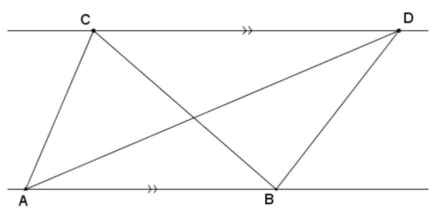
Lesson 4: Comparing the Ratio Method with the Parallel Method

Classwork

Today, our goal is to show that the parallel method and the ratio method are equivalent; that is, given a figure in the plane and a scale factor $r > 0$, the scale drawing produced by the parallel method is congruent to the scale drawing produced by the ratio method. We start with two easy exercises about the areas of two triangles whose bases lie on the same line, which will help show that the two methods are equivalent.

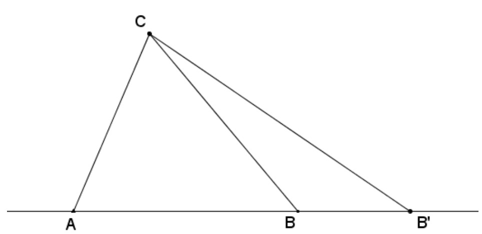
Opening Exercises 1–2

1. Suppose two triangles, $\triangle ABC$ and $\triangle ABD$, share the same base $\overline{AB}$ such that points C and D lie on a line parallel to line $\overleftrightarrow{AB}$. Show that their areas are equal, i.e., $\text{Area}(\triangle ABC) = \text{Area}(\triangle ABD)$. (Hint: Why are the altitudes of each triangle equal in length?)



2. Suppose two triangles have different length bases, $\overline{AB}$ and $\overline{AB'}$, that lie on the same line. Furthermore, suppose they both have the same vertex C opposite these bases. Show that value of the ratio of their areas is equal to the value of the ratio of the lengths of their bases, i.e.,

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle AB'C)} = \frac{AB}{AB'}.$$



Discussion

To show that the parallel and ratio methods are equivalent, we need only look at one of the simplest versions of a scale drawing: scaling segments. First, we need to show that the scale drawing of a segment generated by the parallel method is the same segment that the ratio method would have generated and vice versa. (i.e., that the scaled segment generated by the ratio method is the same segment generated by the parallel method.) That is,

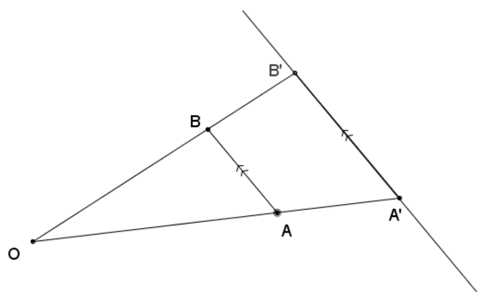
The parallel method $\Longrightarrow$ The ratio method,

and

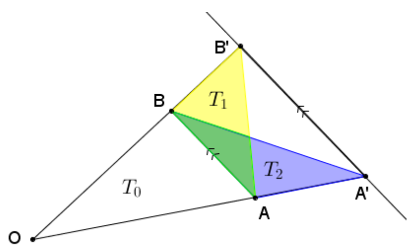
The ratio method $\Longrightarrow$ The parallel method.

The first implication above can be stated as the following theorem:

PARALLEL $\Longrightarrow$ RATIO THEOREM: Given a line segment $\overline{AB}$ and point O not on the line $\overleftrightarrow{AB}$, construct a scale drawing of $\overline{AB}$ with scale factor $r > 0$ using the parallel method: Let $A' = D_{O,r}(A)$, and ℓ be the line parallel to $\overleftrightarrow{AB}$ that passes through A'. Let B' be the point where ray $\overrightarrow{OB}$ intersects ℓ. Then B' is the same point found by the ratio method; that is, $B' = D_{O,r}(B)$.



