Lesson 1: Integer Exponents

Classwork

Opening Exercise

Can you fold a piece of notebook paper in half 10 times?

How thick will the folded paper be?

Will the area of the paper on the top of the folded stack be larger or smaller than a postage stamp?

Exploratory Challenge

a. What are the dimensions of your paper?

b. How thick is one sheet of paper? Explain how you decided on your answer.

c. Describe how you folded the paper.

d. Record data in the following table based on the size and thickness of your paper.

Number of Folds	0	1	2	3	4	5	6	7	8	9	10
Thickness of the Stack (in.)											
Area of the Top of the Stack (sq. in.)											

e. Were you able to fold a piece of notebook paper in half 10 times? Why or why not?

f. Create a formula that approximates the height of the stack after n folds.

g. Create a formula that will give you the approximate area of the top after n folds.

h. Answer the original questions from the Opening Exercise. How do the actual answers compare to your original predictions?

Example 1: Using the Properties of Exponents to Rewrite Expressions

The table below displays the thickness and area of a folded square sheet of gold foil. In 2001, Britney Gallivan, a California high school junior, successfully folded this size sheet of gold foil in half 12 times to earn extra credit in her mathematics class.

Rewrite each of the table entries as a multiple of a power of 2.

Number of Folds	Thickness of the Stack (Millionths of a Meter)	Thickness Using a Power of 2	Area of the Top (Square Inches)	Area Using a Power of 2
0	0.28	$0.28 \cdot 2^0$	100	$100 \cdot 2^0$
1	0.56	$0.28 \cdot 2^1$	50	$100 \cdot 2^{-1}$
2	1.12		25	
3	2.24		12.5	
4	4.48		6.25	
5	8.96		3.125	
6	17.92		1.5625	

Example 2: Applying the Properties of Exponents to Rewrite Expressions

Rewrite each expression in the form of kx^n where k is a real number, n is an integer, and x is a nonzero real number.

a. $5x^5 \cdot -3x^2$

b. $\frac{3x^5}{(2x)^4}$

c. $\frac{3}{(x^2)^{-3}}$

d. $\frac{x^{-3}x^4}{x^8}$

Exercises 1–5

Rewrite each expression in the form of kx^n where k is a real number and n is an integer. Assume $x \neq 0$.

1. $2x^5 \cdot x^{10}$

2. $\frac{1}{3x^8}$

3. $\frac{6x^{-5}}{x^{-3}}$

4. $\left(\frac{3}{x^{-22}}\right)^{-3}$

5. $\left(x^2\right)^n \cdot x^3$

Lesson Summary

The Properties of Exponents

For real numbers x and y with $x \neq 0$, $y \neq 0$, and all integers a and b, the following properties hold:

$$x^a \cdot x^b = x^{a+b}$$

$$(x^a)^b = x^{ab}$$

$$(xy)^a = x^a y^a$$

$$\frac{1}{x^a} = x^{-a}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a}$$

$$x^0 = 1$$

Problem Set

1. Suppose your class tried to fold an unrolled roll of toilet paper. It was originally 4 in. wide and 30 ft. long. Toilet paper is approximately 0.002 in. thick.

   a. Complete each table and represent the area and thickness using powers of 2.

Number of Folds n	Thickness After n Folds (inches)
0	
1	
2	
3	
4	
5	
6	

Number of Folds n	Area on Top After n Folds (square inches)
0	
1	
2	
3	
4	
5	
6	

   b. Create an algebraic function that describes the area in square inches after n folds.

   c. Create an algebraic function that describes the thickness in inches after n folds.

2. In the Exit Ticket, we saw the formulas below. The first formula determines the minimum width, W, of a square piece of paper of thickness T needed to fold it in half n times, alternating horizontal and vertical folds. The second formula determines the minimum length, L, of a long rectangular piece of paper of thickness T needed to fold it in half n times, always folding perpendicular to the long side.

$$W = \pi \cdot T \cdot 2^{\frac{3(n-1)}{2}} \qquad L = \frac{\pi T}{6}(2^n + 4)(2^n - 1)$$

Use the appropriate formula to verify why it is possible to fold a 10 inch by 10 inch sheet of gold foil in half 13 times. Use 0.28 millionths of a meter for the thickness of gold foil.

3. Use the formula from the Exit Ticket to determine if you can fold an unrolled roll of toilet paper in half more than 10 times. Assume that the thickness of a sheet of toilet paper is approximately 0.002 in. and that one roll is 102 ft. long.

4. Apply the properties of exponents to rewrite expressions in the form kx^n, where n is an integer and $x \neq 0$.

   a. $(2x^3)(3x^5)(6x)^2$
   b. $\frac{3x^4}{(-6x)^{-2}}$
   c. $\frac{x^{-3}x^5}{3x^4}$
   d. $5(x^3)^{-3}(2x)^{-4}$
   e. $\left(\frac{x^2}{4x^{-1}}\right)^{-3}$

5. Apply the properties of exponents to verify that each statement is an identity.

   a. $\frac{2^{n+1}}{3^n} = 2\left(\frac{2}{3}\right)^n$ for integer values of n.
   b. $3^{n+1} - 3^n = 2 \cdot 3^n$ for integer values of n.
   c. $\frac{1}{(3^n)^2} \cdot \frac{4^n}{3} = \frac{1}{3}\left(\frac{2}{3}\right)^{2n}$ for integer values of n.

6. Jonah was trying to rewrite expressions using the properties of exponents and properties of algebra for nonzero values of x. In each problem, he made a mistake. Explain where he made a mistake in each part and provide a correct solution.

Jonah's Incorrect Work
a. $(3x^2)^{-3} = -9x^{-6}$
b. $\frac{2}{3x^5} = 6x^{-5}$
c. $\frac{2x - x^3}{3x} = \frac{2}{3} - x^3$

7. If $x = 5a^4$, and $a = 2b^3$, express x in terms of b.

8. If $a = 2b^3$, and $b = -\frac{1}{2}c^{-2}$, express a in terms of c.

9. If $x = 3y^4$, and $y = \frac{s}{2x^3}$, show that $s = 54y^{13}$.

10. Do the following without a calculator:
    a. Express 8^3 as a power of 2.
    b. Divide 4^{15} by 2^{10}.

11. Use powers of 2 to help you perform each calculation.
    a. $\frac{2^7 \cdot 2^5}{16}$
    b. $\frac{512000}{320}$

12. Write the first five terms of each of the following recursively-defined sequences:
    a. $a_{n+1} = 2a_n$, $a_1 = 3$
    b. $a_{n+1} = (a_n)^2$, $a_1 = 3$
    c. $a_{n+1} = 2(a_n)^2$, $a_1 = x$, where x is a real number. Write each term in the form kx^n.
    d. $a_{n+1} = 2(a_n)^{-1}$, $a_1 = y$, $(y \neq 0)$. Write each term in the form kx^n.

13. In Module 1, you established the identity
    $(1 - r)(1 + r + r^2 + \cdots + r^{n-1}) = 1 - r^n$, where r is a real number and n is a positive integer.

    Use this identity to find explicit formulas as specified below.
    a. Rewrite the given identity to isolate the sum $1 + r + r^2 + \cdots r^{n-1}$ for $r \neq 1$.
    b. Find an explicit formula for $1 + 2 + 2^2 + 2^3 + \cdots + 2^{10}$.
    c. Find an explicit formula for $1 + a + a^2 + a^3 + \cdots + a^{10}$ in terms of powers of a.
    d. Jerry simplified the sum $1 + a + a^2 + a^3 + a^4 + a^5$ by writing $1 + a^{15}$. What did he do wrong?
    e. Find an explicit formula for $1 + 2a + (2a)^2 + (2a)^3 + \cdots + (2a)^{12}$ in terms of powers of a.
    f. Find an explicit formula for $3 + 3(2a) + 3(2a)^2 + 3(2a)^3 + \ldots + 3(2a)^{12}$ in terms of powers of a. Hint: Use part (e).
    g. Find an explicit formula for $P + P(1 + r) + P(1 + r)^2 + P(1 + r)^3 + \cdots + P(1 + r)^{n-1}$ in terms of powers $(1 + r)$.

Lesson 2: Base 10 and Scientific Notation

Classwork

Opening Exercise

In the last lesson, you worked with the thickness of a sheet of gold foil (a very small number) and some very large numbers that gave the size of a piece of paper that actually could be folded in half more than 13 times.

a. Convert 0.28 millionths of a meter to centimeters and express your answer as a decimal number.

b. The length of a piece of square notebook paper that could be folded in half 13 times was 3294.2 in. Use this number to calculate the area of a square piece of paper that could be folded in half 14 times. Round your answer to the nearest million.

c. Sort the following numbers into products and single numeric expressions. Then match the equivalent expressions without using a calculator.

3.5×10^5	-6	-6×10^0	0.6	3.5×10^{-6}
$3{,}500{,}000$	$350{,}000$	6×10^{-1}	0.0000035	3.5×10^6

Example 1

Write each number as a product of a decimal number between 1 and 10 and a power of 10.

a. $234{,}000$

b. 0.0035

c. $532{,}100{,}000$

d. 0.0000000012

e. 3.331

Exercises 1–6

For Exercises 1–6, write each decimal in scientific notation.

1. $532{,}000{,}000$

2. 0.0000000000000000123 (16 zeros after the decimal place)

3. $8{,}900{,}000{,}000{,}000{,}000$ (14 zeros after the 9)

4. 0.00003382

5. $34{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000$ (24 zeros after the 4)

6. 0.0000000000000000000004 (21 zeros after the decimal place)

Exercises 7–8

7. Approximate the average distances between the Sun and Earth, Jupiter, and Pluto. Express your answers in scientific notation ($d \times 10^n$), where d is rounded to the nearest tenth.
   a. Sun to Earth:

   b. Sun to Jupiter:

   c. Sun to Pluto:

   d. Earth to Jupiter:

   e. Jupiter to Pluto:

8. Order the numbers in Exercise 7 from smallest to largest. Explain how writing the numbers in scientific notation helps you to quickly compare and order them.

Example 2: Arithmetic Operations with Numbers Written Using Scientific Notation

a. $(2.4 \times 10^{20}) + (4.5 \times 10^{21})$

b. $(7 \times 10^{-9})(5 \times 10^{5})$

c. $\frac{1.2 \times 10^{15}}{3 \times 10^{7}}$

Exercises 9–11

9. Perform the following calculations without rewriting the numbers in decimal form.

a. $(1.42 \times 10^{15}) - (2 \times 10^{13})$

b. $(1.42 \times 10^{15})(2.4 \times 10^{13})$

c. $\frac{1.42 \times 10^{-5}}{2 \times 10^{13}}$

10. Estimate how many times farther Jupiter is from the Sun than Earth is from the Sun. Estimate how many times farther Pluto is from the Sun than Earth is from the Sun.

11. Estimate the distance between Earth and Jupiter and between Jupiter and Pluto.

Problem Set

1. Write the following numbers used in these statements in scientific notation. (Note: Some of these numbers have been rounded.)
   a. The density of helium is 0.0001785 grams per cubic centimeter.
   b. The boiling point of gold is $5200°\text{F}$.
   c. The speed of light is $186{,}000$ miles per second.
   d. One second is 0.000278 hours.
   e. The acceleration due to gravity on the Sun is $900\ \text{ft/s}^2$.
   f. One cubic inch is 0.0000214 cubic yards.
   g. Earth's population in 2012 was $7{,}046{,}000{,}000$ people.
   h. Earth's distance from the sun is $93{,}000{,}000$ miles.
   i. Earth's radius is 4000 miles.
   j. The diameter of a water molecule is $0.000000028\ \text{cm}$.

2. Write the following numbers in decimal form. (Note: Some of these numbers have been rounded.)
   a. A light year is $9.46 \times 10^{15}\ \text{m}$.
   b. Avogadro's number is $6.02 \times 10^{23}\ \text{mol}^{-1}$.
   c. The universal gravitational constant is 6.674×10^{-11} N(m/kg)^2.
   d. Earth's age is 4.54×10^9 years.
   e. Earth's mass is $5.97 \times 10^{24}\ \text{kg}$.
   f. A foot is 1.9×10^{-4} miles.
   g. The population of China in 2014 was 1.354×10^9 people.
   h. The density of oxygen is 1.429×10^{-4} grams per liter.
   i. The width of a pixel on a smartphone is $7.8 \times 10^{-2}\ \text{mm}$.
   j. The wavelength of light used in optic fibers is $1.55 \times 10^{-6}\ \text{m}$.

3. State the necessary value of n that will make each statement true.
   a. $0.000027 = 2.7 \times 10^n$
   b. $-3.125 = -3.125 \times 10^n$
   c. $7{,}540{,}000{,}000 = 7.54 \times 10^n$
   d. $0.033 = 3.3 \times 10^n$
   e. $15 = 1.5 \times 10^n$
   f. $26{,}000 \times 200 = 5.2 \times 10^n$
   g. $3000 \times 0.0003 = 9 \times 10^n$
   h. $0.0004 \times 0.002 = 8 \times 10^n$
   i. $\frac{16000}{80} = 2 \times 10^n$
   j. $\frac{500}{0.002} = 2.5 \times 10^n$

Perform the following calculations without rewriting the numbers in decimal form.

k. $(2.5 \times 10^4) + (3.7 \times 10^3)$

l. $(6.9 \times 10^{-3}) - (8.1 \times 10^{-3})$

m. $(6 \times 10^{11})(2.5 \times 10^{-5})$

n. $\frac{4.5 \times 10^8}{2 \times 10^{10}}$

4. The wavelength of visible light ranges from 650 nanometers to 850 nanometers, where $1 \text{ nm} = 1 \times 10^{-7} \text{ cm}$. Express the wavelength range of visible light in centimeters.

5. In 1694, the Dutch scientist Antonie van Leeuwenhoek was one of the first scientists to see a red blood cell in a microscope. He approximated that a red blood cell was "$25{,}000$ times as small as a grain of sand." Assume a grain of sand is $\frac{1}{2}$ millimeter wide and a red blood cell is approximately 7 micrometers wide. One micrometer is 1×10^{-6} meters. Support or refute Leeuwenhoek's claim. Use scientific notation in your calculations.

6. When the Mars Curiosity Rover entered the atmosphere of Mars on its descent in 2012, it was traveling roughly $13{,}200$ mph. On the surface of Mars, its speed averaged 0.00073 mph. How many times faster was the speed when it entered the atmosphere than its typical speed on the planet's surface? Use scientific notation in your calculations.

7. Earth's surface is approximately 70% water. There is no water on the surface of Mars, and its diameter is roughly half of Earth's diameter. Assume both planets are spherical. The surface area of a sphere is given by the formula $SA = 4\pi r^2$ where r is the radius of the sphere. Which has more land mass, Earth or Mars? Use scientific notation in your calculations.

8. There are approximately 25 trillion (2.5×10^{13}) red blood cells in the human body at any one time. A red blood cell is approximately 7×10^{-6} m wide. Imagine if you could line up all your red blood cells end to end. How long would the line of cells be? Use scientific notation in your calculations.

9. Assume each person needs approximately 100 sq. ft. of living space. Now imagine that we are going to build a giant apartment building that will be 1 mile wide and 1 mile long to house all the people in the United States, estimated to be 313.9 million people in 2012. If each floor of the apartment building is 10 ft. high, how tall will the apartment building be?

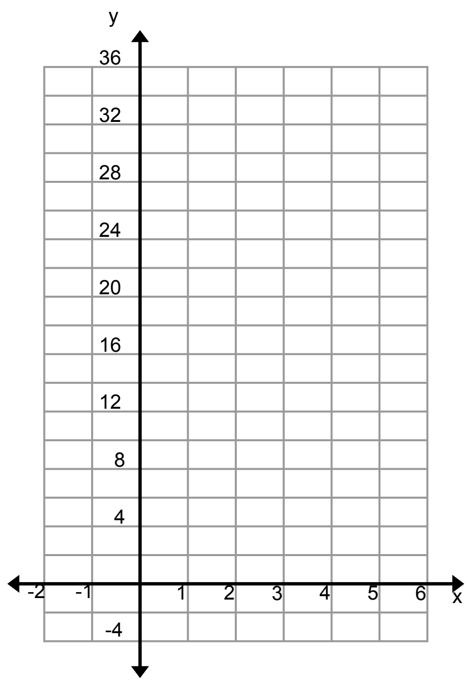
Lesson 3: Rational Exponents—What Are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$?

Classwork

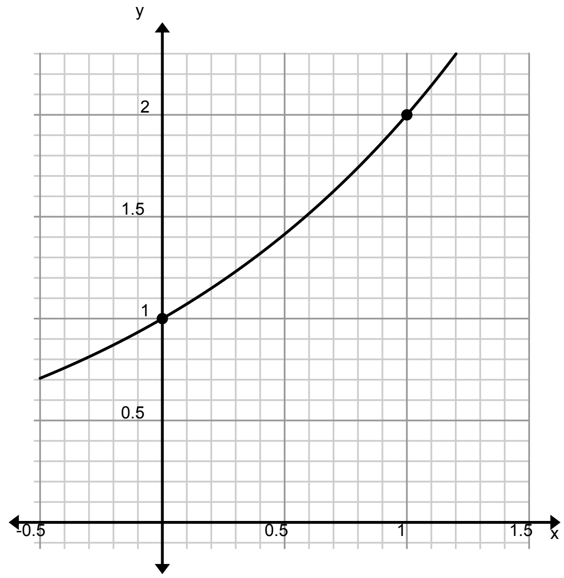
Opening Exercise

a. What is the value of $2^{\frac{1}{2}}$? Justify your answer.

b. Graph $f(x) = 2^x$ for each integer x from $x = -2$ to $x = 5$. Connect the points on your graph with a smooth curve.



The graph at right shows a close-up view of $f(x) = 2^x$ for $-0.5 < x < 1.5$.



c. Find two consecutive integers that are over and under estimates of the value of $2^{\frac{1}{2}}$.

d. Does it appear that $2^{\frac{1}{2}}$ is halfway between the integers you specified in Exercise 1?

e. Use the graph of $f(x) = 2^x$ to estimate the value of $2^{\frac{1}{2}}$.

f. Use the graph of $f(x) = 2^x$ to estimate the value of $2^{\frac{1}{3}}$.

Example 1

a. What is the 4^{th} root of 16?

b. What is the cube root of 125?

c. What is the 5^{th} root of $100{,}000$?

Exercise 1

Evaluate each expression.

a. $\sqrt[4]{81}$

b. $\sqrt[5]{32}$

c. $\sqrt[3]{9} \cdot \sqrt[3]{3}$

d. $\sqrt[4]{25} \cdot \sqrt[4]{100} \cdot \sqrt[4]{4}$

Discussion

If $2^{\frac{1}{2}} = \sqrt{2}$ and $2^{\frac{1}{3}} = \sqrt[3]{2}$, what does $2^{\frac{3}{4}}$ equal? Explain your reasoning.

Exercises 2–8

Rewrite each exponential expression as an n^{th} root.

2. $3^{\frac{1}{2}}$

3. $11^{\frac{1}{5}}$

4. $\left(\frac{1}{4}\right)^{\frac{1}{5}}$

5. $6^{\frac{1}{10}}$

Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.

6. $2^{\frac{3}{2}}$

7. $4^{\frac{5}{2}}$

8. $\left(\frac{1}{8}\right)^{\frac{5}{3}}$

9. Show why the following statement is true:

$$2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}}$$

Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.

10. $4^{-\frac{3}{2}}$

11. $27^{-\frac{2}{3}}$

12. $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$

Lesson Summary

n^{TH} ROOT OF A NUMBER: Let a and b be numbers, and let $n \geq 2$ be a positive integer. If $b = a^n$, then a is an n^{th} *root of* b. If $n = 2$, then the root is a called a *square root*. If $n = 3$, then the root is called a *cube root*.

PRINCIPAL n^{TH} ROOT OF A NUMBER: Let b be a real number that has at least one real n^{th} root. The *principal* n^{th} *root of* b is the real n^{th} root that has the same sign as b and is denoted by a radical symbol: $\sqrt[n]{b}$.

Every positive number has a unique principal n^{th} root. We often refer to the principal n^{th} root of b as just the n^{th} *root of* b. The n^{th} root of 0 is 0.

For any positive integers m and n, and any real number b for which the principal n^{th} root of b exists, we have

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = \left(\sqrt[n]{b}\right)^m$$

$$b^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{b^m}}.$$

Problem Set

1. Select the expression from (A), (B), and (C) that correctly completes the statement.

		(A)	(B)	(C)
a.	$x^{\frac{1}{3}}$ is equivalent to ________.	$\frac{1}{3}x$	$\sqrt[3]{x}$	$\frac{3}{x}$
b.	$x^{\frac{2}{3}}$ is equivalent to ________.	$\frac{2}{3}x$	$\sqrt[3]{x^2}$	$\left(\sqrt{x}\right)^3$
c.	$x^{-\frac{1}{4}}$ is equivalent to ________.	$-\frac{1}{4}x$	$\frac{4}{x}$	$\frac{1}{\sqrt[4]{x}}$
d.	$\left(\frac{4}{x}\right)^{\frac{1}{2}}$ is equivalent to ________.	$\frac{2}{x}$	$\frac{4}{x^2}$	$\frac{2}{\sqrt{x}}$

2. Identify which of the expressions (A), (B), and (C) are equivalent to the given expression.

		(A)	(B)	(C)
a.	$16^{\frac{1}{2}}$	$\left(\frac{1}{16}\right)^{-\frac{1}{2}}$	$8^{\frac{2}{3}}$	$64^{\frac{3}{2}}$
b.	$\left(\frac{2}{3}\right)^{-1}$	$-\frac{3}{2}$	$\left(\frac{9}{4}\right)^{\frac{1}{2}}$	$\frac{27^{\frac{1}{3}}}{6}$

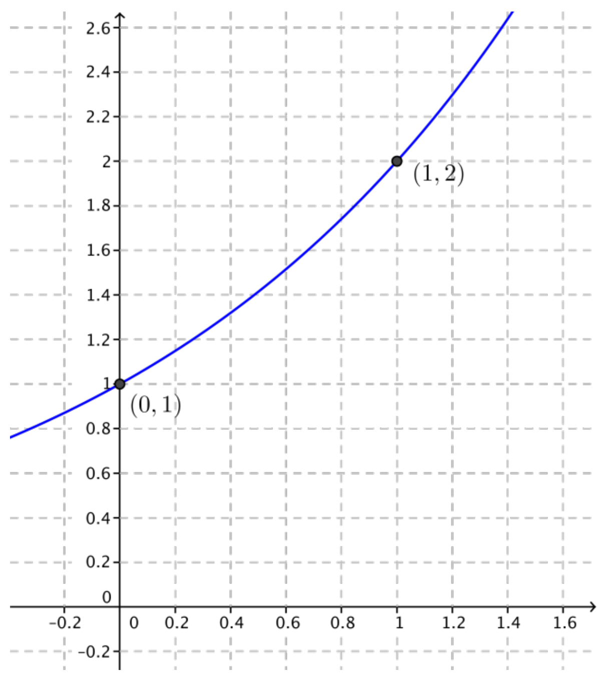
3. Rewrite in radical form. If the number is rational, write it without using radicals.

   a. $6^{\frac{3}{2}}$
   b. $\left(\frac{1}{2}\right)^{\frac{1}{4}}$
   c. $3(8)^{\frac{1}{3}}$
   d. $\left(\frac{64}{125}\right)^{-\frac{2}{3}}$
   e. $81^{-\frac{1}{4}}$

4. Rewrite the following expressions in exponent form.

   a. $\sqrt{5}$
   b. $\sqrt[3]{5^2}$
   c. $\sqrt{5^3}$
   d. $\left(\sqrt[3]{5}\right)^2$

5. Use the graph of $f(x) = 2^x$ shown to the right to estimate the following powers of 2.



   a. $2^{\frac{1}{4}}$
   b. $2^{\frac{2}{3}}$
   c. $2^{\frac{3}{4}}$
   d. $2^{0.2}$
   e. $2^{1.2}$
   f. $2^{-\frac{1}{5}}$

6. Rewrite each expression in the form kx^n, where k is a real number, x is a positive real number, and n is rational.

   a. $\sqrt[4]{16x^3}$
   b. $\frac{5}{\sqrt{x}}$
   c. $\sqrt[3]{1/x^4}$
   d. $\frac{4}{\sqrt[3]{8x^3}}$
   e. $\frac{27}{\sqrt{9x^4}}$
   f. $\left(\frac{125}{x^2}\right)^{-\frac{1}{3}}$

7. Find a value of x for which $2x^{\frac{1}{2}} = 32$.

8. Find a value of x for which $x^{\frac{4}{3}} = 81$.

9. If $x^{\frac{3}{2}} = 64$, find the value of $4x^{-\frac{3}{4}}$.

10. If $b = \frac{1}{9}$, evaluate the following expressions.

   a. $b^{-\frac{1}{2}}$
   b. $b^{\frac{5}{2}}$
   c. $\sqrt[3]{3b^{-1}}$

11. Show that each expression is equivalent to $2x$. Assume x is a positive real number.

   a. $\sqrt[4]{16x^4}$
   b. $\frac{\left(\sqrt[3]{8x^3}\right)^2}{\sqrt{4x^2}}$
   c. $\frac{6x^3}{\sqrt[3]{27x^6}}$

12. Yoshiko said that $16^{\frac{1}{4}} = 4$ because 4 is one-fourth of 16. Use properties of exponents to explain why she is or is not correct.

13. Jefferson said that $8^{\frac{4}{3}} = 16$ because $8^{\frac{1}{3}} = 2$ and $2^4 = 16$. Use properties of exponents to explain why he is or is not correct.

14. Rita said that $8^{\frac{2}{3}} = 128$ because $8^{\frac{2}{3}} = 8^2 \cdot 8^{\frac{1}{3}}$, so $8^{\frac{2}{3}} = 64 \cdot 2$, and then $8^{\frac{2}{3}} = 128$. Use properties of exponents to explain why she is or is not correct.

15. Suppose for some positive real number a that $\left(a^{\frac{1}{4}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{4}}\right)^2 = 3$.
    a. What is the value of a?
    b. Which exponent properties did you use to find your answer to part (a)?

16. In the lesson, you made the following argument:

$$\begin{aligned}\left(2^{\frac{1}{3}}\right)^3 &= 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \\ &= 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \\ &= 2^1 \\ &= 2.\end{aligned}$$

Since $\sqrt[3]{2}$ is a number so that $\left(\sqrt[3]{2}\right)^3 = 2$ and $2^{\frac{1}{3}}$ is a number so that $\left(2^{\frac{1}{3}}\right)^3 = 2$, you concluded that $2^{\frac{1}{3}} = \sqrt[3]{2}$. Which exponent property was used to make this argument?

Lesson 4: Properties of Exponents and Radicals

Classwork

Opening Exercise

Write each exponent as a radical, and then use the definition and properties of radicals to write that expression as an integer.

a. $7^{\frac{1}{2}} \cdot 7^{\frac{1}{2}}$

b. $3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}} \cdot 3^{\frac{1}{3}}$

c. $12^{\frac{1}{2}} \cdot 3^{\frac{1}{2}}$

d. $\left(64^{\frac{1}{3}}\right)^{\frac{1}{2}}$

Examples 1–3

Write each expression in the form $b^{\frac{m}{n}}$ for positive real numbers b and integers m and n with $n > 0$ by applying the properties of radicals and the definition of n^{th} root.

1. $b^{\frac{1}{4}} \cdot b^{\frac{1}{4}}$

2. $b^{\frac{1}{3}} \cdot b^{\frac{4}{3}}$

3. $b^{\frac{1}{5}} \cdot b^{\frac{3}{4}}$

Exercises 1–4

Write each expression in the form $b^{\frac{m}{n}}$. If a numeric expression is a rational number, then write your answer without exponents.

1. $b^{\frac{2}{3}} \cdot b^{\frac{1}{2}}$

2. $\left(b^{-\frac{1}{5}}\right)^{\frac{2}{3}}$

3. $64^{\frac{1}{3}} \cdot 64^{\frac{3}{2}}$

4. $\left(\frac{9^3}{4^2}\right)^{\frac{3}{2}}$

Example 4

Rewrite the radical expression $\sqrt{48x^5y^4z^2}$ so that no perfect square factors remain inside the radical.

Exercise 5

5. If $x = 50$, $y = 12$, and $z = 3$, the following expressions are difficult to evaluate without using properties of radicals or exponents (or a calculator). Use the definition of rational exponents and properties of exponents to rewrite each expression in a form where it can be easily evaluated, and then use that exponential expression to find the value.

   a. $\sqrt{8x^3y^2}$

   b. $\sqrt[3]{54y^7z^2}$

Exercise 6

6. Order these numbers from smallest to largest. Explain your reasoning.

   $16^{2.5}$ $9^{3.1}$ $32^{1.2}$

Lesson Summary

The properties of exponents developed in Grade 8 for integer exponents extend to rational exponents.

That is, for any integers m, n, p, and q, with $n > 0$ and $q > 0$ and any real numbers a and b so that $a^{\frac{1}{n}}$, $b^{\frac{1}{n}}$, and $b^{\frac{1}{q}}$ are defined, we have the following properties of exponents:

$$b^{\frac{m}{n}} \cdot b^{\frac{p}{q}} = b^{\frac{m}{n}+\frac{p}{q}}$$

$$b^{\frac{m}{n}} = \sqrt[n]{b^m}$$

$$\left(b^{\frac{1}{n}}\right)^n = b$$

$$(b^n)^{\frac{1}{n}} = b$$

$$(ab)^{\frac{m}{n}} = a^{\frac{m}{n}} \cdot b^{\frac{m}{n}}$$

$$\left(b^{\frac{m}{n}}\right)^{\frac{p}{q}} = b^{\frac{mp}{nq}}$$

$$b^{-\frac{m}{n}} = \frac{1}{b^{\frac{m}{n}}}$$

Problem Set

1. Evaluate each expression if $a = 27$ and $b = 64$.

   a. $\sqrt[3]{a}\sqrt{b}$

   b. $\left(3\sqrt[3]{a}\sqrt{b}\right)^2$

   c. $\left(\sqrt[3]{a} + 2\sqrt{b}\right)^2$

   d. $a^{-\frac{2}{3}} + b^{\frac{3}{2}}$

   e. $\left(a^{-\frac{2}{3}} \cdot b^{\frac{3}{2}}\right)^{-1}$

   f. $\left(a^{-\frac{2}{3}} - \frac{1}{8}b^{\frac{3}{2}}\right)^{-1}$

2. Rewrite each expression so that each term is in the form kx^n, where k is a real number, x is a positive real number, and n is a rational number.

   a. $x^{-\frac{2}{3}} \cdot x^{\frac{1}{3}}$

   b. $2x^{\frac{1}{2}} \cdot 4x^{-\frac{5}{2}}$

   c. $\frac{10x^{\frac{1}{3}}}{2x^2}$

   d. $\left(3x^{\frac{1}{4}}\right)^{-2}$

   e. $x^{\frac{1}{2}}\left(2x^2 - \frac{4}{x}\right)$

   f. $\sqrt[3]{\frac{27}{x^6}}$

   g. $\sqrt[3]{x} \cdot \sqrt[3]{-8x^2} \cdot \sqrt[3]{27x^4}$

   h. $\frac{2x^4 - x^2 - 3x}{\sqrt{x}}$

   i. $\frac{\sqrt{x} - 2x^{-3}}{4x^2}$

3. Show that $\left(\sqrt{x}+\sqrt{y}\right)^2$ is not equal to x^1+y^1 when $x=9$ and $y=16$.

4. Show that $\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)^{-1}$ is not equal to $\frac{1}{x^{\frac{1}{2}}}+\frac{1}{y^{\frac{1}{2}}}$ when $x=9$ and $y=16$.

5. From these numbers, select (a) one that is negative, (b) one that is irrational, (c) one that is not a real number, and (d) one that is a perfect square:

$$3^{\frac{1}{2}}\cdot 9^{\frac{1}{2}},\ 27^{\frac{1}{3}}\cdot 144^{\frac{1}{2}},\ 64^{\frac{1}{3}}-64^{\frac{2}{3}},\text{ and }\left(4^{-\frac{1}{2}}-4^{\frac{1}{2}}\right)^{\frac{1}{2}}$$

6. Show that the expression $2^n\cdot 4^{n+1}\cdot\left(\frac{1}{8}\right)^n$ is equal to 4.

7. Express each answer as a power of 10.
   a. Multiply 10^n by 10.
   b. Multiply $\sqrt{10}$ by 10^n.
   c. Square 10^n.
   d. Divide $100\cdot 10^n$ by 10^{2n}.
   e. Show that $10^n=11\cdot 10^n-10^{n+1}$

8. Rewrite each of the following radical expressions as an equivalent exponential expression in which each variable occurs no more than once.
   a. $\sqrt{8x^2y}$
   b. $\sqrt[5]{96x^3y^{15}z^6}$

9. Use properties of exponents to find two integers that are upper and lower estimates of the value of $4^{1.6}$.

10. Use properties of exponents to find two integers that are upper and lower estimates of the value of $8^{2.3}$.

11. Kepler's third law of planetary motion relates the average distance, a, of a planet from the Sun to the time t it takes the planet to complete one full orbit around the Sun according to the equation $t^2=a^3$. When the time, t, is measured in Earth years, the distance, a, is measured in astronomical units (AU). (One AU is equal to the average distance from Earth to the Sun.)
   a. Find an equation for t in terms of a and an equation for a in terms of t.
   b. Venus takes about 0.616 Earth years to orbit the Sun. What is its average distance from the Sun?
   c. Mercury is an average distance of 0.387 AU from the Sun. About how long is its orbit in Earth years?

Lesson 5: Irrational Exponents—What Are $2^{\sqrt{2}}$ and 2^{π}?

Classwork

Exercise 1

a. Write the following finite decimals as fractions (you do not need to reduce to lowest terms).

$$1, \quad 1.4, \quad 1.41, \quad 1.414, \quad 1.4142, \quad 1.41421$$

b. Write $2^{1.4}$, $2^{1.41}$, $2^{1.414}$, and $2^{1.4142}$ in radical form ($\sqrt[n]{2^m}$).

c. Compute a decimal approximation to 5 decimal places of the radicals you found in part (b) using your calculator. For each approximation, underline the digits that are also in the previous approximation, starting with 2.00000 done for you below. What do you notice?

$$2^1 = 2 = 2.00000$$

Exercise 2

a. Write six terms of a sequence that a calculator can use to approximate 2^{π}.
(Hint: $\pi = 3.14159 \ldots$)

b. Compute $2^{3.14} = \sqrt[100]{2^{314}}$ and 2^{π} on your calculator. In which digit do they start to differ?

c. How could you improve the accuracy of your estimate of 2^{π}?

Problem Set

1. Is it possible for a number to be both rational and irrational?

2. Use properties of exponents to rewrite the following expressions as a number or an exponential expression with only one exponent.

   a. $\left(2^{\sqrt{3}}\right)^{\sqrt{3}}$

   b. $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$

   c. $\left(3^{1+\sqrt{5}}\right)^{1-\sqrt{5}}$

   d. $3^{\frac{1+\sqrt{5}}{2}} \cdot 3^{\frac{1-\sqrt{5}}{2}}$

   e. $3^{\frac{1+\sqrt{5}}{2}} \div 3^{\frac{1-\sqrt{5}}{2}}$

   f. $3^{2\cos^2(x)} \cdot 3^{2\sin^2(x)}$

3.
   a. Between what two integer powers of 2 does $2^{\sqrt{5}}$ lie?

   b. Between what two integer powers of 3 does $3^{\sqrt{10}}$ lie?

   c. Between what two integer powers of 5 does $5^{\sqrt{3}}$ lie?

4. Use the process outlined in the lesson to approximate the number $2^{\sqrt{5}}$. Use the approximation $\sqrt{5} \approx 2.23606798$.

   a. Find a sequence of five intervals that contain $\sqrt{5}$ whose endpoints get successively closer to $\sqrt{5}$.

   b. Find a sequence of five intervals that contain $2^{\sqrt{5}}$ whose endpoints get successively closer to $2^{\sqrt{5}}$. Write your intervals in the form $2^r < 2^{\sqrt{5}} < 2^s$ for rational numbers r and s.

   c. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (b).

   d. Based on your work in part (c), what is your best estimate of the value of $2^{\sqrt{5}}$?

   e. Can we tell if $2^{\sqrt{5}}$ is rational or irrational? Why or why not?

5. Use the process outlined in the lesson to approximate the number $3^{\sqrt{10}}$. Use the approximation $\sqrt{10} \approx 3.1622777$.

   a. Find a sequence of five intervals that contain $3^{\sqrt{10}}$ whose endpoints get successively closer to $3^{\sqrt{10}}$. Write your intervals in the form $3^r < 3^{\sqrt{10}} < 3^s$ for rational numbers r and s.

   b. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (a).

   c. Based on your work in part (b), what is your best estimate of the value of $3^{\sqrt{10}}$?

6. Use the process outlined in the lesson to approximate the number $5^{\sqrt{7}}$. Use the approximation $\sqrt{7} \approx 2.64575131$.
   a. Find a sequence of seven intervals that contain $5^{\sqrt{7}}$ whose endpoints get successively closer to $5^{\sqrt{7}}$. Write your intervals in the form $5^r < 5^{\sqrt{7}} < 5^s$ for rational numbers r and s.
   b. Use your calculator to find approximations to four decimal places of the endpoints of the intervals in part (a).
   c. Based on your work in part (b), what is your best estimate of the value of $5^{\sqrt{7}}$?

7. Can the value of an irrational number raised to an irrational power ever be rational?

Lesson 6: Euler's Number, e

Classwork

Exercises 1–3

1. Assume that there is initially 1 cm of water in the tank and the height of the water doubles every 10 seconds. Write an equation that could be used to calculate the height $H(t)$ of the water in the tank at any time t.

2. How would the equation in Exercise 1 change if …
   a. The initial depth of water in the tank was 2 cm?

   b. The initial depth of water in the tank was $\frac{1}{2}$ cm?

   c. The initial depth of water in the tank was 10 cm?

   d. The initial depth of water in the tank was A cm, for some positive real number A?

3. How would the equation in Exercise 2, part (d) change if …
   a. The height tripled every ten seconds?

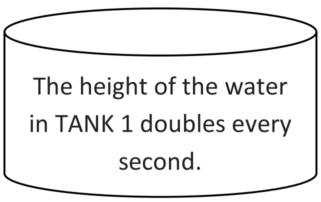
   b. The height doubled every five seconds?

