Lesson 16: Proving Trigonometric Identities

Classwork

Opening Exercise

Which of these statements is a trigonometric identity? Provide evidence to support your claim.

Statement 1: $\sin^2(\theta) = 1 - \cos^2(\theta)$ for θ any real number.

Statement 2: $1 - \cos(\theta) = 1 - \cos(\theta)$ for θ any real number.

Statement 3: $1 - \cos(\theta) = 1 + \cos(\theta)$ for θ any real number.

Using Statements 1 and 2, create a third identity, Statement 4, whose left side is $\frac{\sin^2(\theta)}{1-\cos(\theta)}$.

For which values of θ is this statement valid?

Discuss in pairs what it might mean to "prove" an identity. What might it take to prove, for example, that the following statement is an identity?

$$\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta) \text{ where } \theta \neq 2\pi k, \text{ for all integers } k.$$

To prove an identity, you have to use logical steps to show that one side of the equation in the identity can be transformed into the other side of the equation using already established identities such as the Pythagorean identity or the properties of operation (commutative, associative, and distributive properties). It is not correct to start with what you want to prove and work on both sides of the equation at the same time, as the following exercise shows.

Exercise

Take out your calculators and quickly graph the equations $y = \sin(x) + \cos(x)$ and $y = -\sqrt{1 + 2\sin(x)\cos(x)}$ to determine whether $\sin(\theta) + \cos(\theta) = -\sqrt{1 + 2\sin(\theta)\cos(\theta)}$ for all θ for which both functions are defined is a valid identity. You should see from the graphs that the functions are not equivalent.

Suppose that Charles did not think to graph the equations to see if the given statement was a valid identity, so he set about proving the identity using algebra and a previous identity. His argument is shown below.

First, [1] $\sin(\theta) + \cos(\theta) = -\sqrt{1 + 2\sin(\theta)\cos(\theta)}$ for θ any real number.

Now, using the multiplication property of equality, square both sides, which gives

[2] $\sin^2(\theta) + 2\sin(\theta)\cos(\theta) + \cos^2(\theta) = 1 + 2\sin(\theta)\cos(\theta)$ for θ any real number.

Using the subtraction property of equality, subtract $2\sin(\theta)\cos(\theta)$ from each side, which gives

[3] $\sin^2(\theta) + \cos^2(\theta) = 1$ for θ any real number.

Statement [3] is the Pythagorean identity. So, replace $\sin^2(\theta) + \cos^2(\theta)$ by 1 to get

[4] $1 = 1$, which is definitely true.

Therefore, the original statement must be true.

Does this mean that Charles has proven that Statement [1] is an identity? Discuss with your group whether it is a valid proof. If you decide it is not a valid proof, then discuss with your group how and where his argument went wrong.

Example 1: Two Proofs of our New Identity

Work through these two different ways to approach proving the identity $\frac{\sin^2(\theta)}{1-\cos(\theta)} = 1 + \cos(\theta)$ where $\theta \neq 2\pi k$, for integers k. The proofs make use of some of the following properties of equality and real numbers. Here *a*, *b*, and *c* stand for arbitrary real numbers.

Reflexive property of equality	$a = a$
Symmetric property of equality	If $a = b$, then $b = a$.
Transitive property of equality	If $a = b$ and $b = c$, then $a = c$.
Addition property of equality	If $a = b$, then $a + c = b + c$.
Subtraction property of equality	If $a = b$, then $a - c = b - c$.
Multiplication property of equality	If $a = b$, then $a \times c = b \times c$.
Division property of equality	If $a = b$ and $c \neq 0$, then $a \div c = b \div c$.
Substitution property of equality	If $a = b$, then b may be substituted for a in any expression containing a.
Associative properties	$(a + b) + c = a + (b + c)$ and $a(bc) = (ab)c$.
Commutative properties	$a + b = b + a$ and $ab = ba$.
Distributive property	$a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

Fill in the missing parts of the proofs below.

A. We start with Statement 1 from the opening activity and divide both sides by the same expression, $1 - \cos(\theta)$. This step will introduce division by zero when $1 - \cos(\theta) = 0$ and will change the set of values of θ for which the identity is valid.

