Lesson 1: Thales' Theorem

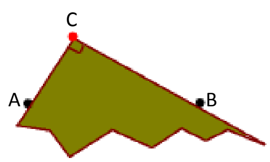
Classwork

Opening Exercise

a. Mark points A and B on the sheet of white paper provided by your teacher.

b. Take the colored paper provided, and "push" that paper up between points A and B on the white sheet.

c. Mark on the white paper the location of the corner of the colored paper, using a different color than black. Mark that point C. See the example below.



d. Do this again, pushing the corner of the colored paper up between the black points but at a different angle. Again, mark the location of the corner. Mark this point D.

e. Do this again and then again, multiple times. Continue to label the points. What curve do the colored points (C, D, …) seem to trace?

Exploratory Challenge

Choose one of the colored points (C, D, ...) that you marked. Draw the right triangle formed by the line segment connecting the original two points A and B and that colored point. Draw a rotated copy of the triangle underneath it.

Label the acute angles in the original triangle as x and y, and label the corresponding angles in the rotated triangle the same.

Todd says $ABCC'$ is a rectangle. Maryam says $ABCC'$ is a quadrilateral, but she's not sure it's a rectangle. Todd is right but doesn't know how to explain himself to Maryam. Can you help him out?

a. What composite figure is formed by the two triangles? How would you prove it?

i. What is the sum of x and y? Why?

ii. How do we know that the figure whose vertices are the colored points (C, D, …) and points A and B is a rectangle?

b. Draw the two diagonals of the rectangle. Where is the midpoint of the segment connecting the two original points A and B? Why?

c. Label the intersection of the diagonals as point P. How does the distance from point P to a colored point (C, D, …) compare to the distance from P to points A and B?

d. Choose another colored point, and construct a rectangle using the same process you followed before. Draw the two diagonals of the new rectangle. How do the diagonals of the new and old rectangle compare? How do you know?

e. How does your drawing demonstrate that all the colored points you marked do indeed lie on a circle?

Example 1

In the Exploratory Challenge, you proved the converse of a famous theorem in geometry. Thales' theorem states: *If A, B, and C are three distinct points on a circle and segment $\overline{AB}$ is a diameter of the circle, then $\angle ACB$ is right.*

Notice that, in the proof in the Exploratory Challenge, you started with a right angle (the corner of the colored paper) and created a circle. With Thales' theorem, you must start with the circle, and then create a right angle.

Prove Thales' theorem.

a. Draw circle P with distinct points A, B, and C on the circle and diameter $\overline{AB}$. Prove that $\angle ACB$ is a right angle.

b. Draw a third radius ($\overline{PC}$). What types of triangles are $\triangle APC$ and $\triangle BPC$? How do you know?

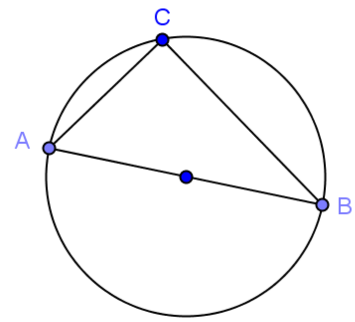
c. Using the diagram that you just created, develop a strategy to prove Thales' theorem.

d. Label the base angles of $\triangle APC$ as $b°$ and the bases of $\triangle BPC$ as $a°$. Express the measure of $\angle ACB$ in terms of $a°$ and $b°$.

e. How can the previous conclusion be used to prove that $\angle ACB$ is a right angle?

Exercises 1–2

1. $\overline{AB}$ is a diameter of the circle shown. The radius is 12.5 cm, and $AC = 7$ cm.

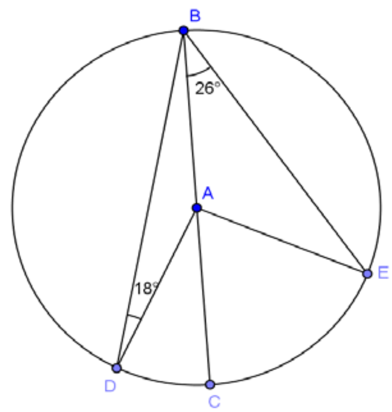


   a. Find $m\angle C$.

   b. Find AB.

   c. Find BC.