PROOF: We prove the case when $r > 1$; the case when $0 < r < 1$ is the same but with a different picture. Construct two line segments $\overline{BA'}$ and $\overline{AB'}$ to form two triangles $\triangle BAB'$ and $\triangle BAA'$, labeled as T_1 and T_2, respectively, in the picture below.



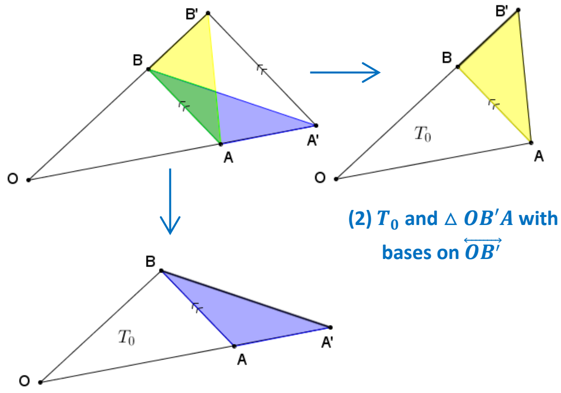
The areas of these two triangles are equal,

$$\text{Area}(T_1) = \text{Area}(T_2),$$

by Exercise 1 (why?). Label $\triangle OAB$ by T_0. Then $\text{Area}(\triangle OA'B) = \text{Area}(\triangle OB'A)$ because areas add:

$$\begin{aligned}\text{Area}(\triangle OA'B) &= \text{Area}(T_0) + \text{Area}(T_2)\\ &= \text{Area}(T_0) + \text{Area}(T_1)\\ &= \text{Area}(\triangle OB'A).\end{aligned}$$

Next, we apply Exercise 2 to two sets of triangles: (1) T_0 and $\triangle\ OA'B$ and (2) T_0 and $\triangle\ OB'A$.



(2) T_0 and $\triangle\ OB'A$ with bases on $\overleftrightarrow{OB'}$

(1) T_0 and $\triangle\ OA'B$ with bases on $\overleftrightarrow{OA'}$

Therefore,

$$\frac{\text{Area}(\triangle\ OA'B)}{\text{Area}(T_0)} = \frac{OA'}{OA}, \text{ and}$$

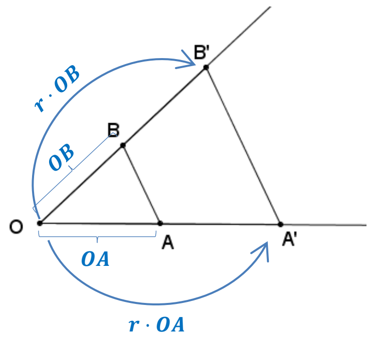
$$\frac{\text{Area}(\triangle\ OB'A)}{\text{Area}(T_0)} = \frac{OB'}{OB}.$$

Since $\text{Area}(\triangle\ OA'B) = \text{Area}(\triangle\ OB'A)$, we can equate the fractions: $\frac{OA'}{OA} = \frac{OB'}{OB}$. Since r is the scale factor used in dilating $\overline{OA}$ to $\overline{OA'}$, we know that $\frac{OA'}{OA} = r$; therefore, $\frac{OB'}{OB} = r$, or $OB' = r \cdot OB$. This last equality implies that B' is the dilation of B from O by scale factor r, which is what we wanted to prove.

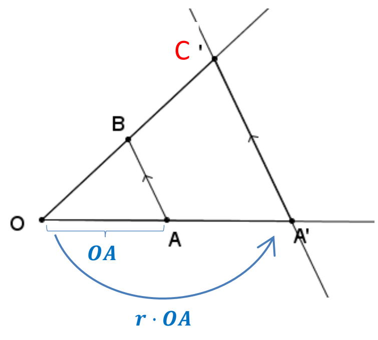
Next, we prove the reverse implication to show that both methods are equivalent to each other.

