Lesson 12: Properties of Logarithms

Classwork

Opening Exercise

Use the approximation \( \log(2) \approx 0.3010 \) to approximate the values of each of the following logarithmic expressions.

a. \( \log(20) \)

b. \( \log(0.2) \)

c. \( \log(2^4) \)

Exercises 1–10

For Exercises 1–6, explain why each statement below is a property of base-10 logarithms.

1. Property 1: \( \log(1) = 0 \).

2. Property 2: \( \log(10) = 1 \).

3. Property 3: For all real numbers \( r \), \( \log(10^r) = r \).

4. Property 4: For any \( x > 0 \), \( 10^{\log(x)} = x \).
5. Property 5: For any positive real numbers \( x \) and \( y \), \( \log(x \cdot y) = \log(x) + \log(y) \).
   Hint: Use an exponent rule as well as Property 4.

6. Property 6: For any positive real number \( x \) and any real number \( r \), \( \log(x^r) = r \cdot \log(x) \).
   Hint: Again, use an exponent rule as well as Property 4.

7. Apply properties of logarithms to rewrite the following expressions as a single logarithm or number.
   a. \( \frac{1}{2} \log(25) + \log(4) \)
   b. \( \frac{1}{3} \log(8) + \log(16) \)
   c. \( 3 \log(5) + \log(0.8) \)

8. Apply properties of logarithms to rewrite each expression as a sum of terms involving numbers, \( \log(x) \), and \( \log(y) \).
   a. \( \log(3x^2y^5) \)
   b. \( \log(\sqrt{x^7y^3}) \)
9. In mathematical terminology, logarithms are well defined because if \( X = Y \), then \( \log(X) = \log(Y) \) for \( X, Y > 0 \). This means that if you want to solve an equation involving exponents, you can apply a logarithm to both sides of the equation, just as you can take the square root of both sides when solving a quadratic equation. You do need to be careful not to take the logarithm of a negative number or zero.

Use the property stated above to solve the following equations.

a. \( 10^{10x} = 100 \)

b. \( 10^{x-1} = \frac{1}{10^{x+1}} \)

c. \( 100^{2x} = 10^{3x-1} \)

10. Solve the following equations.

a. \( 10^x = 2^7 \)

b. \( 10^{x^2+1} = 15 \)

c. \( 4^x = 5^3 \)
Lesson Summary

We have established the following properties for base-10 logarithms, where $x$ and $y$ are positive real numbers and $r$ is any real number:

1. $\log(1) = 0$
2. $\log(10) = 1$
3. $\log(10^r) = r$
4. $10^{\log(x)} = x$
5. $\log(x \cdot y) = \log(x) + \log(y)$
6. $\log(x^r) = r \cdot \log(x)$

Additional properties not yet established are the following:

7. $\log\left(\frac{1}{x}\right) = -\log(x)$
8. $\log\left(\frac{x}{y}\right) = \log(x) - \log(y)$

Also, logarithms are well defined, meaning that for $X, Y > 0$, if $X = Y$, then $\log(X) = \log(Y)$.

Problem Set

1. Use the approximate logarithm values below to estimate each of the following logarithms. Indicate which properties you used.

   \[
   \begin{align*}
   \log(2) &= 0.3010 & \log(3) &= 0.4771 \\
   \log(5) &= 0.6990 & \log(7) &= 0.8451
   \end{align*}
   \]

   a. $\log(6)$
   b. $\log(15)$
   c. $\log(12)$
   d. $\log(10^7)$
   e. $\log\left(\frac{1}{7}\right)$
   f. $\log\left(\frac{2}{7}\right)$
   g. $\log\left(\sqrt{2}\right)$
2. Let $\log(X) = r$, $\log(Y) = s$, and $\log(Z) = t$. Express each of the following in terms of $r$, $s$, and $t$.
   a. $\log\left(\frac{X}{Y}\right)$
   b. $\log(YZ)$
   c. $\log(X^t)$
   d. $\log(\sqrt[3]{Z})$
   e. $\log\left(\sqrt[4]{\frac{s}{t}}\right)$
   f. $\log(XY^2Z^3)$

3. Use the properties of logarithms to rewrite each expression in an equivalent form containing a single logarithm.
   a. $\log\left(\frac{13}{5}\right) + \log\left(\frac{5}{4}\right)$
   b. $\log\left(\frac{5}{6}\right) - \log\left(\frac{2}{3}\right)$
   c. $\frac{1}{2}\log(16) + \log(3) + \log\left(\frac{1}{4}\right)$