2. In the circle shown, $\overline{BC}$ is a diameter with center A.



   a. Find $m\angle DAB$.

   b. Find $m\angle BAE$.

   c. Find $m\angle DAE$.

Lesson Summary

THEOREMS:

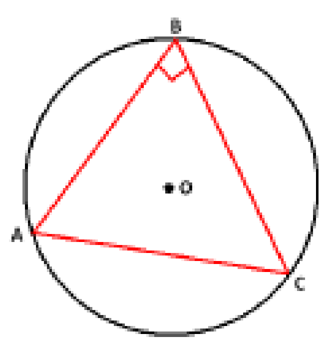
- **THALES' THEOREM**: If A, B, and C are three different points on a circle with $\overline{AB}$ a diameter, then $\angle ACB$ is a right angle.
- **CONVERSE OF THALES' THEOREM**: If ΔABC is a right triangle with $\angle C$ the right angle, then A, B, and C are three distinct points on a circle with $\overline{AB}$ a diameter.
- Therefore, given distinct points A, B, and C on a circle, ΔABC is a right triangle with $\angle C$ the right angle if and only if $\overline{AB}$ is a diameter of the circle.
- Given two points A and B, let point P be the midpoint between them. If C is a point such that $\angle ACB$ is right, then $BP = AP = CP$.

Relevant Vocabulary

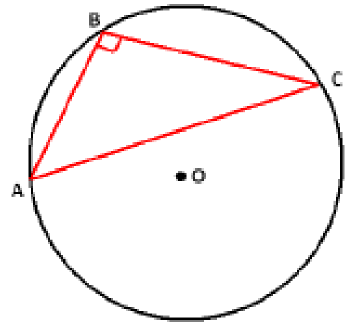
- **CIRCLE**: Given a point C in the plane and a number $r > 0$, the *circle* with center C and radius r is the set of all points in the plane that are distance r from the point C.
- **RADIUS**: May refer either to the line segment joining the center of a circle with any point on that circle (a *radius*) or to the length of this line segment (the *radius*).
- **DIAMETER**: May refer either to the segment that passes through the center of a circle whose endpoints lie on the circle (a *diameter*) or to the length of this line segment (the *diameter*).
- **CHORD**: Given a circle C, and let P and Q be points on C. The segment $\overline{PQ}$ is called a *chord* of C.
- **CENTRAL ANGLE**: A *central angle* of a circle is an angle whose vertex is the center of a circle.

Problem Set

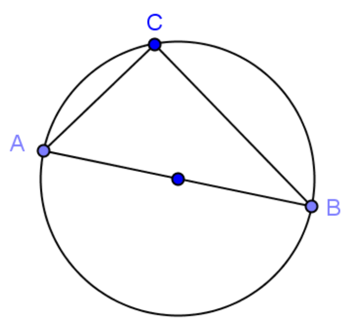
1. A, B, and C are three points on a circle, and angle ABC is a right angle. What's wrong with the picture below? Explain your reasoning.



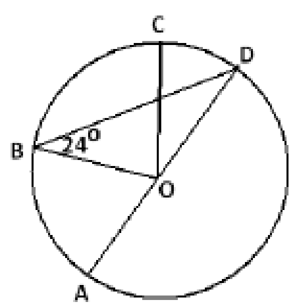
2. Show that there is something mathematically wrong with the picture below.



3. In the figure below, $\overline{AB}$ is the diameter of a circle of radius 17 miles. If $BC = 30$ miles, what is AC?



4. In the figure below, O is the center of the circle, and $\overline{AD}$ is a diameter.



a. Find $m\angle AOB$.

b. If $m\angle AOB : m\angle COD = 3 : 4$, what is $m\angle BOC$?

5. $\overline{PQ}$ is a diameter of a circle, and M is another point on the circle. The point R lies on the line $\overleftrightarrow{MQ}$ such that $RM = MQ$. Show that $m\angle PRM = m\angle PQM$. (Hint: Draw a picture to help you explain your thinking!)

6. Inscribe $\triangle ABC$ in a circle of diameter 1 such that $\overline{AC}$ is a diameter. Explain why:
   a. $\sin(\angle A) = BC$.
   b. $\cos(\angle A) = AB$.