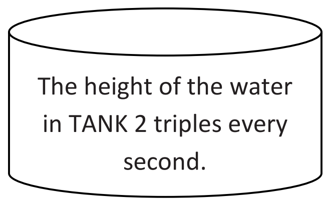
c. The height quadrupled every second?

d. The height halved every ten seconds?

Example 1

1. Consider two identical water tanks, each of which begins with a height of water 1 cm and fills with water at a different rate. Which equations can be used to calculate the height of water in each tank at time t? Use H_1 for tank 1 and H_2 for tank 2.





a. If both tanks start filling at the same time, which one fills first?

b. We want to know the average rate of change of the height of the water in these tanks over an interval that starts at a fixed time T as they are filling up. What is the formula for the average rate of change of a function f on an interval $[a, b]$?

c. What is the formula for the average rate of change of the function H_1 on an interval $[a, b]$?

d. Let's calculate the average rate of change of the function H_1 on the interval $[T, T + 0.1]$, which is an interval one-tenth of a second long starting at an unknown time T.

Exercises 4–8

4. For the second tank, calculate the average change in the height, H_2, from time T seconds to $T + 0.1$ seconds. Express the answer as a number times the value of the original function at time T. Explain the meaning of these findings.

5. For each tank, calculate the average change in height from time T seconds to $\mathrm{T} + 0.001$ seconds. Express the answer as a number times the value of the original function at time T. Explain the meaning of these findings.

6. In Exercise 5, the average rate of change of the height of the water in tank 1 on the interval $[T, T + 0.01]$ can be described by the expression $c_1 \cdot 2^T$, and the average rate of change of the height of the water in tank 2 on the interval $[T, T + 0.01]$ can be described by the expression $c_2 \cdot 3^T$. What are approximate values of c_1 and c_2?

7. As an experiment, let's look for a value of b so that if the height of the water can be described by $H(t) = b^t$, then the expression for the average of change on the interval $[T, T + 0.01]$ is $1 \cdot H(T)$.
   a. Write out the expression for the average rate of change of $H(t) = b^t$ on the interval $[T, T + 0.01]$.

   b. Set your expression in part (a) equal to $1 \cdot H(T)$ and reduce to an expression involving a single b.

   c. Now we want to find the value of b that satisfies the equation you found in part (b), but we do not have a way to explicitly solve this equation. Look back at Exercise 6; which two consecutive integers have b between them?

   d. Use your calculator and a guess-and-check method to find an approximate value of b to 2 decimal places.

8. Verify that for the value of b found in Exercise 7, $\dfrac{H_b(T + 0.001) - H_b(T)}{0.001} = H_b(T)$, where $H_b(T) = b^T$.

Lesson Summary

- Euler's number, e, is an irrational number that is approximately equal to $e \approx 2.7182818284590$.
- **AVERAGE RATE OF CHANGE:** Given a function f whose domain contains the interval of real numbers $[a, b]$ and whose range is a subset of the real numbers, the average rate of change on the interval $[a, b]$ is defined by the number

$$\frac{f(b) - b(a)}{b - a}.$$

Problem Set

1. The product $4 \cdot 3 \cdot 2 \cdot 1$ is called 4 factorial and is denoted by $4!$. Then $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, and for any positive integer n, $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$.

   a. Complete the following table of factorial values:

n	1	2	3	4	5	6	7	8
$n!$								

   b. Evaluate the sum $1 + \frac{1}{1!}$.

   c. Evaluate the sum $1 + \frac{1}{1!} + \frac{1}{2!}$.

   d. Use a calculator to approximate the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$ to 7 decimal places. Do not round the fractions before evaluating the sum.

   e. Use a calculator to approximate the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$ to 7 decimal places. Do not round the fractions before evaluating the sum.

   f. Use a calculator to approximate sums of the form $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!}$ to 7 decimal places for $k = 5, 6, 7, 8, 9, 10$. Do not round the fractions before evaluating the sums with a calculator.

   g. Make a conjecture about the sums $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!}$ for positive integers k as k increases in size.

   h. Would calculating terms of this sequence ever yield an exact value of e? Why or why not?

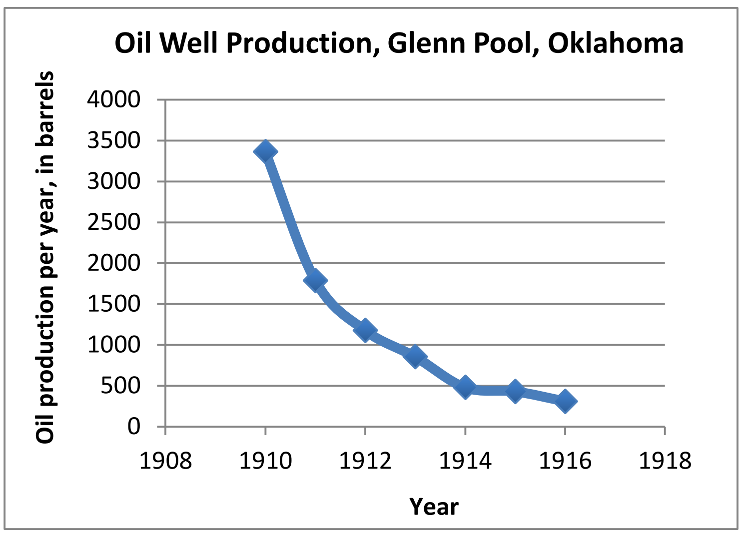
2. Consider the sequence given by the function $a_n = \left(1 + \frac{1}{n}\right)^n$, where $n \geq 1$ is an integer.
   a. Use your calculator to approximate the first 5 terms of this sequence to 7 decimal places.
   b. Does it appear that this sequence settles near a particular value?
   c. Use a calculator to approximate the following terms of this sequence to 7 decimal places.
      i. a_{100}
      ii. a_{1000}
      iii. $a_{10,000}$
      iv. $a_{100,000}$
      v. $a_{1,000,000}$
      vi. $a_{10,000,000}$
      vii. $a_{100,000,000}$
   d. Does it appear that this sequence settles near a particular value?
   e. Compare the results of this exercise with the results of Problem 1. What do you observe?

3. If $x = 5a^4$ and $a = 2e^3$, express x in terms of e and approximate to the nearest whole number.

4. If $a = 2b^3$ and $b = -\frac{1}{2}e^{-2}$, express a in terms of e and approximate to four decimal places.

5. If $x = 3e^4$ and $= \frac{s}{2x^3}$, show that $s = 54e^{13}$ and approximate s to the nearest whole number.

6. The following graph shows the number of barrels of oil produced by the Glenn Pool well in Oklahoma from 1910 to 1916.



Source: Cutler, Willard W., Jr. Estimation of Underground Oil Reserves by Oil-Well Production Curves, U.S. Department of the Interior, 1924.

a. Estimate the average rate of change of the amount of oil produced by the well on the interval $[1910, 1916]$ and explain what that number represents.

b. Estimate the average rate of change of the amount of oil produced by the well on the interval $[1910, 1913]$ and explain what that number represents.

c. Estimate the average rate of change of the amount of oil produced by the well on the interval $[1913, 1916]$ and explain what that number represents.

d. Compare your results for the rates of change in oil production in the first half and the second half of the time period in question in parts (b) and (c). What do those numbers say about the production of oil from the well?

e. Notice that the average rate of change of the amount of oil produced by the well on any interval starting and ending in two consecutive years is always negative. Explain what that means in the context of oil production.

7. The following table lists the number of hybrid electric vehicles (HEVs) sold in the United States between 1999 and 2013.

Year	Number of HEVs sold in U.S.
1999	17
2000	9350
2001	20,282
2002	36,035
2003	47,600
2004	84,199
2005	209,711
2006	252,636

Year	Number of HEVs sold in U.S.
2007	352,274
2008	312,386
2009	290,271
2010	274,210
2011	268,752
2012	434,498
2013	495,685

Source: U.S. Department of Energy, Alternative Fuels and Advanced Vehicle Data Center, 2013.

a. During which one-year interval is the average rate of change of the number of HEVs sold the largest? Explain how you know.

b. Calculate the average rate of change of the number of HEVs sold on the interval $[2003, 2004]$ and explain what that number represents.

c. Calculate the average rate of change of the number of HEVs sold on the interval $[2003, 2008]$ and explain what that number represents.

d. What does it mean if the average rate of change of the number of HEVs sold is negative?

Extension:

8. The formula for the area of a circle of radius r can be expressed as a function $A(r) = \pi r^2$.

a. Find the average rate of change of the area of a circle on the interval $[4, 5]$.

b. Find the average rate of change of the area of a circle on the interval $[4, 4.1]$.

c. Find the average rate of change of the area of a circle on the interval $[4, 4.01]$.

d. Find the average rate of change of the area of a circle on the interval $[4, 4.001]$.

e. What is happening to the average rate of change of the area of the circle as the interval gets smaller and smaller?

f. Find the average rate of change of the area of a circle on the interval $[4, 4+h]$ for some small positive number h.

g. What happens to the average rate of change of the area of the circle on the interval $[4, 4+h]$ as $h \to 0$? Does this agree with your answer to part (d)? Should it agree with your answer to part (e)?

h. Find the average rate of change of the area of a circle on the interval $[r_0, r_0 + h]$ for some positive number r_0 and some small positive number h.

i. What happens to the average rate of change of the area of the circle on the interval $[r_0, r_0 + h]$ as $h \to 0$? Do you recognize the resulting formula?

9. The formula for the volume of a sphere of radius r can be expressed as a function $V(r) = \frac{4}{3}\pi r^3$. As you work through these questions, you will see the pattern develop more clearly if you leave your answers in the form of a coefficient times π. Approximate the coefficient to five decimal places.

a. Find the average rate of change of the volume of a sphere on the interval $[2, 3]$.

b. Find the average rate of change of the volume of a sphere on the interval $[2, 2.1]$.

c. Find the average rate of change of the volume of a sphere on the interval $[2, 2.01]$.

d. Find the average rate of change of the volume of a sphere on the interval $[2, 2.001]$.

e. What is happening to the average rate of change of the volume of a sphere as the interval gets smaller and smaller?

f. Find the average rate of change of the volume of a sphere on the interval $[2, 2+h]$ for some small positive number h.

g. What happens to the average rate of change of the volume of a sphere on the interval $[2, 2+h]$ as $h \to 0$? Does this agree with your answer to part (e)? Should it agree with your answer to part (e)?

h. Find the average rate of change of the volume of a sphere on the interval $[r_0, r_0 + h]$ for some positive number r_0 and some small positive number h.

i. What happens to the average rate of change of the volume of a sphere on the interval $[r_0, r_0 + h]$ as $h \to 0$? Do you recognize the resulting formula?

Lesson 7: Bacteria and Exponential Growth

Classwork

Opening Exercise

Work with your partner or group to solve each of the following equations for x.

a. $2^x = 2$

b. $2^x = 2^3$

c. $2^x = 16$

d. $2^x - 64 = 0$

e. $2^x - 1 = 0$

f. $2^{3x} = 64$

g. $2^{x+1} = 32$

Example

The *Escherichia coli* bacteria (commonly known as *E. coli*), reproduces once every 30 minutes, meaning that a colony of *E. coli* can double every half hour. *Mycobacterium tuberculosis* has a generation time in the range of 12 to 16 hours. Researchers have found evidence that suggests certain bacteria populations living deep below the surface of the earth may grow at extremely slow rates, reproducing once every several thousand years. With this variation in bacterial growth rates, it is reasonable that we assume a 24-hour reproduction time for a hypothetical bacteria colony in the next example.

Suppose we have a bacteria colony that starts with 1 bacterium, and the population of bacteria doubles every day.

What function P can we use to model the bacteria population on day t?

t	$P(t)$

How many days will it take for the bacteria population to reach 8?

How many days will it take for the bacteria population to reach 16?

Roughly how long will it take for the population to reach 10?

We already know from our previous discussion that if $2^d = 10$, then $3 < d < 4$, and the table confirms that. At this point, we have an underestimate of 3 and an overestimate of 4 for d. How can we find better under and over estimates for d?

t	$P(t)$

From our table, we now know another set of under and over estimates for the number d that we seek. What are they?

Continue this process of “squeezing” the number d between two numbers until you are confident you know the value of d to two decimal places.

t	$P(t)$

t	$P(t)$

What if we had wanted to find d to 5 decimal places?

To the nearest minute, when does the population of bacteria become 10?

t	$P(t)$

t	$P(t)$

t	$P(t)$

t	$P(t)$

Exercise

Use the method from the Example to approximate the solution to the equations below to two decimal places.

a. $2^x = 1000$

b. $3^x = 1000$

c. $4^x = 1000$

d. $5^x = 1000$

e. $6^x = 1000$

f. $7^x = 1000$

g. $8^x = 1000$

h. $9^x = 1000$

i. $11^x = 1000$

j. $12^x = 1000$

k. $13^x = 1000$

l. $14^x = 1000$

m. $15^x = 1000$

n. $16^x = 1000$

Problem Set

1. Solve each of the following equations for x using the same technique as was used in the Opening Exercise.
   a. $2^x = 32$
   b. $2^{x-3} = 2^{2x+5}$
   c. $2^{x^2-3x} = 2^{-2}$
   d. $2^x - 2^{4x-3} = 0$
   e. $2^{3x} \cdot 2^5 = 2^7$
   f. $2^{x^2-16} = 1$
   g. $3^{2x} = 27$
   h. $3^{\frac{2}{x}} = 81$
   i. $\frac{3^{x^2}}{3^{5x}} = 3^6$

2. Solve the equation $\frac{2^{2x}}{2^{x+5}} = 1$ algebraically using two different initial steps as directed below.
   a. Write each side as a power of 2.
   b. Multiply both sides by 2^{x+5}.

3. Find consecutive integers that are under and over estimates of the solutions to the following exponential equations.
   a. $2^x = 20$
   b. $2^x = 100$
   c. $3^x = 50$
   d. $10^x = 432{,}901$
   e. $2^{x-2} = 750$
   f. $2^x = 1.35$

4. Complete the following table to approximate the solution to $10^x = 34{,}198$ to two decimal places.

t	$P(t)$	t	$P(t)$	t	$P(t)$	t	$P(t)$
1	10	4.1		4.51		4.531	
2	100	4.2		4.52		4.532	
3	1,000	4.3		4.53		4.533	
4	10,000	4.4		4.54		4.534	
5	100,000	4.5				4.535	
		4.6					

5. Complete the following table to approximate the solution to $2^x = 18$ to two decimal places.

t	$P(t)$

t	$P(t)$

t	$P(t)$

t	$P(t)$

6. Approximate the solution to $5^x = 5555$ to four decimal places.

7. A dangerous bacterial compound forms in a closed environment but is immediately detected. An initial detection reading suggests the concentration of bacteria in the closed environment is one percent of the fatal exposure level. This bacteria is known to double in growth (double in concentration in a closed environment) every hour and canbe modeled by the function $P(t) = 100 \cdot 2^t$, where t is measured in hours.

   a. In the function $P(t) = 100 \cdot 2^t$, what does the 100 mean? What does the 2 mean?

   b. Doctors and toxicology professionals estimate that exposure to two-thirds of the bacteria's fatal concentration level will begin to cause sickness. Without consulting a calculator or other technology, offer a rough time limit for the inhabitants of the infected environment to evacuate in order to avoid sickness in the doctors' estimation. Note that immediate evacuation is not always practical, so offer extra evacuation time if it is affordable.

   c. A more conservative approach is to evacuate the infected environment before bacteria concentration levels reach one-third of fatal levels. Without consulting a calculator or other technology, offer a rough time limit for evacuation in this circumstance.

d. Use the method of the Example to approximate when the infected environment will reach fatal levels (100%) of bacteria concentration, to the nearest minute.

t	2^t

t	2^t

t	2^t

t	2^t

t	2^t

Lesson 8: The "WhatPower" Function

Classwork

Opening Exercise

Evaluate each expression. The first two have been completed for you.

a. $\text{WhatPower}_2(8) = 3$

b. $\text{WhatPower}_3(9) = 2$

c. $\text{WhatPower}_6(36) =$ ______

d. $\text{WhatPower}_2(32) =$ ______

e. $\text{WhatPower}_{10}(1000) =$ ______

f. $\text{WhatPower}_{10}(1{,}000{,}000) =$ ______

g. $\text{WhatPower}_{100}(1{,}000{,}000) =$ ______

h. $\text{WhatPower}_4(64) =$ ______

i. $\text{WhatPower}_2(64) =$ ______

j. $\text{WhatPower}_9(3) =$ ______

k. $\text{WhatPower}_5\left(\sqrt{5}\right) =$ ______

l. $\text{WhatPower}_{\frac{1}{2}}\left(\frac{1}{8}\right) =$ ______

m. $\text{WhatPower}_{42}(1) =$ ______

n. $\text{WhatPower}_{100}(0.01) =$ ______

o. $\text{WhatPower}_2\left(\frac{1}{4}\right) =$ ______

p. $\text{WhatPower}_{\frac{1}{4}}(2) =$ ______

q. With your group members, write a definition for the function WhatPower_b , where b is a number.

Exercises 1–9

Evaluate the following expressions and justify your answers.

2. $\text{WhatPower}_7(49)$

3. $\text{WhatPower}_0(7)$

4. $\text{WhatPower}_5(1)$

5. $\text{WhatPower}_1(5)$

6. $\text{WhatPower}_2(16)$

7. $\text{WhatPower}_{-2}(32)$

8. $\text{WhatPower}_{\frac{1}{3}}(9)$

9. $\text{WhatPower}_{-\frac{1}{3}}(27)$

10. Describe the allowable values of b in the expression $\mathrm{WhatPower}_{\mathrm{b}}(x)$. When can we define a function $f(x) = \mathrm{WhatPower}_b(x)$? Explain how you know.

Examples

1. $\log_2(8) = 3$

2. $\log_3(9) = 2$

3. $\log_6(36) =$ ______

4. $\log_2(32) =$ ______

5. $\log_{10}(1000) =$ ______

6. $\log_{42}(1) =$ ______

7. $\log_{100}(0.01) =$ ______

8. $\log_2\left(\frac{1}{4}\right) =$ ______

Exercise 10

10. Compute the value of each logarithm. Verify your answers using an exponential statement.
    a. $\log_2(32)$

b. $\log_3(81)$

c. $\log_9(81)$

d. $\log_5(625)$

e. $\log_{10}(1{,}000{,}000{,}000)$

f. $\log_{1000}(1{,}000{,}000{,}000)$

g. $\log_{13}(13)$

h. $\log_{13}(1)$

i. $\log_9(27)$

j. $\log_7(\sqrt{7})$

k. $\log_{\sqrt{7}}(7)$

l. $\log_{\sqrt{7}}\left(\frac{1}{49}\right)$

m. $\log_x(x^2)$

Lesson Summary

- If three numbers, L, b, and x are related by $x = b^L$, then L is the *logarithm base* b of x and we write $\log_b(x)$. That is, the value of the expression $L = \log_b(x)$ is the power of b needed to obtain x.
- Valid values of b as a base for a logarithm are $0 < b < 1$ and $b > 1$.

Problem Set

1. Rewrite each of the following in the form $\text{WhatPower}_b(x) = L$.

   a. $3^5 = 243$ b. $6^{-3} = \frac{1}{216}$ c. $9^0 = 1$

2. Rewrite each of the following in the form $\log_b(x) = L$.

   a. $16^{\frac{1}{4}} = 2$ b. $10^3 = 1{,}000$ c. $b^k = r$

3. Rewrite each of the following in the form $b^L = x$.

   a. $\log_5(625) = 4$ b. $\log_{10}(0.1) = -1$ c. $\log_{27}9 = \frac{2}{3}$

4. Consider the logarithms base 2. For each logarithmic expression below, either calculate the value of the expression or explain why the expression does not make sense.
   a. $\log_2(1024)$
   b. $\log_2(128)$
   c. $\log_2(\sqrt{8})$
   d. $\log_2\left(\frac{1}{16}\right)$
   e. $\log_2(0)$
   f. $\log_2\left(-\frac{1}{32}\right)$

5. Consider the logarithms base 3. For each logarithmic expression below, either calculate the value of the expression or explain why the expression does not make sense.
   a. $\log_3(243)$
   b. $\log_3(27)$
   c. $\log_3(1)$
   d. $\log_3\left(\frac{1}{3}\right)$
   e. $\log_3(0)$
   f. $\log_3\left(-\frac{1}{3}\right)$

6. Consider the logarithms base 5. For each logarithmic expression below, either calculate the value of the expression or explain why the expression does not make sense.
   a. $\log_5(3125)$
   b. $\log_5(25)$
   c. $\log_5(1)$
   d. $\log_5\left(\frac{1}{25}\right)$
   e. $\log_5(0)$
   f. $\log_5\left(-\frac{1}{25}\right)$

7. Is there any positive number b so that the expression $\log_b(0)$ makes sense? Explain how you know.

8. Is there any positive number b so that the expression $\log_b(-1)$ makes sense? Explain how you know.

9. Verify each of the following by evaluating the logarithms.
   a. $\log_2(8) + \log_2(4) = \log_2(32)$
   b. $\log_3(9) + \log_3(9) = \log_3(81)$
   c. $\log_4(4) + \log_4(16) = \log_4(64)$
   d. $\log_{10}(10^3) + \log_{10}(10^4) = \log_{10}(10^7)$

10. Looking at the results from Problem 9, do you notice a trend or pattern? Can you make a general statement about the value of $\log_b(x) + \log_b(y)$?

11. To evaluate $\log_2(3)$, Autumn reasoned that since $\log_2(2) = 1$ and $\log_2(4) = 2$, $\log_2(3)$ must be the average of 1 and 2 and therefore $\log_2(3) = 1.5$. Use the definition of logarithm to show that $\log_2(3)$ cannot be 1.5. Why is her thinking not valid?

12. Find the value of each of the following.
    a. If $x = \log_2(8)$ and $y = 2^x$, find the value of y.
    b. If $\log_2(x) = 6$, find the value of x.
    c. If $r = 2^6$ and $s = \log_2(r)$, find the value of s.

Lesson 9: Logarithms—How Many Digits Do You Need?

Classwork

Opening Exercise

a. Evaluate $\text{WhatPower}_2(8)$. State your answer as a logarithm and evaluate it.

b. Evaluate $\text{WhatPower}_5(625)$. State your answer as a logarithm and evaluate it.

Exploratory Challenge

Autumn is starting a new club with eight members including herself. She wants everyone to have a secret identification code made up of only A's and B's. For example, using two characters, her ID code could be AB, which also happens to be her initials.

a. Using A's and B's, can Autumn assign each club member a unique two-character ID using only A's and B's? Justify your answer. Here's what Autumn has so far.

Club Member Name	Secret ID
Autumn	AA
Kris	
Tia	
Jimmy	

Club Member Name	Secret ID
Robert	
Jillian	
Benjamin	
Scott	

b. Using A's and B's, how many characters would be needed to assign each club member a unique ID code? Justify your answer by showing the IDs you would assign to each club member by completing the table above (adjust Autumn's ID if needed).

When the club grew to 16 members, Autumn started noticing a pattern.

Using A's and B's:

i. Two people could be given a secret ID with 1 letter: A and B.
ii. Four people could be given a secret ID with 2 letters: AA, BA, AB, BB.
iii. Eight people could be given a secret ID with 3 letters: AAA, BAA, ABA, BBA, AAB, BAB, ABB, BBB.

c. Complete the following statement and list the secret IDs for the 16 people.

16 people could be given a secret ID with _________ letters using A's and B's.

Club Member Name	Secret ID
Autumn	
Kris	
Tia	
Jimmy	
Robert	
Jillian	
Benjamin	
Scott	

Club Member Name	Secret ID
Gwen	
Jerrod	
Mykel	
Janette	
Nellie	
Serena	
Ricky	
Mia	

d. Describe the pattern in words. What type of function could be used to model this pattern?

Exercises 1–2

In the previous problems, the letters A and B were like the digits in a number. A 4-digit ID for Autumn's club could be any 4-letter arrangement of A's and B's because in her ID system, the only digits are the letters A and B.

1. When Autumn's club grows to include more than 16 people, she will need 5 digits to assign a unique ID to each club member. What is the maximum number of people that could be in the club before she needs to switch to a 6-digit ID? Explain your reasoning.

2. If Autumn has 256 members in her club, how many digits would she need to assign each club member a unique ID using only A's and B's? Show how you got your answers.

Example 1

A thousand people are given unique identifiers made up of the digits $0, 1, 2, \ldots, 9$. How many digits would be needed for each ID number?

Exercises 3–4

3. There are approximately 317 million people in the United States. Compute and use $\log(100{,}000{,}000)$ and $\log(1{,}000{,}000{,}000)$ to explain why Social Security numbers are 9 digits long.

4. There are many more telephones than the number of people in the United States because of people having home phones, cell phones, business phones, fax numbers, etc. Assuming we need at most 10 billion phone numbers in the United States, how many digits would be needed so that each phone number is unique? Is this reasonable? Explain.

Problem Set

1. The student body president needs to assign each officially sanctioned club on campus a unique ID number for purposes of tracking expenses and activities. She decides to use the letters A, B, and C to create a unique three-character code for each club.
   a. How many clubs can be assigned a unique ID according to this proposal?
   b. There are actually over 500 clubs on campus. Assuming the student body president still wants to use the letters A, B, and C, how many characters would be needed to generate a unique ID for each club?

2. Can you use the numbers $1, 2, 3$, and 4 in a combination of 4 digits to assign a unique ID to each of 500 people? Explain your reasoning.

3. Automobile license plates typically have a combination of letters (26) and numbers (10). Over time, the state of New York has used different criteria to assign vehicle license plate numbers.
   a. From 1973 to 1986, the state used a 3-letter and 4-number code where the three letters indicated the county where the vehicle was registered. Essex County had 13 different 3-letter codes in use. How many cars could be registered to this county?
   b. Since 2001, the state has used a 3-letter and 4-number code but no longer assigns letters by county. Is this coding scheme enough to register 10 million vehicles?

4. The Richter scale uses common (base 10) logarithms to assign a magnitude to an earthquake that is based on the amount of force released at the earthquake's source as measured by seismographs in various locations.
   a. Explain the difference between an earthquake that is assigned a magnitude of 5 versus one assigned a magnitude of 7.
   b. A magnitude 2 earthquake can usually be felt by multiple people who are located near the earthquake's origin. The largest recorded earthquake was magnitude 9.5 in Chile in 1960. How many times greater force than a magnitude 2 earthquake was the largest recorded earthquake?
   c. What would be the magnitude of an earthquake whose force was 1000 times greater than a magnitude 4.3 quake?

5. Sound pressure level is measured in decibels (dB) according to the formula $L = 10\log\left(\frac{I}{I_0}\right)$, where I is the intensity of the sound and I_0 is a reference intensity that corresponds to a barely perceptible sound.
   a. Explain why this formula would assign 0 decibels to a barely perceptible sound.
   b. Decibel levels above 120 dB can be painful to humans. What would be the intensity that corresponds to this level?

Lesson 10: Building Logarithmic Tables

Classwork

Opening Exercise

Find the value of the following expressions without using a calculator.

$\text{WhatPower}_{10}(1000)$ $\log_{10}(1000)$

$\text{WhatPower}_{10}(100)$ $\log_{10}(100)$

$\text{WhatPower}_{10}(10)$ $\log_{10}(10)$

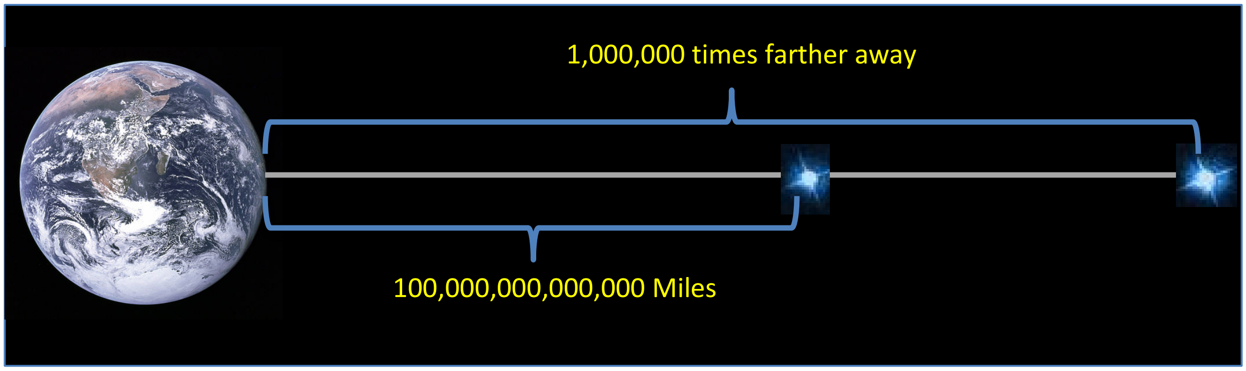
$\text{WhatPower}_{10}(1)$ $\log_{10}(1)$

$\text{WhatPower}_{10}\left(\frac{1}{10}\right)$ $\log_{10}\left(\frac{1}{10}\right)$

$\text{WhatPower}_{10}\left(\frac{1}{100}\right)$ $\log_{10}\left(\frac{1}{100}\right)$

Formulate a rule based on your results above: If k is an integer, then $\log_{10}(10^k) =$ _______.

Example 1



Exercises

1. Find two consecutive powers of 10 so that 30 is between them. That is, find an integer exponent k so that $10^k < 30 < 10^{k+1}$.

2. From your result in Exercise 1, $\log(30)$ is between which two integers?

3. Find a number k to one decimal place so that $10^k < 30 < 10^{k+0.1}$, and use that to find under and over estimates for $\log(30)$.

4. Find a number k to two decimal places so that $10^k < 30 < 10^{k+0.01}$, and use that to find under and over estimates for $\log\,(30)$.

5. Repeat this process to approximate the value of $\log(30)$ to 4 decimal places.

6. Verify your result on your calculator, using the $\boxed{\text{LOG}}$ button.

7. Use your calculator to complete the following table. Round the logarithms to 4 decimal places.

x	$\log(x)$
1	
2	
3	
4	
5	
6	
7	
8	
9	

x	$\log(x)$
10	
20	
30	
40	
50	
60	
70	
80	
90	

x	$\log(x)$
100	
200	
300	
400	
500	
600	
700	
800	
900	

8. What pattern(s) can you see in the table from Exercise 7 as x is multiplied by 10? Write the pattern(s) using logarithmic notation.

9. What pattern would you expect to find for $\log(1000x)$? Make a conjecture and test it to see whether or not it appears to be valid.

10. Use your results from Exercises 8 and 9 to make a conjecture about the value of $\log(10^k \cdot x)$ for any positive integer k.

11. Use your calculator to complete the following table. Round the logarithms to 4 decimal places.

x	$\log(x)$
1	
2	
3	
4	
5	
6	
7	
8	
9	

x	$\log(x)$
0.1	
0.2	
0.3	
0.4	
0.5	
0.6	
0.7	
0.8	
0.9	

x	$\log(x)$
0.01	
0.02	
0.03	
0.04	
0.05	
0.06	
0.07	
0.08	
0.09	

12. What pattern(s) can you see in the table from Exercise 11? Write them using logarithmic notation.

13. What pattern would you expect to find for $\log\left(\frac{x}{1000}\right)$? Make a conjecture and test it to see whether or not it appears to be valid.

14. Combine your results from Exercises 10 and 12 to make a conjecture about the value of the logarithm for a multiple of a power of 10; that is, find a formula for $\log(10^k \cdot x)$ for any integer k.

Lesson Summary

- The notation $\log(x)$ is used to represent $\log_{10}(x)$.
- For integers k, $\log(10^k) = k$.
- For integers m and n, $\log(10^m \cdot 10^n) = \log(10^m) + \log(10^n)$.
- For integers k and positive real numbers x, $\log(10^k \cdot x) = k + \log(x)$.

Problem Set

1. Complete the following table of logarithms without using a calculator; then, answer the questions that follow.

x	$\log(x)$
1,000,000	
100,000	
10,000	
1000	
100	
10	

x	$\log(x)$
0.1	
0.01	
0.001	
0.0001	
0.00001	
0.000001	

   a. What is $\log(1)$? How does that follow from the definition of a base-10 logarithm?
   b. What is $\log(10^k)$ for an integer k? How does that follow from the definition of a base-10 logarithm?
   c. What happens to the value of $\log(x)$ as x gets really large?
   d. For $x > 0$, what happens to the value of $\log(x)$ as x gets really close to zero?

2. Use the table of logarithms below to estimate the values of the logarithms in parts (a)–(h).

x	$\log(x)$
2	0.3010
3	0.4771
5	0.6990
7	0.8451
11	1.0414
13	1.1139

   a. $\log(70{,}000)$
   b. $\log(0.0011)$
   c. $\log(20)$
   d. $\log(0.00005)$
   e. $\log(130{,}000)$
   f. $\log(3000)$
   g. $\log(0.07)$
   h. $\log(11{,}000{,}000)$

3. If $\log(n) = 0.6$, find the value of $\log(10n)$.

4. If m is a positive integer and $\log(m) \approx 3.8$, how many digits are there in m? Explain how you know.

5. If m is a positive integer and $\log(m) \approx 9.6$, how many digits are there in m? Explain how you know.

6. Vivian says $\log(452{,}000) = 5 + \log(4.52)$, while her sister Lillian says that $\log(452{,}000) = 6 + \log(0.452)$. Which sister is correct? Explain how you know.

7. Write the logarithm base 10 of each number in the form $k + \log(x)$, where k is the exponent from the scientific notation, and x is a positive real number.
   a. 2.4902×10^4
   b. 2.58×10^{13}
   c. 9.109×10^{-31}

8. For each of the following statements, write the number in scientific notation and then write the logarithm base 10 of that number in the form $k + \log(x)$, where k is the exponent from the scientific notation, and x is a positive real number.
   a. The speed of sound is 1116 ft/s.
   b. The distance from Earth to the Sun is 93 million miles.
   c. The speed of light is $29{,}980{,}000{,}000$ cm/s .
   d. The weight of the earth is $5{,}972{,}000{,}000{,}000{,}000{,}000{,}000{,}000$ kg.
   e. The diameter of the nucleus of a hydrogen atom is 0.00000000000000175 m.
   f. For each part (a)–(e), you have written each logarithm in the form $k + \log(x)$, for integers k and positive real numbers x. Use a calculator to find the values of the expressions $\log(x)$. Why are all of these values between 0 and 1?

Lesson 11: The Most Important Property of Logarithms

Classwork

Opening Exercise

Use the logarithm table below to calculate the specified logarithms.

x	$\log(x)$
1	0
2	0.3010
3	0.4771
4	0.6021
5	0.6990
6	0.7782
7	0.8451
8	0.9031
9	0.9542

a. $\log(80)$

b. $\log(7000)$

c. $\log(0.00006)$

d. $\log(3.0 \times 10^{27})$

e. $\log(9.0 \times 10^{k})$ for an integer k

Exercises 1–5

1. Use your calculator to complete the following table. Round the logarithms to four decimal places.

x	$\log(x)$
1	0
2	0.3010
3	0.4771
4	0.6021
5	0.6990
6	0.7782
7	0.8451
8	0.9031
9	0.9542

x	$\log(x)$
10	
12	
16	
18	
20	
25	
30	
36	
100	

2. Calculate the following values. Do they appear anywhere else in the table?

   a. $\log(2) + \log(4)$

   b. $\log(2) + \log(6)$

   c. $\log(3) + \log(4)$

   d. $\log(6) + \log(6)$

   e. $\log(2) + \log(18)$

   f. $\log(3) + \log(12)$

3. What pattern(s) can you see in Exercise 2 and the table from Exercise 1? Write them using logarithmic notation.

4. What pattern would you expect to find for $\log(x^2)$? Make a conjecture, and test it to see whether or not it appears to be valid.

5. Make a conjecture for a logarithm of the form $\log(xyz)$, where x, y, and z are positive real numbers. Provide evidence that your conjecture is valid.

Example 1

Use the logarithm table from Exercise 1 to approximate the following logarithms:

a. $\log(14)$

b. $\log(35)$

c. $\log(72)$

d. $\log(121)$

Exercises 6–8

6. Use your calculator to complete the following table. Round the logarithms to four decimal places.

x	$\log(x)$
2	
4	
5	
8	
10	
16	
20	
50	
100	

x	$\log(x)$
0.5	
0.25	
0.2	
0.125	
0.1	
0.0625	
0.05	
0.02	
0.01	

7. What pattern(s) can you see in the table from Exercise 6? Write a conjecture using logarithmic notation.

8. Use the definition of logarithm to justify the conjecture you found in Exercise 7.

Example 2

Use the logarithm tables and the rules we discovered to estimate the following logarithms to four decimal places.

a. $\log(2100)$

b. $\log(0.00049)$

c. $\log(42{,}000{,}000)$

d. $\log\left(\frac{1}{640}\right)$

Lesson Summary

- The notation $\log(x)$ is used to represent $\log_{10}(x)$.
- The most important property of logarithms base 10 is that for positive real numbers x and y,
$$\log(xy) = \log(x) + \log(y).$$
- For positive real numbers x,
$$\log\left(\frac{1}{x}\right) = -\log(x).$$

Problem Set

1. Use the table of logarithms at right to estimate the value of the logarithms in parts (a)–(h).
   a. $\log(25)$
   b. $\log(27)$
   c. $\log(33)$
   d. $\log(55)$
   e. $\log(63)$
   f. $\log(75)$
   g. $\log(81)$
   h. $\log(99)$

x	$\log(x)$
2	0.30
3	0.48
5	0.70
7	0.85
11	1.04
13	1.11

2. Use the table of logarithms at right to estimate the value of the logarithms in parts (a)–(f).
   a. $\log(350)$
   b. $\log(0.0014)$
   c. $\log(0.077)$
   d. $\log(49{,}000)$
   e. $\log(1.69)$
   f. $\log(6.5)$

3. Use the table of logarithms at right to estimate the value of the logarithms in parts (a)–(f).
   a. $\log\left(\frac{1}{30}\right)$
   b. $\log\left(\frac{1}{35}\right)$
   c. $\log\left(\frac{1}{40}\right)$
   d. $\log\left(\frac{1}{42}\right)$
   e. $\log\left(\frac{1}{50}\right)$
   f. $\log\left(\frac{1}{64}\right)$

4. Reduce each expression to a single logarithm of the form $\log(x)$.
   a. $\log(5) + \log(7)$
   b. $\log(3) + \log(9)$
   c. $\log(15) - \log(5)$
   d. $\log(8) + \log\left(\frac{1}{4}\right)$

5. Use properties of logarithms to write the following expressions involving logarithms of only prime numbers.
   a. $\log(2500)$
   b. $\log(0.00063)$
   c. $\log(1250)$
   d. $\log(26{,}000{,}000)$

6. Use properties of logarithms to show that $\log(26) = \log(2) - \log\left(\frac{1}{13}\right)$.

7. Use properties of logarithms to show that $\log(3) + \log(4) + \log(5) - \log(6) = 1$.

8. Use properties of logarithms to show that $-\log(3) = \log\left(\frac{1}{2} - \frac{1}{3}\right) + \log(2)$.

9. Use properties of logarithms to show that $\log\left(\frac{1}{3} - \frac{1}{4}\right) + \left(\log\left(\frac{1}{3}\right) - \log\left(\frac{1}{4}\right)\right) = -2\log(3)$.