4. Use the properties of logarithms to rewrite each expression in an equivalent form containing a single logarithm.
   a. $\log(\sqrt{x}) + \frac{1}{2}\log\left(\frac{1}{x}\right) + 2\log(x)$
   b. $\log(\sqrt[3]{x}) + \log(\sqrt[3]{x^4})$
   c. $\log(x) + 2\log(y) - \frac{1}{2}\log(z)$
   d. $\frac{1}{3}(\log(x) - 3\log(y) + \log(x))$
   e. $2(\log(x) - \log(3y)) + 3(\log(z) - 2\log(x))$

5. Use properties of logarithms to rewrite the following expressions in an equivalent form containing only $\log(x)$, $\log(y)$, $\log(z)$, and numbers.
   a. $\log\left(\frac{3x^2y^4}{\sqrt{z}}\right)$
   b. $\log\left(\frac{42\sqrt{3y}}{x^2z}\right)$
   c. $\log\left(\frac{100x^2}{y^3}\right)$
   d. $\log\left(\frac{r^3y^2}{10z}\right)$
   e. $\log\left(\frac{1}{10x^2z}\right)$

6. Express $\log\left(\frac{1}{x} - \frac{1}{x+1}\right) + \left(\log\left(\frac{1}{x}\right) - \log\left(\frac{1}{x+1}\right)\right)$ as a single logarithm for positive numbers $x$.

7. Show that $\log(x + \sqrt{x^2 - 1}) + \log(x - \sqrt{x^2 - 1}) = 0$ for $x \geq 1$. 

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8. If \(xy = 10^{3.67}\), find the value of \(\log(x) + \log(y)\).

9. Solve the following exponential equations by taking the logarithm base 10 of both sides. Leave your answers stated in terms of logarithmic expressions.
   a. \(10^{x^2} = 320\)
   b. \(10^x = 300\)
   c. \(10^{3x} = 400\)
   d. \(5^{2x} = 200\)
   e. \(3^x = 7^{-3x+2}\)

10. Solve the following exponential equations.
    a. \(10^x = 3\)
    b. \(10^y = 30\)
    c. \(10^z = 300\)
    d. Use the properties of logarithms to justify why \(x, y,\) and \(z\) form an arithmetic sequence whose constant difference is 1.

11. Without using a calculator, explain why the solution to each equation must be a real number between 1 and 2.
    a. \(11^x = 12\)
    b. \(21^x = 30\)
    c. \(100^x = 2000\)
    d. \(\left(\frac{1}{11}\right)^x = 0.01\)
    e. \(\left(\frac{2}{3}\right)^x = \frac{1}{2}\)
    f. \(99^x = 9000\)

12. Express the exact solution to each equation as a base-10 logarithm. Use a calculator to approximate the solution to the nearest 1000th.
    a. \(11^x = 12\)
    b. \(21^x = 30\)
    c. \(100^x = 2000\)
    d. \(\left(\frac{1}{11}\right)^x = 0.01\)
    e. \(\left(\frac{2}{3}\right)^x = \frac{1}{2}\)
    f. \(99^x = 9000\)

13. Show that the value of \(x\) that satisfies the equation \(10^x = 3 \cdot 10^n\) is \(\log(3) + n\).
14. Solve each equation. If there is no solution, explain why.
   a. \(3 \cdot 5^x = 21\)
   b. \(10^{x-3} = 25\)
   c. \(10^x + 10^{x+1} = 11\)
   d. \(8 - 2^x = 10\)

15. Solve the following equation for \(n\): \(A = P(1 + r)^n\).

16. In this exercise, we will establish a formula for the logarithm of a sum. Let \(L = \log(x + y)\), where \(x, y > 0\).
   a. Show \(\log(x) + \log\left(1 + \frac{y}{x}\right) = L\). State as a property of logarithms after showing this is a true statement.
   b. Use part (a) and the fact that \(\log(100) = 2\) to rewrite \(\log(365)\) as a sum.
   c. Rewrite 365 in scientific notation, and use properties of logarithms to express \(\log(365)\) as a sum of an integer and a logarithm of a number between 0 and 10.
   d. What do you notice about your answers to (b) and (c)?
   e. Find two integers that are upper and lower estimates of \(\log(365)\).