RATIO $\Longrightarrow$ PARALLEL THEOREM: Given a line segment $\overline{AB}$ and point O not on the line $\overleftrightarrow{AB}$, construct a scale drawing $\overline{A'B'}$ of $\overline{AB}$ with scale factor $r > 0$ using the ratio method (Find $A' = D_{O,r}(A)$ and $B' = D_{O,r}(B)$, and draw $\overline{A'B'}$). Then B' is the same as the point found using the parallel method.

PROOF: Since both the ratio method and the parallel method start with the same first step of setting $A' = D_{O,r}(A)$, the only difference between the two methods is in how the second point is found. If we use the parallel method, we construct the line ℓ parallel to $\overleftrightarrow{AB}$ that passes through A' and label the point where ℓ intersects $\overrightarrow{OB}$ by C. Then B' is the same as the point found using the parallel method if we can show that $C = B'$.



The ratio method

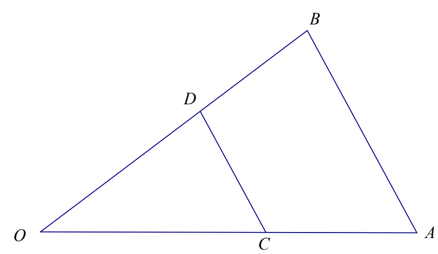


The parallel method

By the parallel $\Longrightarrow$ ratio theorem, we know that $C = D_{O,r}(B)$, i.e., that C is the point on ray $\overrightarrow{OB}$ such that $OC = r \cdot OB$. But B' is also the point on ray $\overrightarrow{OB}$ such that $OB' = r \cdot OB$. Hence, they must be the same point.

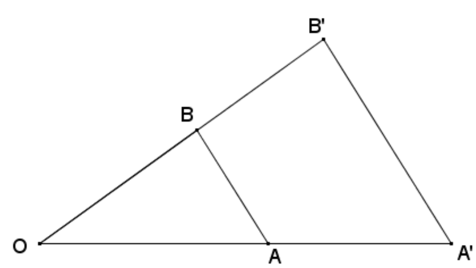
The fact that the ratio and parallel methods are equivalent is often stated as the triangle side splitter theorem. To understand the triangle side splitter theorem, we need a definition:

SIDE SPLITTER: A line segment CD is said to *split the sides of* $\triangle OAB$ *proportionally* if C is a point on $\overline{OA}$, D is a point on $\overline{OB}$, and $\frac{OA}{OC} = \frac{OB}{OD}$ (or equivalently, $\frac{OC}{OA} = \frac{OD}{OB}$). We call line segment CD a *side splitter.*



TRIANGLE SIDE SPLITTER THEOREM: A line segment splits two sides of a triangle proportionally if and only if it is parallel to the third side.

Restatement of the triangle side splitter theorem:

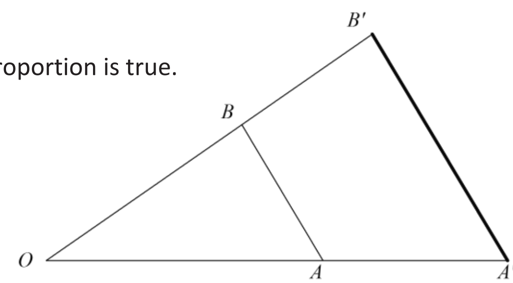


Lesson Summary

THE TRIANGLE SIDE SPLITTER THEOREM: A line segment splits two sides of a triangle proportionally if and only if it is parallel to the third side.

Problem Set

1. Use the diagram to answer each part below.



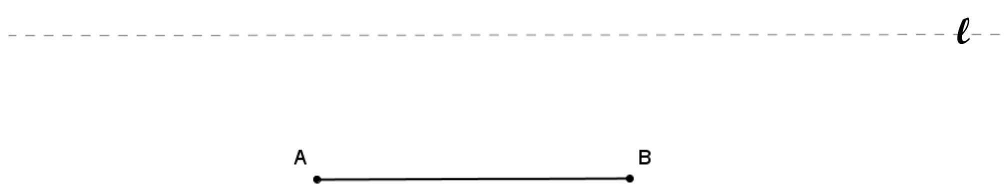
   a. Measure the segments in the figure below to verify that the proportion is true.