Lesson 12: Properties of Logarithms

Classwork

Opening Exercise

Use the approximation $\log(2) \approx 0.3010$ to approximate the values of each of the following logarithmic expressions.

a. $\log(20)$

b. $\log(0.2)$

c. $\log(2^4)$

Exercises 1–10

For Exercises 1–6, explain why each statement below is a property of base-10 logarithms.

1. Property 1: $\log(1) = 0$.

2. Property 2: $\log(10) = 1$.

3. Property 3: For all real numbers r, $\log(10^r) = r$.

4. Property 4: For any $x > 0$, $10^{\log(x)} = x$.

5. Property 5: For any positive real numbers x and y, $\log(x \cdot y) = \log(x) + \log(y)$.
   Hint: Use an exponent rule as well as Property 4.

6. Property 6: For any positive real number x and any real number r, $\log(x^r) = r \cdot \log(x)$.
   Hint: Again, use an exponent rule as well as Property 4.

7. Apply properties of logarithms to rewrite the following expressions as a single logarithm or number.
   a. $\frac{1}{2}\log(25) + \log(4)$

   b. $\frac{1}{3}\log(8) + \log(16)$

   c. $3\log(5) + \log(0.8)$

8. Apply properties of logarithms to rewrite each expression as a sum of terms involving numbers, $\log(x)$, and $\log(y)$.
   a. $\log(3x^2y^5)$

   b. $\log\left(\sqrt{x^7y^3}\right)$

9. In mathematical terminology, logarithms are well defined because if $X = Y$, then $\log(X) = \log(Y)$ for $X, Y > 0$. This means that if you want to solve an equation involving exponents, you can apply a logarithm to both sides of the equation, just as you can take the square root of both sides when solving a quadratic equation. You do need to be careful not to take the logarithm of a negative number or zero.

   Use the property stated above to solve the following equations.

   a. $10^{10x} = 100$

   b. $10^{x-1} = \frac{1}{10^{x+1}}$

   c. $100^{2x} = 10^{3x-1}$

10. Solve the following equations.

   a. $10^{x} = 2^{7}$

   b. $10^{x^2+1} = 15$

   c. $4^{x} = 5^{3}$

Lesson Summary

We have established the following properties for base-10 logarithms, where x and y are positive real numbers and r is any real number:

1. $\log(1) = 0$
2. $\log(10) = 1$
3. $\log(10^r) = r$
4. $10^{\log(x)} = x$
5. $\log(x \cdot y) = \log(x) + \log(y)$
6. $\log(x^r) = r \cdot \log(x)$

Additional properties not yet established are the following:

7. $\log\left(\frac{1}{x}\right) = -\log(x)$
8. $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

Also, logarithms are well defined, meaning that for $X, Y > 0$, if $X = Y$, then $\log(X) = \log(Y)$.

Problem Set

1. Use the approximate logarithm values below to estimate each of the following logarithms. Indicate which properties you used.

$$\log(2) = 0.3010 \qquad \log(3) = 0.4771$$
$$\log(5) = 0.6990 \qquad \log(7) = 0.8451$$

   a. $\log(6)$
   b. $\log(15)$
   c. $\log(12)$
   d. $\log(10^7)$
   e. $\log\left(\frac{1}{5}\right)$
   f. $\log\left(\frac{3}{7}\right)$
   g. $\log(\sqrt[4]{2})$

2. Let $\log(X) = r$, $\log(Y) = s$, and $\log(Z) = t$. Express each of the following in terms of r, s, and t.
   a. $\log\left(\frac{X}{Y}\right)$
   b. $\log(YZ)$
   c. $\log(X^r)$
   d. $\log\left(\sqrt[3]{Z}\right)$
   e. $\log\left(\sqrt[4]{\frac{Y}{Z}}\right)$
   f. $\log(XY^2Z^3)$

3. Use the properties of logarithms to rewrite each expression in an equivalent form containing a single logarithm.
   a. $\log\left(\frac{13}{5}\right) + \log\left(\frac{5}{4}\right)$
   b. $\log\left(\frac{5}{6}\right) - \log\left(\frac{2}{3}\right)$
   c. $\frac{1}{2}\log(16) + \log(3) + \log\left(\frac{1}{4}\right)$

4. Use the properties of logarithms to rewrite each expression in an equivalent form containing a single logarithm.
   a. $\log\left(\sqrt{x}\right) + \frac{1}{2}\log\left(\frac{1}{x}\right) + 2\log(x)$
   b. $\log\left(\sqrt[5]{x}\right) + \log\left(\sqrt[5]{x^4}\right)$
   c. $\log(x) + 2\log(y) - \frac{1}{2}\log(z)$
   d. $\frac{1}{3}\left(\log(x) - 3\log(y) + \log(z)\right)$
   e. $2(\log(x) - \log(3y)) + 3(\log(z) - 2\log(x))$

5. Use properties of logarithms to rewrite the following expressions in an equivalent form containing only $\log(x)$, $\log(y)$, $\log(z)$, and numbers.
   a. $\log\left(\frac{3x^2y^4}{\sqrt{z}}\right)$
   b. $\log\left(\frac{42\sqrt[3]{xy^7}}{x^2z}\right)$
   c. $\log\left(\frac{100x^2}{y^3}\right)$
   d. $\log\left(\sqrt{\frac{x^3y^2}{10z}}\right)$
   e. $\log\left(\frac{1}{10x^2z}\right)$

6. Express $\log\left(\frac{1}{x} - \frac{1}{x+1}\right) + \left(\log\left(\frac{1}{x}\right) - \log\left(\frac{1}{x+1}\right)\right)$ as a single logarithm for positive numbers x.

7. Show that $\log\left(x + \sqrt{x^2 - 1}\right) + \log\left(x - \sqrt{x^2 - 1}\right) = 0$ for $x \geq 1$.

8. If $xy = 10^{3.67}$, find the value of $\log(x) + \log(y)$.

9. Solve the following exponential equations by taking the logarithm base 10 of both sides. Leave your answers stated in terms of logarithmic expressions.
   a. $10^{x^2} = 320$
   b. $10^{\frac{x}{8}} = 300$
   c. $10^{3x} = 400$
   d. $5^{2x} = 200$
   e. $3^x = 7^{-3x+2}$

10. Solve the following exponential equations.
   a. $10^x = 3$
   b. $10^y = 30$
   c. $10^z = 300$
   d. Use the properties of logarithms to justify why x, y, and z form an arithmetic sequence whose constant difference is 1.

11. Without using a calculator, explain why the solution to each equation must be a real number between 1 and 2.
   a. $11^x = 12$
   b. $21^x = 30$
   c. $100^x = 2000$
   d. $\left(\frac{1}{11}\right)^x = 0.01$
   e. $\left(\frac{2}{3}\right)^x = \frac{1}{2}$
   f. $99^x = 9000$

12. Express the exact solution to each equation as a base-10 logarithm. Use a calculator to approximate the solution to the nearest 1000^{th}.
   a. $11^x = 12$
   b. $21^x = 30$
   c. $100^x = 2000$
   d. $\left(\frac{1}{11}\right)^x = 0.01$
   e. $\left(\frac{2}{3}\right)^x = \frac{1}{2}$
   f. $99^x = 9000$

13. Show that the value of x that satisfies the equation $10^x = 3 \cdot 10^n$ is $\log(3) + n$.

14. Solve each equation. If there is no solution, explain why.
    a. $3 \cdot 5^x = 21$
    b. $10^{x-3} = 25$
    c. $10^x + 10^{x+1} = 11$
    d. $8 - 2^x = 10$

15. Solve the following equation for n: $A = P(1+r)^n$.

16. In this exercise, we will establish a formula for the logarithm of a sum. Let $L = \log(x+y)$, where $x, y > 0$.
    a. Show $\log(x) + \log\left(1 + \frac{y}{x}\right) = L$. State as a property of logarithms after showing this is a true statement.
    b. Use part (a) and the fact that $\log(100) = 2$ to rewrite $\log(365)$ as a sum.
    c. Rewrite 365 in scientific notation, and use properties of logarithms to express $\log(365)$ as a sum of an integer and a logarithm of a number between 0 and 10.
    d. What do you notice about your answers to (b) and (c)?
    e. Find two integers that are upper and lower estimates of $\log(365)$.

Lesson 13: Changing the Base

Classwork

Exercises

1. Assume that x, a, and b are all positive real numbers, so that $a \neq 1$ and $b \neq 1$. What is $\log_b(x)$ in terms of $\log_a(x)$? The resulting equation allows us to change the base of a logarithm from a to b.

2. Approximate each of the following logarithms to four decimal places. Use the $\boxed{\text{LOG}}$ key on your calculator rather than logarithm tables, first changing the base of the logarithm to 10 if necessary.
   a. $\log(3^2)$

   b. $\log_3(3^2)$

   c. $\log_2(3^2)$

3. In Lesson 12, we justified a number of properties of base 10 logarithms. Working in pairs, justify the following properties of base b logarithms.

   a. $\log_b(1) = 0$

   b. $\log_b(b) = 1$

   c. $\log_b(\mathrm{b}^r) = r$

   d. $b^{\log_b(x)} = x$

   e. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$

f. $\log_b(x^r) = r \cdot \log_b(x)$

g. $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$

h. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

4. Find each of the following to four decimal places. Use the $\boxed{\text{LN}}$ key on your calculator rather than a table.

   a. $\ln(3^2)$

   b. $\ln(2^4)$

5. Write as a single logarithm:

   a. $\ln(4) - 3\ln\left(\frac{1}{3}\right) + \ln(2)$.

   b. $\ln(5) + \frac{3}{5}\ln(32) - \ln(4)$.

6. Write each expression as a sum or difference of constants and logarithms of simpler terms.

   a. $\ln\left(\frac{\sqrt{5x^3}}{e^2}\right)$

   b. $\ln\left(\frac{(x+y)^2}{x^2+y^2}\right)$

Lesson Summary

We have established a formula for changing the base of logarithms from b to a:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}.$$

In particular, the formula allows us to change logarithms base b to common or natural logarithms, which are the only two kinds of logarithms that calculators compute:

$$\log_{\mathrm{b}}(x) = \frac{\log(x)}{\log(b)} = \frac{\ln(x)}{\ln(b)}.$$

We have also established the following properties for base b logarithms. If x, y, a, and b are all positive real numbers with $a \neq 1$ and $b \neq 1$ and r is any real number, then:

1. $\log_b(1) = 0$
2. $\log_b(b) = 1$
3. $\log_b(b^r) = r$
4. $b^{\log_b(x)} = x$
5. $\log_b(x \cdot y) = \log_b(x) + \log_b(y)$
6. $\log_b(x^r) = r \cdot \log_b(x)$
7. $\log_b\left(\frac{1}{x}\right) = -\log_b(x)$
8. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$

Problem Set

1. Evaluate each of the following logarithmic expressions, approximating to four decimal places if necessary. Use the LN or LOG key on your calculator rather than a table.
   a. $\log_8(16)$
   b. $\log_7(11)$
   c. $\log_3(2) + \log_2(3)$

2. Use logarithmic properties and the fact that $\ln(2) \approx 0.69$ and $\ln(3) \approx 1.10$ to approximate the value of each of the following logarithmic expressions. Do not use a calculator.
   a. $\ln(e^4)$
   b. $\ln(6)$
   c. $\ln(108)$
   d. $\ln\left(\frac{8}{3}\right)$

3. Compare the values of $\log_{\frac{1}{9}}(10)$ and $\log_9\left(\frac{1}{10}\right)$ without using a calculator.

4. Show that for any positive numbers a and b with $a \neq 1$ and $b \neq 1$, $\log_a(b) \cdot \log_b(a) = 1$.

5. Express x in terms of a, e, and y if $\ln(x) - \ln(y) = 2a$.

6. Rewrite each expression in an equivalent form that only contains one base 10 logarithm.
   a. $\log_2(800)$
   b. $\log_x\left(\frac{1}{10}\right)$, for positive real values of $x \neq 1$
   c. $\log_5(12{,}500)$
   d. $\log_3(0.81)$

7. Write each number in terms of natural logarithms, and then use the properties of logarithms to show that it is a rational number.
   a. $\log_9\left(\sqrt{27}\right)$
   b. $\log_8(32)$
   c. $\log_4\left(\frac{1}{8}\right)$

8. Write each expression as an equivalent expression with a single logarithm. Assume x, y, and z are positive real numbers.
   a. $\ln(x) + 2\ln(y) - 3\ln(z)$
   b. $\frac{1}{2}(\ln(x+y) - \ln(z))$
   c. $(x+y) + \ln(z)$

9. Rewrite each expression as sums and differences in terms of $\ln(x)$, $\ln(y)$, and $\ln(z)$.
   a. $\ln(xyz^3)$
   b. $\ln\left(\frac{e^3}{xyz}\right)$
   c. $\ln\left(\sqrt{\frac{x}{y}}\right)$

10. Solve the following equations in terms of base 5 logarithms. Then, use the change of base properties and a calculator to estimate the solution to the nearest 1000^{th}. If the equation has no solution, explain why.
   a. $5^{2x} = 20$
   b. $75 = 10 \cdot 5^{x-1}$
   c. $5^{2+x} - 5^x = 10$
   d. $5^{x^2} = 0.25$

11. In Lesson 6, you discovered that $\log(x \cdot 10^k) = k + \log(x)$ by looking at a table of logarithms. Use the properties of logarithms to justify this property for an arbitrary base $b > 0$ with $b \neq 1$. That is, show that $\log_b(x \cdot b^k) = k + \log_b(x)$.

12. Larissa argued that since $\log_2(2) = 1$ and $\log_2(4) = 2$, then it must be true that $\log_2(3) = 1.5$. Is she correct? Explain how you know.

13. Extension: Suppose that there is some positive number b so that

$$\log_b(2) = 0.36$$
$$\log_b(3) = 0.57$$
$$\log_b(5) = 0.84.$$

   a. Use the given values of $\log_b(2)$, $\log_b(3)$, and $\log_b(5)$ to evaluate the following logarithms.
      i. $\log_b(6)$
      ii. $\log_b(8)$
      iii. $\log_b(10)$
      iv. $\log_b(600)$

   b. Use the change of base formula to convert $\log_b(10)$ to base 10, and solve for b. Give your answer to four decimal places.

14. Solve the following exponential equations.
   a. $2^{3x} = 16$
   b. $2^{x+3} = 4^{3x}$
   c. $3^{4x-2} = 27^{x+2}$
   d. $4^{2x} = \left(\frac{1}{4}\right)^{3x}$
   e. $5^{0.2x+3} = 625$

15. Solve each exponential equation.
   a. $3^{2x} = 81$
   b. $6^{3x} = 36^{x+1}$
   c. $625 = 5^{3x}$
   d. $25^{4-x} = 5^{3x}$
   e. $32^{x-1} = \frac{1}{2}$
   f. $\frac{4^{2x}}{2^{x-3}} = 1$
   g. $\frac{1}{8^{2x-4}} = 1$
   h. $2^x = 81$
   i. $8 = 3^x$
   j. $6^{x+2} = 12$
   k. $10^{x+4} = 27$
   l. $2^{x+1} = 3^{1-x}$
   m. $3^{2x-3} = 2^{x+4}$
   n. $e^{2x} = 5$
   o. $e^{x-1} = 6$

16. In Problem 9(e) of Lesson 12, you solved the equation $3^x = 7^{-3x+2}$ using the logarithm base 10.
    a. Solve $3^x = 7^{-3x+2}$ using the logarithm base 3.
    b. Apply the change of base formula to show that your answer to part (a) agrees with your answer to Problem 9(e) of Lesson 12.
    c. Solve $3^x = 7^{-3x+2}$ using the logarithm base 7.
    d. Apply the change of base formula to show that your answer to part (c) also agrees with your answer to Problem 9(e) of Lesson 12.

17. Pearl solved the equation $2^x = 10$ as follows:

$$\log(2^x) = \log(10)$$
$$x\log(2) = 1$$
$$x = \frac{1}{\log(2)}.$$

Jess solved the equation $2^x = 10$ as follows:

$$\log_2(2^x) = \log_2(10)$$
$$x\log_2(2) = \log_2(10)$$
$$x = \log_2(10).$$

Is Pearl correct? Is Jess correct? Explain how you know.

Lesson 14: Solving Logarithmic Equations

Classwork

Opening Exercises

Convert the following logarithmic equations to exponential form:

a. $\log(10{,}000) = 4$

b. $\log\left(\sqrt{10}\right) = \frac{1}{2}$

c. $\log_2(256) = 8$

d. $\log_4(256) = 4$

e. $\ln(1) = 0$

f. $\log(x + 2) = 3$

Examples 1–3

Write each of the following equations as an equivalent exponential equation, and solve for x.

1. $\log(3x+7)=0$

2. $\log_2(x+5)=4$

3. $\log(x+2)+\log(x+5)=1$

Exercises

1. Drew said that the equation $\log_2[(x+1)^4] = 8$ cannot be solved because he expanded $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$ and realized that he cannot solve the equation $x^4 + 4x^3 + 6x^2 + 4x + 1 = 2^8$. Is he correct? Explain how you know.

Solve the equations in Exercises 2–4 for x.

2. $\ln((4x)^5) = 15$

3. $\log((2x+5)^2) = 4$

4. $\log_2((5x+7)^{19}) = 57$

Solve the logarithmic equations in Exercises 5–9, and identify any extraneous solutions.

5. $\log(x^2 + 7x + 12) - \log(x + 4) = 0$

6. $\log_2(3x) + \log_2(4) = 4$

7. $2\ln(x+2) - \ln(-x) = 0$

8. $\log(x) = 2 - \log(x)$

9. $\ln(x+2) = \ln(12) - \ln(x+3)$

Problem Set

1. Solve the following logarithmic equations.
   a. $\log(x) = \frac{5}{2}$
   b. $5\log(x+4) = 10$
   c. $\log_2(1-x) = 4$
   d. $\log_2(49x^2) = 4$
   e. $\log_2(9x^2 + 30x + 25) = 8$

2. Solve the following logarithmic equations.
   a. $\ln(x^6) = 36$
   b. $\log[(2x^2 + 45x - 25)^5] = 10$
   c. $\log[(x^2 + 2x - 3)^4] = 0$

3. Solve the following logarithmic equations.
   a. $\log(x) + \log(x-1) = \log(3x+12)$
   b. $\ln(32x^2) - 3\ln(2) = 3$
   c. $\log(x) + \log(-x) = 0$
   d. $\log(x+3) + \log(x+5) = 2$
   e. $\log(10x+5) - 3 = \log(x-5)$
   f. $\log_2(x) + \log_2(2x) + \log_2(3x) + \log_2(36) = 6$

4. Solve the following equations.
   a. $\log_2(x) = 4$
   b. $\log_6(x) = 1$
   c. $\log_3(x) = -4$
   d. $\log_{\sqrt{2}}(x) = 4$
   e. $\log_{\sqrt{5}}(x) = 3$
   f. $\log_3(x^2) = 4$
   g. $\log_2(y^{-3}) = 12$
   h. $\log_3(8x+9) = 4$
   i. $2 = \log_4(3x-2)$
   j. $\log_5(3-2x) = 0$
   k. $\ln(2x) = 3$
   l. $\log_3(x^2 - 3x + 5) = 2$
   m. $\log((x^2+4)^5) = 10$
   n. $\log(x) + \log(x+21) = 2$
   o. $\log_4(x-2) + \log_4(2x) = 2$
   p. $\log(x) - \log(x+3) = -1$
   q. $\log_4(x+3) - \log_4(x-5) = 2$
   r. $\log(x) + 1 = \log(x+9)$
   s. $\log_3(x^2-9) - \log_3(x+3) = 1$
   t. $1 - \log_8(x-3) = \log_8(2x)$
   u. $\log_2(x^2-16) - \log_2(x-4) = 1$
   v. $\log\left(\sqrt{(x+3)^3}\right) = \frac{3}{2}$
   w. $\ln(4x^2 - 1) = 0$
   x. $\ln(x+1) - \ln(2) = 1$

Lesson 15: Why Were Logarithms Developed?

Classwork

Exercises

1. Solve the following equations. Remember to check for extraneous solutions because logarithms are only defined for positive real numbers.

   a. $\log(x^2) = \log(49)$

   b. $\log(x+1) + \log(x-2) = \log(7x-17)$

   c. $\log(x^2+1) = \log\big(x(x-2)\big)$

d. $\log(x+4) + \log(x-1) = \log(3x)$

e. $\log(x^2 - x) - \log(x-2) = \log(x-3)$

f. $\log(x) + \log(x-1) + \log(x+1) = 3\log(x)$

g. $\log(x-4) = -\log(x-2)$

2. How do you know if you need to use the definition of logarithm to solve an equation involving logarithms as we did in Lesson 15 or if you can use the methods of this lesson?

Lesson Summary

A table of base 10 logarithms can be used to simplify multiplication of multi-digit numbers:

1. To compute $A \times B$ for positive real numbers A and B, look up the values $\log(A)$ and $\log(B)$ in the logarithm table.
2. Add $\log(A)$ and $\log(B)$. The sum can be written as $k + d$, where k is an integer and $0 \le d < 1$ is the decimal part.
3. Look back at the table and find the entry closest to the decimal part, d.
4. The product of that entry and 10^k is an approximation to $A \times B$.

A similar process simplifies division of multi-digit numbers:

1. To compute $A \div B$ for positive real numbers A and B, look up the values $\log(A)$ and $\log(B)$ in the logarithm table.
2. Calculate $\log(A) - \log(B)$. The difference can be written as $k + d$, where k is an integer and $0 \le d < 1$ is the decimal part.
3. Look back at the table to find the entry closest to the decimal part, d.
4. The product of that entry and 10^k is an approximation to $A \div B$.

For any positive values X and Y, if $\log_b(X) = \log_b(Y)$, we can conclude that $X = Y$. This property is the essence of how a logarithm table works, and it allows us to solve equations with logarithmic expressions on both sides of the equation.

Problem Set

1. Use the table of logarithms to approximate solutions to the following logarithmic equations.
   a. $\log(x) = 0.5044$
   b. $\log(x) = -0.5044$ [Hint: Begin by writing -0.5044 as $[(-0.5044) + 1] - 1$.]
   c. $\log(x) = 35.5044$
   d. $\log(x) = 4.9201$

2. Use logarithms and the logarithm table to evaluate each expression.
   a. $\sqrt{2.33}$
   b. $13{,}500 \cdot 3{,}600$
   c. $\dfrac{7.2 \times 10^9}{1.3 \times 10^5}$

3. Solve for x: $\log(3) + 2\log(x) = \log(27)$.

4. Solve for x: $\log(3x) + \log(x + 4) = \log(15)$.

5. Solve for x.
   a. $\log(x) = \log(y + z) + \log(y - z)$
   b. $\log(x) = \big(\log(y) + \log(z)\big) + \big(\log(y) - \log(z)\big)$

6. If x and y are positive real numbers, and $\log(y) = 1 + \log(x)$, express y in terms of x.

7. If x, y, and z are positive real numbers, and $\log(x) - \log(y) = \log(y) - \log(z)$, express y in terms of x and z.

8. If x and y are positive real numbers, and $\log(x) = y\big(\log(y + 1) - \log(y)\big)$, express x in terms of y.

9. If x and y are positive real numbers, and $\log(y) = 3 + 2\log(x)$, express y in terms of x.

10. If x, y, and z are positive real numbers, and $\log(z) = \log(y) + 2\log(x) - 1$, express z in terms of x and y.

11. Solve the following equations.
    a. $\ln(10) - \ln(7 - x) = \ln(x)$
    b. $\ln(x + 2) + \ln(x - 2) = \ln(9x - 24)$
    c. $\ln(x + 2) + \ln(x - 2) = \ln(-2x - 1)$

12. Suppose the formula $P = P_0(1 + r)^t$ gives the population of a city P growing at an annual percent rate r, where P_0 is the population t years ago.
    a. Find the time t it takes this population to double.
    b. Use the structure of the expression to explain why populations with lower growth rates take a longer time to double.
    c. Use the structure of the expression to explain why the only way to double the population in one year is if there is a 100 percent growth rate.

13. If $x > 0$, $a + b > 0$, $a > b$, and $\log(x) = \log(a + b) + \log(a - b)$, find x in terms of a and b.

14. Jenn claims that because $\log(1) + \log(2) + \log(3) = \log(6)$, then $\log(2) + \log(3) + \log(4) = \log(9)$.
    a. Is she correct? Explain how you know.
    b. If $\log(a) + \log(b) + \log(c) = \log(a + b + c)$, express c in terms of a and b. Explain how this result relates to your answer to part (a).
    c. Find other values of a, b, and c so that $\log(a) + \log(b) + \log(c) = \log(a + b + c)$.

15. In Problem 7 of the Lesson 12 Problem Set, you showed that for $x \geq 1$, $\log\left(x+\sqrt{x^2-1}\right)+\log\left(x-\sqrt{x^2-1}\right)=0$. It follows that $\log\left(x+\sqrt{x^2-1}\right)=-\log\left(x-\sqrt{x^2-1}\right)$. What does this tell us about the relationship between the expressions $x+\sqrt{x^2-1}$ and $x-\sqrt{x^2-1}$?

16. Use the change of base formula to solve the following equations.
    a. $\log(x)=\log_{100}(x^2-2x+6)$
    b. $\log(x-2)=\log_{100}(14-x)$
    c. $\log_2(x+1)=\log_4(x^2+3x+4)$
    d. $\log_2(x-1)=\log_8(x^3-2x^2-2x+5)$

17. Solve the following equation:

$$\log(9x)=\frac{2\ln(3)+\ln(x)}{\ln(10)}.$$

Lesson 16: Rational and Irrational Numbers

Classwork

Opening Exercise

a. Explain how to use a number line to add the fractions $\frac{7}{5} + \frac{9}{4}$.

b. Convert $\frac{7}{5}$ and $\frac{9}{4}$ to decimals, and explain the process for adding them together.

Exercises

1. According to the calculator, $\log(4) = 0.6020599913\ldots$ and $\log(25) = 1.3979400087\ldots$. Find an approximation of $\log(4) + \log(25)$ to one decimal place, that is, to an accuracy of 10^{-1}.

2. Find the value of $\log(4) + \log(25)$ to an accuracy of 10^{-2}.

3. Find the value of $\log(4) + \log(25)$ to an accuracy of 10^{-8}.

4. Make a conjecture: Is $\log(4) + \log(25)$ a rational or an irrational number?

5. Why is your conjecture in Exercise 4 true?

Remember that the calculator gives the following values: $\log(4) = 0.6020599913\ldots$ and $\log(25) = 1.3979400087\ldots$.

6. Find the value of $\log(4) \cdot \log(25)$ to three decimal places.

7. Find the value of $\log(4) \cdot \log(25)$ to five decimal places.

8. Does your conjecture from the above discussion appear to be true?

Lesson Summary

- Irrational numbers occur naturally and frequently.
- The n^{th} roots of most integers and rational numbers are irrational.
- Logarithms of most positive integers or positive rational numbers are irrational.
- We can locate an irrational number on the number line by trapping it between lower and upper approximations. The infinite process of squeezing the irrational number in smaller and smaller intervals locates exactly where the irrational number is on the number line.
- We can perform arithmetic operations such as addition and multiplication with irrational numbers using lower and upper approximations and squeezing the result of the operation in smaller and smaller intervals between two rational approximations to the result.

Problem Set

1. Given that $\sqrt{5} \approx 2.2360679775$ and $\pi \approx 3.1415926535$, find the sum $\sqrt{5} + \pi$ to an accuracy of 10^{-8}, without using a calculator.

2. Put the following numbers in order from least to greatest.
$$\sqrt{2}, \pi, 0, e, \frac{22}{7}, \frac{\pi^2}{3}, 3.14, \sqrt{10}$$

3. Find a rational number between the specified two numbers.
   a. $\frac{4}{13}$ and $\frac{5}{13}$
   b. $\frac{3}{8}$ and $\frac{5}{9}$
   c. 1.7299999 and 1.73
   d. $\frac{\sqrt{2}}{7}$ and $\frac{\sqrt{2}}{9}$
   e. π and $\sqrt{10}$

4. Knowing that $\sqrt{2}$ is irrational, find an irrational number between $\frac{1}{2}$ and $\frac{5}{9}$.

5. Give an example of an irrational number between e and π.

6. Given that $\sqrt{2}$ is irrational, which of the following numbers are irrational?
$$\frac{\sqrt{2}}{2}, 2+\sqrt{2}, \frac{\sqrt{2}}{2\sqrt{2}}, \frac{2}{\sqrt{2}}, \left(\sqrt{2}\right)^2$$

7. Given that π is irrational, which of the following numbers are irrational?

$$\frac{\pi}{2}, \frac{\pi}{2\pi}, \sqrt{\pi}, \pi^2$$

8. Which of the following numbers are irrational?

$$1, 0, \sqrt{5}, \sqrt[3]{64}, e, \pi, \frac{\sqrt{2}}{2}, \frac{\sqrt{8}}{\sqrt{2}}, \cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right)$$

9. Find two irrational numbers x and y so that their average is rational.

10. Suppose that $\frac{2}{3}x$ is an irrational number. Explain how you know that x must be an irrational number. (Hint: What would happen if there were integers a and b so that $x = \frac{a}{b}$?)

11. If r and s are rational numbers, prove that $r + s$ and $r - s$ are also rational numbers.

12. If r is a rational number and x is an irrational number, determine whether the following numbers are always rational, sometimes rational, or never rational. Explain how you know.
    a. $r + x$
    b. $r - x$
    c. rx
    d. x^r

13. If x and y are irrational numbers, determine whether the following numbers are always rational, sometimes rational, or never rational. Explain how you know.
    a. $x + y$
    b. $x - y$
    c. xy
    d. $\frac{x}{y}$

Lesson 17: Graphing the Logarithm Function

Classwork

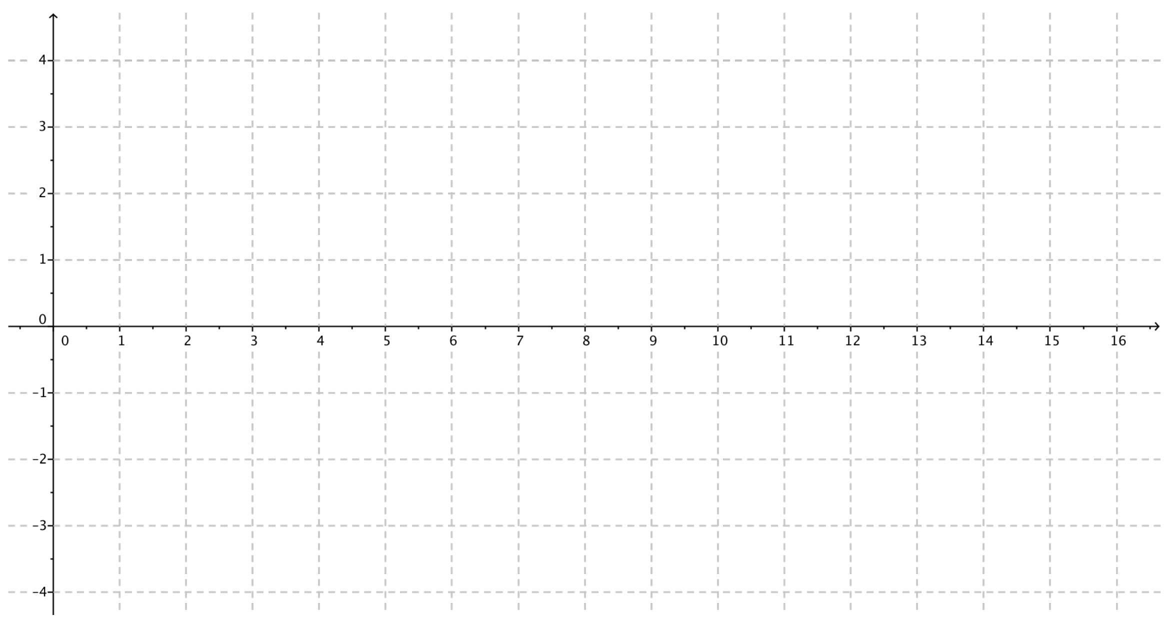
Opening Exercise

Graph the points in the table for your assigned function $f(x) = \log(x)$, $g(x) = \log_2(x)$, or $h(x) = \log_5(x)$ for $0 < x \leq 16$. Then, sketch a smooth curve through those points and answer the questions that follow.

10-team $f(x) = \log(x)$	
x	$f(x)$
0.0625	-1.20
0.125	-0.90
0.25	-0.60
0.5	-0.30
1	0
2	0.30
4	0.60
8	0.90
16	1.20

2-team $g(x) = \log_2(x)$	
x	$g(x)$
0.0625	-4
0.125	-3
0.25	-2
0.5	-1
1	0
2	1
4	2
8	3
16	4

5-team $h(x) = \log_5(x)$	
x	$h(x)$
0.0625	-1.72
0.125	-1.29
0.25	-0.86
0.5	-0.43
1	0
2	0.43
4	0.86
8	1.29
16	1.72



a. What does the graph indicate about the domain of your function?

b. Describe the x-intercepts of the graph.

c. Describe the y-intercepts of the graph.

d. Find the coordinates of the point on the graph with y-value 1.

e. Describe the behavior of the function as $x \to 0$.

f. Describe the end behavior of the function as $x \to \infty$.

g. Describe the range of your function.

h. Does this function have any relative maxima or minima? Explain how you know.

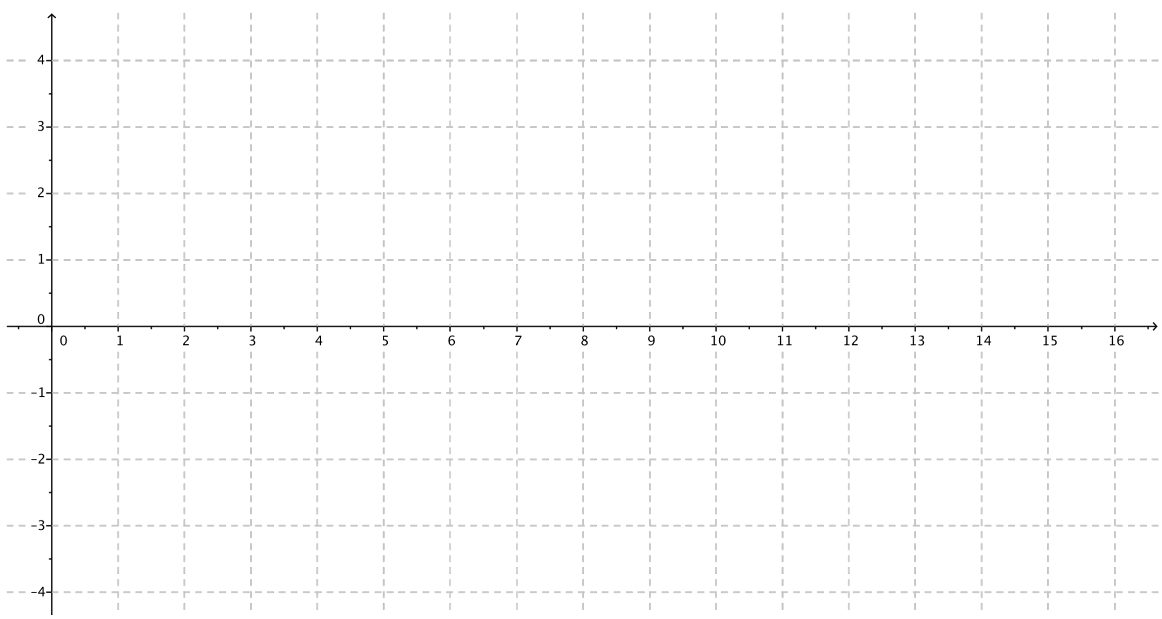
Exercises

1. Graph the points in the table for your assigned function $r(x) = \log_{\frac{1}{10}}(x)$, $s(x) = \log_{\frac{1}{2}}(x)$, or $t(x) = \log_{\frac{1}{5}}(x)$ for $0 < x \leq 16$. Then sketch a smooth curve through those points, and answer the questions that follow.

10-team $r(x) = \log_{\frac{1}{10}}(x)$	
x	$r(x)$
0.0625	1.20
0.125	0.90
0.25	0.60
0.5	0.30
1	0
2	-0.30
4	-0.60
8	-0.90
16	-1.20

2-team $s(x) = \log_{\frac{1}{2}}(x)$	
x	$s(x)$
0.0625	4
0.125	3
0.25	2
0.5	1
1	0
2	-1
4	-2
8	-3
16	-4

e-team $t(x) = \log_{\frac{1}{5}}(x)$	
x	$t(x)$
0.0625	1.72
0.125	1.29
0.25	0.86
0.5	0.43
1	0
2	-0.43
4	-0.86
8	-1.29
16	-1.72



a. What is the relationship between your graph in the Opening Exercise and your graph from this exercise?

b. Why does this happen? Use the change of base formula to justify what you have observed in part (a).

