Lesson 10: The Power of Algebra—Finding Pythagorean Triples

Classwork

Opening Exercise

Sam and Jill decide to explore a city. Both begin their walk from the same starting point.

- Sam walks 1 block north, 1 block east, 3 blocks north, and 3 blocks west.
- Jill walks 4 blocks south, 1 block west, 1 block north, and 4 blocks east.

If all city blocks are the same length, who is the farthest distance from the starting point?

Example 1

Prove that if $x > 1$, then a triangle with side lengths $x^2 - 1$, $2x$, and $x^2 + 1$ is a right triangle.

Example 2

Next we describe an easy way to find Pythagorean triples using the expressions from Example 1. Look at the multiplication table below for $\{1, 2, \ldots, 9\}$. Notice that the square numbers $\{1, 4, 9, \ldots, 81\}$ lie on the diagonal of this table.

a. What value of x is used to generate the Pythagorean triple $(15,8,17)$ by the formula $(x^2 - 1,\ 2x,\ x^2 + 1)$? How do the numbers $(1, 4, 4, 16)$ at the corners of the shaded square in the table relate to the values 15, 8, and 17?

×	1	2	3	4	5	6	7	8	9	10
1	**1**	2	3	**4**	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	**4**	8	12	**16**	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

b. Now you try one. Form a square on the multiplication table below whose left-top corner is the 1 (as in the example above) and whose bottom-right corner is a square number. Use the sums or differences of the numbers at the vertices of your square to form a Pythagorean triple. Check that the triple you generate is a Pythagorean triple.

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

Let's generalize this square to any square in the multiplication table where two opposite vertices of the square are square numbers.

c. How can you use the sums or differences of the numbers at the vertices of the shaded square to get a triple $(16, 30, 34)$? Is this a Pythagorean triple?

×	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	**9**	12	**15**	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	**15**	20	**25**	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

d. Using x instead of 5 and y instead of 3 in your calculations in part (c), write down a formula for generating Pythagorean triples in terms of x and y.

Relevant Facts and Vocabulary

Pythagorean Theorem: If a right triangle has legs of length a and b units and hypotenuse of length c units, then $a^2 + b^2 = c^2$.

Converse to the Pythagorean Theorem: If the lengths a, b, c of the sides of a triangle are related by $a^2 + b^2 = c^2$, then the angle opposite the side of length c is a right angle.

Pythagorean Triple: A *Pythagorean triple* is a triplet of positive integers (a, b, c) such that $a^2 + b^2 = c^2$. The triplet $(3, 4, 5)$ is a Pythagorean triple but $(1, 1, \sqrt{2})$ is not, even though the numbers are side lengths of an isosceles right triangle.

Problem Set

1. Rewrite each expression as a sum or difference of terms.
   a. $(x-3)(x+3)$
   b. $(x^2-3)(x^2+3)$
   c. $(x^{15}+3)(x^{15}-3)$
   d. $(x-3)(x^2+9)(x+3)$
   e. $(x^2+y^2)(x^2-y^2)$
   f. $(x^2+y^2)^2$
   g. $(x-y)^2(x+y)^2$
   h. $(x-y)^2(x^2+y^2)^2(x+y)^2$

2. Tasha used a clever method to expand and simplify $(a+b+c)(a+b-c)$. She grouped the addends together like this $[(a+b)+c][(a+b)-c]$ and then expanded them to get the difference of two squares:
$$(a+b+c)(a+b-c) = [(a+b)+c][(a+b)-c] = (a+b)^2 - c^2 = a^2 + 2ab + b^2 - c^2.$$
   a. Is Tasha's method correct? Explain why or why not.
   b. Use a version of her method to find $(a+b+c)(a-b-c)$.
   c. Use a version of her method to find $(a+b-c)(a-b+c)$.

3. Use the difference of two squares identity to factor each of the following expressions.
   a. x^2-81
   b. $(3x+y)^2-(2y)^2$
   c. $4-(x-1)^2$
   d. $(x+2)^2-(y+2)^2$

4. Show that the expression $(x+y)(x-y)-6x+9$ may be written as the difference of two squares, and then factor the expression.

5. Show that $(x+y)^2-(x-y)^2 = 4xy$ for all real numbers x and y.

6. Prove that a triangle with side lengths $2xy$, x^2-y^2, and x^2+y^2 with $x > y > 0$ is a right triangle.

7. Complete the table below to find Pythagorean triples (the first row is done for you).

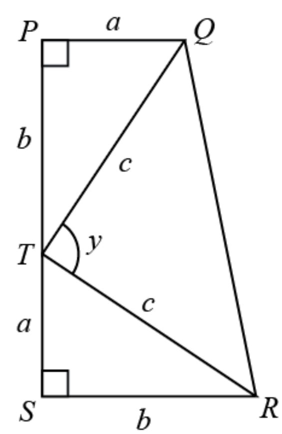
x	y	x^2-y^2	$2xy$	x^2+y^2	Check: Is it a Pythagorean Triple?
2	1	3	4	5	Yes: $3^2+4^2=25$, $5^2=25$
3	1				
3	2				
4	1				
4	2				
4	3				
5	1				

8. Answer the following parts about the triple $(9,12,15)$.
   a. Show that $(9, 12, 15)$ is a Pythagorean triple.
   b. Prove that neither $(9, 12, 15)$ nor $(12,9,15)$ can be found by choosing a pair of integers x and y with $x > y$ and computing $(x^2 - y^2, 2xy, x^2 + y^2)$.
   (Hint: What are the possible values of x and y if $2xy = 12$? What about if $2xy = 9$?)
   c. Wouldn't it be nice if all Pythagorean triples were generated by the expressions $x^2 - y^2$, $2xy$, $x^2 + y^2$? Research Pythagorean triples on the Internet to discover what is known to be true about generating all Pythagorean triples using these three expressions.

9. Follow the steps below to prove the identity $\left(a^2 + b^2\right)(x^2 + y^2) = (ax - by)^2 + (bx + ay)^2$.
   a. Multiply $\left(a^2 + b^2\right)(x^2 + y^2)$.
   b. Square both binomials in $(ax - by)^2 + (bx + ay)^2$ and collect like terms.
   c. Use your answers from part (a) and part (b) to prove the identity.

10. Many U.S. presidents took great delight in studying mathematics. For example, President James Garfield, while still a congressman, came up with a proof of the Pythagorean Theorem based upon the ideas presented below,

    In the diagram, two congruent right triangles with side lengths a, b, and hypotenuse c, are used to form a trapezoid $PQRS$ composed of three triangles.



   a. Explain why $\angle QTR$ is a right angle.
   b. What are the areas of ΔSTR, ΔPTQ, and ΔQTR in terms of a, b, and c?
   c. Using the formula for the area of a trapezoid, what is the total area of trapezoid $PQRS$ in terms of a and b?
   d. Set the sum of the areas of the three triangles from part (b) equal to the area of the trapezoid you found in part (c), and simplify the equation to derive a relationship between a, b, and c. Conclude that a right triangle with legs of length a and b and hypotenuse of length c must satisfy the relationship $a^2 + b^2 = c^2$.