Lesson 21: Ptolemy's Theorem

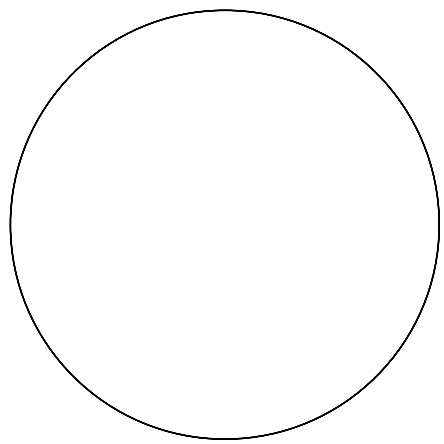
Classwork

Opening Exercise

Ptolemy's theorem says that for a cyclic quadrilateral $ABCD$, $AC \cdot BD = AB \cdot CD + BC \cdot AD$.

With ruler and a compass, draw an example of a cyclic quadrilateral. Label its vertices A, B, C, and D.

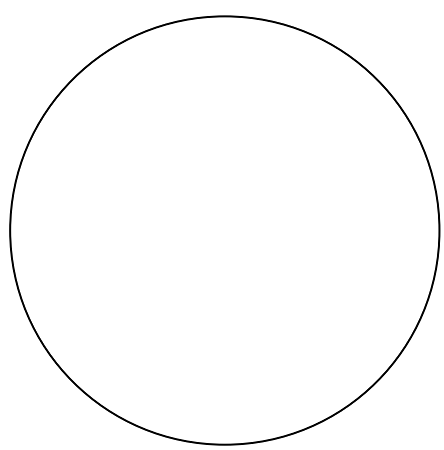
Draw the two diagonals $\overline{AC}$ and $\overline{BD}$.



With a ruler, test whether or not the claim that $AC \cdot BD = AB \cdot CD + BC \cdot AD$ seems to hold true.

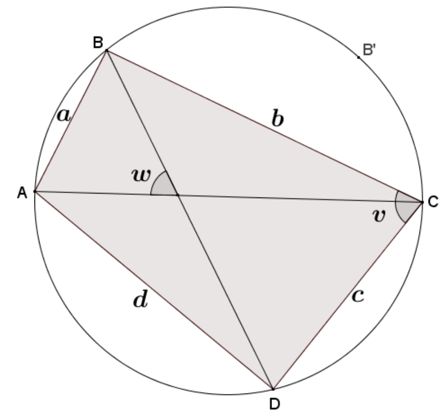
Repeat for a second example of a cyclic quadrilateral.

Challenge: Draw a cyclic quadrilateral with one side of length zero. What shape is the this cyclic quadrilateral? Does Ptolemy's claim hold true for it?



Exploratory Challenge: A Journey to Ptolemy's Theorem

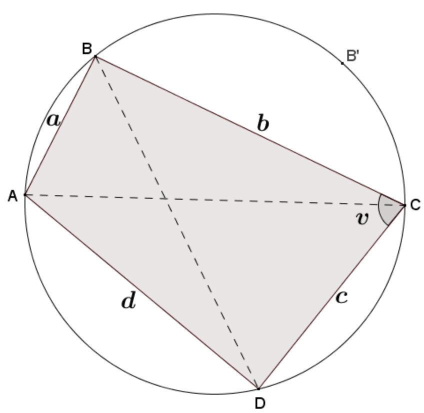
The diagram shows cyclic quadrilateral $ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting to form an acute angle with degree measure w. $AB = a$, $BC = b$, $CD = c$, and $DA = d$.



a. From last lesson, what is the area of quadrilateral $ABCD$ in terms of the lengths of its diagonals and the angle w? Remember this formula for later on!

b. Explain why one of the angles, $\angle BCD$ or $\angle BAD$, has a measure less than or equal to $90°$.

c. Let's assume that $\angle BCD$ in our diagram is the angle with a measure less than or equal to $90°$. Call its measure v degrees. What is the area of triangle BCD in terms of b, c, and v? What is the area of triangle BAD in terms of a, d, and v? What is the area of quadrilateral $ABCD$ in terms of a, b, c, d, and v?



d. We now have two different expressions representing the area of the same cyclic quadrilateral $ABCD$. Does it seem to you that we are close to a proof of Ptolemy's claim?

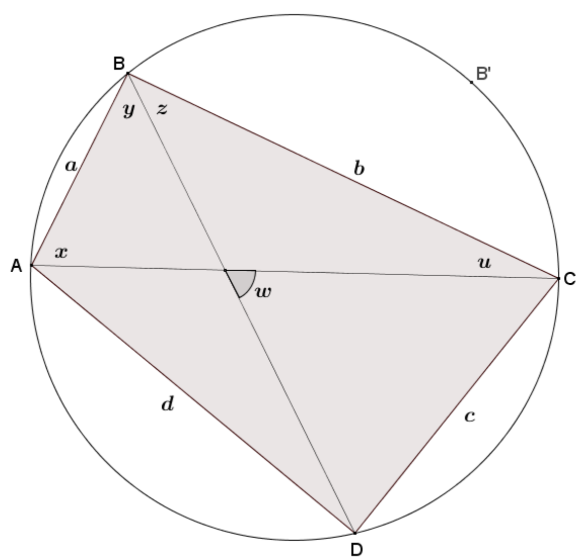
e. Trace the circle and points A, B, C, and D onto a sheet of patty paper. Reflect triangle ABC about the perpendicular bisector of diagonal $\overline{AC}$. Let A', B', and C' be the images of the points A, B, and C, respectively.