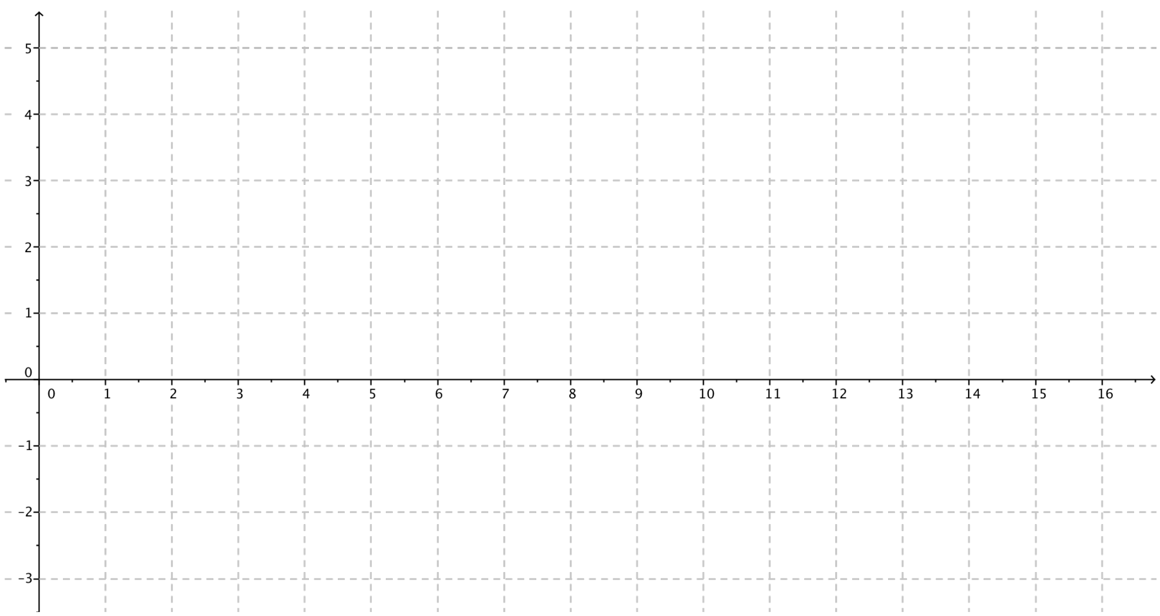
2. In general, what is the relationship between the graph of a function $y = f(x)$ and the graph of $y = f(kx)$ for a constant k?

3. Graph the points in the table for your assigned function $u(x) = \log(10x)$, $v(x) = \log_2(2x)$, or $w(x) = \log_5(5x)$ for $0 < x \leq 16$. Then sketch a smooth curve through those points, and answer the questions that follow.

10-team $u(x) = \log(10x)$	
x	$u(x)$
0.0625	-0.20
0.125	0.10
0.25	0.40
0.5	0.70
1	1
2	1.30
4	1.60
8	1.90
16	2.20

2-team $v(x) = \log_2(2x)$	
x	$v(x)$
0.0625	-3
0.125	-2
0.25	-1
0.5	0
1	1
2	2
4	3
8	4
16	5

5-team $w(x) = \log_5(5x)$	
x	$w(x)$
0.0625	-0.72
0.125	-0.29
0.25	0.14
0.5	0.57
1	1
2	1.43
4	1.86
8	2.29
16	2.72



a. Describe a transformation that takes the graph of your team's function in this exercise to the graph of your team's function in the Opening Exercise.

b. Do your answers to Exercise 2 and part (a) agree? If not, use properties of logarithms to justify your observations in part (a).

Lesson Summary

The function $f(x) = \log_b(x)$ is defined for irrational and rational numbers. Its domain is all positive real numbers. Its range is all real numbers.

The function $f(x) = \log_b(x)$ goes to negative infinity as x goes to zero. It goes to positive infinity as x goes to positive infinity.

The larger the base b, the more slowly the function $f(x) = \log_b(x)$ increases.

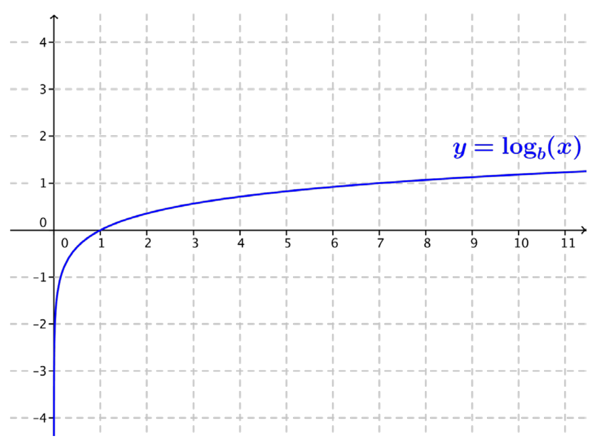
By the change of base formula, $\log_{\frac{1}{b}}(x) = -\log_b(x)$.

Problem Set

1. The function $Q(x) = \log_b(x)$ has function values in the table at right.
   a. Use the values in the table to sketch the graph of $y = Q(x)$.
   b. What is the value of b in $Q(x) = \log_b(x)$? Explain how you know.
   c. Identify the key features in the graph of $y = Q(x)$.

x	$Q(x)$
0.1	1.66
0.3	0.87
0.5	0.50
1.00	0.00
2.00	-0.50
4.00	-1.00
6.00	-1.29
10.00	-1.66
12.00	-1.79

2. Consider the logarithmic functions $f(x) = \log_b(x)$, $g(x) = \log_5(x)$, where b is a positive real number, and $b \neq 1$. The graph of f is given at right.
   a. Is $b > 5$, or is $b < 5$? Explain how you know.
   b. Compare the domain and range of functions f and g.
   c. Compare the x-intercepts and y-intercepts of f and g.
   d. Compare the end behavior of f and g.



3. Consider the logarithmic functions $f(x) = \log_b(x)$, $g(x) = \log_{\frac{1}{2}}(x)$, where b is a positive real number and $b \neq 1$. A table of approximate values of f is given below.

   a. Is $b > \frac{1}{2}$, or is $b < \frac{1}{2}$? Explain how you know.
   b. Compare the domain and range of functions f and g.
   c. Compare the x-intercepts and y-intercepts of f and g.
   d. Compare the end behavior of f and g.

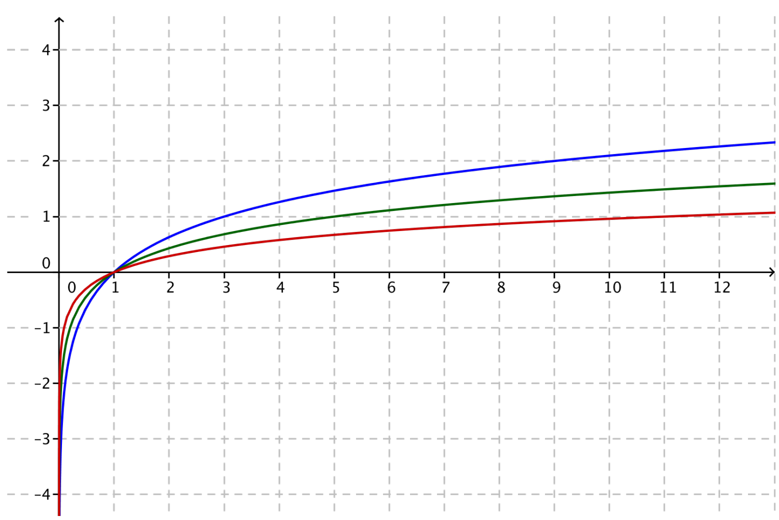
x	$f(x)$
$\frac{1}{4}$	0.86
$\frac{1}{2}$	0.43
1	0
2	-0.43
4	-0.86

4. On the same set of axes, sketch the functions $f(x) = \log_2(x)$ and $g(x) = \log_2(x^3)$.
   a. Describe a transformation that takes the graph of f to the graph of g.
   b. Use properties of logarithms to justify your observations in part (a).

5. On the same set of axes, sketch the functions $f(x) = \log_2(x)$ and $g(x) = \log_2\left(\frac{x}{4}\right)$.
   a. Describe a transformation that takes the graph of f to the graph of g.
   b. Use properties of logarithms to justify your observations in part (a).

6. On the same set of axes, sketch the functions $f(x) = \log_{\frac{1}{2}}(x)$ and $g(x) = \log_2\left(\frac{1}{x}\right)$.
   a. Describe a transformation that takes the graph of f to the graph of g.
   b. Use properties of logarithms to justify your observations in part (a).

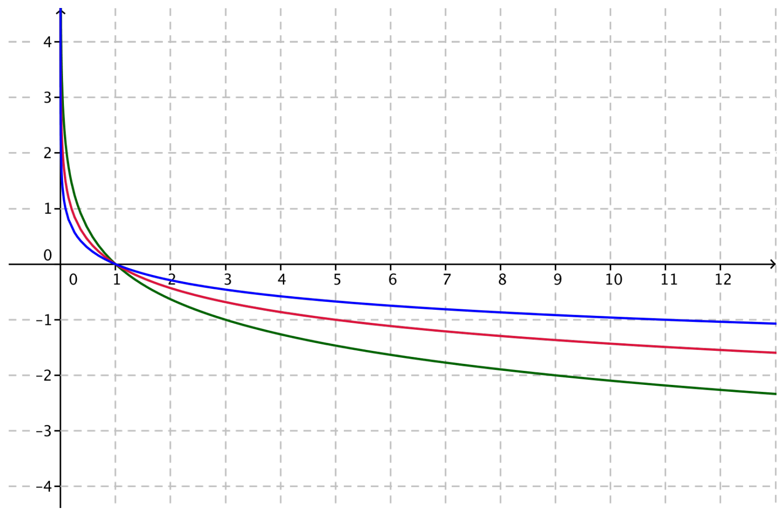
7. The figure below shows graphs of the functions $f(x) = \log_3(x)$, $g(x) = \log_5(x)$, and $h(x) = \log_{11}(x)$.
   a. Identify which graph corresponds to which function. Explain how you know.
   b. Sketch the graph of $k(x) = \log_7(x)$ on the same axes.



8. The figure below shows graphs of the functions $f(x) = \log_{\frac{1}{3}}(x)$, $g(x) = \log_{\frac{1}{5}}(x)$, and $h(x) = \log_{\frac{1}{11}}(x)$.

   a. Identify which graph corresponds to which function. Explain how you know.

   b. Sketch the graph of $k(x) = \log_{\frac{1}{7}}(x)$ on the same axes.



9. For each function f, find a formula for the function h in terms of x. Part (a) has been done for you.

   a. If $f(x) = x^2 + x$, find $h(x) = f(x + 1)$.

   b. If $f(x) = \sqrt{x^2 + \frac{1}{4}}$, find $h(x) = f\left(\frac{1}{2}x\right)$.

   c. If $f(x) = \log(x)$, find $h(x) = f\left(\sqrt[3]{10x}\right)$ when $x > 0$.

   d. If $f(x) = 3^x$, find $h(x) = f(\log_3(x^2 + 3))$.

   e. If $f(x) = x^3$, find $h(x) = f\left(\frac{1}{x^3}\right)$ when $x \neq 0$.

   f. If $f(x) = x^3$, find $h(x) = f\left(\sqrt[3]{x}\right)$.

   g. If $f(x) = \sin(x)$, find $h(x) = f\left(x + \frac{\pi}{2}\right)$.

   h. If $f(x) = x^2 + 2x + 2$, find $h(x) = f(\cos(x))$.

10. For each of the functions f and g below, write an expression for (i) $f\big(g(x)\big)$, (ii) $g\big(f(x)\big)$, and (iii) $f\big(f(x)\big)$ in terms of x. Part (a) has been done for you.

    a. $f(x) = x^2$, $g(x) = x + 1$

       i. $$f\big(g(x)\big) = f(x+1) \\ = (x+1)^2$$

       ii. $$g\big(f(x)\big) = g(x^2) \\ = x^2 + 1$$

       iii. $$f\big(f(x)\big) = f(x^2) \\ = (x^2)^2 \\ = x^4$$

    b. $f(x) = \frac{1}{4}x - 8$, $g(x) = 4x + 1$

    c. $f(x) = \sqrt[3]{x+1}$, $g(x) = x^3 - 1$

    d. $f(x) = x^3$, $g(x) = \frac{1}{x}$

    e. $f(x) = |x|$, $g(x) = x^2$

Extension:

11. Consider the functions $f(x) = \log_2(x)$ and $(x) = \sqrt{x-1}$.

    a. Use a calculator or other graphing utility to produce graphs of $f(x) = \log_2(x)$ and $g(x) = \sqrt{x-1}$ for $x \leq 17$.

    b. Compare the graph of the function $f(x) = \log_2(x)$ with the graph of the function $g(x) = \sqrt{x-1}$. Describe the similarities and differences between the graphs.

    c. Is it always the case that $\log_2(x) > \sqrt{x-1}$ for $x > 2$?

12. Consider the functions $f(x) = \log_2(x)$ and $(x) = \sqrt[3]{x-1}$.

    a. Use a calculator or other graphing utility to produce graphs of $f(x) = \log_2(x)$ and $h(x) = \sqrt[3]{x-1}$ for $x \leq 28$.

    b. Compare the graph of the function $f(x) = \log_2(x)$ with the graph of the function $h(x) = \sqrt[3]{x-1}$. Describe the similarities and differences between the graphs.

    c. Is it always the case that $\log_2(x) > \sqrt[3]{x-1}$ for $x > 2$?

Lesson 18: Graphs of Exponential Functions and Logarithmic Functions

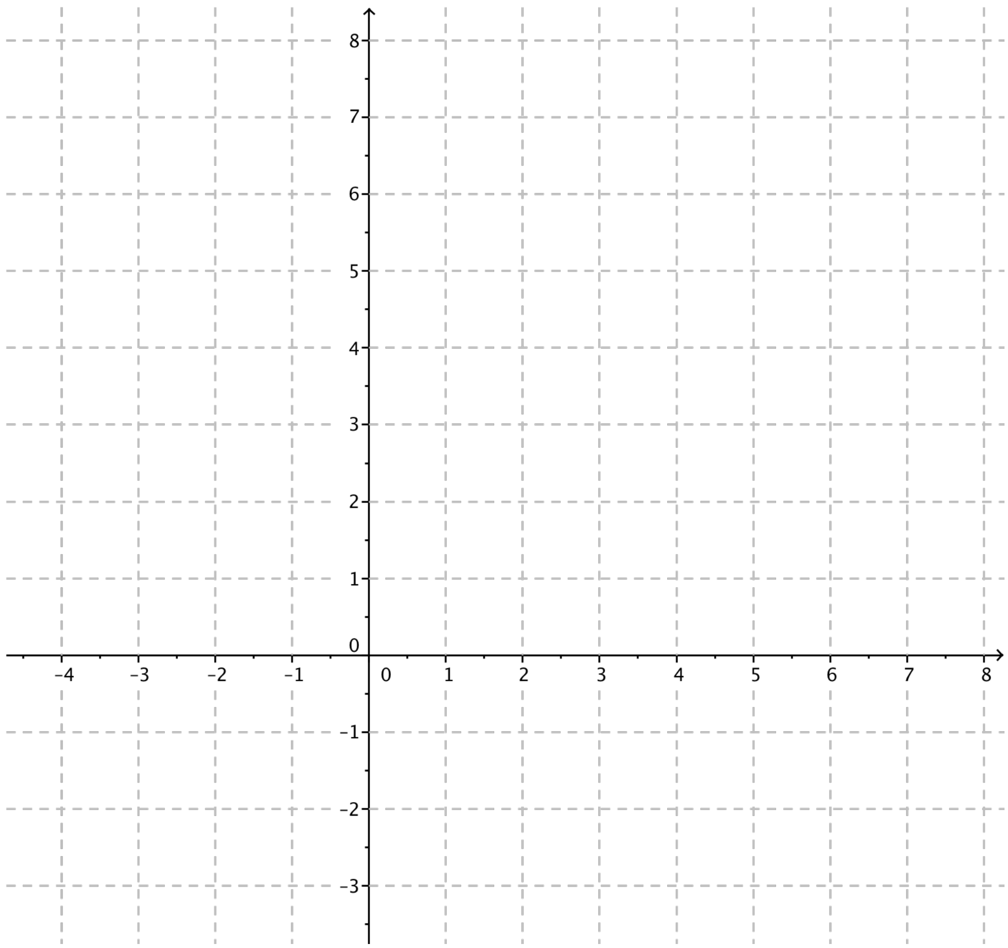
Classwork

Opening Exercise

Complete the following table of values of the function $f(x) = 2^x$. We want to sketch the graph of $y = f(x)$ and then reflect that graph across the diagonal line with equation $y = x$.

x	$y = 2^x$	Point (x, y) on the graph of $y = 2^x$
-3		
-2		
-1		
0		
1		
2		
3		

On the set of axes below, plot the points from the table and sketch the graph of $y = 2^x$. Next, sketch the diagonal line with equation $y = x$, and then reflect the graph of $y = 2^x$ across the line.

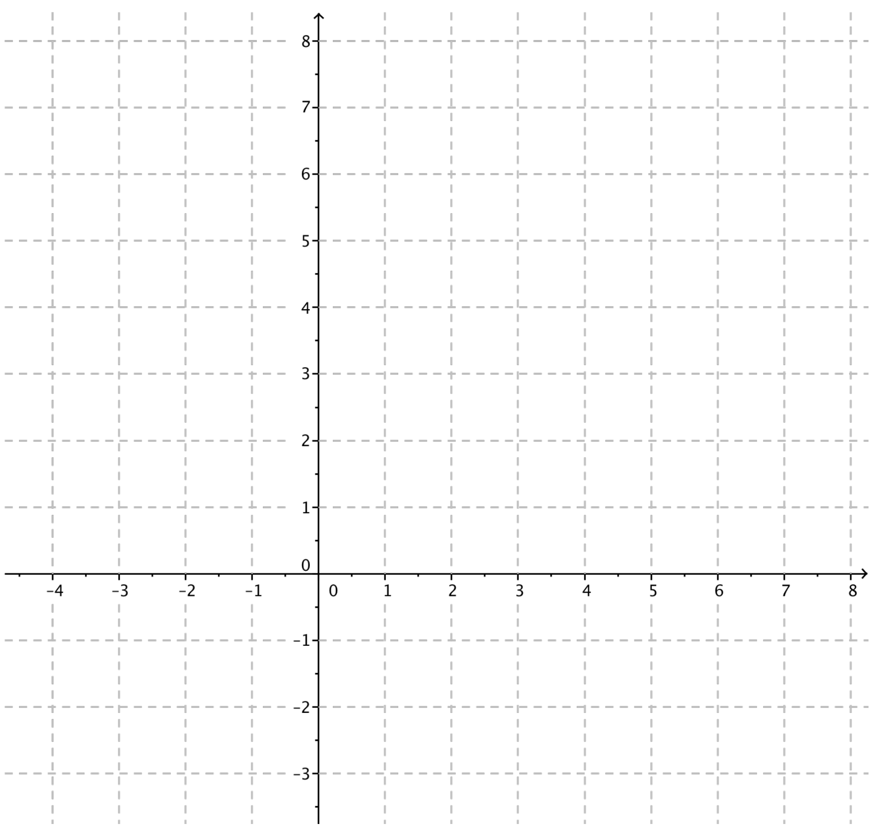


Exercises

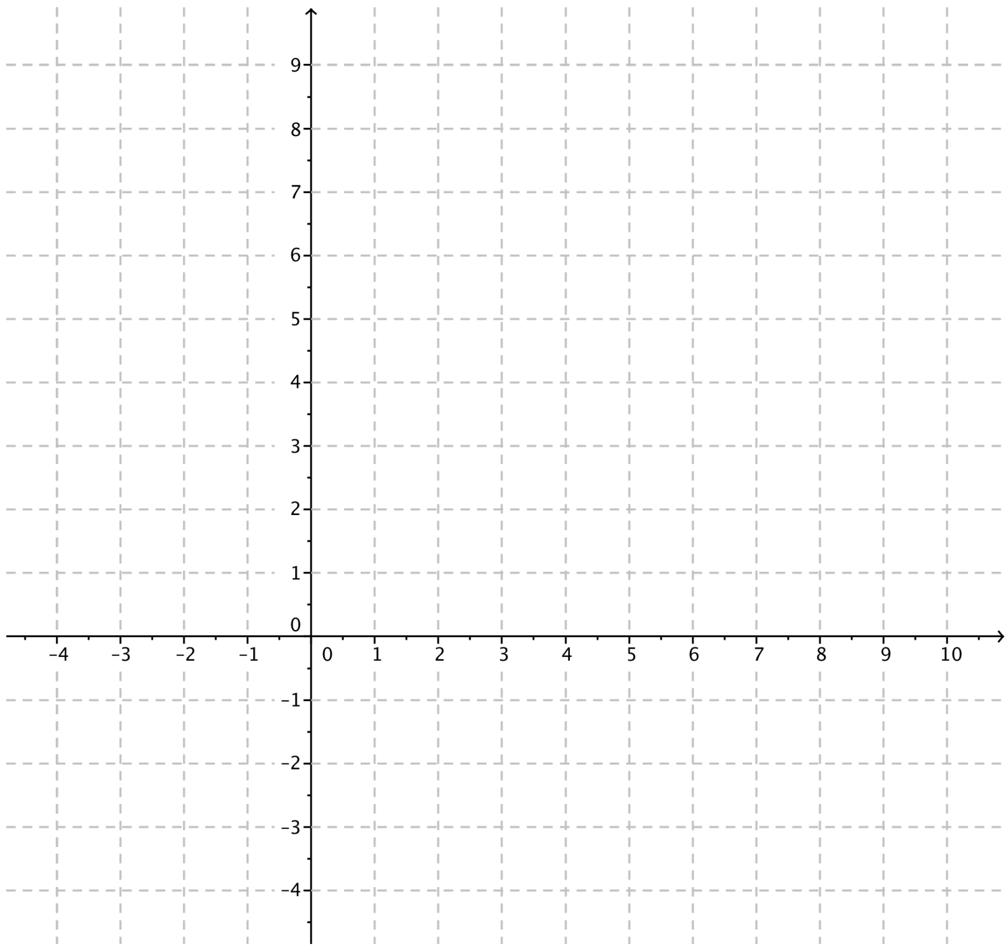
1. Complete the following table of values of the function $g(x) = \log_2(x)$. We want to sketch the graph of $y = g(x)$ and then reflect that graph across the diagonal line with equation $y = x$.

x	$y = \log_2(x)$	Point (x, y) on the graph of $y = \log_2(x)$
$-\frac{1}{8}$		
$-\frac{1}{4}$		
$-\frac{1}{2}$		
1		
2		
4		
8		

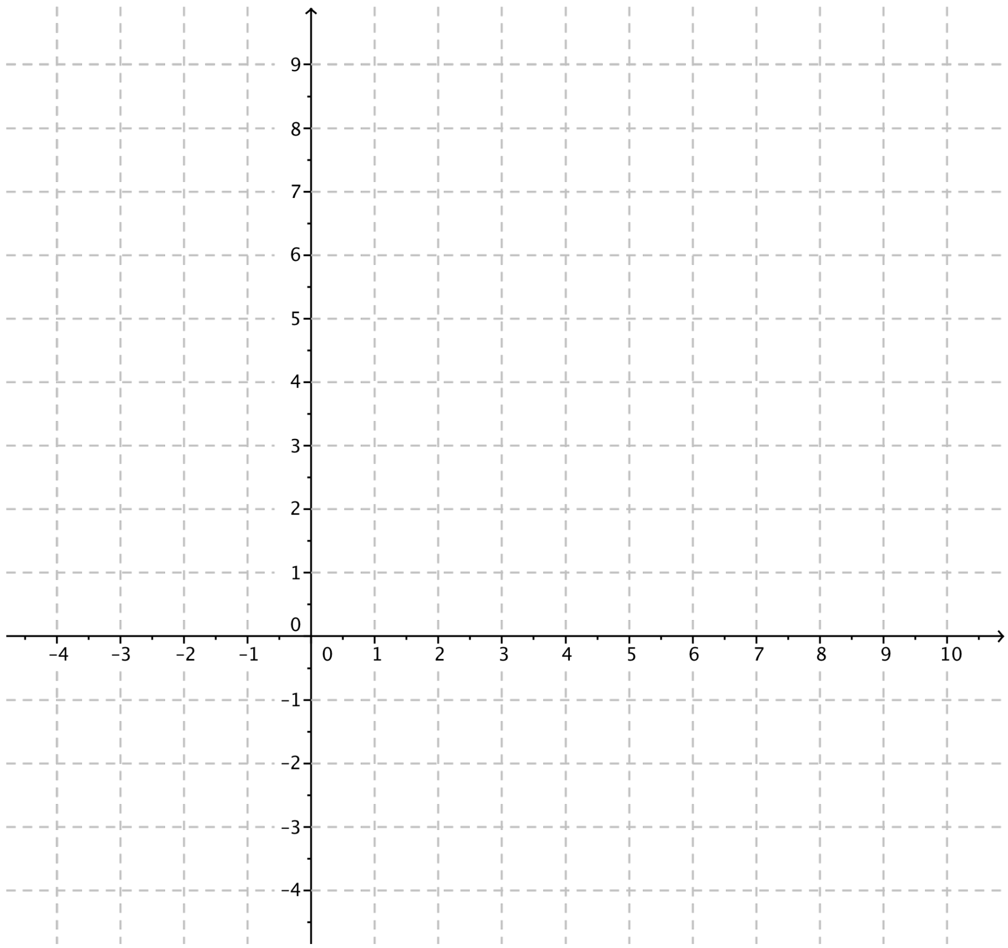
On the set of axes below, plot the points from the table and sketch the graph of $y = \log_2(x)$. Next, sketch the diagonal line with equation $y = x$, and then reflect the graph of $y = \log_2(x)$ across the line.



2. Working independently, predict the relation between the graphs of the functions $f(x) = 3^x$ and $g(x) = \log_3(x)$. Test your predictions by sketching the graphs of these two functions. Write your prediction in your notebook, provide justification for your prediction, and compare your prediction with that of your neighbor.



3. Now let's compare the graphs of the functions $f_2(x) = 2^x$ and $f_3(x) = 3^x$; sketch the graphs of the two exponential functions on the same set of axes; then, answer the questions below.



a. Where do the two graphs intersect?

b. For which values of x is $2^x < 3^x$?

c. For which values of x is $2^x > 3^x$?

d. What happens to the values of the functions f_2 and f_3 as $x \to \infty$?

e. What happens to the values of the functions f_2 and f_3 as $x \to -\infty$?

f. Does either graph ever intersect the x-axis? Explain how you know.

4. Add sketches of the two logarithmic functions $g_2(x) = \log_2(x)$ and $g_3(x) = \log_3(x)$ to the axes with the graphs of the exponential functions; then, answer the questions below.

a. Where do the two logarithmic graphs intersect?

b. For which values of x is $\log_2(x) < \log_3(x)$?

c. For which values of x is $\log_2(x) > \log_3(x)$?

d. What happens to the values of the functions f_2 and f_3 as $x \to \infty$?

e. What happens to the values of the functions f_2 and f_3 as $x \to 0$?

f. Does either graph ever intersect the y-axis? Explain how you know.

g. Describe the similarities and differences in the behavior of $f_2(x)$ and $g_2(x)$ as $x \to \infty$.

Problem Set

1. Sketch the graphs of the functions $f(x) = 5^x$ and $g(x) = \log_5(x)$.

2. Sketch the graphs of the functions $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = \log_{\frac{1}{2}}(x)$.

3. Sketch the graphs of the functions $f_1(x) = \left(\frac{1}{2}\right)^x$ and $f_2(x) = \left(\frac{3}{4}\right)^x$ on the same sheet of graph paper and answer the following questions.
   a. Where do the two exponential graphs intersect?
   b. For which values of x is $\left(\frac{1}{2}\right)^x < \left(\frac{3}{4}\right)^x$?
   c. For which values of x is $\left(\frac{1}{2}\right)^x > \left(\frac{3}{4}\right)^x$?
   d. What happens to the values of the functions f_1 and f_2 as $x \to \infty$?
   e. What are the domains of the two functions f_1 and f_2?

4. Use the information from Problem 3 together with the relationship between graphs of exponential and logarithmic functions to sketch the graphs of the functions $g_1(x) = \log_{\frac{1}{2}}(x)$ and $g_2(x) = \log_{\frac{3}{4}}(x)$ on the same sheet of graph paper. Then, answer the following questions.
   a. Where do the two logarithmic graphs intersect?
   b. For which values of x is $\log_{\frac{1}{2}}(x) < \log_{\frac{3}{4}}(x)$?
   c. For which values of x is $\log_{\frac{1}{2}}(x) > \log_{\frac{3}{4}}(x)$?
   d. What happens to the values of the functions g_1 and g_2 as $x \to \infty$?
   e. What are the domains of the two functions g_1 and g_2?

5. For each function f, find a formula for the function h in terms of x.
   a. If $f(x) = x^3$, find $h(x) = 128f\left(\frac{1}{4}x\right) + f(2x)$.
   b. If $f(x) = x^2 + 1$, find $h(x) = f(x + 2) - f(2)$.
   c. If $f(x) = x^3 + 2x^2 + 5x + 1$, find $h(x) = \frac{f(x) + f(-x)}{2}$.
   d. If $f(x) = x^3 + 2x^2 + 5x + 1$, find $h(x) = \frac{f(x) - f(-x)}{2}$.

6. In Problem 5, parts (c) and (d), list at least two aspects about the formulas you found as they relate to the function $f(x) = x^3 + 2x^2 + 5x + 1$.

7. For each of the functions f and g below, write an expression for (i) $f\big(g(x)\big)$, (ii) $g\big(f(x)\big)$, and (iii) $f\big(f(x)\big)$ in terms of x.

   a. $f(x) = x^{\frac{2}{3}},\ g(x) = x^{12}$

   b. $f(x) = \frac{b}{x-a},\ g(x) = \frac{b}{x} + a$ for two numbers a and b, when x is not equal to 0 or a

   c. $f(x) = \frac{x+1}{x-1},\ g(x) = \frac{x+1}{x-1}$, when x is not equal to 1 or -1

   d. $f(x) = 2^x,\ g(x) = \log_2(x)$

   e. $f(x) = \ln(x),\ g(x) = e^x$

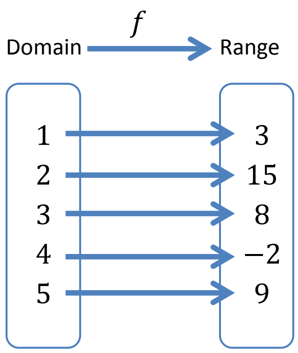
   f. $f(x) = 2 \cdot 100^x,\ g(x) = \frac{1}{2}\log\left(\frac{1}{2}x\right)$

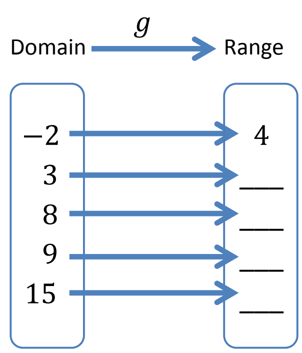
Lesson 19: The Inverse Relationship Between Logarithmic and Exponential Functions

Classwork

Opening Exercise

a. Consider the mapping diagram of the function f below. Fill in the blanks of the mapping diagram of g to construct a function that "undoes" each output value of f by returning the original input value of f. (The first one is done for you.)





b. Write the set of input-output pairs for the functions f and g by filling in the blanks below. (The set F for the function f has been done for you.)
$F = \{(1,3), (2,15), (3,8), (4,-2), (5,9)\}$

$G = \{(-2,4),$ ________, ________, ________, ________ $\}$

c. How can the points in the set G be obtained from the points in F?

d. Peter studied the mapping diagrams of the functions f and g above and exclaimed, "I can get the mapping diagram for g by simply taking the mapping diagram for f and reversing all of the arrows!" Is he correct?

Exercises

For each function f in Exercises 1–5, find the formula for the corresponding inverse function g. Graph both functions on a calculator to check your work.

1. $f(x) = 1 - 4x$

2. $f(x) = x^3 - 3$

3. $f(x) = 3\log(x^2)$ for $x > 0$

4. $f(x) = 2^{x-3}$

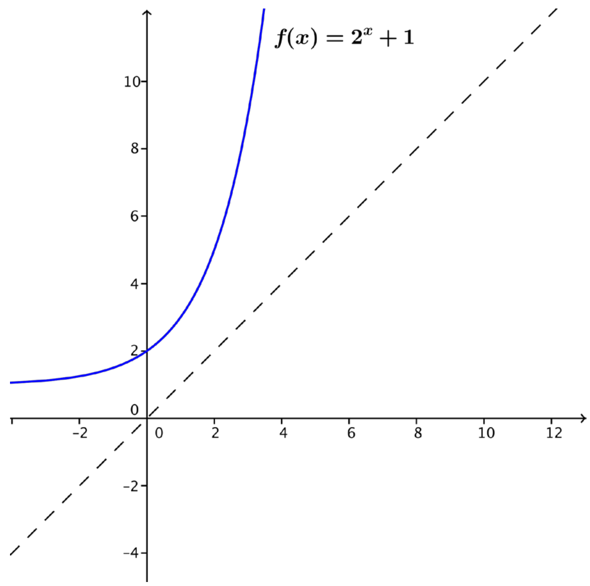
5. $f(x) = \frac{x+1}{x-1}$ for $x \neq 1$

6. Cindy thinks that the inverse of $f(x) = x - 2$ is $g(x) = 2 - x$. To justify her answer, she calculates $f(2) = 0$ and then substitutes the output 0 into g to get $g(0) = 2$, which gives back the original input. Show that Cindy is incorrect by using other examples from the domain and range of f.

7. After finding the inverse for several functions, Henry claims that every function must have an inverse. Rihanna says that his statement is not true and came up with the following example: If $f(x) = |x|$ has an inverse, then because $f(3)$ and $f(-3)$ both have the same output 3, the inverse function g would have to map 3 to both 3 and -3 simultaneously, which violates the definition of a function. What is another example of a function without an inverse?

Example

Consider the function $f(x) = 2^x + 1$, whose graph is shown at right.



a. What are the domain and range of f?

b. Sketch the graph of the inverse function g on the graph. What type of function do you expect g to be?

c. What are the domain and range of g? How does that relate to your answer in part (a)?

d. Find the formula for g.

Lesson Summary

- **INVERTIBLE FUNCTION:** Let f be a function whose domain is the set X and whose image is the set Y. Then f is *invertible* if there exists a function g with domain Y and image X such that f and g satisfy the property:

  For all x in X and y in Y, $f(x) = y$ if and only if $g(y) = x$.
- The function g is called the *inverse* of f.
- If two functions whose domain and range are a subset of the real numbers are inverses, then their graphs are reflections of each other across the diagonal line given by $y = x$ in the Cartesian plane.
- If f and g are inverses of each other, then
  - The domain of f is the same set as the range of g.
  - The range of f is the same set as the domain of g.
- In general, to find the formula for an inverse function g of a given function f:
  - Write $y = f(x)$ using the formula for f.
  - Interchange the symbols x and y to get $x = f(y)$.
  - Solve the equation for y to write y as an expression in x.
  - Then, the formula for g is the expression in x found in step (iii).
- The functions $f(x) = \log_b(x)$ and $g(x) = b^x$ are inverses of each other.

Problem Set

1. For each function h below, find two functions f and g such that $h(x) = f\big(g(x)\big)$. (There are many correct answers.)

   a. $h(x) = (3x + 7)^2$
   b. $h(x) = \sqrt[3]{x^2 - 8}$
   c. $h(x) = \frac{1}{2x - 3}$
   d. $h(x) = \frac{4}{(2x - 3)^3}$
   e. $h(x) = (x + 1)^2 + 2(x + 1)$
   f. $h(x) = (x + 4)^{\frac{4}{5}}$
   g. $h(x) = \sqrt[3]{\log(x^2 + 1)}$
   h. $h(x) = \sin(x^2 + 2)$
   i. $h(x) = \ln(\sin(x))$

2. Let f be the function that assigns to each student in your class his or her biological mother.
   a. Use the definition of function to explain why f is a function.
   b. In order for f to have an inverse, what condition must be true about the students in your class?
   c. If we enlarged the domain to include all students in your school, would this larger domain function have an inverse?

3. The table below shows a partially filled-out set of input-output pairs for two functions f and h that have the same finite domain of $\{0, 5, 10, 15, 20, 25, 30, 35, 40\}$.

x	0	5	10	15	20	25	30	35	40
$f(x)$	0	0.3	1.4		2.1		2.7	6	
$h(x)$	0	0.3	1.4		2.1		2.7	6	

   a. Complete the table so that f is invertible but h is definitely not invertible.
   b. Graph both functions and use their graphs to explain why f is invertible and h is not.

4. Find the inverse of each of the following functions. In each case, indicate the domain and range of both the original function and its inverse.
   a. $f(x) = \frac{3x-7}{5}$
   b. $f(x) = \frac{5+x}{6-2x}$
   c. $f(x) = e^{x-5}$
   d. $f(x) = 2^{5-8x}$
   e. $f(x) = 7\log(1+9x)$
   f. $f(x) = 8 + \ln(5 + \sqrt[3]{x})$
   g. $f(x) = \log\left(\frac{100}{3x+2}\right)$
   h. $f(x) = \ln(x) - \ln(x+1)$
   i. $f(x) = \frac{2^x}{2^x+1}$

5. Unlike square roots that do not have any real principal square roots for negative numbers, principal cube roots do exist for negative numbers: $\sqrt[3]{-8}$ is the real number -2 since it satisfies $-2 \cdot -2 \cdot -2 = -8$. Use the identities $\sqrt[3]{x^3} = x$ and $\left(\sqrt[3]{x}\right)^3 = x$ for any real number x to find the inverse of each of the functions below. In each case, indicate the domain and range of both the original function and its inverse.
   a. $f(x) = \sqrt[3]{2x}$ for any real number x.
   b. $f(x) = \sqrt[3]{2x-3}$ for any real number x.
   c. $f(x) = (x-1)^3 + 3$ for any real number x.

6. Suppose that the inverse of a function is the function itself. For example, the inverse of the function $f(x) = \frac{1}{x}$ (for $x \neq 0$) is just itself again, $g(x) = \frac{1}{x}$ (for $x \neq 0$). What symmetry must the graphs of all such functions have? (Hint: Study the graph of Exercise 5 in the lesson.)

7. When traveling abroad, you will find that daily temperatures in other countries are often reported in Celsius. The sentence, "It will be 25°C today in Paris," does not mean it will be freezing in Paris. It will often be necessary for you to convert temperatures reported in degrees Celsius to degrees Fahrenheit, the scale we use in the U.S. for reporting daily temperatures.

