Lesson 6: Euler's Number, e

Classwork

Exercises 1–3

1. Assume that there is initially 1 cm of water in the tank and the height of the water doubles every 10 seconds. Write an equation that could be used to calculate the height $H(t)$ of the water in the tank at any time t.

2. How would the equation in Exercise 1 change if …
   a. The initial depth of water in the tank was 2 cm?

   b. The initial depth of water in the tank was $\frac{1}{2} \text{ cm}$?

   c. The initial depth of water in the tank was 10 cm?

   d. The initial depth of water in the tank was A cm, for some positive real number A?

3. How would the equation in Exercise 2, part (d) change if …
   a. The height tripled every ten seconds?

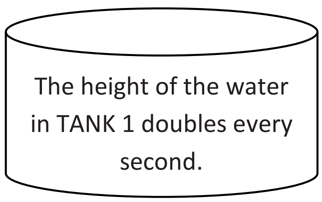
   b. The height doubled every five seconds?

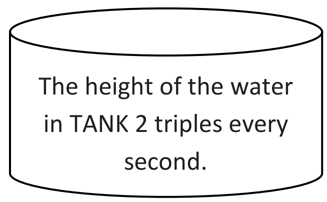
c. The height quadrupled every second?

d. The height halved every ten seconds?

Example 1

1. Consider two identical water tanks, each of which begins with a height of water 1 cm and fills with water at a different rate. Which equations can be used to calculate the height of water in each tank at time t? Use H_1 for tank 1 and H_2 for tank 2.





a. If both tanks start filling at the same time, which one fills first?

b. We want to know the average rate of change of the height of the water in these tanks over an interval that starts at a fixed time T as they are filling up. What is the formula for the average rate of change of a function f on an interval $[a, b]$?

c. What is the formula for the average rate of change of the function H_1 on an interval $[a, b]$?

d. Let's calculate the average rate of change of the function H_1 on the interval $[T, T + 0.1]$, which is an interval one-tenth of a second long starting at an unknown time T.

Exercises 4–8

4. For the second tank, calculate the average change in the height, H_2, from time T seconds to $T + 0.1$ seconds. Express the answer as a number times the value of the original function at time T. Explain the meaning of these findings.

5. For each tank, calculate the average change in height from time T seconds to $\mathrm{T} + 0.001$ seconds. Express the answer as a number times the value of the original function at time T. Explain the meaning of these findings.

6. In Exercise 5, the average rate of change of the height of the water in tank 1 on the interval $[T, T + 0.01]$ can be described by the expression $c_1 \cdot 2^T$, and the average rate of change of the height of the water in tank 2 on the interval $[T, T + 0.01]$ can be described by the expression $c_2 \cdot 3^T$. What are approximate values of c_1 and c_2?

7. As an experiment, let's look for a value of b so that if the height of the water can be described by $H(t) = b^t$, then the expression for the average of change on the interval $[T, T + 0.01]$ is $1 \cdot H(T)$.

   a. Write out the expression for the average rate of change of $H(t) = b^t$ on the interval $[T, T + 0.01]$.

   b. Set your expression in part (a) equal to $1 \cdot H(T)$ and reduce to an expression involving a single b.

   c. Now we want to find the value of b that satisfies the equation you found in part (b), but we do not have a way to explicitly solve this equation. Look back at Exercise 6; which two consecutive integers have b between them?

   d. Use your calculator and a guess-and-check method to find an approximate value of b to 2 decimal places.

8. Verify that for the value of b found in Exercise 7, $\frac{H_b(T + 0.001) - H_b(T)}{0.001} = H_b(T)$, where $H_b(T) = b^T$.

