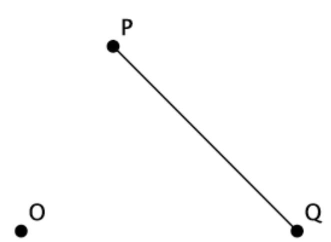
Lesson 7: How Do Dilations Map Segments?

Classwork

Opening Exercise

a. Is a dilated segment still a segment? If the segment is transformed under a dilation, explain how.

b. Dilate the segment PQ by a scale factor of 2 from center O.



i. Is the dilated segment $P'Q'$ a segment?

ii. How, if at all, has the segment PQ been transformed?

Example 1

Case 1. Consider the case where the scale factor of dilation is $r = 1$. Will a dilation from center O map segment PQ to a segment $P'Q'$? Explain.

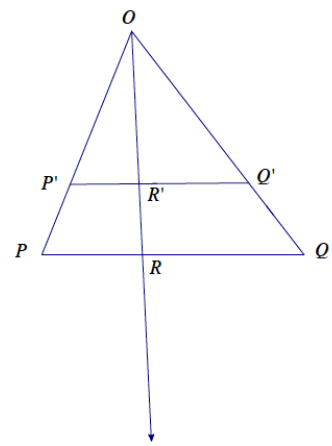
Example 2

Case 2. Consider the case where a line PQ contains the center of the dilation. Will a dilation from the center with scale factor $r \neq 1$ map the segment PQ to a segment $P'Q'$? Explain.

Example 3

Case 3. Consider the case where $\overleftrightarrow{PQ}$ does not contain the center O of the dilation and the scale factor r of the dilation is not equal to 1; then we have the situation where the key points O, P, and Q form $\triangle OPQ$. The scale factor not being equal to 1 means that we must consider scale factors such that $0 < r < 1$ and $r > 1$. However, the proofs for each are similar, so we will focus on the case when $0 < r < 1$.

When we dilate points P and Q from center O by scale factor $0 < r < 1$, as shown, what do we know about points P' and Q'?



We will use the fact that the line segment $P'Q'$ splits the sides of $\triangle OPQ$ proportionally and that the lines containing $\overline{PQ}$ and $\overline{P'Q'}$ are parallel to prove that a dilation maps segments to segments. Because we already know what happens when points P and Q are dilated, consider another point R that is on the segment $\overline{PQ}$. After dilating R from center O by scale factor r to get the point R', does R' lie on the segment $\overline{P'Q'}$?

Putting together the preliminary dilation theorem for segments with the dilation theorem, we get:

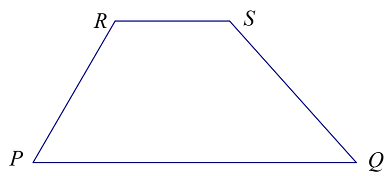
DILATION THEOREM FOR SEGMENTS: A dilation $D_{O,r}$ maps a line segment PQ to a line segment $P'Q'$ sending the endpoints to the endpoints so that $P'Q' = rPQ$. Whenever the center O does not lie in line PQ and $r \neq 1$, we conclude $\overleftrightarrow{PQ} \parallel \overleftrightarrow{P'Q'}$. Whenever the center O lies in $\overleftrightarrow{PQ}$ or if $r = 1$, we conclude $\overleftrightarrow{PQ} = \overleftrightarrow{P'Q'}$.

As an aside, observe that dilation maps parallel line segments to parallel line segments. Further, a dilation maps a directed line segment to a directed line segment that points in the same direction.

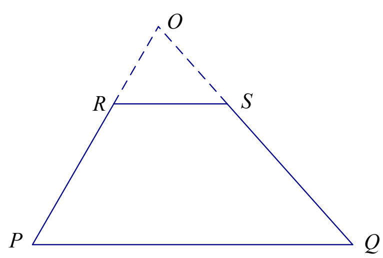
Example 4

Now look at the converse of the dilation theorem for segments: If $\overline{PQ}$ and $\overline{RS}$ are line segments of different lengths in the plane, then there is a dilation that maps one to the other if and only if $\overleftrightarrow{PQ} = \overleftrightarrow{RS}$ or $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$.

Based on Examples 2 and 3, we already know that a dilation maps a segment PQ to another line segment, say $\overline{RS}$, so that $\overleftrightarrow{PQ} = \overleftrightarrow{RS}$ (Example 2) or $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$ (Example 3). If $\overleftrightarrow{PQ} \parallel \overleftrightarrow{RS}$, then, because $\overline{PQ}$ and $\overline{RS}$ are different lengths in the plane, they are bases of a trapezoid, as shown.



Since $\overline{PQ}$ and $\overline{RS}$ are segments of different lengths, then the non-base sides of the trapezoid are not parallel, and the lines containing them will meet at a point O as shown.