$$\frac{OA'}{OA} = \frac{OB'}{OB}$$

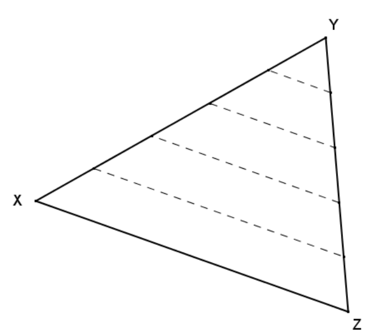
   b. Is the proportion $\frac{OA}{OA'} = \frac{OB}{OB'}$ also true? Explain algebraically.

   c. Is the proportion $\frac{AA'}{OA'} = \frac{BB'}{OB'}$ also true? Explain algebraically.

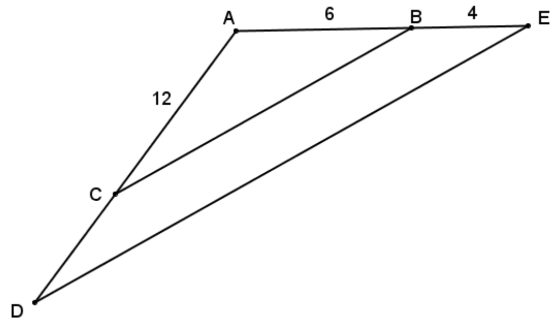
2. Given the diagram below, $AB = 30$, line ℓ is parallel to $\overline{AB}$, and the distance from $\overline{AB}$ to ℓ is 25. Locate point C on line ℓ such that $\triangle ABC$ has the greatest area. Defend your answer.



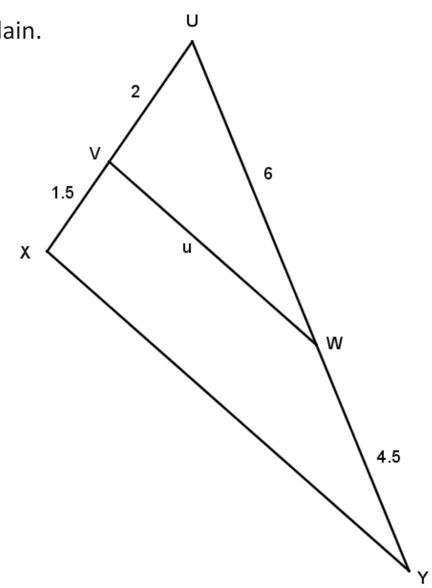
3. Given $\triangle XYZ$, $\overline{XY}$ and $\overline{YZ}$ are partitioned into equal length segments by the endpoints of the dashed segments as shown. What can be concluded about the diagram?



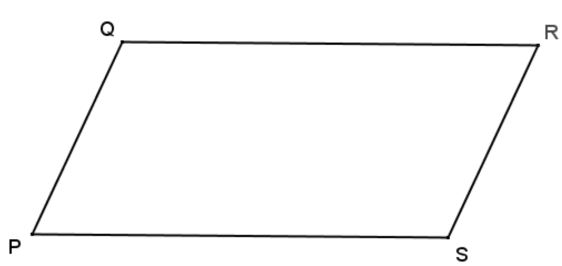
4. Given the diagram, $AC = 12$, $AB = 6$, $BE = 4$, $\angle ACB = x°$, and $\angle D = x°$, find CD.



5. What conclusions can be drawn from the diagram shown to the right? Explain.



6. Parallelogram $PQRS$ is shown. Two triangles are formed by a diagonal within the parallelogram. Identify those triangles and explain why they are guaranteed to have the same areas.



7. In the diagram to the right, $HI = 36$ and $GJ = 42$. If the ratio of the areas of the triangles is $\frac{\text{Area } \Delta GHI}{\text{Area } \Delta JHI} = \frac{5}{9}$, find JH, GH, GI, and JI.