Lesson Summary

- Euler's number, e, is an irrational number that is approximately equal to $e \approx 2.7182818284590$.
- **AVERAGE RATE OF CHANGE:** Given a function f whose domain contains the interval of real numbers $[a, b]$ and whose range is a subset of the real numbers, the average rate of change on the interval $[a, b]$ is defined by the number

$$\frac{f(b) - b(a)}{b - a}.$$

Problem Set

1. The product $4 \cdot 3 \cdot 2 \cdot 1$ is called 4 factorial and is denoted by $4!$. Then $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, and for any positive integer n, $n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$.

   a. Complete the following table of factorial values:

n	1	2	3	4	5	6	7	8
$n!$								

   b. Evaluate the sum $1 + \frac{1}{1!}$.

   c. Evaluate the sum $1 + \frac{1}{1!} + \frac{1}{2!}$.

   d. Use a calculator to approximate the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!}$ to 7 decimal places. Do not round the fractions before evaluating the sum.

   e. Use a calculator to approximate the sum $1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!}$ to 7 decimal places. Do not round the fractions before evaluating the sum.

   f. Use a calculator to approximate sums of the form $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!}$ to 7 decimal places for $k = 5, 6, 7, 8, 9, 10$. Do not round the fractions before evaluating the sums with a calculator.

   g. Make a conjecture about the sums $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{k!}$ for positive integers k as k increases in size.

   h. Would calculating terms of this sequence ever yield an exact value of e? Why or why not?

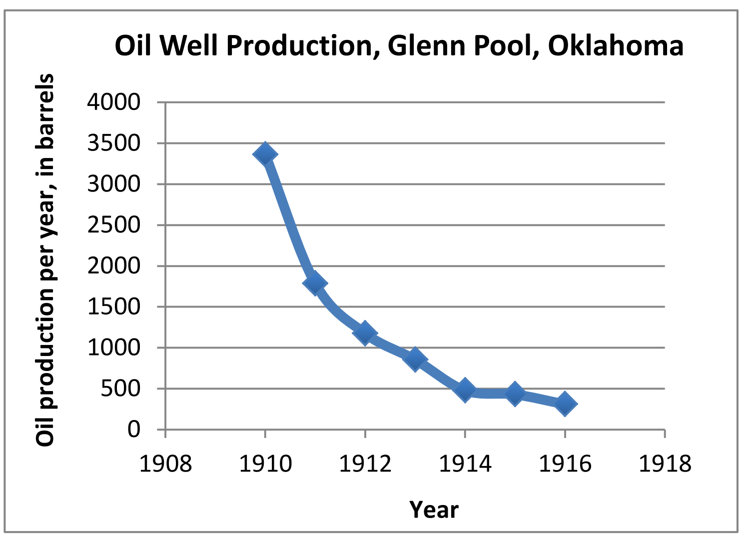
2. Consider the sequence given by the function $a_n = \left(1+\frac{1}{n}\right)^n$, where $n \geq 1$ is an integer.
   a. Use your calculator to approximate the first 5 terms of this sequence to 7 decimal places.
   b. Does it appear that this sequence settles near a particular value?
   c. Use a calculator to approximate the following terms of this sequence to 7 decimal places.
      i. a_{100}
      ii. a_{1000}
      iii. $a_{10,000}$
      iv. $a_{100,000}$
      v. $a_{1,000,000}$
      vi. $a_{10,000,000}$
      vii. $a_{100,000,000}$
   d. Does it appear that this sequence settles near a particular value?
   e. Compare the results of this exercise with the results of Problem 1. What do you observe?

3. If $x = 5a^4$ and $a = 2e^3$, express x in terms of e and approximate to the nearest whole number.

4. If $a = 2b^3$ and $b = -\frac{1}{2}e^{-2}$, express a in terms of e and approximate to four decimal places.

5. If $x = 3e^4$ and $= \frac{s}{2x^3}$, show that $s = 54e^{13}$ and approximate s to the nearest whole number.

6. The following graph shows the number of barrels of oil produced by the Glenn Pool well in Oklahoma from 1910 to 1916.



Source: Cutler, Willard W., Jr. Estimation of Underground Oil Reserves by Oil-Well Production Curves, U.S. Department of the Interior, 1924.

a. Estimate the average rate of change of the amount of oil produced by the well on the interval $[1910, 1916]$ and explain what that number represents.
b. Estimate the average rate of change of the amount of oil produced by the well on the interval $[1910, 1913]$ and explain what that number represents.
c. Estimate the average rate of change of the amount of oil produced by the well on the interval $[1913, 1916]$ and explain what that number represents.
d. Compare your results for the rates of change in oil production in the first half and the second half of the time period in question in parts (b) and (c). What do those numbers say about the production of oil from the well?
e. Notice that the average rate of change of the amount of oil produced by the well on any interval starting and ending in two consecutive years is always negative. Explain what that means in the context of oil production.