   Let f be the function that inputs a temperature measure in degrees Celsius and outputs the corresponding temperature measure in degrees Fahrenheit.

   a. Assuming that f is linear, we can use two points on the graph of f to determine a formula for f. In degrees Celsius, the freezing point of water is 0, and its boiling point is 100. In degrees Fahrenheit, the freezing point of water is 32, and its boiling point is 212. Use this information to find a formula for the function f. (Hint: Plot the points and draw the graph of f first, keeping careful track of the meaning of values on the x-axis and y-axis.)
   b. What temperature will Paris be in degrees Fahrenheit if it is reported that it will be 25°C?
   c. Find the inverse of the function f and explain its meaning in terms of degree scales that its domain and range represent.
   d. The graphs of f and its inverse are two lines that intersect in one point. What is that point? What is its significance in terms of degrees Celsius and degrees Fahrenheit?

Extension: Use the fact that, for $b > 1$, the functions $f(x) = b^x$ and $g(x) = \log_b(x)$ are increasing to solve the following problems. Recall that an increasing function f has the property that if both a and b are in the domain of f and $a < b$, then $f(a) < f(b)$.

8. For which values of x is $2^x < \frac{1}{1{,}000{,}000}$?

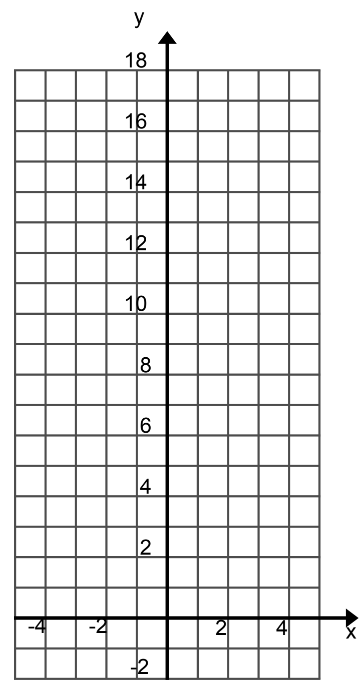
9. For which values of x is $\log_2(x) < -1{,}000{,}000$?

Lesson 20: Transformations of the Graphs of Logarithmic and Exponential Functions

Classwork

Opening Exercise

a. Sketch the graphs of the three functions $f(x) = x^2$, $g(x) = (2x)^2 + 1$, and $h(x) = 4x^2 + 1$.



i. Describe the transformations that will take the graph of $f(x) = x^2$ to the graph of $g(x) = (2x)^2 + 1$.

ii. Describe the transformations that will take the graph of $f(x) = x^2$ to the graph of $h(x) = 4x^2 + 1$.

iii. Explain why g and h from parts (i) and (ii) are equivalent functions.

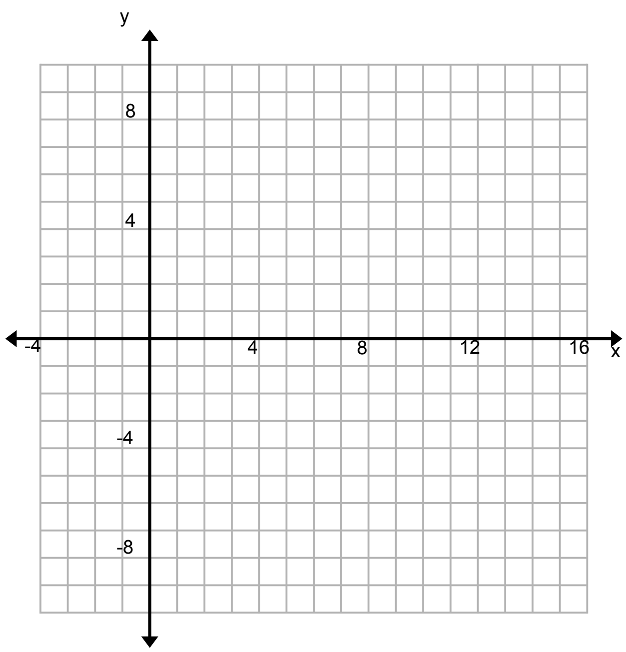
b. Describe the transformations that will take the graph of $f(x) = \sin(x)$ to the graph of $g(x) = \sin(2x) - 3$.

c. Describe the transformations that will take the graph of $f(x) = \sin(x)$ to the graph of $h(x) = 4\sin(x) - 3$.

d. Explain why g and h from parts (b)–(c) are *not* equivalent functions.

Exploratory Challenge

a. Sketch the graph of $f(x) = \log_2(x)$ by identifying and plotting at least five key points. Use the table below to help you get started.



b. Describe the transformations that will take the graph of f to the graph of $g(x) = \log_2(4x)$.

c. Describe the transformations that will take the graph of f to the graph of $h(x) = 2 + \log_2(x)$.

d. Complete the table below for f, g, and h and describe any patterns that you notice.

x	$f(x)$	$g(x)$	$h(x)$
$\frac{1}{4}$			
$\frac{1}{2}$			
1			
2			
4			
8			

e. Graph the three functions on the same coordinate axes and describe any patterns that you notice. Use a property of logarithms to show that g and h are equivalent.

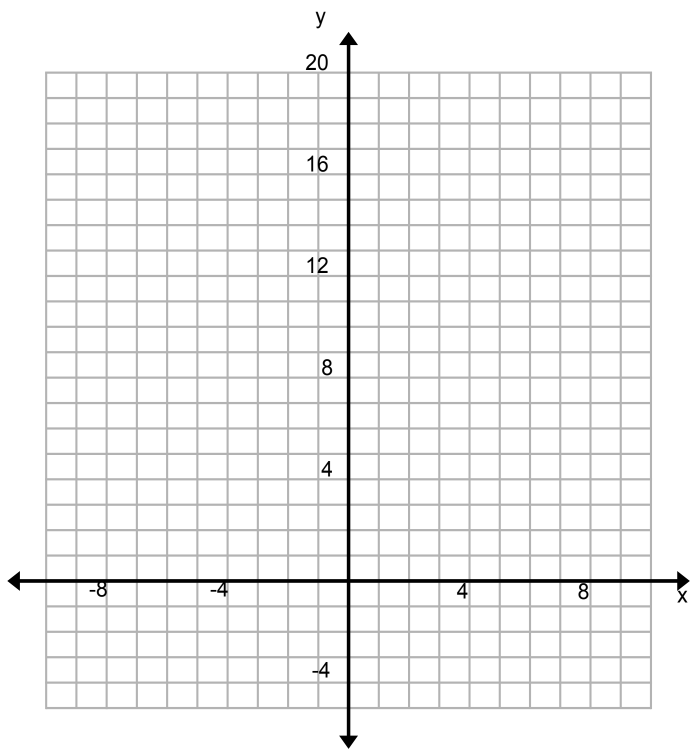
f. Describe the graph of $g(x) = \log_2\left(\frac{x}{4}\right)$ as a vertical translation of the graph of $f(x) = \log_2(x)$. Justify your response.

g. Describe the graph of $h(x) = \log_2(x) + 3$ as a horizontal scaling of the graph of $f(x) = \log_2(x)$. Justify your response.

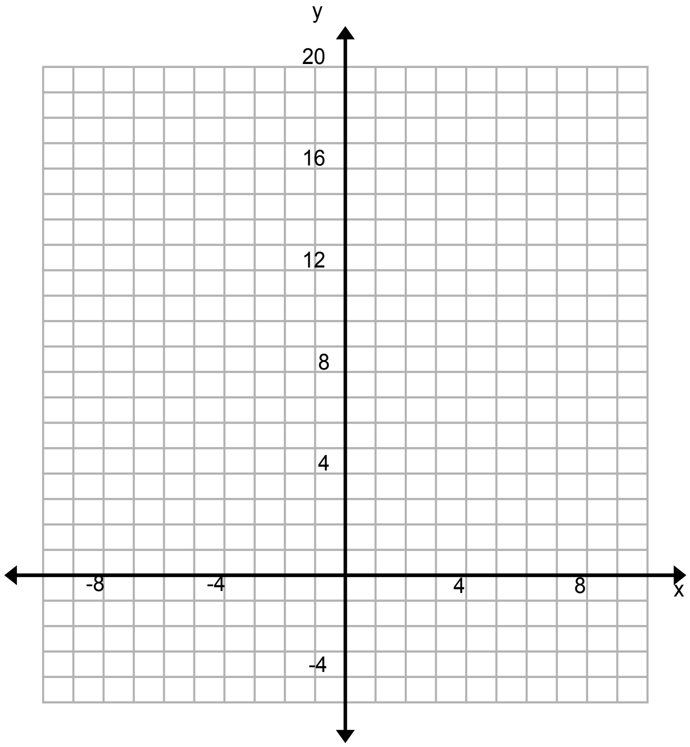
h. Do the functions $f(x) = \log_2(x) + \log_2(4)$ and $g(x) = \log_2(x + 4)$ have the same graphs? Justify your reasoning.

i. Use the properties of exponents to explain why the graphs of $f(x) = 4^x$ and $g(x) = 2^{2x}$ are identical.

j. Use the properties of exponents to predict what the graphs of $f(x) = 4 \cdot 2^x$ and $g(x) = 2^{x+2}$ will look like compared to one another. Describe the graphs of f and g as transformations of the graph of $f = 2^x$. Confirm your prediction by graphing f and g on the same coordinate axes.



k. Graph $f(x) = 2^x$, $g(x) = 2^{-x}$, and $h(x) = \left(\frac{1}{2}\right)^x$ on the same coordinate axes. Describe the graphs of g and h as transformations of the graph of f. Use the properties of exponents to explain why g and h are equivalent.

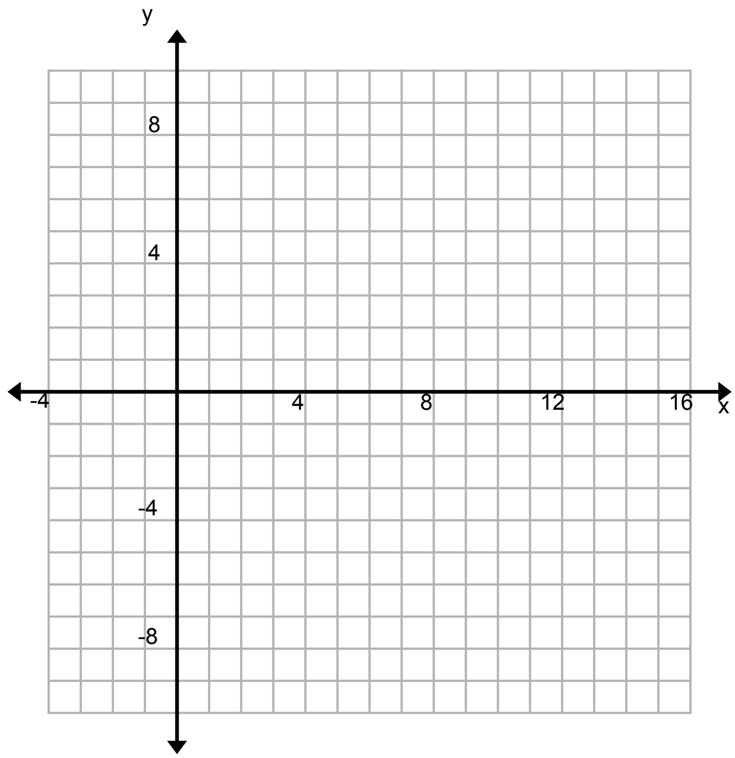


Example 1: Graphing Transformations of the Logarithm Functions

The general form of a logarithm function is given by $f(x) = k + a\log_b(x - h)$, where a, b, k, and h are real numbers such that b is a positive number not equal to 1, and $x - h > 0$.

a. Given $g(x) = 3 + 2\log(x - 2)$, describe the graph of g as a transformation of the common logarithm function.

b. Graph the common logarithm function and g on the same coordinate axes.



Example 2: Graphing Transformations of Exponential Functions

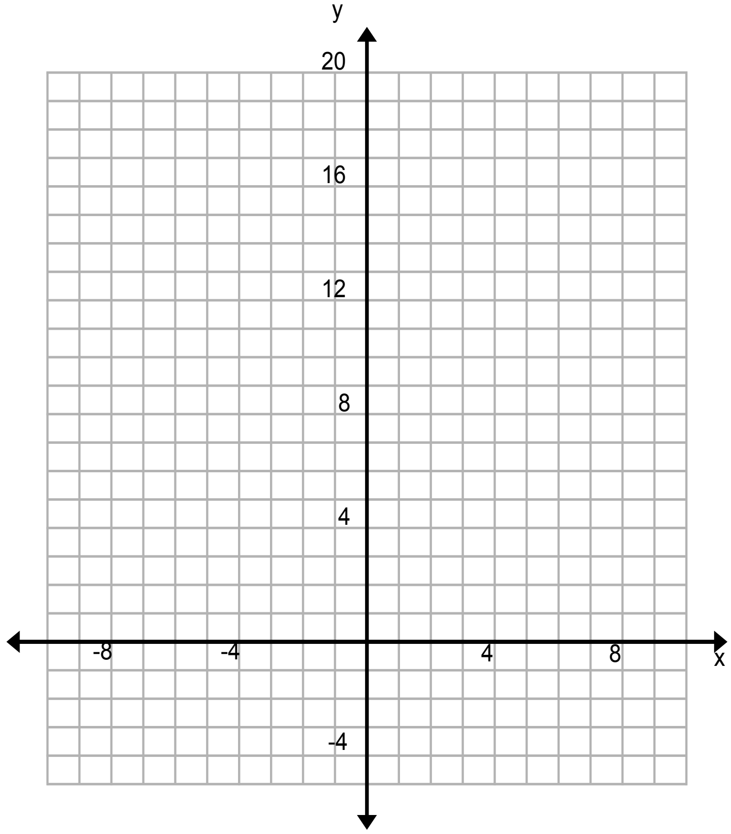
The general form of the exponential function is given by $f(x) = a \cdot b^x + k$, where a, b, and k are real numbers such that b is a positive number not equal to 1.

a. Use the properties of exponents to transform the function $g(x) = 3^{2x+1} - 2$ to the general form, and then graph it. What are the values of a, b, and k?

b. Describe the graph of g as a transformation of the graph of $h(x) = 9^x$.

c. Describe the graph of g as a transformation of the graph of $h(x) = 3^x$.

d. Graph g using transformations.



Exercises 1–4

Graph each pair of functions by first graphing f and then graphing g by applying transformations of the graph of f. Describe the graph of g as a transformation of the graph of f.

1. $f(x) = \log_3(x)$ and $g(x) = 2\log_3(x-1)$

2. $f(x) = \log(x)$ and $g(x) = \log(100x)$

3. $f(x) = \log_5 x$ and $g(x) = -\log_5\big(5(x+2)\big)$

4. $f(x) = 3^x$ and $g(x) = -2 \cdot 3^{x-1}$

Lesson Summary

GENERAL FORM OF A LOGARITHMIC FUNCTION: $f(x) = k + a\log_b(x-h)$ such that a, h, and k are real numbers, b is any positive number not equal to 1, and $x - h > 0$.

GENERAL FORM OF AN EXPONENTIAL FUNCTION: $f(x) = a \cdot b^x + k$ such that a and k are real numbers, and b is any positive number not equal to 1.

The properties of logarithms and exponents can be used to rewrite expressions for functions in equivalent forms that can then be graphed by applying transformations.

Problem Set

1. Describe each function as a transformation of the graph of a function in the form $f(x) = \log_b(x)$. Sketch the graph of f and the graph of g by hand. Label key features such as intercepts, increasing or decreasing intervals, and the equation of the vertical asymptote.

   a. $g(x) = \log_2(x-3)$

   b. $g(x) = \log_2(16x)$

   c. $g(x) = \log_2\left(\frac{8}{x}\right)$

   d. $g(x) = \log_2((x-3)^2)$

2. Each function graphed below can be expressed as a transformation of the graph of $f(x) = \log(x)$. Write an algebraic function for g and h and state the domain and range.

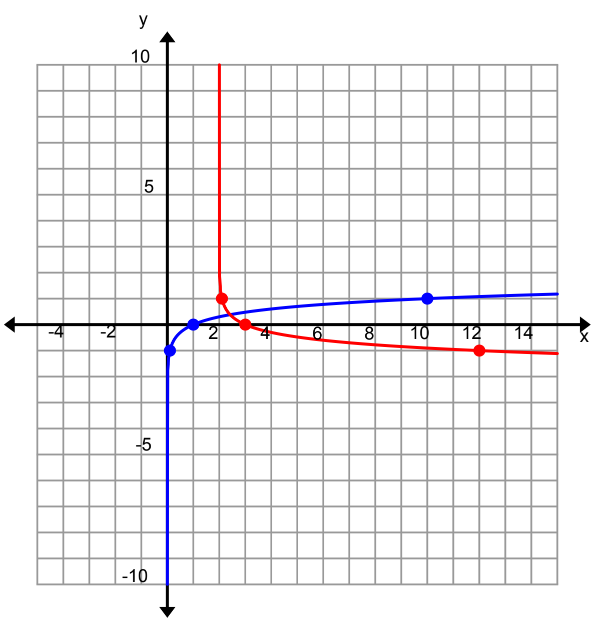


Figure 1: Graphs of $f(x) = \log(x)$ and the function g

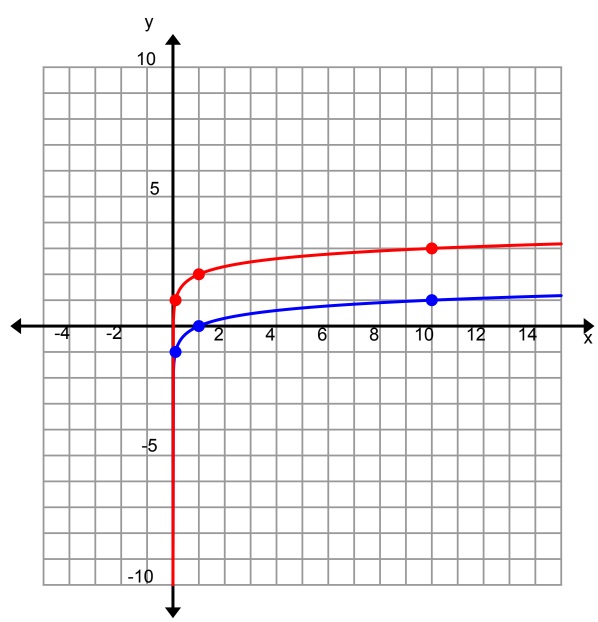


Figure 2: Graphs of $f(x) = \log(x)$ and the function h

3. Describe each function as a transformation of the graph of a function in the form $f(x) = b^x$. Sketch the graph of f and the graph of g by hand. Label key features such as intercepts, increasing or decreasing intervals, and the horizontal asymptote. (Estimate when needed from the graph.)
   a. $g(x) = 2 \cdot 3^x - 1$
   b. $g(x) = 2^{2x} + 3$
   c. $g(x) = 3^{x-2}$
   d. $g(x) = -9^{\frac{x}{2}} + 1$

4. Using the function $f(x) = 2^x$, create a new function g whose graph is a series of transformations of the graph of f with the following characteristics:
   - The graph of g is decreasing for all real numbers.
   - The equation for the horizontal asymptote is $y = 5$.
   - The y-intercept is 7.

5. Using the function $f(x) = 2^x$, create a new function g whose graph is a series of transformations of the graph of f with the following characteristics:
   - The graph of g is increasing for all real numbers.
   - The equation for the horizontal asymptote is $y = 5$.
   - The y-intercept is 4.

6. Given the function $g(x) = \left(\frac{1}{4}\right)^{x-3}$:
   a. Write the function g as an exponential function with base 4. Describe the transformations that would take the graph of $f(x) = 4^x$ to the graph of g.
   b. Write the function g as an exponential function with base 2. Describe two different series of transformations that would take the graph of $f(x) = 2^x$ to the graph of g.

7. Explore the graphs of functions in the form $f(x) = \log(x^n)$ for $n > 1$. Explain how the graphs of these functions change as the values of n increase. Use a property of logarithms to support your reasoning.

8. Use a graphical approach to solve each equation. If the equation has no solution, explain why.
   a. $\log(x) = \log(x - 2)$
   b. $\log(x) = \log(2x)$
   c. $\log(x) = \log\left(\frac{2}{x}\right)$
   d. Show algebraically that the exact solution to the equation in part (c) is $\sqrt{2}$.

9. Make a table of values for $f(x) = x^{\frac{1}{\log(x)}}$ for $x > 1$. Graph this function for $x > 1$. Use properties of logarithms to explain what you see in the graph and the table of values.

Lesson 21: The Graph of the Natural Logarithm Function

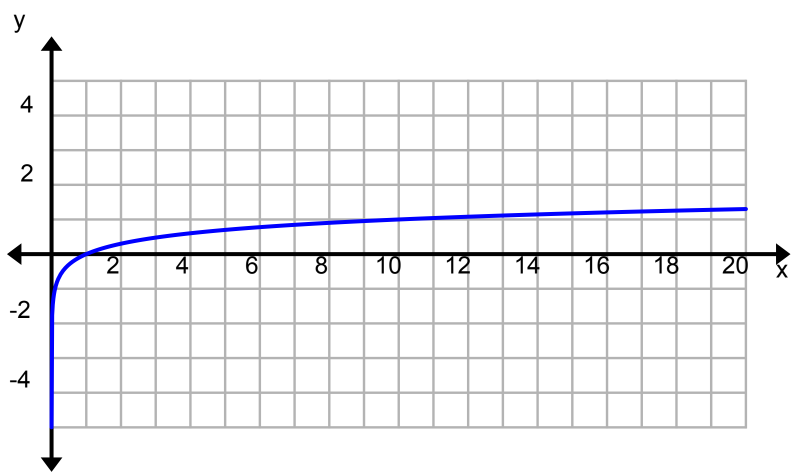
Classwork

Exploratory Challenge

Your task is to compare graphs of base b logarithm functions to the graph of the common logarithm function $f(x) = \log(x)$ and summarize your results with your group. Recall that the base of the common logarithm function is 10. A graph of f is provided below.

a. Select at least one base value from this list: $\frac{1}{10}, \frac{1}{2}, 2, 5, 20, 100$. Write a function in the form $g(x) = \log_b(x)$ for your selected base value, b.

b. Graph the functions f and g in the same viewing window using a graphing calculator or other graphing application, and then add a sketch of the graph of g to the graph of f shown below.



c. Describe how the graph of g for the base you selected compares to the graph of $f(x) = \log(x)$.

d. Share your results with your group and record observations on the graphic organizer below. Prepare a group presentation that summarizes the group's findings.

How does the graph of $g(x) = \log_b(x)$ compare to the graph of $f(x) = \log(x)$ for various values of b?	
$0 < b < 1$	
$1 < b < 10$	
$b > 10$	

Exercise 1

Use the change of base property to rewrite each function as a common logarithm.

<u>Base b</u> <u>Base 10 (Common Logarithm)</u>

$g(x) = \log_{\frac{1}{4}}(x)$

$g(x) = \log_{\frac{1}{2}}(x)$

$g(x) = \log_2(x)$

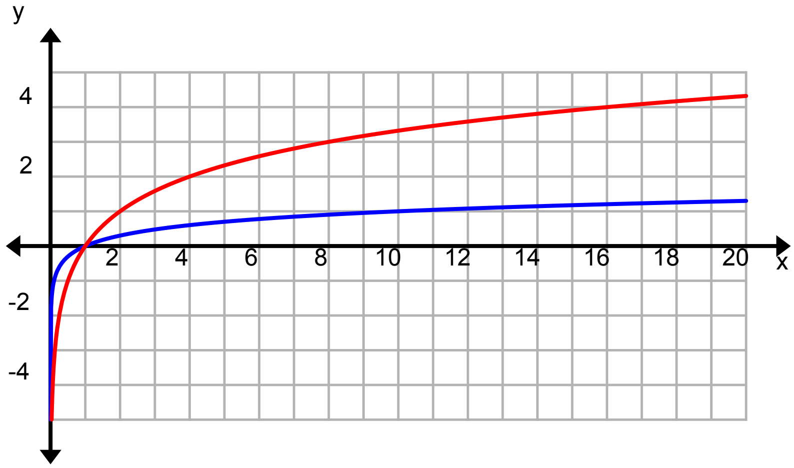
$g(x) = \log_5(x)$

$g(x) = \log_{20}(x)$

$g(x) = \log_{100}(x)$

Example 1: The Graph of the Natural Logarithm Function $f(x) = \ln(x)$

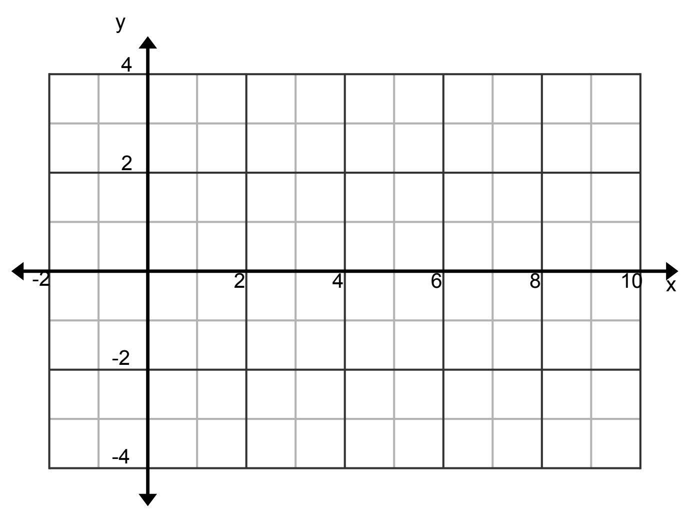
Graph the natural logarithm function below to demonstrate where it sits in relation to the base 2 and base 10 logarithm functions.



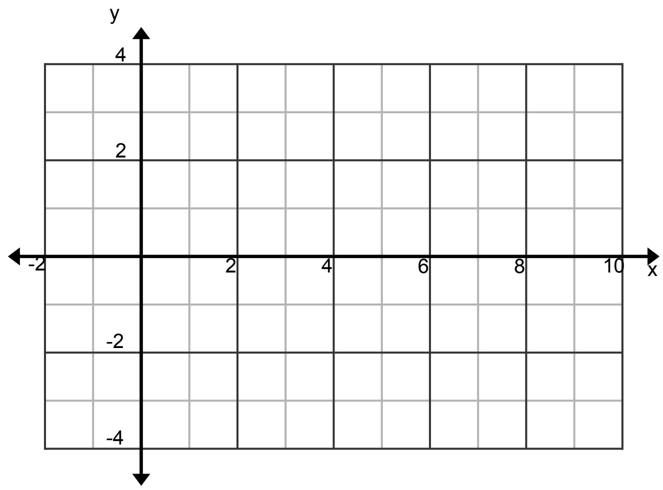
Example 2

Graph each function by applying transformations of the graphs of the natural logarithm function.

a. $f(x) = 3\ln(x - 1)$



b. $g(x) = \log_6(x) - 2$



Problem Set

1. Rewrite each logarithm function as a natural logarithm function.
   a. $f(x) = \log_5(x)$
   b. $f(x) = \log_2(x - 3)$
   c. $f(x) = \log_2\left(\frac{x}{3}\right)$
   d. $f(x) = 3 - \log(x)$
   e. $f(x) = 2\log(x + 3)$
   f. $f(x) = \log_5(25x)$

2. Describe each function as a transformation of the natural logarithm function $f(x) = \ln(x)$.
   a. $g(x) = 3\ln(x + 2)$
   b. $g(x) = -\ln(1 - x)$
   c. $g(x) = 2 + \ln(e^2 x)$
   d. $g(x) = \log_5(25x)$

3. Sketch the graphs of each function in Problem 2 and identify the key features including intercepts, decreasing or increasing intervals, and the vertical asymptote.

4. Solve the equation $e^{-x} = \ln(x)$ graphically.

5. Use a graphical approach to explain why the equation $\log(x) = \ln(x)$ has only one solution.

6. Juliet tried to solve this equation as shown below using the change of base property and concluded there is no solution because $\ln(10) \neq 1$. Construct an argument to support or refute her reasoning.

$$\log(x) = \ln(x)$$
$$\frac{\ln(x)}{\ln(10)} = \ln(x)$$
$$\left(\frac{\ln(x)}{\ln(10)}\right)\frac{1}{\ln(x)} = (\ln(x))\frac{1}{\ln(x)}$$
$$\frac{1}{\ln(10)} = 1$$

7. Consider the function f given by $f(x) = \log_x(100)$ for $x > 0$ and $x \neq 1$.
   a. What are the values of $f(100)$, $f(10)$, and $f(\sqrt{10})$?
   b. Why is the value 1 excluded from the domain of this function?
   c. Find a value x so that $f(x) = 0.5$.
   d. Find a value w so that $f(w) = -1$.
   e. Sketch a graph of $y = \log_x(100)$ for $x > 0$ and $x \neq 1$.

Lesson 22: Choosing a Model

Classwork

Opening Exercise

a. You are working on a team analyzing the following data gathered by your colleagues:

$$(-1.1, 5), (0, 105), (1.5, 178), (4.3, 120).$$

Your coworker Alexandra says that the model you should use to fit the data is

$$k(t) = 100 \cdot \sin(1.5t) + 105.$$

Sketch Alexandra's model on the axes at left on the next page.

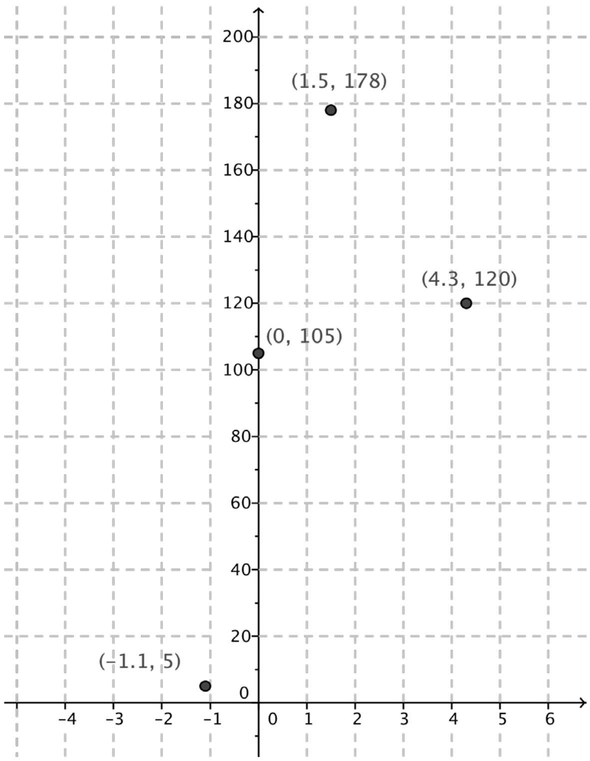
b. How does the graph of Alexandra's model $k(t) = 100 \cdot \sin(1.5t) + 105$ relate to the four points? Is her model a good fit to this data?

c. Another teammate Randall says that the model you should use to fit the data is

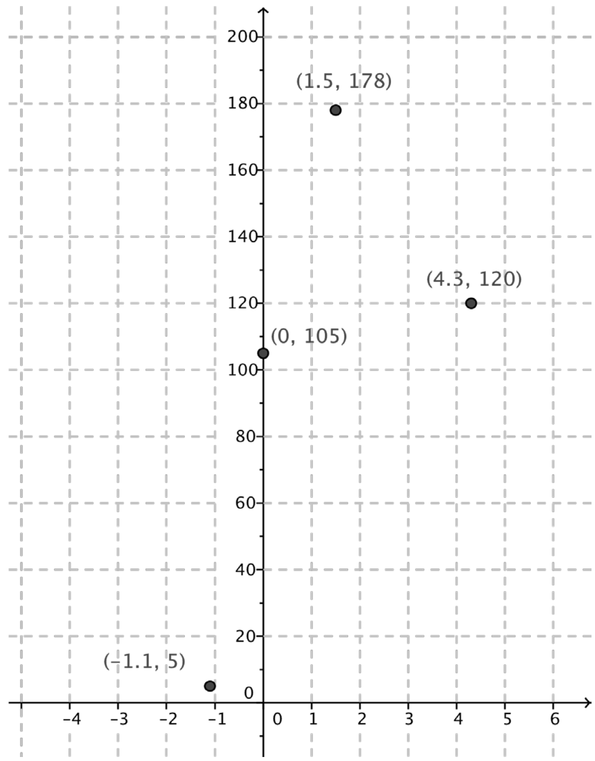
$$g(t) = -16t^2 + 72t + 105.$$

Sketch Randall's model on the axes at right on the next page.

d. How does the graph of Randall's model $g(t) = -16t^2 + 72t + 105$ relate to the four points? Is his model a good fit to the data?



Alexandra's Model



Randall's Model

e. Suppose the four points represent positions of a projectile fired into the air. Which of the two models is more appropriate in that situation, and why?

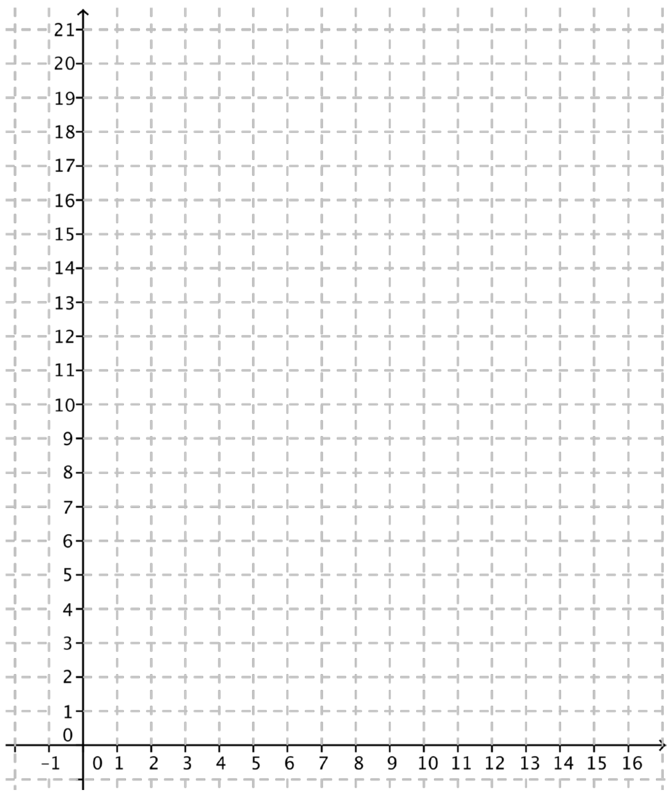
f. In general, how do we know which model to choose?

Exercises

1. The table below contains the number of daylight hours in Oslo, Norway, on the specified dates.

Date	Hours and Minutes	Hours
August 1	$16:56$	16.82
September 1	$14:15$	14.25
October 1	$11:33$	11.55
November 1	$8:50$	8.90

a. Plot the data on the grid provided. You will need to decide how to best represent it.



b. Looking at the data, what type of function appears to be the best fit?

c. Looking at the context in which the data was gathered, what type of function should be used to model the data?

d. Do you have enough information to find a model that is appropriate for this data? Either find a model or explain what other information you would need to do so.

2. The goal of the U.S. Centers for Disease Control and Prevention (CDC) is to protect public health and safety through the control and prevention of disease, injury, and disability. Suppose that 45 people have been diagnosed with a new strain of the flu virus, and scientists estimate that each person with the virus will infect 5 people every day with the flu.

   a. What type of function should the scientists at the CDC use to model the initial spread of this strain of flu to try to prevent an epidemic? Explain how you know.

   b. Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.

3. An artist is designing posters for a new advertising campaign. The first poster takes 10 hours to design, but each subsequent poster takes roughly 15 minutes less time than the previous one as he gets more practice.

   a. What type of function models the amount of time needed to create n posters, for $n \leq 20$? Explain how you know.

   b. Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.

4. A homeowner notices that her heating bill is the lowest in the month of July and increases until it reaches its highest amount in the month of February. After February, the amount of the heating bill slowly drops back to the level it was in July, when it begins to increase again. The amount of the bill in February is roughly four times the amount of the bill in July.
   a. What type of function models the amount of the heating bill in a particular month? Explain how you know.

   b. Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.

5. An online merchant sells used books for $\$5$ each, and the sales tax rate is 6% of the cost of the books. Shipping charges are a flat rate of $\$4$ plus an additional $\$1$ per book.
   a. What type of function models the total cost, including the shipping costs, of a purchase of x books? Explain how you know.

   b. Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.

6. A stunt woman falls from a tall building in an action-packed movie scene. Her speed increases by 32 ft/s for every second that she is falling.
   a. What type of function models her distance from the ground at time t seconds? Explain how you know.

   b. Do you have enough information to find a model that is appropriate for this situation? Either find a model or explain what other information you would need to do so.

Lesson Summary

- If we expect from the context that each new term in the sequence of data is a constant added to the previous term, then we try a linear model.
- If we expect from the context that the second differences of the sequence are constant (meaning that the rate of change between terms either grows or shrinks linearly), then we try a quadratic model.
- If we expect from the context that each new term in the sequence of data is a constant multiple of the previous term, then we try an exponential model.
- If we expect from the context that the sequence of terms is periodic, then we try a sinusoidal model.

Model	Equation of Function	Rate of Change
Linear	$f(t) = at + b$ for $a \neq 0$	Constant
Quadratic	$g(t) = at^2 + bt + c$ for $a \neq 0$	Changing linearly
Exponential	$h(t) = ab^{ct}$ for $0 < b < 1$ or $b > 1$	A multiple of the current value
Sinusoidal	$k(t) = A\sin\big(w(t-h)\big) + k$ for $A, w \neq 0$	Periodic

Problem Set

1. A new car depreciates at a rate of about 20% per year, meaning that its resale value decreases by roughly 20% each year. After hearing this, Brett said that if you buy a new car this year, then after 5 years the car has a resale value of $\$0$. Is his reasoning correct? Explain how you know.

2. Alexei just moved to Seattle, and he keeps track of the average rainfall for a few months to see if the city deserves its reputation as the rainiest city in the United States.

Month	Average rainfall
July	0.93 in.
September	1.61 in.
October	3.24 in.
December	6.06 in.

What type of function should Alexei use to model the average rainfall in month t?

3. Sunny, who wears her hair long and straight, cuts her hair once per year on January 1, always to the same length. Her hair grows at a constant rate of 2 cm per month. Is it appropriate to model the length of her hair with a sinusoidal function? Explain how you know.