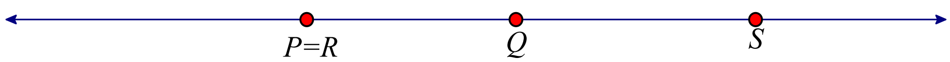
Recall that we want to show that a dilation will map $\overline{PQ}$ to $\overline{RS}$. Explain how to show it.

The case when the segments $\overline{PQ}$ and $\overline{RS}$ are such that $\overleftrightarrow{PQ} = \overleftrightarrow{RS}$ is left as an exercise.

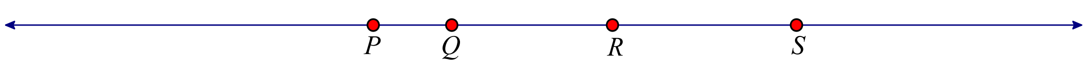
Exercises 1–2

In the following exercises, you will consider the case where the segment and its dilated image belong to the same line; that is, when $\overline{PQ}$ and $\overline{RS}$ are such that $\overleftrightarrow{PQ} = \overleftrightarrow{RS}$.

1. Consider points P, Q, R, and S on a line, where $P = R$, as shown below. Show there is a dilation that maps $\overline{PQ}$ to $\overline{RS}$. Where is the center of the dilation?



2. Consider points P, Q, R, and S on a line as shown below where $PQ \neq RS$. Show there is a dilation that maps $\overline{PQ}$ to $\overline{RS}$. Where is the center of the dilation?

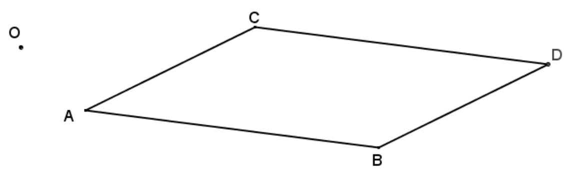


Lesson Summary

- When a segment is dilated by a scale factor of $r = 1$, then the segment and its image would be the same length.
- When the points P and Q are on a line containing the center, then the dilated points P' and Q' will also be collinear with the center producing an image of the segment that is a segment.
- When the points P and Q are not collinear with the center, and the segment is dilated by a scale factor of $r \neq 1$, then the point P' lies on the ray OP' with $OP' = r \cdot OP$ *and* Q' *lies on ray* OQ *with* $Q' = r \cdot OQ$.

Problem Set

1. Draw the dilation of parallelogram $ABCD$ from center O using the scale factor $r = 2$, and then answer the questions that follow.



   a. Is the image $A'B'C'D'$ also a parallelogram? Explain.
   b. What do parallel lines seem to map to under a dilation?

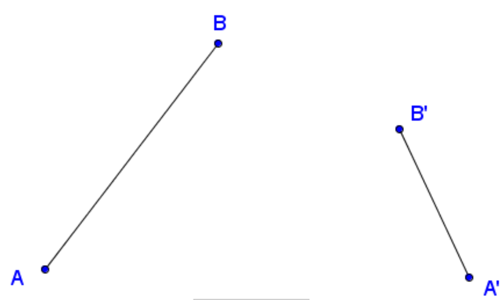
2. Given parallelogram $ABCD$ with $A(-8,1)$, $B(2,-4)$, $C(-3,-6)$, and $D(-13,-1)$, perform a dilation of the plane centered at the origin using the following scale factors.
   a. Scale factor $\frac{1}{2}$
   b. Scale factor 2
   c. Are the images of parallel line segments under a dilation also parallel? Use your graphs to support your answer.

3. In Lesson 7, Example 3, we proved that a line segment $\overline{PQ}$, where O, P, and Q are the vertices of a triangle, maps to a line segment $\overline{P'Q'}$ under a dilation with a scale factor $r < 1$. Using a similar proof, prove that for O not on $\overleftrightarrow{PQ}$, a dilation with center O and scale factor $r > 1$ maps a point R on $\overline{PQ}$ to a point R' on line $\overleftrightarrow{PQ}$.

4. On the plane, $\overline{AB} \parallel \overline{A'B'}$ and $AB \neq A'B'$. Describe a dilation mapping $\overline{AB}$ to $\overline{A'B'}$. (Hint: There are 2 cases to consider.)

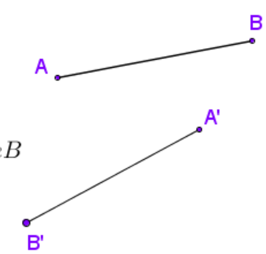
5. Only one of Figures A, B, or C below contains a dilation that maps A to A' and B to B'. Explain for each figure why the dilation does or does not exist. For each figure, assume that $AB \neq A'B'$.

   a.



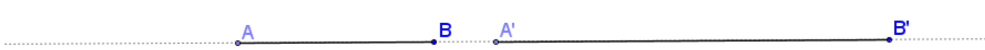
$Figure\ A$

   b.



$Figure B$

   c.



$Figure\ C$