7. The following table lists the number of hybrid electric vehicles (HEVs) sold in the United States between 1999 and 2013.

Year	Number of HEVs sold in U.S.
1999	17
2000	9350
2001	20,282
2002	36,035
2003	47,600
2004	84,199
2005	209,711
2006	252,636

Year	Number of HEVs sold in U.S.
2007	352,274
2008	312,386
2009	290,271
2010	274,210
2011	268,752
2012	434,498
2013	495,685

Source: U.S. Department of Energy, Alternative Fuels and Advanced Vehicle Data Center, 2013.

a. During which one-year interval is the average rate of change of the number of HEVs sold the largest? Explain how you know.
b. Calculate the average rate of change of the number of HEVs sold on the interval $[2003, 2004]$ and explain what that number represents.
c. Calculate the average rate of change of the number of HEVs sold on the interval $[2003, 2008]$ and explain what that number represents.
d. What does it mean if the average rate of change of the number of HEVs sold is negative?

Extension:

8. The formula for the area of a circle of radius r can be expressed as a function $A(r) = \pi r^2$.
a. Find the average rate of change of the area of a circle on the interval $[4, 5]$.
b. Find the average rate of change of the area of a circle on the interval $[4, 4.1]$.
c. Find the average rate of change of the area of a circle on the interval $[4, 4.01]$.
d. Find the average rate of change of the area of a circle on the interval $[4, 4.001]$.

e. What is happening to the average rate of change of the area of the circle as the interval gets smaller and smaller?

f. Find the average rate of change of the area of a circle on the interval $[4, 4+h]$ for some small positive number h.

g. What happens to the average rate of change of the area of the circle on the interval $[4, 4+h]$ as $h \to 0$? Does this agree with your answer to part (d)? Should it agree with your answer to part (e)?

h. Find the average rate of change of the area of a circle on the interval $[r_0, r_0+h]$ for some positive number r_0 and some small positive number h.

i. What happens to the average rate of change of the area of the circle on the interval $[r_0, r_0+h]$ as $h \to 0$? Do you recognize the resulting formula?

9. The formula for the volume of a sphere of radius r can be expressed as a function $V(r) = \frac{4}{3}\pi r^3$. As you work through these questions, you will see the pattern develop more clearly if you leave your answers in the form of a coefficient times π. Approximate the coefficient to five decimal places.

   a. Find the average rate of change of the volume of a sphere on the interval $[2, 3]$.

   b. Find the average rate of change of the volume of a sphere on the interval $[2, 2.1]$.

   c. Find the average rate of change of the volume of a sphere on the interval $[2, 2.01]$.

   d. Find the average rate of change of the volume of a sphere on the interval $[2, 2.001]$.

   e. What is happening to the average rate of change of the volume of a sphere as the interval gets smaller and smaller?

   f. Find the average rate of change of the volume of a sphere on the interval $[2, 2+h]$ for some small positive number h.

   g. What happens to the average rate of change of the volume of a sphere on the interval $[2, 2+h]$ as $h \to 0$? Does this agree with your answer to part (e)? Should it agree with your answer to part (e)?

   h. Find the average rate of change of the volume of a sphere on the interval $[r_0, r_0+h]$ for some positive number r_0 and some small positive number h.

   i. What happens to the average rate of change of the volume of a sphere on the interval $[r_0, r_0+h]$ as $h \to 0$? Do you recognize the resulting formula?