4. On average, it takes 2 minutes for a customer to order and pay for a cup of coffee.
   a. What type of function models the amount of time you will wait in line as a function of how many people are in front of you? Explain how you know.
   b. Find a model that is appropriate for this situation.

5. An online ticket-selling service charges $\$50$ for each ticket to an upcoming concert. In addition, the buyer must pay 8% sales tax and a convenience fee of $\$6.00$ for the purchase.
   a. What type of function models the total cost of the purchase of n tickets in a single transaction?
   b. Find a model that is appropriate for this situation.

6. In a video game, the player must earn enough points to pass one level and progress to the next as shown in the table below.

To pass this level …	You need this many total points …
1	5,000
2	15,000
3	35,000
4	65,000

   That is, the increase in the required number of points increases by $10{,}000$ points at each level.
   a. What type of function models the total number of points you need to pass to level n? Explain how you know.
   b. Find a model that is appropriate for this situation.

7. The southern white rhinoceros reproduces roughly once every three years, giving birth to one calf each time. Suppose that a nature preserve houses 100 white rhinoceroses, 50 of which are female. Assume that half of the calves born are female and that females can reproduce as soon as they are 1 year old.
   a. What type of function should be used to model the population of female white rhinoceroses in the preserve?
   b. Assuming that there is no death in the rhinoceros population, find a function to model the population of female white rhinoceroses in the preserve.
   c. Realistically, not all of the rhinoceroses will survive each year, so we will assume a 5% death rate of all rhinoceroses. Now what type of function should be used to model the population of female white rhinoceroses in the preserve?
   d. Find a function to model the population of female white rhinoceroses in the preserve, taking into account the births of new calves and the 5% death rate.

Lesson 23: Bean Counting

Classwork

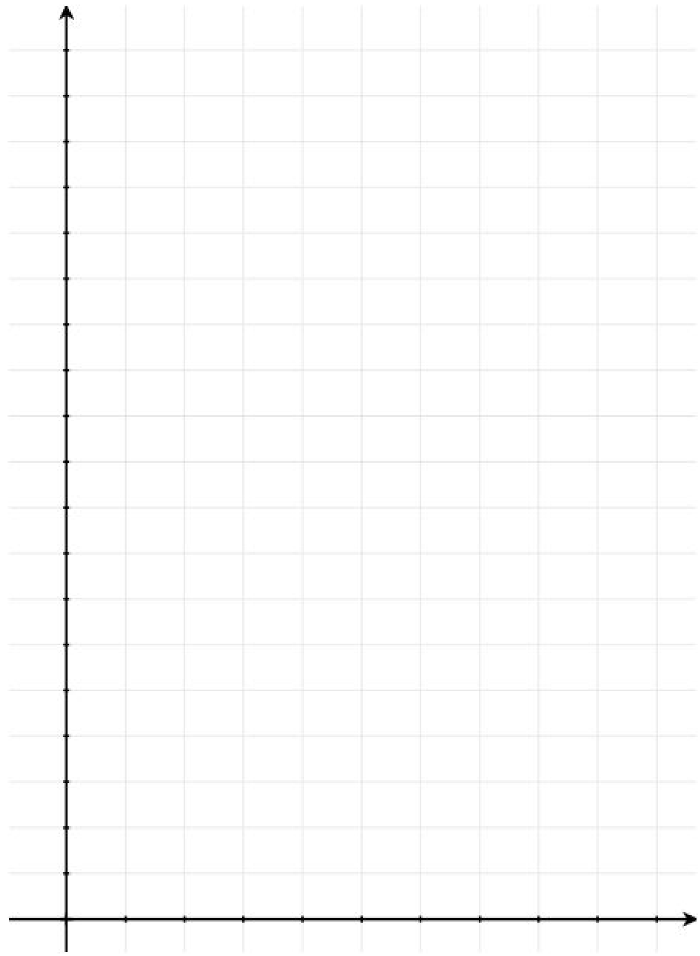
Mathematical Modeling Exercises

1. Working with a partner, you are going to gather some data, analyze it, and find a function to use to model it. Be prepared to justify your choice of function to the class.
   a. Gather your data: For each trial, roll the beans from the cup to the paper plate. Count the number of beans that land marked side up, and add that many beans to the cup. Record the data in the table below. Continue until you have either completed 10 trials or the number of beans at the start of the trial exceeds the number that you have.

Trial Number, t	Number of Beans at Start of Trial	Number of Beans That Landed Marked-Side Up
1	1	
2		
3		
4		
5		
6		
7		
8		
9		
10		

   b. Based on the context in which you gathered this data, which type of function would best model your data points?

c. Plot the data: Plot the trial number on the horizontal axis and the number of beans in the cup at the start of the trial on the vertical axis. Be sure to label the axes appropriately and to choose a reasonable scale for the axes.



d. Analyze the data: Which type of function would best fit your data? Explain your reasoning.

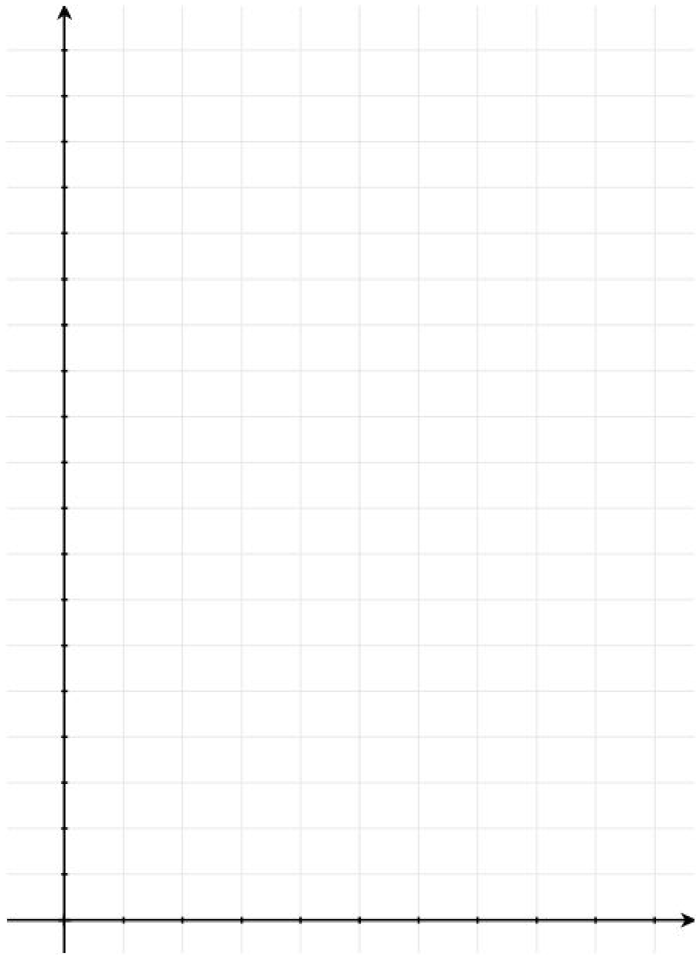
e. Model the data: Enter the data into the calculator and use the appropriate type of regression to find an equation that fits this data. Round the constants to two decimal places.

2. This time, we are going to start with 50 beans in your cup. Roll the beans onto the plate and remove any beans that land marked-side up. Repeat until you have no beans remaining.

   a. Gather your data: For each trial, roll the beans from the cup to the paper plate. Count the number of beans that land marked-side up, and remove that many beans from the plate. Record the data in the table below. Repeat until you have no beans remaining.

Trial Number, t	Number of Beans at Start of Trial	Number of Beans That Landed Marked-Side Up
1	50	
2		
3		
4		
5		
6		
7		
8		
9		
10		

b. Plot the data: Plot the trial number on the horizontal axis and the number of beans in the cup at the start of the trial on the vertical axis. Be sure to label the axes appropriately and choose a reasonable scale for the axes.



c. Analyze the data: Which type of function would best fit your data? Explain your reasoning.

d. Make a prediction: What do you expect the values of a and b to be for your function? Explain your reasoning.

e. Model the data: Enter the data into the calculator. Do not enter your final data point of 0 beans. Use the appropriate type of regression to find an equation that fits this data. Round the constants to two decimal places.

Problem Set

1. For this exercise, we will consider three scenarios for which data has been collected and functions have been found to model the data, where a, b, c, d, p, q, r, s, t, and u are positive real number constants.

   (i) The function $f(t) = a \cdot b^t$ models the original bean activity (Mathematical Modeling Exercise 1). Each bean is painted or marked on one side, and we start with one bean in the cup. A trial consists of throwing the beans in the cup and adding one more bean for each bean that lands marked side up.

   (ii) The function $g(t) = c \cdot d^t$ models a modified bean activity. Each bean is painted or marked on one side, and we start with one bean in the cup. A trial consists of throwing the beans in the cup and adding two more beans for each bean that lands marked side up.

   (iii) The function $h(t) = p \cdot q^t$ models the dice activity from the Exit Ticket. Start with one six-sided die in the cup. A trial consists of rolling the dice in the cup and adding one more die to the cup for each die that lands with a 6 showing.

   (iv) The function $j(t) = r \cdot s^t$ models a modified dice activity. Start with one six-sided die in the cup. A trial consists of rolling the dice in the cup and adding one more die to the cup for each die that lands with a 5 or a 6 showing.

   (v) The function $k(t) = u \cdot v^t$ models a modified dice activity. Start with one six-sided die in the cup. A trial consists of rolling the dice in the cup and adding one more die to the cup for each die that lands with an even number showing.

   a. What values do you expect for a, c, p, r, and u?

   b. What value do you expect for the base b in the function $f(t) = a \cdot b^t$?

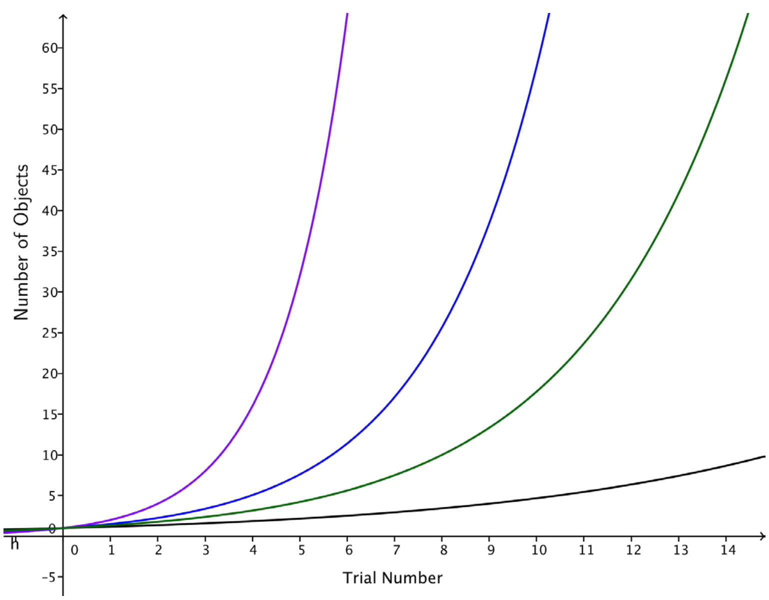
   c. What value do you expect for the base d in the function $g(t) = c \cdot d^t$?

   d. What value do you expect for the base q in the function $h(t) = p \cdot q^t$?

   e. What value do you expect for the base s in the function $j(t) = r \cdot s^t$?

   f. What value do you expect for the base v in the function $k(t) = u \cdot v^t$?

   g. The following graphs represent the four functions f, g, h, and j. Identify which graph represents which function.



2. Teams 1, 2, and 3 gathered data as shown in the tables below, and each team modeled their data using an exponential function of the form $f(t) = a \cdot b^t$.

   a. Which team should have the highest value of b? Which team should have the lowest value of b? Explain how you know.

Team 1	
Trial Number, t	Number of Beans
0	1
1	1
2	2
3	2
4	4
5	6
6	8
7	14
8	22
9	41
10	59

Team 2	
Trial Number, t	Number of Beans
0	1
1	1
2	1
3	2
4	2
5	3
6	5
7	7
8	12
9	18
10	27

Team 3	
Trial Number, t	Number of Beans
0	1
1	2
2	3
3	5
4	8
5	14
6	26
7	46
8	76
9	
10	

   b. Use a graphing calculator to find the equation that best fits each set of data. Do the equations of the functions provide evidence that your answer in part (a) is correct?

3. Omar has devised an activity in which he starts with 15 dice in his cup. A trial consists of rolling the dice in the cup and adding one more die to the cup for each die that lands with a 1, 2, or 3 showing.

   a. Find a function $f(t) = a(b^t)$ that Omar would expect to model his data.

   b. Solve the equation $f(t) = 30$. What does the solution mean?

   c. Omar wants to know in advance how many trials it should take for his initial quantity of 15 dice to double. He uses properties of exponents and logarithms to rewrite the function from part (a) as an exponential function of the form $f(t) = a\left(2^{t \cdot \log_2(b)}\right)$.

   d. Has Omar correctly applied the properties of exponents and logarithms to obtain an equivalent expression for his original equation in part (a)? Explain how you know.

   e. Explain how the modified formula from part (c) allows Omar to easily find the expected amount of time, t, for the initial quantity of dice to double.

4. Brenna has devised an activity in which she starts with 10 dice in her cup. A trial consists of rolling the dice in the cup and adding one more die to the cup for each die that lands with a 6 showing.

   a. Find a function $f(t) = a(b^t)$ that you would expect to model her data.

   b. Solve the equation $f(t) = 30$. What does your solution mean?

   c. Brenna wants to know in advance how many trials it should take for her initial quantity of 10 dice to triple. Use properties of exponents and logarithms to rewrite your function from part (a) as an exponential function of the form $f(t) = a(3^{ct})$.

   d. Explain how your formula from part (c) allows you to easily find the expected amount of time, t, for the initial quantity of dice to triple.

   e. Rewrite the formula for the function f using a base 10 exponential function.

   f. Use your formula from part (e) to find out how many trials it should take for the quantity of dice to grow to 100 dice.

5. Suppose that one bacteria population can be modeled by the function $P_1(t) = 500(2^t)$ and a second bacteria population can be modeled by the function $P_2(t) = 500(2.83^t)$, where t measures time in hours. Keep four digits of accuracy for decimal approximations of logarithmic values.

   a. What does the 500 mean in each function?

   b. Which population should double first? Explain how you know.

   c. How many hours and minutes will it take until the first population doubles?

   d. Rewrite the formula for $P_2(t)$ in the form $P_2(t) = a(2^{ct})$, for some real numbers a and c.

   e. Use your formula in part (d) to find the time, t, in hours and minutes until the second population doubles.

6. Copper has antibacterial properties, and it has been shown that direct contact with copper alloy C11000 at 20°C kills 99.9% of all methicillin-resistant *Staphylococcus aureus* (MRSA) bacteria in about 75 minutes. Keep four digits of accuracy for decimal approximations of logarithmic values.

   a. A function that models a population of $1{,}000$ MRSA bacteria t minutes after coming in contact with copper alloy C11000 is $P(t) = 1000(0.912)^t$. What does the base 0.912 mean in this scenario?

   b. Rewrite the formula for P as an exponential function with base $\frac{1}{2}$.

   c. Explain how your formula from part (b) allows you to easily find the time it takes for the population of MRSA to be reduced by half.

Lesson 24: Solving Exponential Equations

Classwork

Opening Exercise

In Lesson 7, we modeled a population of bacteria that doubled every day by the function $P(t) = 2^t$, where t was the time in days. We wanted to know the value of t when there were 10 bacteria. Since we didn't yet know about logarithms, we approximated the value of t numerically and we found that $P(t) = 10$ at approximately $t \approx 3.32$ days.

Use your knowledge of logarithms to find an exact value for t when $P(t) = 10$, and then use your calculator to approximate that value to four decimal places.

Exercises

1. Fiona modeled her data from the bean-flipping experiment in Lesson 23 by the function $f(t) = 1.263(1.357)^t$, and Gregor modeled his data with the function $g(t) = 0.972(1.629)^t$.
   a. Without doing any calculating, determine which student, Fiona or Gregor, accumulated 100 beans first. Explain how you know.

   b. Using Fiona's model …
      i. How many trials would be needed for her to accumulate 100 beans?

ii. How many trials would be needed for her to accumulate $1{,}000$ beans?

c. Using Gregor's model …

i. How many trials would be needed for him to accumulate 100 beans?

ii. How many trials would be needed for him to accumulate $1{,}000$ beans?

d. Was your prediction in part (a) correct? If not, what was the error in your reasoning?

2. Fiona wants to know when her model $f(t) = 1.263(1.357)^t$ predicts accumulations of 500, $5{,}000$, and $50{,}000$ beans, but she wants to find a way to figure it out without doing the same calculation three times.

   a. Let the positive number c represent the number of beans that Fiona wants to have. Then solve the equation $1.263(1.357)^t = c$ for t.

   b. Your answer to part (a) can be written as a function M of the number of beans c, where $c > 0$. Explain what this function represents.

   c. When does Fiona's model predict that she will accumulate …

      i. 500 beans?

      ii. 5000 beans?

      iii. $50{,}000$ beans?

3. Gregor states that the function g that he found to model his bean-flipping data can be written in the form $g(t) = 0.972\left(10^{\log(1.629)t}\right)$. Since $\log(1.629) \approx 0.2119$, he is using $g(t) = 0.972(10^{0.2119t})$ as his new model.

   a. Is Gregor correct? Is $g(t) = 0.972\left(10^{\log(1.629)t}\right)$ an equivalent form of his original function? Use properties of exponents and logarithms to explain how you know.

   b. Gregor also wants to find a function that will help him to calculate the number of trials his function g predicts it will take to accumulate 500, $5{,}000$, and $50{,}000$ beans. Let the positive number c represent the number of beans that Gregor wants to have. Solve the equation $0.972(10^{0.2119t}) = c$ for t.

   c. Your answer to part (b) can be written as a function N of the number of beans c, where $c > 0$. Explain what this function represents.

   d. When does Gregor's model predict that he will accumulate …

      i. 500 beans?

ii. 5,000 beans?

iii. 50,000 beans?

4. Helena and Karl each change the rules for the bean experiment. Helena started with four beans in her cup and added one bean for each that landed marked-side up for each trial. Karl started with one bean in his cup but added two beans for each that landed marked-side up for each trial.

   a. Helena modeled her data by the function $h(t) = 4.127(1.468^t)$. Explain why her values of $a = 4.127$ and $b = 1.468$ are reasonable.

   b. Karl modeled his data by the function $k(t) = 0.897(1.992^t)$. Explain why his values of $a = 0.897$ and $b = 1.992$ are reasonable.

c. At what value of t do Karl and Helena have the same number of beans?

d. Use a graphing utility to graph $y = h(t)$ and $y = k(t)$ for $0 < t < 10$.

e. Explain the meaning of the intersection point of the two curves $y = h(t)$ and $y = k(t)$ in the context of this problem.

f. Which student reaches 20 beans first? Does the reasoning you used with whether Gregor or Fiona would get to 100 beans first hold true here? Why or why not?

For the following functions f and g, solve the equation $f(x) = g(x)$. Express your solutions in terms of logarithms.

5. $f(x) = 10(3.7)^{x+1}$, $g(x) = 5(7.4)^{x}$

6. $f(x) = 135(5)^{3x+1}$, $g(x) = 75(3)^{4-3x}$

7. $f(x) = 100^{x^3+x^2-4x}$, $g(x) = 10^{2x^2-6x}$

8. $f(x) = 48\left(4^{x^2+3x}\right)$, $g(x) = 3\left(8^{x^2+4x+4}\right)$

9. $f(x) = e^{\sin^2(x)}$, $g(x) = e^{\cos^2(x)}$

10. $f(x) = (0.49)^{\cos(x)+\sin(x)}$, $g(x) = (0.7)^{2\sin(x)}$

Problem Set

1. Solve the following equations.
   a. $2 \cdot 5^{x+3} = 6250$
   b. $3 \cdot 6^{2x} = 648$
   c. $5 \cdot 2^{3x+5} = 10240$
   d. $4^{3x-1} = 32$
   e. $3 \cdot 2^{5x} = 216$
   f. $5 \cdot 11^{3x} = 120$
   g. $7 \cdot 9^{x} = 5405$
   h. $\sqrt{3} \cdot 3^{3x} = 9$
   i. $\log(400) \cdot 8^{5x} = \log(160000)$

2. Lucy came up with the model $f(t) = 0.701(1.382)^t$ for the first bean activity. When does her model predict that she would have $1{,}000$ beans?

3. Jack came up with the model $g(t) = 1.033(1.707)^t$ for the first bean activity. When does his model predict that he would have $50{,}000$ beans?

4. If instead of beans in the first bean activity you were using fair coins, when would you expect to have $\$1{,}000{,}000$?

5. Let $f(x) = 2 \cdot 3^x$ and $g(x) = 3 \cdot 2^x$.
   a. Which function is growing faster as x increases? Why?
   b. When will $f(x) = g(x)$?

6. A population of *E. coli* bacteria can be modeled by the function $E(t) = 500(11.547)^t$, and a population of *Salmonella* bacteria can be modeled by the function $S(t) = 4000(3.668)^t$, where t measures time in hours.
   a. Graph these two functions on the same set of axes. At which value of t does it appear that the graphs intersect?
   b. Use properties of logarithms to find the time t when these two populations are the same size. Give your answer to two decimal places.

7. Chain emails contain a message suggesting you will have bad luck if you do not forward the email to others. Suppose a student started a chain email by sending the message to 10 friends and asking those friends to each send the same email to 3 more friends exactly one day after receiving the message. Assuming that everyone that gets the email participates in the chain, we can model the number of people who will receive the email on the n^{th} day by the formula $E(n) = 10(3^n)$, where $n = 0$ indicates the day the original email was sent.
   a. If we assume the population of the United States is 318 million people and everyone who receives the email sends it to 3 people who have not received it previously, how many days until there are as many emails being sent out as there are people in the United States?
   b. The population of Earth is approximately 7.1 billion people. On what day will 7.1 billion emails be sent out?

8. Solve the following exponential equations.
   a. $10^{(3x-5)} = 7^x$
   b. $3^{\frac{x}{5}} = 2^{4x-2}$
   c. $10^{x^2+5} = 100^{2x^2+x+2}$
   d. $4^{x^2-3x+4} = 2^{5x-4}$

9. Solve the following exponential equations.
   a. $(2^x)^x = 8^x$
   b. $(3^x)^x = 12$

10. Solve the following exponential equations.
   a. $10^{x+1} - 10^{x-1} = 1287$
   b. $2(4^x) + 4^{x+1} = 342$

11. Solve the following exponential equations.
   a. $(10^x)^2 - 3(10^x) + 2 = 0$ Hint: Let $u = 10^x$ and solve for u before solving for x.
   b. $(2^x)^2 - 3(2^x) - 4 = 0$
   c. $3(e^x)^2 - 8(e^x) - 3 = 0$
   d. $4^x + 7(2^x) + 12 = 0$
   e. $(10^x)^2 - 2(10^x) - 1 = 0$

12. Solve the following systems of equations.
   a. $2^{x+2y} = 8$
   $4^{2x+y} = 1$
   b. $2^{2x+y-1} = 32$
   $4^{x-2y} = 2$
   c. $2^{3x} = 8^{2y+1}$
   $9^{2y} = 3^{3x-9}$

13. Because $f(x) = \log_b(x)$ is an increasing function, we know that if $p < q$, then $\log_b(p) < \log_b(q)$. Thus, if we take logarithms of both sides of an inequality, then the inequality is preserved. Use this property to solve the following inequalities.
   a. $4^x > \frac{5}{3}$
   b. $\left(\frac{2}{7}\right)^x > 9$
   c. $4^x > 8^{x-1}$
   d. $3^{x+2} > 5^{3-2x}$
   e. $\left(\frac{3}{4}\right)^x > \left(\frac{4}{3}\right)^{x+1}$

Lesson 25: Geometric Sequences and Exponential Growth and Decay

Classwork

Opening Exercise

Suppose a ball is dropped from an initial height h_0 and that each time it rebounds, its new height is 60% of its previous height.

a. What are the first four rebound heights h_1, h_2, h_3, and h_4 after being dropped from a height of $h_0 = 10$ ft.?

b. Suppose the initial height is A ft. What are the first four rebound heights? Fill in the following table:

Rebound	Height (ft.)
1	
2	
3	
4	

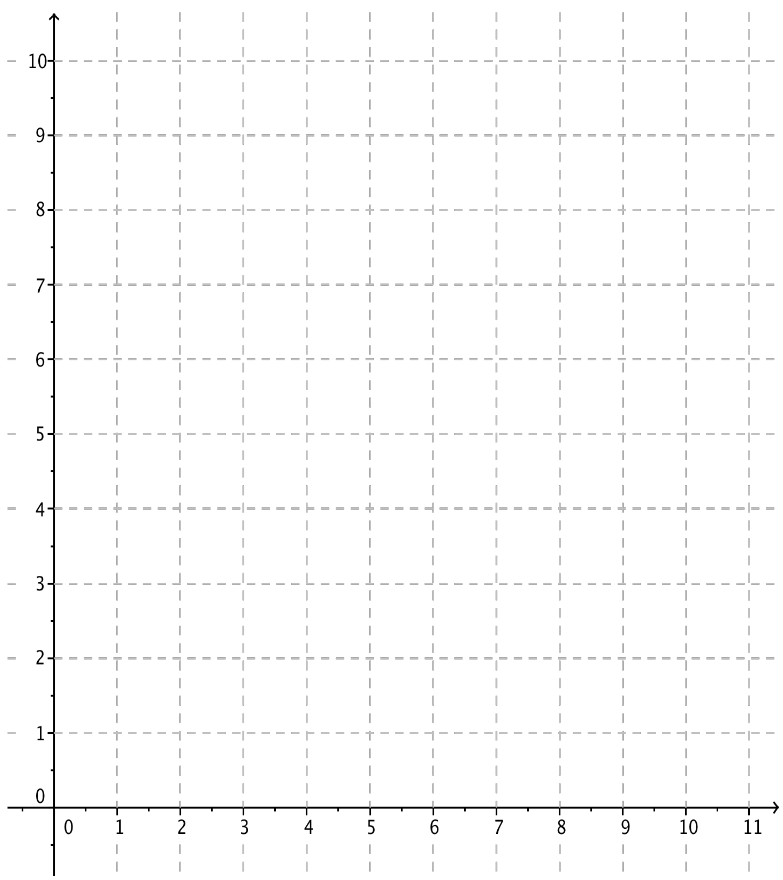
c. How is each term in the sequence related to the one that came before it?

d. Suppose the initial height is A ft. and that each rebound, rather than being 60% of the previous height, is r times the previous height, where $0 < r < 1$. What are the first four rebound heights? What is the n^{th} rebound height?

e. What kind of sequence is the sequence of rebound heights?

f. Suppose that we define a function f with domain all real numbers so that $f(1)$ is the first rebound height, $f(2)$ is the second rebound height, and continuing so that $f(k)$ is the k^{th} rebound height for positive integers k. What type of function would you expect f to be?

g. On the coordinate plane below, sketch the height of the bouncing ball when $A = 10$ and $r = 0.60$, assuming that the highest points occur at $x = 1, 2, 3, 4, \ldots$.



h. Does the exponential function $f(x) = 10(0.60)^x$ for real numbers x model the height of the bouncing ball? Explain how you know.

i. What does the function $f(n) = 10(0.60)^n$ for integers $n \geq 0$ model?

Exercises

1.

a. Jane works for a video game development company that pays her a starting salary of $\$100$ a day, and each day she works, she earns $\$100$ more than the day before. How much does she earn on day 5?

b. If you were to graph the growth of her salary for the first 10 days she worked, what would the graph look like?

c. What kind of sequence is the sequence of Jane's earnings each day?

2. A laboratory culture begins with $1{,}000$ bacteria at the beginning of the experiment, which we will denote by time 0 hours. By time 2 hours, there were $2{,}890$ bacteria.

   a. If the number of bacteria is increasing by a common factor each hour, how many bacteria were there at time 1 hour? At time 3 hours?

   b. Find the explicit formula for term P_n of the sequence in this case.

   c. How would you find term P_{n+1} if you know term P_n? Write a recursive formula for P_{n+1} in terms of P_n.

   d. If P_0 is the initial population, the growth of the population P_n at time n hours can be modeled by the sequence $P_n = P(n)$, where P is an exponential function with the following form:

   $$P(n) = P_0 2^{kn}, \text{ where } k > 0.$$

   Find the value of k and write the function P in this form. Approximate k to four decimal places.

   e. Use the function in part (d) to determine the value of t when the population of bacteria has doubled.

f. If P_0 is the initial population, the growth of the population P at time t can be expressed in the following form:

$$P(n) = P_0 e^{kn}, \text{ where } k > 0.$$

Find the value of k, and write the function P in this form. Approximate k to four decimal places.

g. Use the formula in part (d) to determine the value of t when the population of bacteria has doubled.

3. The first term a_0 of a geometric sequence is -5, and the common ratio r is -2.

 a. What are the terms a_0, a_1, and a_2?

 b. Find a recursive formula for this sequence.

 c. Find an explicit formula for this sequence.

 d. What is term a_9?

 e. What is term a_{10}?

4. Term a_4 of a geometric sequence is 5.8564, and term a_5 is -6.44204.
   a. What is the common ratio r?

   b. What is term a_0?

   c. Find a recursive formula for this sequence.

   d. Find an explicit formula for this sequence.

5. The recursive formula for a geometric sequence is $a_{n+1} = 3.92(a_n)$ with $a_0 = 4.05$. Find an explicit formula for this sequence.

6. The explicit formula for a geometric sequence is $a_n = 147(2.1)^{3n}$. Find a recursive formula for this sequence.

Lesson Summary

ARITHMETIC SEQUENCE: A sequence is called *arithmetic* if there is a real number d such that each term in the sequence is the sum of the previous term and d.

- *Explicit formula:* Term a_n of an arithmetic sequence with first term a_0 and common difference d is given by $a_n = a_0 + nd$, for $n \geq 0$.
- *Recursive formula:* Term a_{n+1} of an arithmetic sequence with first term a_0 and common difference d is given by $a_{n+1} = a_n + d$, for $n \geq 0$.

GEOMETRIC SEQUENCE: A sequence is called *geometric* if there is a real number r such that each term in the sequence is a product of the previous term and r.

- *Explicit formula:* Term a_n of a geometric sequence with first term a_0 and common ratio r is given by $a_n = a_0 r^n$, for $n \geq 0$.
- *Recursive formula:* Term a_{n+1} of a geometric sequence with first term a_0 and common ratio r is given by $a_{n+1} = a_n r$.

Problem Set

1. Convert the following recursive formulas for sequences to explicit formulas.
   a. $a_{n+1} = 4.2 + a_n$ with $a_0 = 12$
   b. $a_{n+1} = 4.2a_n$ with $a_0 = 12$
   c. $a_{n+1} = \sqrt{5}\, a_n$ with $a_0 = 2$
   d. $a_{n+1} = \sqrt{5} + a_n$ with $a_0 = 2$
   e. $a_{n+1} = \pi\, a_n$ with $a_0 = \pi$

2. Convert the following explicit formulas for sequences to recursive formulas.
   a. $a_n = \frac{1}{5}(3^n)$ for $n \geq 0$
   b. $a_n = 16 - 2n$ for $n \geq 0$
   c. $a_n = 16\left(\frac{1}{2}\right)^n$ for $n \geq 0$
   d. $a_n = 71 - \frac{6}{7}n$ for $n \geq 0$
   e. $a_n = 190(1.03)^n$ for $n \geq 0$

3. If a geometric sequence has $a_1 = 256$ and $a_8 = 512$, find the exact value of the common ratio r.

4. If a geometric sequence has $a_2 = 495$ and $a_6 = 311$, approximate the value of the common ratio r to four decimal places.

5. Find the difference between the terms a_{10} of an arithmetic sequence and a geometric sequence, both of which begin at term a_0 and have $a_2 = 4$ and $a_4 = 12$.

6. Given the geometric series defined by the following values of a_0 and r, find the value of n so that a_n has the specified value.

   a. $a_0 = 64, r = \frac{1}{2}, a_n = 2$
   b. $a_0 = 13, r = 3, a_n = 85293$
   c. $a_0 = 6.7, r = 1.9, a_n = 7804.8$
   d. $a_0 = 10958, r = 0.7, a_n = 25.5$

7. Jenny planted a sunflower seedling that started out 5 cm tall, and she finds that the average daily growth is 3.5 cm.

   a. Find a recursive formula for the height of the sunflower plant on day n.
   b. Find an explicit formula for the height of the sunflower plant on day $n \geq 0$.

8. Kevin modeled the height of his son (in inches) at age n years for $n = 2, 3, \ldots, 8$ by the sequence $h_n = 34 + 3.2(n - 2)$. Interpret the meaning of the constants 34 and 3.2 in his model.

9. Astrid sells art prints through an online retailer. She charges a flat rate per order for an order processing fee, sales tax, and the same price for each print. The formula for the cost of buying n prints is given by $P_n = 4.5 + 12.6\,n$.

   a. Interpret the number 4.5 in the context of this problem.
   b. Interpret the number 12.6 in the context of this problem.
   c. Find a recursive formula for the cost of buying n prints.

10. A bouncy ball rebounds to 90% of the height of the preceding bounce. Craig drops a bouncy ball from a height of 20 ft.

    a. Write out the sequence of the heights h_1, h_2, h_3, and h_4 of the first four bounces, counting the initial height as $h_0 = 20$.
    b. Write a recursive formula for the rebound height of a bouncy ball dropped from an initial height of 20 ft.
    c. Write an explicit formula for the rebound height of a bouncy ball dropped from an initial height of 20 ft.
    d. How many bounces will it take until the rebound height is under 6 ft.?
    e. Extension: Find a formula for the minimum number of bounces needed for the rebound height to be under y ft., for a real number $0 < y < 20$.

11. Show that when a quantity $a_0 = A$ is increased by $x\%$, its new value is $a_1 = A\left(1 + \frac{x}{100}\right)$. If this quantity is again increased by $x\%$, what is its new value a_2? If the operation is performed n times in succession, what is the final value of the quantity a_n?

12. When Eli and Daisy arrive at their cabin in the woods in the middle of winter, the internal temperature is $40°\text{F}$.
    a. Eli wants to turn up the thermostat by $2°\text{F}$ every 15 minutes. Find an explicit formula for the sequence that represents the thermostat settings using Eli's plan.
    b. Daisy wants to turn up the thermostat by 4% every 15 minutes. Find an explicit formula for the sequence that represents the thermostat settings using Daisy's plan.
    c. Which plan will get the thermostat to $60°\text{F}$ most quickly?
    d. Which plan will get the thermostat to $72°\text{F}$ most quickly?

13. In nuclear fission, one neutron splits an atom causing the release of two other neutrons, each of which splits an atom and produces the release of two more neutrons, and so on.
    a. Write the first few terms of the sequence showing the numbers of atoms being split at each stage after a single atom splits. Use $a_0 = 1$.
    b. Find the explicit formula that represents your sequence in part (a).
    c. If the interval from one stage to the next is one-millionth of a second, write an expression for the number of atoms being split at the end of one second.
    d. If the number from part (c) were written out, how many digits would it have?

Lesson 26: Percent Rate of Change

Classwork

Exercise

Answer the following questions.

The youth group from Example 1 is given the option of investing their money at 2.976% interest per year, compounded monthly.

a. After two years, how much would be in each account with an initial deposit of $\$800$?

b. Compare the total amount from part (a) to how much they would have made using the interest rate of 3% compounded yearly for two years. Which account would you recommend the youth group invest its money in? Why?

Lesson Summary

- For application problems involving a percent rate of change represented by the unit rate r, we can write $F(t) = P(1+r)^t$, where F is the future value (or ending amount), P is the present amount, and t is the number of time units. When the percent rate of change is negative, r is negative and the quantity decreases with time.
- The nominal APR is the percent rate of change per compounding period times the number of compounding periods per year. If the nominal APR is given by the unit rate r and is compounded n times a year, then function $F(t) = P\left(1+\frac{r}{n}\right)^{nt}$ describes the future value at time t of an account given that nominal APR and an initial value of P.
- For continuous compounding, we can write $F = Pe^{rt}$, where e is Euler's number and r is the unit rate associated to the percent rate of change.

Problem Set

1. Write each recursive sequence in explicit form. Identify each sequence as arithmetic, geometric, or neither.
   a. $a_1 = 3$, $a_{n+1} = a_n + 5$
   b. $a_1 = -1$, $a_{n+1} = -2a_n$
   c. $a_1 = 30$, $a_{n+1} = a_n - 3$
   d. $a_1 = \sqrt{2}$, $a_{n+1} = \frac{a_n}{\sqrt{2}}$
   e. $a_1 = 1$, $a_{n+1} = \cos(\pi a_n)$

2. Write each sequence in recursive form. Assume the first term is when $n = 1$.
   a. $a_n = \frac{3}{2}n + 3$
   b. $a_n = 3\left(\frac{3}{2}\right)^n$
   c. $a_n = n^2$
   d. $a_n = \cos(2\pi n)$

3. Consider two bank accounts. Bank A gives simple interest on an initial investment in savings accounts at a rate of 3% per year. Bank B gives compound interest on savings accounts at a rate of 2.5% per year. Fill out the following table.

Number of Years, n	Bank A Balance, a_n	Bank B Balance, b_n
0	$\$1000.00$	$\$1000.00$
1		
2		
3		
4		
5		

a. What type of sequence do the Bank A balances represent?
b. Give both a recursive and an explicit formula for the Bank A balances.
c. What type of sequence do the Bank B balances represent?
d. Give both a recursive and an explicit formula for the Bank B balances.
e. Which bank account balance is increasing faster in the first five years?
f. If you were to recommend a bank account for a long-term investment, which would you recommend?
g. At what point is the balance in Bank B larger than the balance in Bank A?

4. You decide to invest your money in a bank that uses continuous compounding at 5.5% interest per year. You have $500.

a. Ja'mie decides to invest $1,000 in the same bank for one year. She predicts she will have double the amount in her account than you will have. Is this prediction correct? Explain.
b. Jonas decides to invest $500 in the same bank as well, but for two years. He predicts that after two years he will have double the amount of cash that you will after one year. Is this prediction correct? Explain.

