# Lesson 25: Geometric Sequences and Exponential Growth and Decay

## Classwork

### Opening Exercise

Suppose a ball is dropped from an initial height $h_0$ and that each time it rebounds, its new height is $60\%$ of its previous height.

a. What are the first four rebound heights $h_1$, $h_2$, $h_3$, and $h_4$ after being dropped from a height of $h_0 = 10$ ft.?

b. Suppose the initial height is $A$ ft. What are the first four rebound heights? Fill in the following table:

| Rebound | Height (ft.) |
|---|---|
| $1$ | |
| $2$ | |
| $3$ | |
| $4$ | |

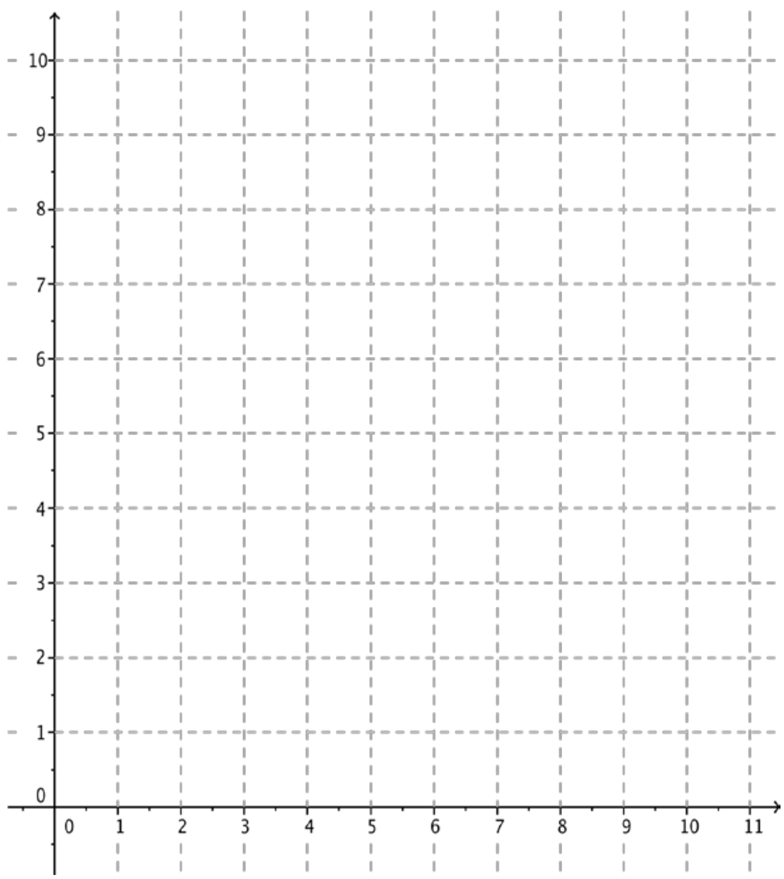
c. How is each term in the sequence related to the one that came before it?

d. Suppose the initial height is $A$ ft. and that each rebound, rather than being $60\%$ of the previous height, is $r$ times the previous height, where $0 < r < 1$. What are the first four rebound heights? What is the $n^{\text{th}}$ rebound height?

e. What kind of sequence is the sequence of rebound heights?

f. Suppose that we define a function $f$ with domain all real numbers so that $f(1)$ is the first rebound height, $f(2)$ is the second rebound height, and continuing so that $f(k)$ is the $k^{\text{th}}$ rebound height for positive integers $k$. What type of function would you expect $f$ to be?

g. On the coordinate plane below, sketch the height of the bouncing ball when $A = 10$ and $r = 0.60$, assuming that the highest points occur at $x = 1, 2, 3, 4, \ldots$.



h. Does the exponential function $f(x) = 10(0.60)^x$ for real numbers $x$ model the height of the bouncing ball? Explain how you know.

i. What does the function $f(n) = 10(0.60)^n$ for integers $n \geq 0$ model?

## Exercises

1.

a. Jane works for a video game development company that pays her a starting salary of $\$100$ a day, and each day she works, she earns $\$100$ more than the day before. How much does she earn on day $5$?

b. If you were to graph the growth of her salary for the first $10$ days she worked, what would the graph look like?

c. What kind of sequence is the sequence of Jane's earnings each day?

