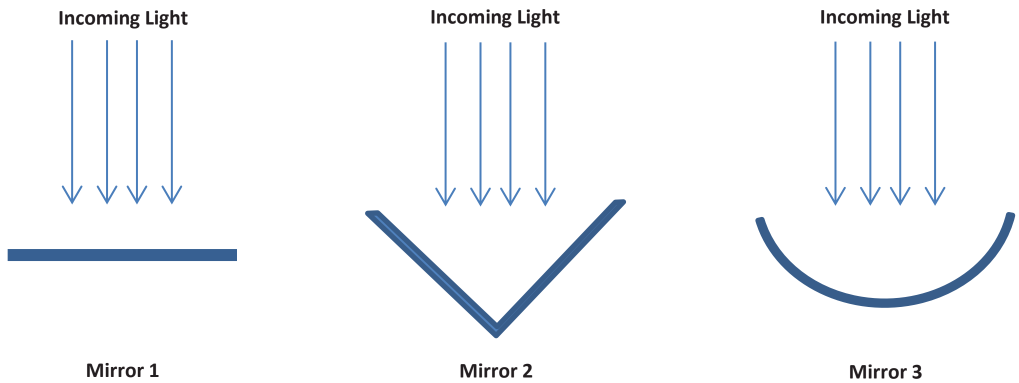
Lesson 33: The Definition of a Parabola

Classwork

Opening Exercise

Suppose you are viewing the cross-section of a mirror. Where would the incoming light be reflected in each type of design? Sketch your ideas below.



Discussion

When Newton designed his reflector telescope he understood two important ideas. Figure 1 shows a diagram of this type of telescope.

- The curved mirror needs to focus all the light to a single point that we will call the focus. An angled flat mirror is placed near this point and reflects the light to the eyepiece of the telescope.
- The reflected light needs to arrive at the focus at the same time, otherwise the image is distorted.

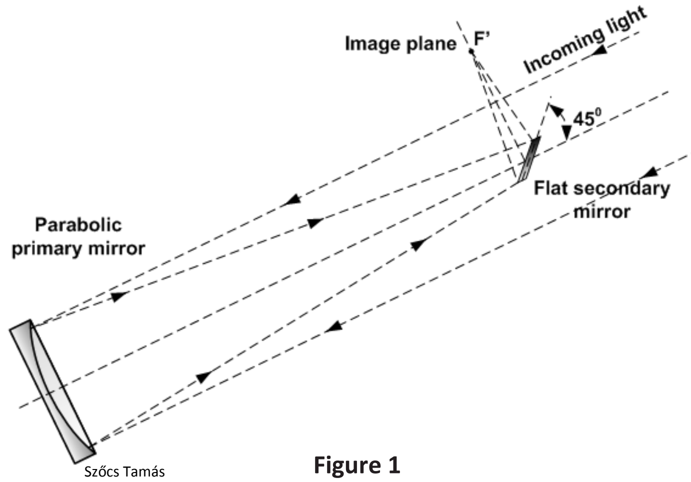


Figure 1

Definition: A *parabola with directrix* L *and focus point* F is the set of all points in the plane that are equidistant from the point F and line L.

Figure 2 to the right illustrates this definition of a parabola. In this diagram, $FP_1 = P_1Q_1$, $FP_2 = P_2Q_2$, $FP_3 = P_3Q_3$ showing that for any point P on the parabola, the distance between P and F is equal to the distance between P and the line L.

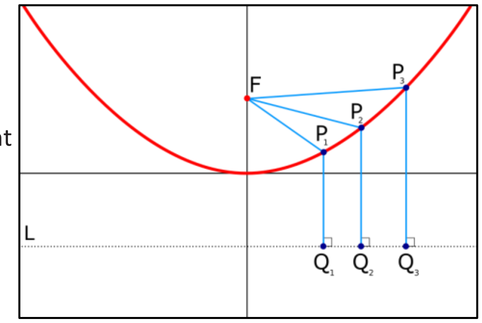


Figure 2

All parabolas have the reflective property illustrated in Figure 3 Rays parallel to the axis will reflect off the parabola and through the focus point, F.