5. Use the properties of exponents to identify the percent rate of change of the functions below, and classify them as representing exponential growth or decay. (The first two problems are done for you.)

a. $f(t) = (1.02)^t$
b. $(t) = (1.01)^{12t}$
c. $f(t) = (0.97)^t$
d. $f(t) = 1000(1.2)^t$
e. $f(t) = \frac{(1.07)^t}{1000}$
f. $f(t) = 100 \cdot 3^t$
g. $f(t) = 1.05 \cdot \left(\frac{1}{2}\right)^t$
h. $f(t) = 80 \cdot \left(\frac{49}{64}\right)^{\frac{1}{2}t}$
i. $f(t) = 1.02 \cdot (1.13)^{\pi t}$

6. The effective rate of an investment is the percent rate of change per year associated to the nominal APR. The effective rate is very useful in comparing accounts with different interest rates and compounding periods. In general, the effective rate can be found with the following formula: $r_E = \left(1 + \frac{r}{k}\right)^k - 1$. The effective rate presented here is the interest rate needed for annual compounding to be equal to compounding n times per year.
   a. For investing, which account is better: an account earning a nominal APR of 7% compounded monthly or an account earning a nominal APR of 6.875% compounded daily? Why?
   b. The effective rate formula for an account compounded continuously is $r_E = e^r - 1$. Would an account earning 6.875% interest compounded continuously be better than the accounts in part (a)?

7. Radioactive decay is the process in which radioactive elements decay into more stable elements. A half-life is the time it takes for half of an amount of an element to decay into a more stable element. For instance, the half-life for half of an amount of uranium-235 to transform into lead-207 is 704 million years. Thus, after 704 million years, only half of any sample of uranium-235 will remain, and the rest will have changed into lead-207. We will assume that radioactive decay is modeled by exponential decay with a constant decay rate.
   a. Suppose we have a sample of A g of uranium-235. Write an exponential formula that gives the amount of uranium-235 remaining after m half-lives.
   b. Does the formula that you wrote in part (a) work for any radioactive element? Why?
   c. Suppose we have a sample of A g of uranium-235. What is the decay rate per million years? Write an exponential formula that gives the amount of uranium-235 remaining after t million years.
   d. How would you calculate the number of years it takes to get to a specific percentage of the original amount of material? For example, how many years will it take for there to be 80% of the original amount of uranium-235 remaining?
   e. How many millions of years would it take 2.35 kg of uranium-235 to decay to 1 kg of uranium?

8. Doug drank a cup of tea with 130 mg of caffeine. Each hour, the caffeine in Doug's body diminishes by about 12%. (This rate varies between 6% and 14% depending on the person.)
   a. Write a formula to model the amount of caffeine remaining in Doug's system after each hour.
   b. About how long will it take for the level of caffeine in Doug's system to drop below 30 mg?
   c. The time it takes for the body to metabolize half of a substance is called a half-life. To the nearest 5 minutes, how long is the half-life for Doug to metabolize caffeine?
   d. Write a formula to model the amount of caffeine remaining in Doug's system after m half-lives.

9. A study done from 1950 through 2000 estimated that the world population increased on average by 1.77% each year. In 1950, the world population was 2.519 billion.
   a. Write a function p for the world population t years after 1950.
   b. If this trend continued, when should the world population have reached 7 billion?
   c. The world population reached 7 billion October 31, 2011, according to the United Nations. Is the model reasonably accurate?
   d. According to the model, when will the world population be greater than 12 billion people?

10. A particular mutual fund offers 4.5% nominal APR compounded monthly. Trevor wishes to deposit $\$1,000$.

    a. What is the percent rate of change per month for this account?

    b. Write a formula for the amount Trevor will have in the account after m months.

    c. *Doubling time* is the amount of time it takes for an investment to double. What is the doubling time of Trevor's investment?

11. When paying off loans, the monthly payment first goes to any interest owed before being applied to the remaining balance. Accountants and bankers use tables to help organize their work.

    a. Consider the situation that Fred is paying off a loan of $\$125,000$ with an interest rate of 6% per year compounded monthly. Fred pays $\$749.44$ every month. Complete the following table:

Payment	Interest Paid	Principal Paid	Remaining Principal
$\$749.44$			
$\$749.44$			
$\$749.44$			

    b. Fred's loan is supposed to last for 30 years. How much will Fred end up paying if he pays $\$749.44$ every month for 30 years? How much of this is interest if his loan was originally for $\$125,000$?

Lesson 27: Modeling with Exponential Functions

Classwork

Opening Exercise

The following table contains U.S. population data for the two most recent census years, 2000 and 2010.

Census Year	U.S. Population (in millions)
2000	281.4
2010	308.7

a. Steve thinks the data should be modeled by a linear function.

   i. What is the average rate of change in population per year according to this data?

   ii. Write a formula for a linear function, L, that will estimate the population t years since the year 2000.

b. Phillip thinks the data should be modeled by an exponential function.

   i. What is the growth rate of the population per year according to this data?

   ii. Write a formula for an exponential function, E, that will estimate the population t years since the year 2000.

c. Who has the correct model? How do you know?

Mathematical Modeling Exercise/Exercises 1–14

In this challenge, you will continue to examine U.S. census data to select and refine a model for the population of the United States over time.

1. The following table contains additional U.S. census population data. Would it be more appropriate to model this data with a linear or an exponential function? Explain your reasoning.

Census Year	U.S. Population (in millions of people)
1900	76.2
1910	92.2
1920	106.0
1930	122.8
1940	132.2
1950	150.7
1960	179.3
1970	203.3
1980	226.5
1990	248.7
2000	281.4
2010	308.7

2. Use a calculator's regression capability to find a function, f, that models the U.S. Census Bureau data from 1900 to 2010.

3. Find the growth factor for each 10-year period and record it in the table below. What do you observe about these growth factors?

Census Year	U.S. Population (in millions of people)	Growth Factor (10-year period)
1900	76.2	--
1910	92.2	
1920	106.0	
1930	122.8	
1940	132.2	
1950	150.7	
1960	179.3	
1970	203.3	
1980	226.5	
1990	248.7	
2000	281.4	
2010	308.7	

4. For which decade is the 10-year growth factor the lowest? What factors do you think caused that decrease?

5. Find an average 10-year growth factor for the population data in the table. What does that number represent? Use the average growth factor to find an exponential function, g, that can model this data.

6. You have now computed three potential models for the population of the United States over time: functions E, f, and g. Which one do you expect would be the most accurate model based on how they were created? Explain your reasoning.

7. Summarize the three formulas for exponential models that you have found so far: Write the formula, the initial populations, and the growth rates indicated by each function. What is different between the structures of these three functions?

8. Rewrite the functions E, f, and g as needed in terms of an annual growth rate.

9. Transform the functions as needed so that the time $t = 0$ represents the same year in functions E, f, and g. Then compare the values of the initial populations and annual growth rates indicated by each function.

10. Which of the three functions is the best model to use for the U.S. census data from 1900 to 2010? Explain your reasoning.

11. The U.S. Census Bureau website http://www.census.gov/popclock displays the current estimate of both the United States and world populations.
    a. What is today's current estimated population of the U.S.?

    b. If time $t = 0$ represents the year 1900, what is the value of t for today's date? Give your answer to two decimal places.

    c. Which of the functions E, f, and g gives the best estimate of today's population? Does that match what you expected? Justify your reasoning.

    d. With your group, discuss some possible reasons for the discrepancy between what you expected in Exercise 8 and the results of part (c) above.

12. Use the model that most accurately predicted today's population in Exercise 9, part (c) to predict when the U.S. population will reach half a billion.

13. Based on your work so far, do you think this is an accurate prediction? Justify your reasoning.

14. Here is a graph of the U.S. population since the census began in 1790. Which type of function would best model this data? Explain your reasoning.

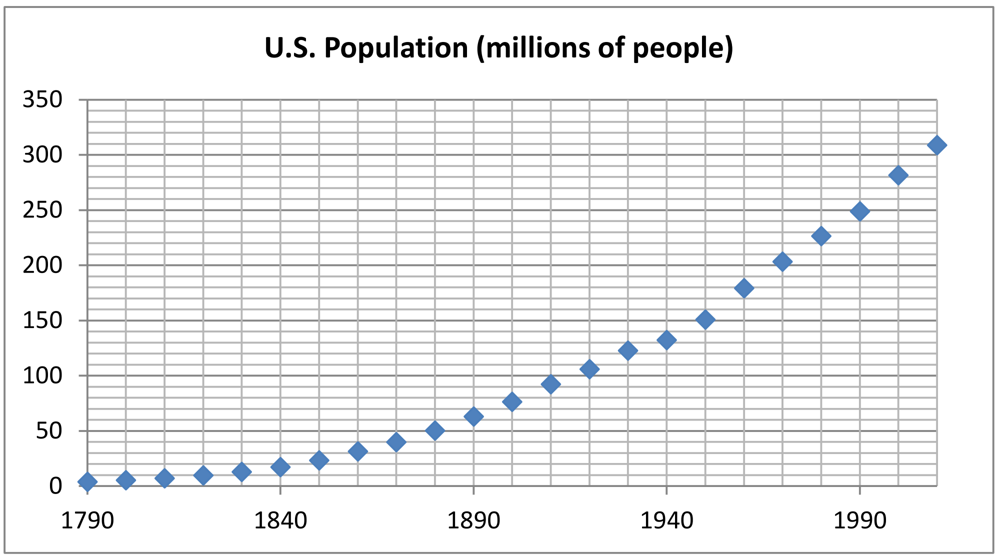
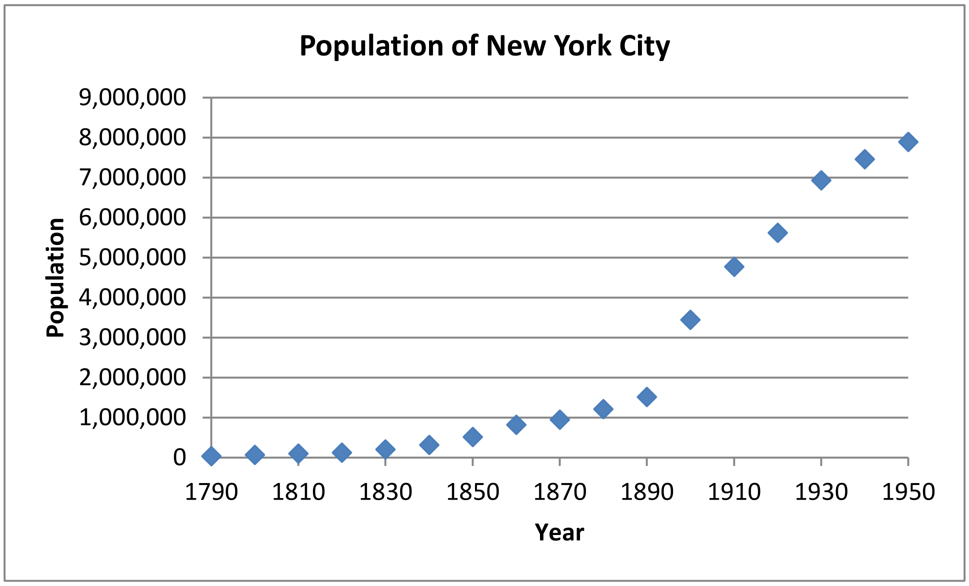


Figure 1: Source U.S. Census Bureau

15. The graph below shows the population of New York City during a time of rapid population growth.



Finn averaged the 10-year growth rates and wrote the function $f(t) = 33131(1.44)^{\frac{t}{10}}$, where t is the time in years since 1790.

Gwen used the regression features on a graphing calculator and got the function $g(t) = 48661(1.036)^t$, where t is the time in years since 1790.

a. Rewrite each function to determine the annual growth rate for Finn's model and Gwen's model.

b. What is the predicted population in the year 1790 for each model?

c. Lenny calculated an exponential regression using his graphing calculator and got the same growth rate as Gwen, but his initial population was very close to 0. Explain what data Lenny may have used to find his function.

d. When does Gwen's function predict the population will reach $1{,}000{,}000$? How does this compare to the graph?

e. Based on the graph, do you think an exponential growth function would be useful for predicting the population of New York in the years after 1950?

16. Suppose each function below represents the population of a different U.S. city since the year 1900.

a. Complete the table below. Use the properties of exponents to rewrite expressions as needed to help support your answers.

City Population Function (t is years since 1900)	Population in the Year 1900	Annual Growth/Decay Rate	Predicted in 2000	Between Which Years Did the Population Double?
$A(t) = 3000(1.1)^{\frac{t}{5}}$				
$B(t) = \dfrac{(1.5)^{2t}}{2.25}$				
$C(t) = 10000(1 - 0.01)^t$				
$D(t) = 900(1.02)^t$				

b. Could the function $(t) = 6520(1.219)^{\frac{t}{10}}$, where t is years since 2000 also represent the population of one of these cities? Use the properties of exponents to support your answer.

c. Which cities are growing in size and which are decreasing according to these models?

d. Which of these functions might realistically represent city population growth over an extended period of time?

Lesson Summary

To model data with an exponential function:

- Examine the data to see if there appears to be a constant growth or decay factor.
- Determine a growth factor and a point in time to correspond to $t = 0$.
- Create a function to model the situation $f(t) = a \cdot b^{ct}$, where b is the growth factor every $\frac{1}{c}$ years and a is the value of f when $t = 0$.

Logarithms can be used to solve for t when you know the value of $f(t)$ in an exponential function model.

Problem Set

1. Does each pair of formulas described below represent the same sequence? Justify your reasoning.
   a. $a_{n+1} = \frac{2}{3}a_n$, $a_0 = -1$ and $b_n = -\left(\frac{2}{3}\right)^n$ for $n \geq 0$.
   b. $a_n = 2a_{n-1} + 3$, $a_0 = 3$ and $b_n = 2(n-1)^3 + 4(n-1) + 3$ for $n \geq 1$.
   c. $a_n = \frac{1}{3}(3)^n$ for $n \geq 0$ and $b_n = 3^{n-2}$ for $n \geq 0$.

2. Tina is saving her babysitting money. She has $\$500$ in the bank, and each month she deposits another $\$100$. Her account earns 2% interest compounded monthly.
   a. Complete the table showing how much money she has in the bank for the first four months.

Month	Amount
1	
2	
3	
4	

   b. Write a recursive sequence for the amount of money she has in her account after n months.

3. Assume each table represents values of an exponential function of the form $f(t) = a(b)^{ct}$, where b is a positive real number and a and c are real numbers. Use the information in each table to write a formula for f in terms of t for parts (a)–(d).

a.

t	$f(t)$
0	10
4	50

b.

t	$f(t)$
0	1000
5	750

c.

t	$f(t)$
6	25
8	45

d.

t	$f(t)$
3	50
6	40

e. Rewrite the expressions for each function in parts (a)–(d) to determine the annual growth or decay rate.

f. For parts (a) and (c), determine when the value of the function is double its initial amount.

g. For parts (b) and (d), determine when the value of the function is half its initial amount.

4. When examining the data in Example 1, Juan noticed the population doubled every five years and wrote the formula $P(t) = 100(2)^{\frac{t}{5}}$. Use the properties of exponents to show that both functions grow at the same rate per year.

5. The growth of a tree seedling over a short period of time can be modeled by an exponential function. Suppose the tree starts out 3 ft. tall and its height increases by 15% per year. When will the tree be 25 ft. tall?

6. Loggerhead turtles reproduce every 2–4 years, laying approximately 120 eggs in a clutch. Studying the local population, a biologist records the following data in the second and fourth years of her study:

Year	Population
2	50
4	1250

a. Find an exponential model that describes the loggerhead turtle population in year t.

b. According to your model, when will the population of loggerhead turtles be over $5{,}000$? Give your answer in years and months.

7. The radioactive isotope seaborgium-266 has a half-life of 30 seconds, which means that if you have a sample of A g of seaborgium-266, then after 30 seconds half of the sample has decayed (meaning it has turned into another element) and only $\frac{A}{2}$ g of seaborgium-266 remain. This decay happens continuously.

a. Define a sequence $a_0, a_1, a_2, \ldots$ so that a_n represents the amount of a 100 g sample that remains after n minutes.

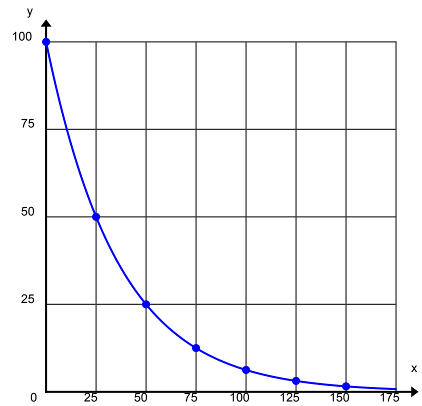
b. Define a function $a(t)$ that describes the amount of seaborgium-266 that remains of a 100 g sample after t minutes.

c. Does your sequence from part (a) and your function from part (b) model the same thing? Explain how you know.

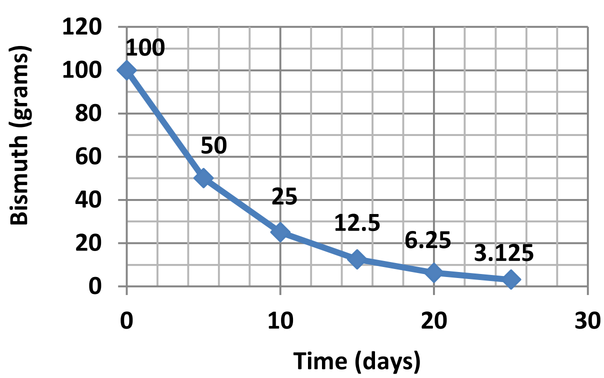
d. How many minutes does it take for less than 1 g of seaborgium-266 to remain from the original 100 g sample? Give your answer to the nearest minute.

8. Compare the data for the amount of substance remaining for each element: strontium-90, magnesium-28, and bismuth.

Strontium-90 (grams) vs. time (hours)



Radioactive Decay of Magnesium-28	
R	t hours
1	0
0.5	21
0.25	42
0.125	63
0.0625	84



a. Which element decays most rapidly? How do you know?

b. Write an exponential function for each element that shows how much of a 100 g sample will remain after t days. Rewrite each expression to show precisely how their exponential decay rates compare to confirm your answer to part (a).

9. The growth of two different species of fish in a lake can be modeled by the functions shown below where t is time in months since January 2000. Assume these models will be valid for at least 5 years.

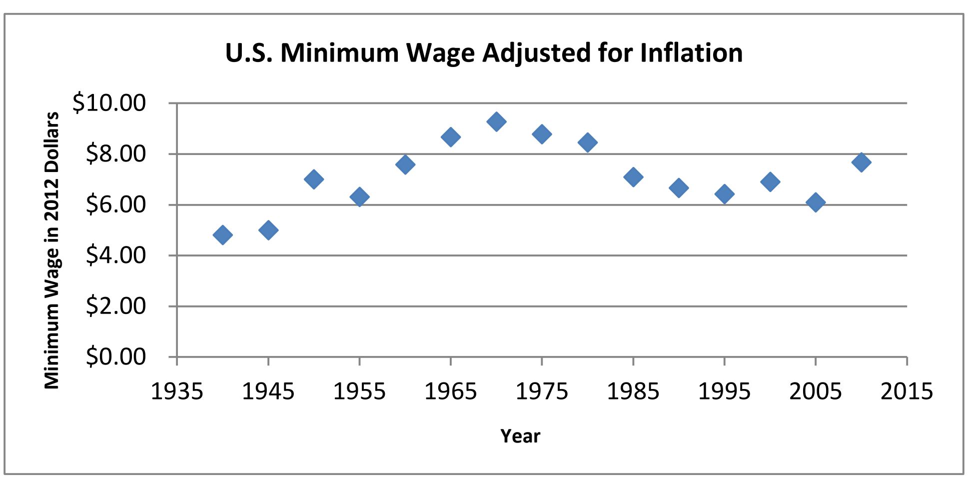
Fish A: $f(t) = 5000(1.3)^t$

Fish B: $g(t) = 10{,}000(1.1)^t$

According to these models, explain why the fish population modeled by function f will eventually catch up to the fish population modeled by function g. Determine precisely when this will occur.

10. When looking at U.S. minimum wage data, you can consider the nominal minimum wage, which is the amount paid in dollars for an hour of work in the given year. You can also consider the minimum wage adjusted for inflation. Below is a table showing the nominal minimum wage and a graph of the data when the minimum wage is adjusted for inflation. Do you think an exponential function would be an appropriate model for either situation? Explain your reasoning.

Year	Nominal Minimum Wage
1940	\$0.30
1945	\$0.40
1950	\$0.75
1955	\$0.75
1960	\$1.00
1965	\$1.25
1970	\$1.60
1975	\$2.10
1980	\$3.10
1985	\$3.35
1990	\$3.80
1995	\$4.25
2000	\$5.15
2005	\$5.15
2010	\$7.25



11. A dangerous bacterial compound forms in a closed environment but is immediately detected. An initial detection reading suggests the concentration of bacteria in the closed environment is one percent of the fatal exposure level. Two hours later, the concentration has increased to four percent of the fatal exposure level.

    a. Develop an exponential model that gives the percent of fatal exposure level in terms of the number of hours passed.

    b. Doctors and toxicology professionals estimate that exposure to two-thirds of the bacteria's fatal concentration level will begin to cause sickness. Provide a rough time limit (to the nearest 15 minutes) for the inhabitants of the infected environment to evacuate in order to avoid sickness.

    c. A prudent and more conservative approach is to evacuate the infected environment before bacteria concentration levels reach 45% of the fatal level. Provide a rough time limit (to the nearest 15 minutes) for evacuation in this circumstance.

    d. When will the infected environment reach 100% of the fatal level of bacteria concentration (to the nearest minute)?

12. Data for the number of users at two different social media companies is given below. Assuming an exponential growth rate, which company is adding users at a faster annual rate? Explain how you know.

Social Media Company A	
Year	Number of Users (Millions)
2010	54
2012	185

Social Media Company B	
Year	Number of Users (Millions)
2009	360
2012	$1{,}056$

Lesson 28: Newton's Law of Cooling, Revisited

Classwork

Newton's law of cooling is used to model the temperature of an object of some temperature placed in an environment of a different temperature. The temperature of the object t hours after being placed in the new environment is modeled by the formula

$$T(t) = T_a + (T_0 - T_a) \cdot e^{-kt},$$

where:

$T(t)$ is the temperature of the object after a time of t hours has elapsed,

T_a is the ambient temperature (the temperature of the surroundings), assumed to be constant and not impacted by the cooling process,

T_0 is the initial temperature of the object, and

k is the decay constant.

Mathematical Modeling Exercise 1

A crime scene investigator is called to the scene of a crime where a dead body has been found. He arrives at the scene and measures the temperature of the dead body at $9{:}30$ p.m. to be $78.3°\text{F}$. He checks the thermostat and determines that the temperature of the room has been kept at $74°\text{F}$. At $10{:}30$ p.m., the investigator measures the temperature of the body again. It is now $76.8°\text{F}$. He assumes that the initial temperature of the body was $98.6°\text{F}$ (normal body temperature). Using this data, the crime scene investigator proceeds to calculate the time of death. According to the data he collected, what time did the person die?

a. Can we find the time of death using only the temperature measured at $9{:}30$ p.m.? Explain.

b. Set up a system of two equations using the data.

d. Find the value of the decay constant, k.

e. What was the time of death?

Mathematical Modeling Exercise 2

A pot of tea is heated to $90°\text{C}$. A cup of the tea is poured into a mug and taken outside where the temperature is $18°\text{C}$. After 2 minutes, the temperature of the cup of tea is approximately $65°\text{C}$.

a. Determine the value of the decay constant, k.

b. Write a function for the temperature of the tea in the mug, T, in $°\text{C}$, as a function of time, t, in minutes.

c. Graph the function T.

d. Use the graph of T to describe how the temperature decreases over time.

e. Use properties of exponents to rewrite the temperature function in the form $T(t) = 18 + 72(1 + r)^t$.

f. In Lesson 26, we saw that the value of r represents the percent change of a quantity that is changing according to an exponential function of the form $f(t) = A(1 + r)^t$. Describe what r represents in the context of the cooling tea.

g. As more time elapses, what temperature does the tea approach? Explain using both the context of the problem and the graph of the function T.

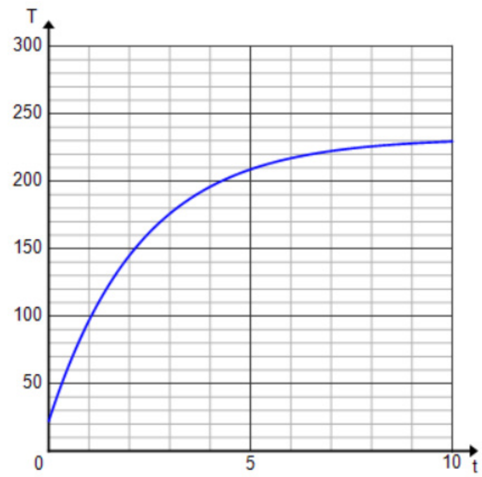
Mathematical Modeling Exercise 3

Two thermometers are sitting in a room that is $22°\text{C}$. When each thermometer reads $22°\text{C}$, the thermometers are placed in two different ovens. Select data for the temperature T of each thermometer (in $°\text{C}$) t minutes after being placed in the oven is provided below.

Thermometer 1:

t (minutes)	0	2	5	8	10	14
T ($°\text{C}$)	22	75	132	173	175	176

Thermometer 2:



a. Do the table and graph given for each thermometer support the statement that Newton's law of cooling also applies when the surrounding temperature is warmer? Explain.

b. Which thermometer was placed in a hotter oven? Explain.

c. Using a generic decay constant, k, without finding its value, write an equation for each thermometer expressing the temperature as a function of time.

d. How do the equations differ when the surrounding temperature is warmer than the object rather than cooler as in previous examples?

e. How do the graphs differ when the surrounding temperature is warmer than the object rather than cooler as in previous examples?

Problem Set

1. Experiments with a covered cup of coffee show that the temperature (in degrees Fahrenheit) of the coffee can be modeled by the following equation:
$$f(t) = 112e^{-0.08t} + 68,$$
where the time is measured in minutes after the coffee was poured into the cup.
   a. What is the temperature of the coffee at the beginning of the experiment?
   b. What is the temperature of the room?
   c. After how many minutes is the temperature of the coffee $140°\text{F}$? Give your answer to 3 decimal places.
   d. What is the temperature of the coffee after how many minutes have elapsed?
   e. What is the percent rate of change of the difference between the temperature of the room and the temperature of the coffee?

2. Suppose a frozen package of hamburger meat is removed from a freezer that is set at $0°\text{F}$ and placed in a refrigerator that is set at $38°\text{F}$. Six hours after being placed in the refrigerator, the temperature of the meat is $12°\text{F}$.
   a. Determine the decay constant, k.
   b. Write a function for the temperature of the meat, T in Fahrenheit, as a function of time, t in hours.
   c. Graph the function T.
   d. Describe the transformations required to graph the function T beginning with the graph of the natural exponential function $f(t) = e^t$.
   e. How long will it take the meat to thaw (reach a temperature above $32°\text{F}$)? Give answer to three decimal places.
   f. What is the percent rate of change of the difference between the temperature of the refrigerator and the temperature of the meat?

3. The table below shows the temperature of biscuits that were removed from an oven at time $t = 0$.

t (min)	0	10	20	30	40	50	60
T (°C)	100	34.183	22.514	20.446	20.079	20.014	20.002

   a. What is the initial temperature of the biscuits?
   b. What does the ambient temperature (room temperature) appear to be?
   c. Use the temperature at $t = 10$ minutes to find the decay constant, k.
   d. Confirm the value of k by using another data point from the table.
   e. Write a function for the temperature of the biscuits (in Celsius) as a function of time in minutes.
   f. Graph the function T.

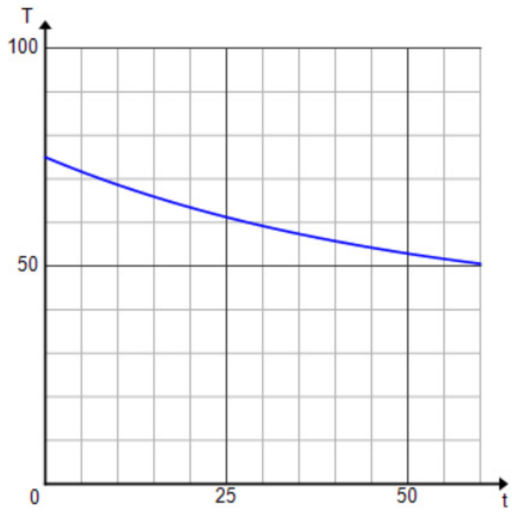
4. Match each verbal description with its correct graph and write a possible equation expressing temperature as a function of time.

   a. A pot of liquid is heated to a boil and then placed on a counter to cool.

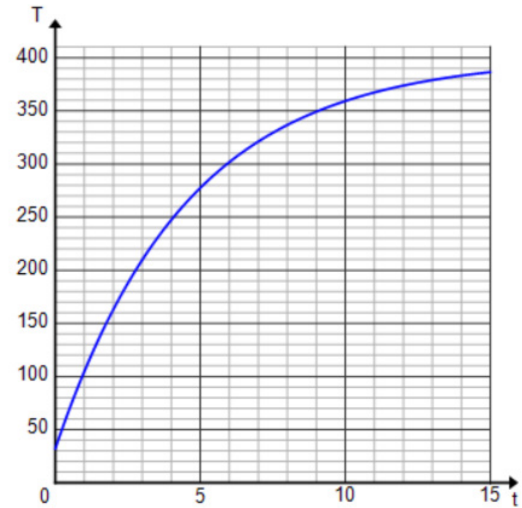
   b. A frozen dinner is placed in a preheated oven to cook.

   c. A can of room-temperature soda is placed in a refrigerator.

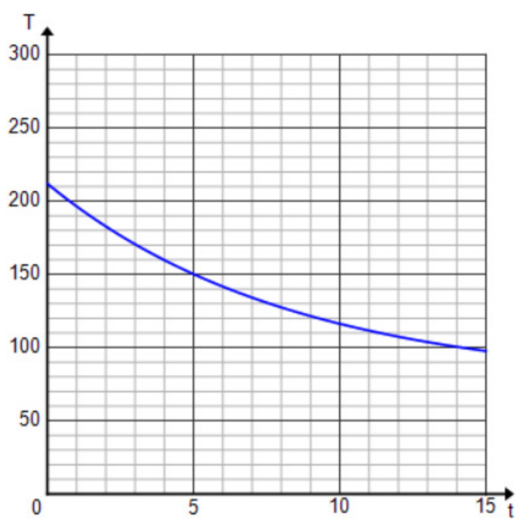
(i)



(ii)



(iii)



Lesson 29: The Mathematics Behind a Structured Savings Plan

Classwork

Opening Exercise

Suppose you invested $\$1,000$ in an account that paid an annual interest rate of 3% compounded monthly. How much would you have after 1 year?

Example 1

Let $a, ar, ar^2, ar^3, ar^4, \ldots$ be a geometric sequence with first term a and common ratio r. Show that the sum S_n of the first n terms of the geometric series

$$S_n = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} \qquad (r \neq 1)$$

is given by the equation

$$S_n = a\frac{1-r^n}{1-r}.$$

Exercises 1–3

1. Find the sum of the geometric series $3 + 6 + 12 + 24 + 48 + 96 + 192$.

2. Find the sum of the geometric series $40 + 40(1.005) + 40(1.005)^2 + \cdots + 40(1.005)^{11}$.

3. Describe a situation that might lead to calculating the sum of the geometric series in Exercise 2.

Example 2

A $\$100$ deposit is made at the end of every month for 12 months in an account that earns interest at an annual interest rate of 3% compounded monthly. How much will be in the account immediately after the last payment?

Discussion

An *annuity* is a series of payments made at fixed intervals of time. Examples of annuities include structured savings plans, lease payments, loans, and monthly home mortgage payments. The term annuity sounds like it is only a yearly payment, but annuities are often monthly, quarterly, or semiannually. The *future amount of the annuity,* denoted A_f, is the sum of all the individual payments made plus all the interest generated from those payments over the specified period of time.

We can generalize the structured savings plan example above to get a generic formula for calculating the future value of an annuity A_f in terms of the recurring payment R, interest rate i, and number of payment periods n. In the example above, we had a recurring payment of $R = 100$, an interest rate per time period of $i = 0.025$, and 12 payments, so $n = 12$. To make things simpler, we always assume that the payments and the time period in which interest is compounded are at the same time. That is, we do not consider plans where deposits are made halfway through the month with interest compounded at the end of the month.

In the example, the amount A_f of the structured savings plan annuity was the sum of all payments plus the interest accrued for each payment:

$$A_f = R + R(1+i)^1 + R(1+i)^2 + \cdots + R(1+i)^{n-1}.$$

This, of course, is a geometric series with n terms, $a = R$, and $r = 1 + i$, which after substituting into the formula for a geometric series and rearranging is

$$A_f = R \cdot \frac{(1+i)^n - 1}{i}.$$

Exercises 4–5

4. Write the sum without using summation notation, and find the sum.

   a. $\sum_{k=0}^{5} k$

   b. $\sum_{j=5}^{7} j^2$

   c. $\sum_{i=2}^{4} \frac{1}{i}$

5. Write each sum using summation notation.

   a. $1^4 + 2^4 + 3^4 + 4^4 + 5^4 + 6^4 + 7^4 + 8^4 + 9^4$

   b. $1 + \cos(\pi) + \cos(2\pi) + \cos(3\pi) + \cos(4\pi) + \cos(5\pi)$

   c. $2 + 4 + 6 + \cdots + 1000$

Lesson Summary

- **SERIES**: Let $a_1, a_2, a_3, a_4, \ldots$ be a sequence of numbers. A sum of the form
$$a_1 + a_2 + a_3 + \cdots + a_n$$
for some positive integer n is called a *series* (or *finite series*) and is denoted S_n. The a_i's are called the *terms* of the series. The number S_n that the series adds to is called the *sum* of the series.
- **GEOMETRIC SERIES:** A *geometric series* is a series whose terms form a geometric sequence.
- **SUM OF A FINITE GEOMETRIC SERIES**: The sum S_n of the first n terms of the geometric series $S_n = a + ar + \cdots + ar^{n-1}$ (when $r \neq 1$) is given by
$$S_n = a\frac{1 - r^n}{1 - r}.$$
The sum of a finite geometric series can be written in summation notation as
$$\sum_{k=0}^{n-1} ar^k = a \cdot \frac{1 - r^n}{1 - r}.$$
- The generic formula for calculating the future value of an annuity A_f in terms of the recurring payment R, interest rate i, and number of periods n is given by
$$A_f = R \cdot \frac{(1 + i)^n - 1}{i}.$$

Problem Set

1. A car loan is one of the first secured loans most Americans obtain. Research used car prices and specifications in your area to find a reasonable used car that you would like to own (under $\$10{,}000$). If possible, print out a picture of the car you selected.
   a. What is the year, make, and model of your vehicle?
   b. What is the selling price for your vehicle?

c. The following table gives the monthly cost per $\$1000$ financed on a 5-year auto loan. Assume you can get a 5% annual interest rate. What is the monthly cost of financing the vehicle you selected? (A formula will be developed to find the monthly payment of a loan in Lesson 30.)

Five-Year (60-month) Loan	
Interest Rate	Amount per $1,000 Financed
1.0%	$\$17.09$
1.5%	$\$17.31$
2.0%	$\$17.53$
2.5%	$\$17.75$
3.0%	$\$17.97$
3.5%	$\$18.19$
4.0%	$\$18.41$
4.5%	$\$18.64$
5.0%	$\$18.87$
5.5%	$\$19.10$
6.0%	$\$19.33$
6.5%	$\$19.56$
7.0%	$\$19.80$
7.5%	$\$20.04$
8.0%	$\$20.28$
8.5%	$\$20.52$
9.0%	$\$20.76$

d. What is the gas mileage for your vehicle?

e. If you drive 120 miles per week and gas is $\$4$ per gallon, then how much will gas cost per month?

2. Write the sum without using summation notation, and find the sum.

a. $\sum_{k=1}^{8} k$

b. $\sum_{k=-8}^{8} k$

c. $\sum_{k=1}^{4} k^3$

d. $\sum_{m=0}^{6} 2m$

e. $\sum_{m=0}^{6} 2m+1$

f. $\sum_{k=2}^{5} \frac{1}{k}$

g. $\sum_{j=0}^{3} (-4)^{j-2}$

h. $\sum_{m=1}^{4} 16\left(\frac{3}{2}\right)^m$

j. $\sum_{j=0}^{3} \frac{105}{2j+1}$

k. $\sum_{p=1}^{3} p \cdot 3^p$

l. $\sum_{j=1}^{6} 100$

m. $\sum_{k=0}^{4} \sin\left(\frac{k\pi}{2}\right)$

n. $\sum_{k=1}^{9} \log\left(\frac{k}{k+1}\right)$

(Hint: You do not need a calculator to find the sum.)

3. Write the sum without using sigma notation (you do not need to find the sum).

   a. $\sum_{k=0}^{4} \sqrt{k+3}$

   b. $\sum_{i=0}^{8} x^i$

   c. $\sum_{j=1}^{6} jx^{j-1}$

   d. $\sum_{k=0}^{9} (-1)^k x^k$

4. Write each sum using summation notation.

   a. $1 + 2 + 3 + 4 + \cdots + 1000$

   b. $2 + 4 + 6 + 8 + \cdots + 100$

   c. $1 + 3 + 5 + 7 + \cdots + 99$

   d. $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots + \frac{99}{100}$

   e. $1^2 + 2^2 + 3^2 + 4^2 + \cdots + 10{,}000^2$

   f. $1 + x + x^2 + x^3 + \cdots + x^{200}$

   g. $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{49\cdot 50}$

   h. $1\ln(1) + 2\ln(2) + 3\ln(3) + \cdots + 10\ln(10)$

5. Find the sum of the geometric series.

   a. $1 + 3 + 9 + \cdots + 2187$

   b. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{512}$

   c. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots - \frac{1}{512}$

   d. $0.8 + 0.64 + 0.512 + \cdots + 0.32768$

   e. $1 + \sqrt{3} + 3 + 3\sqrt{3} + \cdots + 243$

   f. $\sum_{k=0}^{5} 2^k$

   g. $\sum_{m=1}^{4} 5\left(\frac{3}{2}\right)^m$

   h. $1 - x + x^2 - x^3 + \cdots + x^{30}$ in terms of x

   i. $\sum_{m=0}^{11} 4^{\frac{m}{3}}$

   j. $\sum_{n=0}^{14} \left(\sqrt[5]{6}\right)^n$

   k. $\sum_{k=0}^{6} 2 \cdot \left(\sqrt{3}\right)^k$

6. Let a_i represent the sequence of even natural numbers $\{2,4,6,8,\ldots\}$, and evaluate the following expressions.

   a. $\displaystyle\sum_{i=1}^{5} a_i$

   b. $\displaystyle\sum_{i=1}^{4} a_{2i}$

   c. $\displaystyle\sum_{i=1}^{5} (a_i - 1)$

7. Let a_i represent the sequence of integers giving the yardage gained per rush in a high school football game $\{3,-2,17,4,-8,19,2,3,3,4,0,1,-7\}$.

   a. Evaluate $\sum_{i=1}^{13} a_i$. What does this sum represent in the context of the situation?

   b. Evaluate $\frac{\sum_{i=1}^{13} a_i}{13}$. What does this expression represent in the context of the situation?

   c. In general, if a_n describes any sequence of numbers, what does $\frac{\sum_{i=1}^{n} a_i}{n}$ represent?