2. A laboratory culture begins with $1{,}000$ bacteria at the beginning of the experiment, which we will denote by time $0$ hours. By time $2$ hours, there were $2{,}890$ bacteria.

   a. If the number of bacteria is increasing by a common factor each hour, how many bacteria were there at time $1$ hour? At time $3$ hours?

   b. Find the explicit formula for term $P_n$ of the sequence in this case.

   c. How would you find term $P_{n+1}$ if you know term $P_n$? Write a recursive formula for $P_{n+1}$ in terms of $P_n$.

   d. If $P_0$ is the initial population, the growth of the population $P_n$ at time $n$ hours can be modeled by the sequence $P_n = P(n)$, where $P$ is an exponential function with the following form:

   $$P(n) = P_0 2^{kn}, \text{ where } k > 0.$$

   Find the value of $k$ and write the function $P$ in this form. Approximate $k$ to four decimal places.

   e. Use the function in part (d) to determine the value of $t$ when the population of bacteria has doubled.

f. If $P_0$ is the initial population, the growth of the population $P$ at time $t$ can be expressed in the following form:

$$P(n) = P_0 e^{kn}, \text{ where } k > 0.$$

Find the value of $k$, and write the function $P$ in this form. Approximate $k$ to four decimal places.

g. Use the formula in part (d) to determine the value of $t$ when the population of bacteria has doubled.

3. The first term $a_0$ of a geometric sequence is $-5$, and the common ratio $r$ is $-2$.

a. What are the terms $a_0, a_1$, and $a_2$?

b. Find a recursive formula for this sequence.

c. Find an explicit formula for this sequence.

d. What is term $a_9$?

e. What is term $a_{10}$?

4. Term $a_4$ of a geometric sequence is $5.8564$, and term $a_5$ is $-6.44204$.
   a. What is the common ratio $r$?

   b. What is term $a_0$?

   c. Find a recursive formula for this sequence.

   d. Find an explicit formula for this sequence.

5. The recursive formula for a geometric sequence is $a_{n+1} = 3.92(a_n)$ with $a_0 = 4.05$. Find an explicit formula for this sequence.

6. The explicit formula for a geometric sequence is $a_n = 147(2.1)^{3n}$. Find a recursive formula for this sequence.

### Lesson Summary

**ARITHMETIC SEQUENCE:** A sequence is called *arithmetic* if there is a real number $d$ such that each term in the sequence is the sum of the previous term and $d$.

- *Explicit formula:* Term $a_n$ of an arithmetic sequence with first term $a_0$ and common difference $d$ is given by $a_n = a_0 + nd$, for $n \geq 0$.
- *Recursive formula:* Term $a_{n+1}$ of an arithmetic sequence with first term $a_0$ and common difference $d$ is given by $a_{n+1} = a_n + d$, for $n \geq 0$.

**GEOMETRIC SEQUENCE:** A sequence is called *geometric* if there is a real number $r$ such that each term in the sequence is a product of the previous term and $r$.

- *Explicit formula:* Term $a_n$ of a geometric sequence with first term $a_0$ and common ratio $r$ is given by $a_n = a_0 r^n$, for $n \geq 0$.
- *Recursive formula:* Term $a_{n+1}$ of a geometric sequence with first term $a_0$ and common ratio $r$ is given by $a_{n+1} = a_n r$.

## Problem Set

1. Convert the following recursive formulas for sequences to explicit formulas.
   a. $a_{n+1} = 4.2 + a_n$ with $a_0 = 12$
   b. $a_{n+1} = 4.2a_n$ with $a_0 = 12$
   c. $a_{n+1} = \sqrt{5}\, a_n$ with $a_0 = 2$
   d. $a_{n+1} = \sqrt{5} + a_n$ with $a_0 = 2$
   e. $a_{n+1} = \pi\, a_n$ with $a_0 = \pi$

2. Convert the following explicit formulas for sequences to recursive formulas.
   a. $a_n = \frac{1}{5}(3^n)$ for $n \geq 0$
   b. $a_n = 16 - 2n$ for $n \geq 0$
   c. $a_n = 16\left(\frac{1}{2}\right)^n$ for $n \geq 0$
   d. $a_n = 71 - \frac{6}{7}n$ for $n \geq 0$
   e. $a_n = 190(1.03)^n$ for $n \geq 0$

3. If a geometric sequence has $a_1 = 256$ and $a_8 = 512$, find the exact value of the common ratio $r$.

4. If a geometric sequence has $a_2 = 495$ and $a_6 = 311$, approximate the value of the common ratio $r$ to four decimal places.