Thus, a mirror shaped like a rotated parabola would satisfy Newton's requirements for his telescope design.

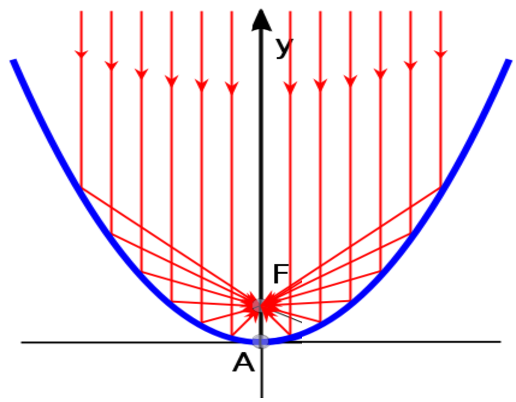


Figure 3

Figure 4 below shows several different line segments representing the reflected light with one endpoint on the curved mirror that is a parabola and the other endpoint at the focus. Anywhere the light hits this type of curved surface, it always reflects to the focus, F, at exactly the same time.

Figure 5 shows the same image with a directrix. Imagine for a minute that the mirror was not there. Then, the light would arrive at the directrix all at the same time. Since the distance from each point on the parabolic mirror to the directrix is the same as the distance from the point on the mirror to the focus, and the speed of light is constant, it takes the light the same amount of time to travel to the focus as it would have taken it to travel to the directrix. In the diagram, this means that $AF = AF_A$, $BF = BF_B$, and so on. Thus, the light rays arrive at the focus at the same time, and the image is not distorted.

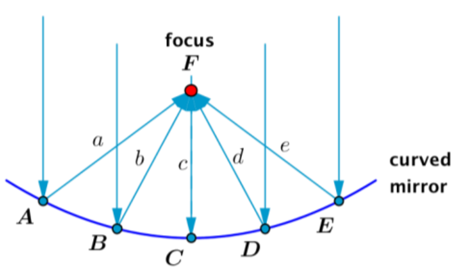


Figure 4

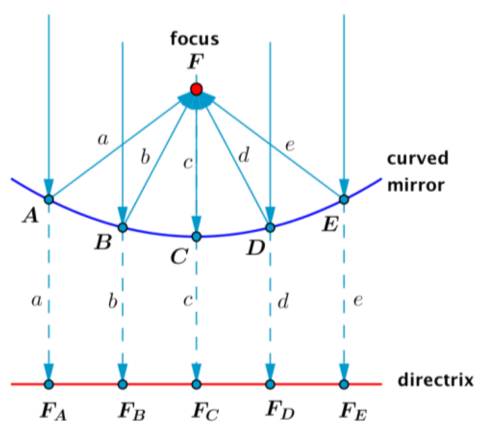
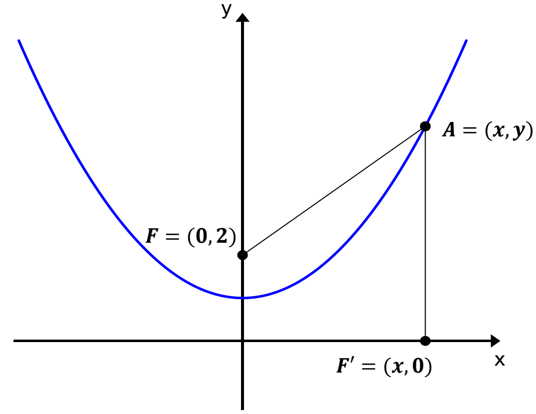


Figure 5

Example 1

Given a focus and a directrix, create an equation for a parabola.



Focus: $F = (0,2)$

Directrix: x-axis

Parabola:
$P = \{(x, y) |\ (x, y)$ is equidistant to F and to the x-axis.$\}$

Let A be any point (x, y) on the parabola P. Let F' be a point on the directrix with the same x-coordinate as point A.

What is the length of AF'?

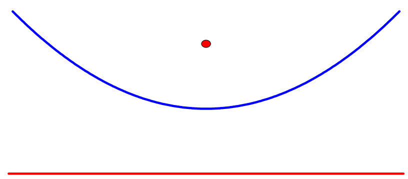
Use the distance formula to create an expression that represents the length of AF.