PROOF:

Step	Left Side of Equation		Equivalent Right Side	Domain	Reason
1	$\sin^2(\theta) + \cos^2(\theta)$	$=$	1	θ any real number	Pythagorean identity
2	$\sin^2(\theta)$	$=$	$1 - \cos^2(\theta)$	θ any real number	
3		$=$	$(1 - \cos(\theta))(1 + \cos(\theta))$	θ any real number	
4		$=$	$\frac{(1 - \cos(\theta))(1 + \cos(\theta))}{1 - \cos(\theta)}$		
5	$\frac{\sin^2(\theta)}{1 - \cos(\theta)}$	$=$		$\theta \neq 2\pi k$ for all integers k	Substitution property of equality using $\frac{1-\cos(\theta)}{1-\cos(\theta)} = 1$

B. Or, we can start with the more complicated side of the identity we want to prove and use algebra and prior trigonometric definitions and identities to transform it to the other side. In this case, the more complicated expression is $\frac{\sin^2(\theta)}{1-\cos(\theta)}$.

PROOF:

Step	Left Side of Equation		Equivalent Right Side	Domain	Reason
1	$\frac{\sin^2(\theta)}{1-\cos(\theta)}$	$=$	$\frac{1-\cos^2(\theta)}{1-\cos(\theta)}$	$\theta \neq 2\pi k$ for all integers k	Substitution property of equality using $\sin^2(\theta) = 1-\cos^2(\theta)$
2		$=$	$\frac{(1-\cos(\theta))(1+\cos(\theta))}{1-\cos(\theta)}$		Distributive property
3	$\frac{\sin^2(\theta)}{1-\cos(\theta)}$	$=$	$1+\cos(\theta)$		

Exercises 1–2

Prove that the following are trigonometric identities, beginning with the side of the equation that seems to be more complicated and starting the proof by restricting x to values where the identity is valid. Make sure that the complete identity statement is included at the end of the proof.

1. $\tan(x) = \frac{\sec(x)}{\csc(x)}$ for real numbers $x \neq \frac{\pi}{2} + \pi k$, for all integers k.

2. $\cot(x) + \tan(x) = \sec(x)\csc(x)$ for all real $x \neq \frac{\pi}{2}n$ for integer n.

Problem Set

1. Does $\sin(x+y)$ equal $\sin(x)+\sin(y)$ for all real numbers x and y?
   a. Find each of the following: $\sin\left(\frac{\pi}{2}\right)$, $\sin\left(\frac{\pi}{4}\right)$, $\sin\left(\frac{3\pi}{4}\right)$.
   b. Are $\sin\left(\frac{\pi}{2}+\frac{\pi}{4}\right)$ and $\sin\left(\frac{\pi}{2}\right)+\sin\left(\frac{\pi}{4}\right)$ equal?
   c. Are there any values of x and y for which $\sin(x+y)=\sin(x)+\sin(y)$?

2. Use $\tan(x)=\frac{\sin(x)}{\cos(x)}$ and identities involving the sine and cosine functions to establish the following identities for the tangent function. Identify the values of x where the equation is an identity.
   a. $\tan(\pi-x)=\tan(x)$
   b. $\tan(x+\pi)=\tan(x)$
   c. $\tan(2\pi-x)=-\tan(x)$
   d. $\tan(-x)=-\tan(x)$

3. Rewrite each of the following expressions as a single term. Identify the values of theta for which the original expression and your expression are equal:
   a. $\cot(\theta)\sec(\theta)\sin(\theta)$
   b. $\left(\frac{1}{1-\sin(x)}\right)\left(\frac{1}{1+\sin(x)}\right)$
   c. $\frac{1}{\cos^2(x)}-\frac{1}{\cot^2(x)}$
   d. $\frac{(\tan(x)-\sin(x))(1+\cos(x))}{\sin^3(x)}$

4. Prove that for any two real numbers a and b,
$$\sin^2(a)-\sin^2(b)+\cos^2(a)\sin^2(b)-\sin^2(a)\cos^2(b)=0.$$

5. Prove that the following statements are identities for all values of θ for which both sides are defined, and describe that set.
   a. $\cot(\theta)\sec(\theta)=\csc(\theta)$
   b. $\left(\csc(\theta)+\cot(\theta)\right)\left(1-\cos(\theta)\right)=\sin(\theta)$
   c. $\tan^2(\theta)-\sin^2(\theta)=\tan^2(\theta)\sin^2(\theta)$
   d. $\frac{4+\tan^2(x)-\sec^2(x)}{\csc^2(x)}=3\sin^2(x)$
   e. $\frac{(1+\sin(\theta))^2+\cos^2(\theta)}{1+\sin(\theta)}=2$

6. Prove that the value of the following expression does not depend on the value of y:
$$\cot(y)\frac{\tan(x)+\tan(y)}{\cot(x)+\cot(y)}.$$