8. Let b_n represent the sequence given by the following recursive formula: $b_1 = 10$, $b_n = b_{n-1} \cdot 5$.

   a. Write the first 4 terms of this sequence.

   b. Expand the sum $\sum_{i=1}^{4} b_i$. Is it easier to add this series, or is it easier to use the formula for the sum of a finite geometric sequence? Explain your answer. Evaluate $\sum_{i=1}^{4} b_i$.

   c. Write an explicit form for b_n.

   d. Evaluate $\sum_{i=1}^{10} b_i$.

9. Consider the sequence given by $a_1 = 20$, $a_n = \frac{1}{2} \cdot a_{n-1}$.

   a. Evaluate $\sum_{i=1}^{10} a_i$, $\sum_{i=1}^{100} a_i$, and $\sum_{i=1}^{1000} a_i$.

   b. What value does it appear this series is approaching as n continues to increase? Why might it seem like the series is bounded?

10. The sum of a geometric series with four terms is 60, and the common ratio is $r = \frac{1}{2}$. Find the first term.

11. The sum of the first four terms of a geometric series is 203, and the common ratio is 0.4. Find the first term.

12. The third term in a geometric series is $\frac{27}{2}$, and the sixth term is $\frac{729}{16}$. Find the common ratio.

13. The second term in a geometric series is 10, and the seventh term is 10240. Find the sum of the first six terms.

14. Find the interest earned and the future value of an annuity with monthly payments of \$200 for two years into an account that pays 6% interest per year compounded monthly.

15. Find the interest earned and the future value of an annuity with annual payments of $\$1{,}200$ for 15 years into an account that pays 4% interest per year.

16. Find the interest earned and the future value of an annuity with semiannual payments of $\$1{,}000$ for 20 years into an account that pays 7% interest per year compounded semiannually.

17. Find the interest earned and the future value of an annuity with weekly payments of $\$100$ for three years into an account that pays 5% interest per year compounded weekly.

18. Find the interest earned and the future value of an annuity with quarterly payments of $\$500$ for 12 years into an account that pays 3% interest per year compounded quarterly.

19. How much money should be invested every month with 8% interest per year compounded monthly in order to save up $\$10{,}000$ in 15 months?

20. How much money should be invested every year with 4% interest per year in order to save up $\$40{,}000$ in 18 years?

21. Julian wants to save up to buy a car. He is told that a loan for a car will cost $\$274$ a month for five years, but Julian does not need a car presently. He decides to invest in a structured savings plan for the next three years. Every month Julian invests $\$274$ at an annual interest rate of 2% compounded monthly.
    a. How much will Julian have at the end of three years?
    b. What are the benefits of investing in a structured savings plan instead of taking a loan out? What are the drawbacks?

22. An *arithmetic series* is a series whose terms form an arithmetic sequence. For example, $2 + 4 + 6 + \cdots + 100$ is an arithmetic series since $2,4,6,8, \dots, 100$ is an arithmetic sequence with constant difference 2.

    The most famous arithmetic series is $1 + 2 + 3 + 4 + \cdots + n$ for some positive integer n. We studied this series in Algebra I and showed that its sum is $S_n = \frac{n(n+1)}{2}$. It can be shown that the general formula for the sum of an arithmetic series $a + (a + d) + (a + 2d) + \cdots + [a + (n-1)d]$ is

$$S_n = \frac{n}{2}[2a + (n-1)d],$$

    where a is the first term and d is the constant difference.
    a. Use the general formula to show that the sum of $1 + 2 + 3 + \cdots + n$ is $S_n = \frac{n(n+1)}{2}$.
    b. Use the general formula to find the sum of $2 + 4 + 6 + 8 + 10 + \cdots + 100$.

23. The sum of the first five terms of an arithmetic series is 25, and the first term is 2. Find the constant difference.

24. The sum of the first nine terms of an arithmetic series is 135, and the first term is 17. Find the ninth term.

25. The sum of the first and 100^{th} terms of an arithmetic series is 101. Find the sum of the first 100 terms.

Lesson 30: Buying a Car

Classwork

Opening Exercise

Write a sum to represent the future amount of a structured savings plan (i.e., annuity) if you deposit $\$250$ into an account each month for 5 years that pays 3.6% interest per year, compounded monthly. Find the future amount of your plan at the end of 3 years.

Example

Jack wanted to buy a $\$9{,}000$ 2-door sports coupe but could not pay the full price of the car all at once. He asked the car dealer if she could give him a loan where he paid a monthly payment. She told him she could give him a loan for the price of the car at an annual interest rate of 3.6% compounded monthly for 60 months (5 years).

The problems below exhibit how Jack's car dealer used the information above to figure out how much his monthly payment of R dollars per month should be.

a. First, the car dealer imagined how much she would have in an account if she deposited $\$9{,}000$ into the account and left it there for 60 months at an annual interest rate of 3.6% compounded monthly. Use the compound interest formula $F = P(1+i)^n$ to calculate how much she would have in that account after 5 years. This is the amount she would have in the account after 5 years if Jack gave her $\$9{,}000$ for the car, and she immediately deposited it.

b. Next, she figured out how much would be in an account after 5 years if she took each of Jack's payments of R dollars and deposited it into a bank that earned 3.6% per year (compounded monthly). Write a sum to represent the future amount of money that would be in the annuity after 5 years in terms of R, and use the sum of a geometric series formula to rewrite that sum as an algebraic expression.

c. The car dealer then reasoned that, to be fair to her and Jack, the two final amounts in both accounts should be the same—that is, she should have the same amount in each account at the end of 60 months either way. Write an equation in the variable R that represents this equality.

d. She then solved her equation to get the amount R that Jack would have to pay monthly. Solve the equation in part (c) to find out how much Jack needed to pay each month.

Exercise

A college student wants to buy a car and can afford to pay $\$200$ per month. If she plans to take out a loan at 6% interest per year with a recurring payment of $\$200$ per month for four years, what price car can she buy?

Mathematical Modeling Exercise

In the Problem Set of Lesson 29, you researched the price of a car that you might like to own. In this exercise, you will determine how much a car payment would be for that price for different loan options.

If you did not find a suitable car, select a car and selling price from the list below:

Car	Selling Price
2005 Pickup Truck	$\$9000$
2007 Two-Door Small Coupe	$\$7500$
2003 Two-Door Luxury Coupe	$\$10{,}000$
2006 Small SUV	$\$8000$
2008 Four-Door Sedan	$\$8500$

a. When you buy a car, you must pay sales tax and licensing and other fees. Assume that sales tax is 6% of the selling price and estimated license/title/fees will be 2% of the selling price. If you put a $\$1{,}000$ down payment on your car, how much money will you need to borrow to pay for the car and taxes and other fees?

b. Using the loan amount you computed above, calculate the monthly payment for the different loan options shown below:

Loan 1	36-month loan at 2%
Loan 2	48-month loan at 3%
Loan 3	60-month loan at 5%

c. Which plan, if any, will keep your monthly payment under $\$175$? Of the plans under $\$175$ per month, why might you choose a plan with fewer months even though it costs more per month?

Lesson Summary

The total cost of car ownership includes many different costs in addition to the selling price, such as sales tax, insurance, fees, maintenance, interest on loans, gasoline, etc.

The present value of an annuity formula can be used to calculate monthly loan payments given a total amount borrowed, the length of the loan, and the interest rate. The present value A_p (i.e., loan amount) of an annuity consisting of n recurring equal payments of size R and interest rate i per time period is

$$A_p = R \cdot \frac{1-(1+i)^{-n}}{i}.$$

Amortization tables and online loan calculators can also help you plan for buying a car.

The amount of your monthly payment depends on the interest rate, the down payment, and the length of the loan.

Problem Set

1. Benji is 24 years old and plans to drive his new car about 200 miles per week. He has qualified for first-time buyer financing, which is a 60-month loan with 0% down at an interest rate of 4%. Use the information below to estimate the monthly cost of each vehicle.

CAR A: 2010 Pickup Truck for \$12,000, 22 miles per gallon

CAR B: 2006 Luxury Coupe for \$11,000, 25 miles per gallon

Gasoline: \$4.00 per gallon New vehicle fees: \$80 Sales Tax: 4.25%

Maintenance Costs:

(2010 model year or newer): 10% of purchase price annually

(2009 model year or older): 20% of purchase price annually

Insurance:

Average Rate Ages 25–29	\$100 per month
If you are male If you are female	Add \$10 per month Subtract \$10 per month
Type of Car Pickup Truck Small Two-Door Coupe or Four-Door Sedan Luxury Two- or Four-Door Coupe	 Subtract \$10 per month Subtract \$10 per month Add \$15 per month
Ages 18–25	Double the monthly cost

a. How much money will Benji have to borrow to purchase each car?

b. What is the monthly payment for each car?

c. What are the annual maintenance costs and insurance costs for each car?

d. Which car should Benji purchase? Explain your choice.

2. Use the total initial cost of buying your car from the lesson to calculate the monthly payment for the following loan options.

Option	Number of Months	Down Payment	Interest Rate	Monthly Payment
Option A	48 months	$0	2.5%	
Option B	60 months	$500	3.0%	
Option C	60 months	$0	4.0%	
Option D	36 months	$1000	0.9%	

a. For each option, what is the total amount of money you will pay for your vehicle over the life of the loan?

b. Which option would you choose? Justify your reasoning.

3. Many lending institutions will allow you to pay additional money toward the principal of your loan every month. The table below shows the monthly payment for an $8,000 loan using Option A above if you pay an additional $25 per month.

Month/ Year	Payment	Principal Paid	Interest Paid	Total Interest	Balance
Aug. 2014	$ 200.31	$ 183.65	$ 16.67	$ 16.67	$ 7,816.35
Sept. 2014	$ 200.31	$ 184.03	$ 16.28	$ 32.95	$ 7,632.33
Oct. 2014	$ 200.31	$ 184.41	$ 15.90	$ 48.85	$ 7,447.91
Nov. 2014	$ 200.31	$ 184.80	$ 15.52	$ 64.37	$ 7,263.12
Dec. 2014	$ 200.31	$ 185.18	$ 15.13	$ 79.50	$ 7,077.94
Jan. 2015	$ 200.31	$ 185.57	$ 14.75	$ 94.25	$ 6,892.37
Feb. 2015	$ 200.31	$ 185.95	$ 14.36	$ 108.60	$ 6,706.42
Mar. 2015	$ 200.31	$ 186.34	$ 13.97	$ 122.58	$ 6,520.08
April 2015	$ 200.31	$ 186.73	$ 13.58	$ 136.16	$ 6,333.35
May 2015	$ 200.31	$ 187.12	$ 13.19	$ 149.35	$ 6,146.23
June 2015	$ 200.31	$ 187.51	$ 12.80	$ 162.16	$ 5,958.72
July 2015	$ 200.31	$ 187.90	$ 12.41	$ 174.57	$ 5,770.83
Aug. 2015	$ 200.31	$ 188.29	$ 12.02	$ 186.60	$ 5,582.54
Sept. 2015	$ 200.31	$ 188.68	$ 11.63	$ 198.23	$ 5,393.85
Oct. 2015	$ 200.31	$ 189.08	$ 11.24	$ 209.46	$ 5,204.78
Nov. 2015	$ 200.31	$ 189.47	$ 10.84	$ 220.31	$ 5,015.31
Dec. 2015	$ 200.31	$ 189.86	$ 10.45	$ 230.75	$ 4,825.45

Note: The months from January 2016 to December 2016 are not shown.

Jan. 2017	$ 200.31	$ 195.07	$ 5.24	$ 330.29	$ 2,320.92
Feb. 2017	$ 200.31	$ 195.48	$ 4.84	$ 335.12	$ 2,125.44
Mar. 2017	$ 200.31	$ 195.88	$ 4.43	$ 339.55	$ 1,929.56
April 2017	$ 200.31	$ 196.29	$ 4.02	$ 343.57	$ 1,733.27
May 2017	$ 200.31	$ 196.70	$ 3.61	$ 347.18	$ 1,536.57
June 2017	$ 200.31	$ 197.11	$ 3.20	$ 350.38	$ 1,339.45
July 2017	$ 200.31	$ 197.52	$ 2.79	$ 353.17	$ 1,141.93
Aug. 2017	$ 200.31	$ 197.93	$ 2.38	$ 355.55	$ 944.00
Sept. 2017	$ 200.31	$ 198.35	$ 1.97	$ 357.52	$ 745.65
Oct. 2017	$ 200.31	$ 198.76	$ 1.55	$ 359.07	$ 546.90
Nov. 2017	$ 200.31	$ 199.17	$ 1.14	$ 360.21	$ 347.72
Dec. 2017	$ 200.31	$ 199.59	$ 0.72	$ 360.94	$ 148.13
Jan. 2018	$ 148.44	$ 148.13	$ 0.31	$ 361.25	$ 0.00

How much money would you save over the life of an $8,000 loan using Option A if you paid an extra $25 per month compared to the same loan without the extra payment toward the principal?

4. Suppose you can afford only $200 a month in car payments and your best loan option is a 60-month loan at 3%. How much money could you spend on a car? That is, calculate the present value of the loan with these conditions.

5. Would it make sense for you to pay an additional amount per month toward your car loan? Use an online loan calculator to support your reasoning.

6. What is the sum of each series?
   a. $900 + 900(1.01)^1 + 900(1.01)^2 + \cdots + 900(1.01)^{59}$
   b. $\sum_{n=0}^{47} 15{,}000\left(1 + \frac{0.04}{12}\right)^n$

7. Gerald wants to borrow $12,000 in order to buy an engagement ring. He wants to repay the loan by making monthly installments for two years. If the interest rate on this loan is $9\frac{1}{2}\%$ per year, compounded monthly, what is the amount of each payment?

8. Ivan plans to surprise his family with a new pool using his Christmas bonus of $4200 as a down payment. If the price of the pool is $9,500 and Ivan can finance it at an interest rate of $2\frac{7}{8}\%$ per year, compounded quarterly, how long is the loan for if he pays $285.45 per quarter?

9. Jenny wants to buy a car by making payments of $120 per month for three years. The dealer tells her that she will need to put a down payment of $3,000 on the car in order to get a loan with those terms at a 9% interest rate per year, compounded monthly. How much is the car that Jenny wants to buy?

10. Kelsey wants to refinish the floors in her house and estimates that it will cost $\$39{,}000$ to do so. She plans to finance the entire amount at $3\frac{1}{4}\%$ interest per year, compounded monthly for 10 years. How much is her monthly payment?

11. Lawrence coaches little league baseball and needs to purchase all new equipment for his team. He has $\$489$ in donations, and the team's sponsor will take out a loan at $4\frac{1}{2}\%$ interest per year, compounded monthly for one year, paying up to $\$95$ per month. What is the most that Lawrence can purchase using the donations and loan?

Lesson 31: Credit Cards

Classwork

Mathematical Modeling Exercise

You have charged $1,500 for the down payment on your car to a credit card that charges 19.99% annual interest, and you plan to pay a fixed amount toward this debt each month until it is paid off. We will denote the balance owed after the n^{th} payment has been made as b_n.

a. What is the monthly interest rate, i? Approximate i to 5 decimal places.

b. You have been assigned to either the 50-team, the 100-team, or the 150-team, where the number indicates the size of the monthly payment R you will make toward your debt. What is your value of R?

c. Remember that you can make any size payment toward a credit card debt, as long as it is at least the minimum payment specified by the lender. Your lender calculates the minimum payment as the sum of 1% of the outstanding balance and the total interest that has accrued over the month or $25, whichever is greater. Under these stipulations, what is the minimum payment? Is your monthly payment R at least as large as the minimum payment?

d. Complete the following table to show 6 months of payments.

Month, n	Interest Due	Payment, R	Paid to Principal	Balance, b_n
0				1500.00
1				
2				
3				
4				
5				
6				

e. Write a recursive formula for the balance b_n in month n in terms of the balance b_{n-1}.

f. Write an explicit formula for the balance b_n in month n, leaving the expression $1 + i$ in symbolic form.

g. Rewrite your formula in part (f) using r to represent the quantity $(1 + i)$.

h. What can you say about your formula in (g)? What term do we use to describe r in this formula?

i. Write your formula from part (g) in summation notation using Σ.

j. Apply the appropriate formula from Lesson 29 to rewrite your formula from part (g).

k. Find the month when your balance is paid off.

l. Calculate the total amount paid over the life of the debt. How much was paid solely to interest?

Problem Set

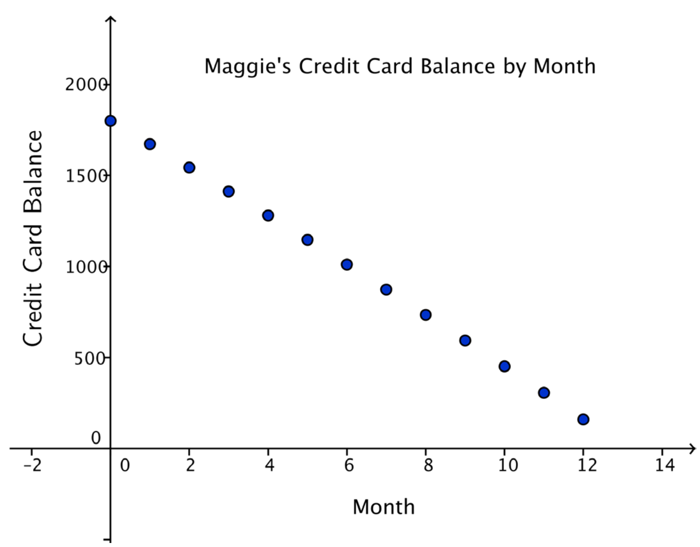
1. Suppose that you have a $2,000 balance on a credit card with a 29.99% annual interest rate, compounded monthly, and you can afford to pay $150 per month toward this debt.
   a. Find the amount of time it will take to pay off this debt. Give your answer in months and years.
   b. Calculate the total amount paid over the life of the debt.
   c. How much money was paid entirely to the interest on this debt?

2. Suppose that you have a $2,000 balance on a credit card with a 14.99% annual interest rate, and you can afford to pay $150 per month toward this debt.
   a. Find the amount of time it will take to pay off this debt. Give your answer in months and years.
   b. Calculate the total amount paid over the life of the debt.
   c. How much money was paid entirely to the interest on this debt?

3. Suppose that you have a $2,000 balance on a credit card with a 7.99% annual interest rate, and you can afford to pay $150 per month toward this debt.
   a. Find the amount of time it will take to pay off this debt. Give your answer in months and years.
   b. Calculate the total amount paid over the life of the debt.
   c. How much money was paid entirely to the interest on this debt?

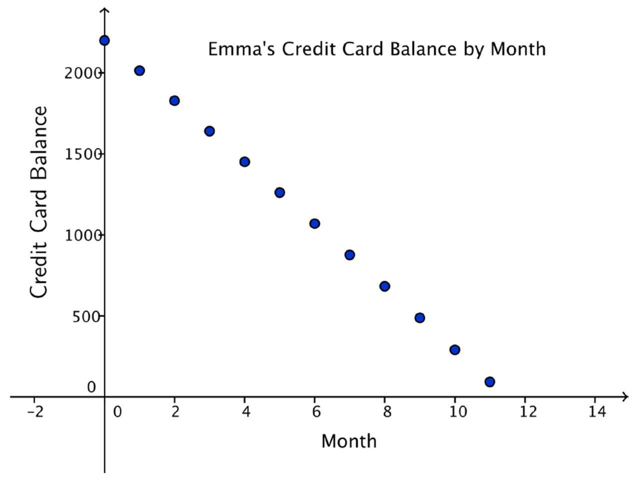
4. Summarize the results of Problems 1, 2, and 3.

5. Brendan owes $1,500 on a credit card with an interest rate of 12%. He is making payments of $100 every month to pay this debt off. Maggie is also making regular payments to a debt owed on a credit card, and she created the following graph of her projected balance over the next 12 months.
   a. Who has the higher initial balance? Explain how you know.
   b. Who will pay their debt off first? Explain how you know.



6. Alan and Emma are both making $200 monthly payments toward balances on credit cards. Alan has prepared a table to represent his projected balances, and Emma has prepared a graph.

Alan's Credit Card Balance			
Month, n	Interest	Payment	Balance, b_n
0	——	——	2000.00
1	41.65	200	1841.65
2	38.35	200	1680.00
3	34.99	200	1514.99
4	31.55	200	1346.54
5	28.04	200	1174.58
6	24.46	200	999.04
7	20.81	200	819.85
8	17.07	200	636.92
9	13.26	200	450.18
10	9.37	200	259.55
11	5.41	200	64.96



a. What is the annual interest rate on Alan's debt? Explain how you know.

b. Who has the higher initial balance? Explain how you know.

c. Who will pay their debt off first? Explain how you know.

d. What do your answers to parts (a), (b), and (c) tell you about the interest rate for Emma's debt?

7. Both Gary and Helena are paying regular monthly payments to a credit card balance. The balance on Gary's credit card debt can be modeled by the recursive formula $g_n = g_{n-1}(1.01666) - 200$ with $g_0 = 2500$, and the balance on Helena's credit card debt can be modeled by the explicit formula $h_n = 2000(1.01666)^n - 250\left(\frac{1.01666^n - 1}{0.01666}\right)$ for $n \geq 0$.

a. Who has the higher initial balance? Explain how you know.

b. Who has the higher monthly payment? Explain how you know.

c. Who will pay their debt off first? Explain how you know.

8. In the next lesson, we will apply the mathematics we have learned to the purchase of a house. In preparation for that task, you need to come to class prepared with an idea of the type of house you would like to buy.

a. Research the median housing price in the county where you live or where you wish to relocate.

b. Find the range of prices that are within 25% of the median price from part (a). That is, if the price from part (a) was P, then your range is $0.75P$ to $1.25P$.

c. Look at online real estate websites, and find a house located in your selected county that falls into the price range specified in part (b). You will be modeling the purchase of this house in Lesson 32, so bring a printout of the real estate listing to class with you.

9. Select a career that interests you from the following list of careers. If the career you are interested in is not on this list, check with your teacher to obtain permission to perform some independent research. Once it has been selected, you will use the career to answer questions in Lesson 32 and Lesson 33.

Occupation	Median Starting Salary	Education Required
Entry-level full-time (waitstaff, office clerk, lawn care worker, etc.)	$18,000	High school diploma or GED
Accountant	$54,630	4-year college degree
Athletic Trainer	$36,560	4-year college degree
Chemical Engineer	$78,860	4-year college degree
Computer Scientist	$93,950	4-year college degree or more
Database Administrator	$64,600	4-year college degree
Dentist	$136,960	Graduate degree
Desktop Publisher	$34,130	4-year college degree
Electrical Engineer	$75,930	4-year college degree
Graphic Designer	$39,900	2- or 4-year college degree
HR Employment Specialist	$42,420	4-year college degree
HR Compensation Manager	$66,530	4-year college degree
Industrial Designer	$54,560	4-year college degree or more
Industrial Engineer	$68,620	4-year college degree
Landscape Architect	$55,140	4-year college degree
Lawyer	$102,470	Law degree
Occupational Therapist	$60,470	Master's degree
Optometrist	$91,040	Master's degree
Physical Therapist	$66,200	Master's degree
Physician—Anesthesiology	$259,948	Medical degree
Physician—Family Practice	$137,119	Medical degree
Physician's Assistant	$74,980	2 years college plus 2-year program
Radiology Technician	$47,170	2-year degree
Registered Nurse	$57,280	2- or 4-year college degree plus
Social Worker—Hospital	$48,420	Master's degree
Teacher—Special Education	$47,650	Master's degree
Veterinarian	$71,990	Veterinary degree

Lesson 32: Buying a House

Classwork

Mathematical Modeling Exercise

Now that you have studied the mathematics of structured savings plans, buying a car, and paying down a credit card debt, it's time to think about the mathematics behind the purchase of a house. In the Problem Set in Lesson 31, you selected a future career and a home to purchase. The question of the day is this: Can you buy the house you have chosen on the salary of the career you have chosen? You need to adhere to the following constraints:

- Mortgages are loans that are usually offered with 30-, 20-, or 15-year repayment options. You will start with a 30-year mortgage.
- The annual interest rate for your mortgage will be 5%.
- Your payment includes the payment of the loan for the house and payments into an account called an *escrow account*, which is used to pay for taxes and insurance on your home. We will approximate the annual payment to escrow as 1.2% of the home's selling price.
- The bank will only approve a mortgage if the total monthly payment for your house, including the payment to the escrow account, does not exceed 30% of your monthly salary.
- You have saved up enough money to put a 10% down payment on this house.

1. Will the bank approve you for a 30-year mortgage on the house that you have chosen?

2. Answer either (a) or (b) as appropriate.
   a. If your bank approved you for a 30-year mortgage, do you meet the criteria for a 20-year mortgage? If you could get a mortgage for any number of years that you want, what is the shortest term for which you would qualify?
   b. If your bank did not approve you for the 30-year mortgage, what is the maximum price of a house that fits your budget?

Problem Set

1. Use the house you selected to purchase in the Problem Set from Lesson 31 for this problem.
   a. What was the selling price of this house?
   b. Calculate the total monthly payment, R, for a 15-year mortgage at 5% annual interest, paying 10% as a down payment and an annual escrow payment that is 1.2% of the full price of the house.

2. In the summer of 2014, the average listing price for homes for sale in the Hollywood Hills was $\$2,663,995$.
   a. Suppose you want to buy a home at that price with a 30-year mortgage at 5.25% annual interest, paying 10% as a down payment and with an annual escrow payment that is 1.2% of the full price of the home. What is your total monthly payment on this house?
   b. How much is paid in interest over the life of the loan?

3. Suppose that you would like to buy a home priced at $\$200,000$. You will make a payment of 10% of the purchase price and pay 1.2% of the purchase price into an escrow account annually.
   a. Compute the total monthly payment and the total interest paid over the life of the loan for a 30-year mortgage at 4.8% annual interest.
   b. Compute the total monthly payment and the total interest paid over the life of the loan for a 20-year mortgage at 4.8% annual interest.
   c. Compute the total monthly payment and the total interest paid over the life of the loan for a 15-year mortgage at 4.8% annual interest.

4. Suppose that you would like to buy a home priced at $\$180,000$. You will qualify for a 30-year mortgage at 4.5% annual interest and pay 1.2% of the purchase price into an escrow account annually.
   a. Calculate the total monthly payment and the total interest paid over the life of the loan if you make a 3% down payment.
   b. Calculate the total monthly payment and the total interest paid over the life of the loan if you make a 10% down payment.
   c. Calculate the total monthly payment and the total interest paid over the life of the loan if you make a 20% down payment.
   d. Summarize the results of parts (a), (b), and (c) in the chart below.

Percent down payment	Amount of down payment	Total interest paid
3%		
10%		
20%		

5. The following amortization table shows the amount of payments to principal and interest on a $100,000 mortgage at the beginning and the end of a 30-year loan. These payments do not include payments to the escrow account.

Month/ Year	Payment	Principal Paid	Interest Paid	Total Interest	Balance
Sept. 2014	$ 477.42	$ 144.08	$ 333.33	$ 333.33	$ 99,855.92
Oct. 2014	$ 477.42	$ 144.56	$ 332.85	$ 666.19	$ 99,711.36
Nov. 2014	$ 477.42	$ 145.04	$ 332.37	$ 998.56	$ 99,566.31
Dec. 2014	$ 477.42	$ 145.53	$ 331.89	$ 1,330.45	$ 99,420.78
Jan. 2015	$ 477.42	$ 146.01	$ 331.40	$ 1,661.85	$ 99,274.77
⋮					
Mar. 2044	$ 477.42	$ 467.98	$ 9.44	$ 71,845.82	$ 2,363.39
April 2044	$ 477.42	$ 469.54	$ 7.88	$ 71,853.70	$ 1,893.85
May 2044	$ 477.42	$ 471.10	$ 6.31	$ 71,860.01	$ 1,422.75
June 2044	$ 477.42	$ 472.67	$ 4.74	$ 71,864.75	$ 950.08
July 2044	$ 477.42	$ 474.25	$ 3.17	$ 71,867.92	$ 475.83
Aug. 2044	$ 477.42	$ 475.83	$ 1.59	$ 71,869.51	$ 0.00

a. What is the annual interest rate for this loan? Explain how you know.

b. Describe the changes in the amount of principal paid each month as the month n gets closer to 360.

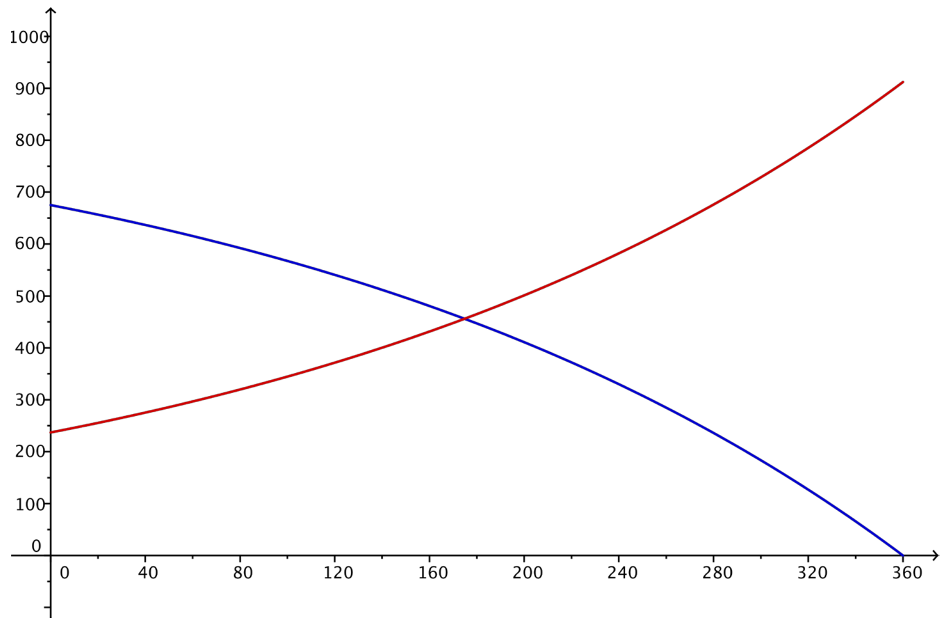
c. Describe the changes in the amount of interest paid each month as the month n gets closer to 360.

6. Suppose you want to buy a $\$200{,}000$ home with a 30-year mortgage at 4.5% annual interest paying 10% down with an annual escrow payment that is 1.2% of the price of the home.

    a. Disregarding the payment to escrow, how much do you pay toward the loan on the house each month?

    b. What is the total monthly payment on this house?

    c. The graph below depicts the amount of your payment from part (b) that goes to the interest on the loan and the amount that goes to the principal on the loan. Explain how you can tell which graph is which.



7. Student loans are very similar to both car loans and mortgages. The same techniques used for car loans and mortgages can be used for student loans. The difference between student loans and other types of loans is that usually students are not required to pay anything until 6 months after they stop being full-time students.

    a. An unsubsidized student loan will accumulate interest while a student remains in school. Sal borrows $\$9{,}000$ his first term in school at an interest rate of 5.95% per year compounded monthly and never makes a payment. How much will he owe $4\frac{1}{2}$ years later? How much of that amount is due to compounded interest?

    b. If Sal pays the interest on his student loan every month while he is in school, how much money has he paid?

    c. Explain why the answer to part (a) is different than the answer to part (b).

8. Consider the sequence $a_0 = 10000$, $a_n = a_{n-1} \cdot \frac{1}{10}$ for $n \geq 1$.

    a. Write the explicit form for the n^{th} term of the sequence.

    b. Evaluate $\sum_{k=0}^{4} a_k$.

    c. Evaluate $\sum_{k=0}^{6} a_k$.

    d. Evaluate $\sum_{k=0}^{8} a_k$ using the sum of a geometric series formula.

    e. Evaluate $\sum_{k=0}^{10} a_k$ using the sum of a geometric series formula.

    f. Describe the value of $\sum_{k=0}^{n} a_k$ for any value of $n \geq 4$.

Lesson 33: The Million Dollar Problem

Classwork

Opening Exercise

In Problem 1 of the Problem Set of Lesson 32, you calculated the monthly payment for a 15-year mortgage at a 5% annual interest rate for the house you chose. You will need that monthly payment for these questions.

a. About how much do you expect your home to be worth in 15 years?

b. For $0 \leq x \leq 15$, plot the graph of the function $f(x) = P(1+r)^x$ where r is the appreciation rate and P is the initial value of your home.

c. Compare the image of the graph you plotted in part (b) with a partner, and write your observations of the differences and similarities. What do you think is causing the differences that you see in the graphs? Share your observations with another group to see if your conclusions are correct.

Your friend Julia bought a home at the same time as you but chose to finance the loan over 30 years. Julia also was able to avoid a down payment and financed the entire value of her home. This allowed her to purchase a more expensive home, but 15 years later she still has not paid off the loan. Consider the following amortization table representing Julia's mortgage, and answer the following questions by comparing the table with your graph.

Payment #	Beginning Balance	Payment on Interest	Payment on Principal
1	$145000	$543.75	$190.94
⋮	⋮	⋮	⋮
178	$96,784.14	$362.94	$371.75
179	$96,412.38	$361.55	$373.15
180	$96,039.23	$360.15	$374.55

d. In Julia's neighborhood, her home has grown in value at around 2.95% per year. Considering how much she still owes the bank, how much of her home does she own after 15 years (the equity in her home)? Express your answer in dollars to the nearest thousand and as a percent of the value of her home.

e. Reasoning from your graph in part (b) and the table above, if both you and Julia sell your homes in 15 years at the homes' appreciated values, who would have more equity?

f. How much more do you need to save over 15 years to have assets over $1,000,000?

Mathematical Modeling Exercises

Assume you can earn 7% interest annually, compounded monthly, in an investment account. Develop a savings plan so that you will have $\$1$ million in assets in 15 years (including the equity in your paid-off house).

1. Use your answer to Opening Exercise, part (g) as the future value of your savings plan.
   a. How much will you have to save every month to save up $\$1$ million in assets?

   b. Recall the monthly payment to pay off your home in 15 years (from Problem 1 of the Problem Set of Lesson 32). How much are the two together? What percentage of your monthly income is this for the profession you chose?

2. Write a report supported by the calculations you did above on how to save $1 million (or more) in your lifetime.

Problem Set

1. Consider the following scenario: You would like to save up $\$50{,}000$ after 10 years and plan to set up a structured savings plan to make monthly payments at 4.125% interest annually, compounded monthly.
   a. What lump sum amount would you need to invest at this interest rate in order to have $\$50{,}000$ after 10 years?
   b. Use an online amortization calculator to find the monthly payment necessary to take a loan for the amount in part (a) at this interest rate and for this time period.
   c. Use $A_f = R\left(\frac{(1+i)^n - 1}{i}\right)$ to solve for R.
   d. Compare your answers to part (b) and part (c). What do you notice? Why did this happen?

2. For structured savings plans, the future value of the savings plan as a function of the number of payments made at that point is an interesting function to examine. Consider a structured savings plan with a recurring payment of $\$450$ made monthly and an annual interest rate of 5.875% compounded monthly.
   a. State the formula for the future value of this structured savings plan as a function of the number of payments made. Use f for the function name.
   b. Graph the function you wrote in part (a) for $0 \le x \le 216$.
   c. State any trends that you notice for this function.
   d. What is the approximate value of the function f for $x = 216$?
   e. What is the domain of f? Explain.
   f. If the domain of the function is restricted to natural numbers, is the function a geometric sequence? Why or why not?
   g. Recall that the n^{th} partial sums of a geometric sequence can be represented with S_n. It is true that $f(x) = S_x$ for positive integers x, since it is a geometric sequence; that is, $S_x = \sum_{i=1}^{x} ar^i$. State the geometric sequence whose sums of the first x terms are represented by this function.
   h. April has been following this structured savings plan for 18 years. April says that taking out the money and starting over will not affect the total money earned because the interest rate does not change. Explain why April is incorrect in her reasoning.

3. Henry plans to have $\$195{,}000$ in property in 14 years and would like to save up to $\$1$ million by depositing $\$3{,}068.95$ each month at 6% interest per year, compounded monthly. Tina's structured savings plan over the same time span is described in the following table:
   a. Who has the higher interest rate? Who pays more every month?
   b. At the end of 14 years, who has more money from their structured savings plan? Does this agree with what you expected? Why or why not?
   c. At the end of 40 years, who has more money from their structured savings plan?

Deposit #	Amount Saved
30	\$110,574.77
31	\$114,466.39
32	\$118,371.79
33	\$122,291.02
34	\$126,224.14
⋮	⋮
167	\$795,266.92
168	\$801,583.49

4. Edgar and Paul are two brothers that both get an inheritance of $\$150{,}000$. Both plan to save up over $\$1{,}000{,}000$ in 25 years. Edgar takes his inheritance and deposits the money into an investment account earning 8% interest annually, compounded monthly, payable at the end of 25 years. Paul spends his inheritance but uses a structured savings plan that is represented by the sequence $b_n = 1275 + b_{n-1} \cdot \left(1 + \frac{0.0775}{12}\right)$ with $b_0 = 1275$ in order to save up over $\$1{,}000{,}000$.

   a. Which of the two has more money at the end of 25 years?

   b. What are the pros and cons of both brothers' plans? Which would you rather do? Why?