Create an equation that relates the two lengths and solve it for y.

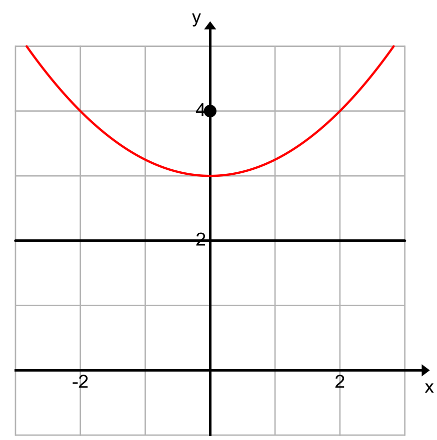
Verify that this equation appears to match the graph shown.

Exercises 1–2

1. Demonstrate your understanding of the definition of a parabola by drawing several pairs of congruent segments given the parabola, its focus, and directrix. Measure the segments that you drew to confirm the accuracy of your sketches in either centimeters or inches.



2. Derive the analytic equation of a parabola given the focus of $(0,4)$ and the directrix $y = 2$. Use the diagram to help you work this problem.



   a. Label a point (x, y) anywhere on the parabola.

   b. Write an expression for the distance from the point (x, y) to the directrix.

   c. Write an expression for the distance from the point (x, y) to the focus.

   d. Apply the definition of a parabola to create an equation in terms of x and y. Solve this equation for y.

   e. What is the translation that takes the graph of this parabola to the graph of the equation derived in Example 1?

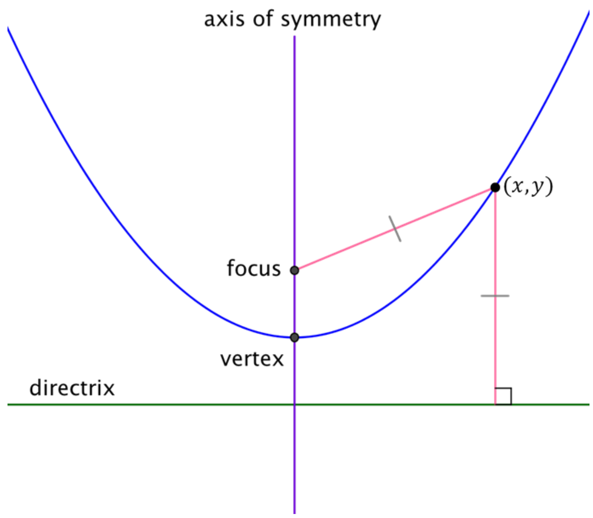
Lesson Summary

Parabola: A parabola with directrix line L and focus point F is the set of all points in the plane that are equidistant from the point F and line L

Axis of symmetry: The axis of symmetry of a parabola given by a focus point and a directrix is the perpendicular line to the directrix that passes through the focus

Vertex of a parabola: The vertex of a parabola is the point where the axis of symmetry intersects the parabola.

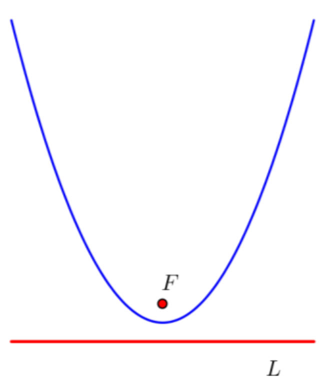
In the Cartesian plane, the distance formula can help us to derive an analytic equation for the parabola.



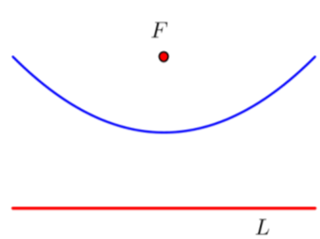
Problem Set

1. Demonstrate your understanding of the definition of a parabola by drawing several pairs of congruent segments given each parabola, its focus, and directrix. Measure the segments that you drew in either inches or centimeters to confirm the accuracy of your sketches.