5. Find the difference between the terms $a_{10}$ of an arithmetic sequence and a geometric sequence, both of which begin at term $a_0$ and have $a_2 = 4$ and $a_4 = 12$.

6. Given the geometric series defined by the following values of $a_0$ and $r$, find the value of $n$ so that $a_n$ has the specified value.

   a. $a_0 = 64, r = \frac{1}{2}, a_n = 2$

   b. $a_0 = 13, r = 3, a_n = 85293$

   c. $a_0 = 6.7, r = 1.9, a_n = 7804.8$

   d. $a_0 = 10958, r = 0.7, a_n = 25.5$

7. Jenny planted a sunflower seedling that started out 5 cm tall, and she finds that the average daily growth is $3.5$ cm.

   a. Find a recursive formula for the height of the sunflower plant on day $n$.

   b. Find an explicit formula for the height of the sunflower plant on day $n \geq 0$.

8. Kevin modeled the height of his son (in inches) at age $n$ years for $n = 2, 3, \ldots, 8$ by the sequence $h_n = 34 + 3.2(n - 2)$. Interpret the meaning of the constants 34 and 3.2 in his model.

9. Astrid sells art prints through an online retailer. She charges a flat rate per order for an order processing fee, sales tax, and the same price for each print. The formula for the cost of buying $n$ prints is given by $P_n = 4.5 + 12.6\,n$.

   a. Interpret the number $4.5$ in the context of this problem.

   b. Interpret the number $12.6$ in the context of this problem.

   c. Find a recursive formula for the cost of buying $n$ prints.

10. A bouncy ball rebounds to $90\%$ of the height of the preceding bounce. Craig drops a bouncy ball from a height of 20 ft.

    a. Write out the sequence of the heights $h_1$, $h_2$, $h_3$, and $h_4$ of the first four bounces, counting the initial height as $h_0 = 20$.

    b. Write a recursive formula for the rebound height of a bouncy ball dropped from an initial height of 20 ft.

    c. Write an explicit formula for the rebound height of a bouncy ball dropped from an initial height of 20 ft.

    d. How many bounces will it take until the rebound height is under 6 ft.?

    e. Extension: Find a formula for the minimum number of bounces needed for the rebound height to be under $y$ ft., for a real number $0 < y < 20$.

11. Show that when a quantity $a_0 = A$ is increased by $x\%$, its new value is $a_1 = A\left(1 + \frac{x}{100}\right)$. If this quantity is again increased by $x\%$, what is its new value $a_2$? If the operation is performed $n$ times in succession, what is the final value of the quantity $a_n$?

12. When Eli and Daisy arrive at their cabin in the woods in the middle of winter, the internal temperature is $40°\text{F}$.

    a. Eli wants to turn up the thermostat by $2°\text{F}$ every $15$ minutes. Find an explicit formula for the sequence that represents the thermostat settings using Eli's plan.

    b. Daisy wants to turn up the thermostat by $4\%$ every $15$ minutes. Find an explicit formula for the sequence that represents the thermostat settings using Daisy's plan.

    c. Which plan will get the thermostat to $60°\text{F}$ most quickly?

    d. Which plan will get the thermostat to $72°\text{F}$ most quickly?

13. In nuclear fission, one neutron splits an atom causing the release of two other neutrons, each of which splits an atom and produces the release of two more neutrons, and so on.

    a. Write the first few terms of the sequence showing the numbers of atoms being split at each stage after a single atom splits. Use $a_0 = 1$.

    b. Find the explicit formula that represents your sequence in part (a).

    c. If the interval from one stage to the next is one-millionth of a second, write an expression for the number of atoms being split at the end of one second.

    d. If the number from part (c) were written out, how many digits would it have?