the product of two polynomials. He claims that for $n \geq 2$, the $(n+1)^{th}$ differences of the product of a degree n polynomial and a linear polynomial are the same as the n^{th} differences of the degree n polynomial. Use what you know about second and third differences (from Examples 2 and 3) and the polynomial $(5x^2 + x + 1)(2x + 3)$ to show that Jason's generalization is incorrect.

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