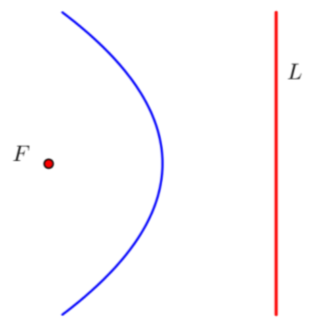
a.



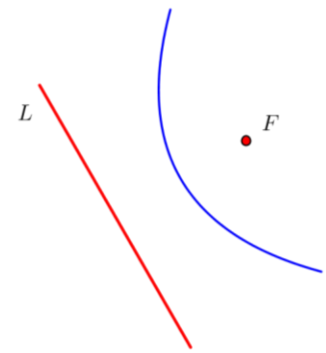
b.



c.



d.



2. Find the distance from the point $(4,2)$ to the point $(0,1)$.

3. Find the distance from the point $(4,2)$ to the line $y = -2$.

4. Find the distance from the point $(-1,3)$ to the point $(3,-4)$.

5. Find the distance from the point $(-1,3)$ to the line $y = 5$.

6. Find the distance from the point $(x, 4)$ to the line $y = -1$.

7. Find the distance from the point $(x, -3)$ to the line $y = 2$.

8. Find the values of x for which the point $(x, 4)$ is equidistant from $(0,1)$ and the line $y = -1$.

9. Find the values of x for which the point $(x, -3)$ is equidistant from $(1, -2)$ and the line $y = 2$.

10. Consider the equation $y = x^2$.
    a. Find the coordinates of the three points on the graph of $y = x^2$ whose x-values are 1, 2, and 3.
    b. Show that each of the three points in part (a) is equidistant from the point $\left(0, \frac{1}{4}\right)$ and the line $y = -\frac{1}{4}$.
    c. Show that if the point with coordinates (x, y) is equidistant from the point $\left(0, \frac{1}{4}\right)$, and the line $y = -\frac{1}{4}$, then $y = x^2$.

11. Given the equation $y = \frac{1}{2}x^2 - 2x$,

    a. Find the coordinates of the three points on the graph of $y = \frac{1}{2}x^2 - 2x$ whose x-values are -2, 0, and 4.

    b. Show that each of the three points in part (a) is equidistant from the point $\left(2, -\frac{3}{2}\right)$, and the line $y = -\frac{5}{2}$.

    c. Show that if the point with coordinates (x, y) is equidistant from the point $\left(2, -\frac{3}{2}\right)$ and the line $y = -\frac{5}{2}$ then $y = \frac{1}{2}x^2 - 2x$.

12. Derive the analytic equation of a parabola with focus $(1,3)$ and directrix $y = 1$. Use the diagram to help you work this problem.

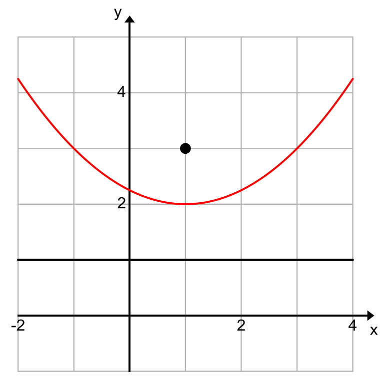
    a. Label a point (x, y) anywhere on the parabola.

    b. Write an expression for the distance from the point (x, y) to the directrix.

    c. Write an expression for the distance from the point (x, y) to the focus $(1,3)$.

    d. Apply the definition of a parabola to create an equation in terms of x and y. Solve this equation for y.

    e. Describe a sequence of transformations that would take this parabola to the parabola with equation $y = \frac{1}{4}x^2 + 1$ derived in Example 1.



13. Consider a parabola with focus $(0, -2)$ and directrix on the x-axis.

    a. Derive the analytic equation for this parabola.

    b. Describe a sequence of transformations that would take the parabola with equation $y = \frac{1}{4}x^2 + 1$ derived in Example 1 to the graph of the parabola in part (a).

14. Derive the analytic equation of a parabola with focus $(0,10)$ and directrix on the x-axis.