   i. What does the reflection do with points A and C?

   ii. Is it correct to draw B' as on the circle? Explain why or why not.

   iii. Explain why quadrilateral $AB'CD$ has the same area as quadrilateral $ABCD$.

f. The diagram shows angles having degree measures u, w, x, y, and z. Find and label any other angles having degree measures u, w, x, y, or z, and justify your answers.



g. Explain why $w = u + z$ in your diagram from part (f).

h. Identify angles of measures u, x, y, z, and w in your diagram of the cyclic quadrilateral $AB'CD$ from part (e).

i. Write a formula for the area of triangle $B'AD$ in terms of b, d, and w. Write a formula for the area of triangle $B'CD$ in terms of a, c, and w.

j. Based on the results of part (i), write a formula for the area of cyclic quadrilateral $ABCD$ In terms of a, b, c, d, and w.

k. Going back to part (a), now establish Ptolemy's theorem.

Lesson Summary

Theorems

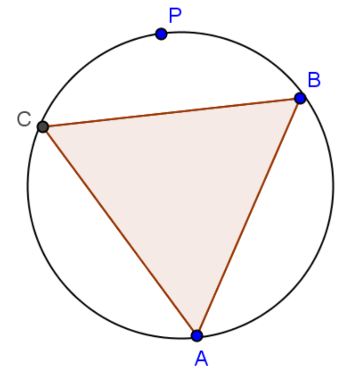
PTOLEMY'S THEOREM: For a cyclic quadrilateral $ABCD$, $AC \cdot BD = AB \cdot CD + BC \cdot AD$.

Relevant Vocabulary

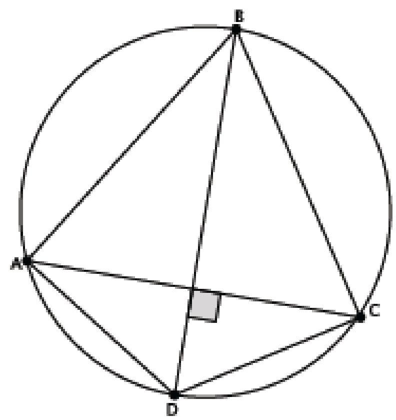
CYCLIC QUADRILATERAL: A quadrilateral with all vertices lying on a circle is known as a cyclic quadrilateral.

Problem Set

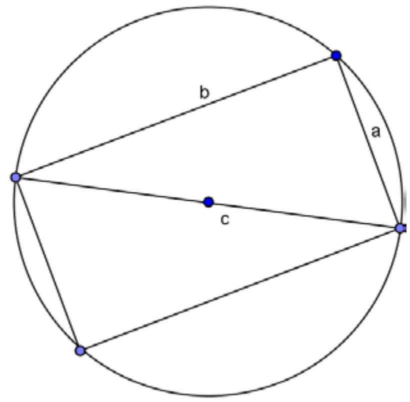
1. An equilateral triangle is inscribed in a circle. If P is a point on the circle, what does Ptolemy's theorem have to say about the distances from this point to the three vertices of the triangle?



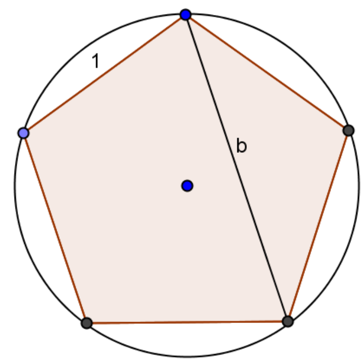
2. Kite $ABCD$ is inscribed in a circle. The kite has an area of 108 sq. in., and the ratio of the lengths of the non-congruent adjacent sides is $3 : 1$. What is the perimeter of the kite?



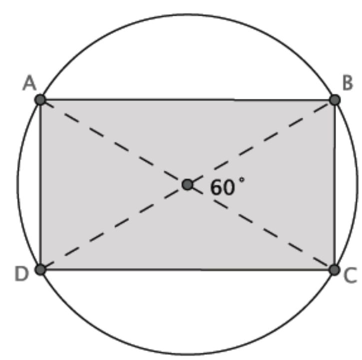
3. Draw a right triangle with leg lengths a and b, and hypotenuse length c. Draw a rotated copy of the triangle such that the figures form a rectangle. What does Ptolemy have to say about this rectangle?



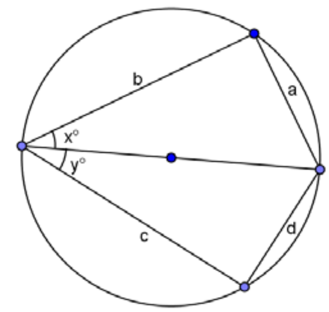
4. Draw a regular pentagon of side length 1 in a circle. Let b be the length of its diagonals. What does Ptolemy's theorem say about the quadrilateral formed by four of the vertices of the pentagon?



5. The area of the inscribed quadrilateral is $\sqrt{300}\ \mathrm{mm}^2$. Determine the circumference of the circle.



6. Extension: Suppose x and y are two acute angles, and the circle has a diameter of 1 unit. Find a, b, c, and d in terms of x and y. Apply Ptolemy's theorem, and determine the exact value of $\sin(75°)$.



   a. Explain why $\frac{a}{\sin(x)}$ equals the diameter of the circle.

   b. If the circle has a diameter of 1, what is a?

   c. Use Thales' theorem to write the side lengths in the original diagram in terms of x and y.

   d. If one diagonal of the cyclic quadrilateral is 1, what is the other?

   e. What does Ptolemy's theorem give?

   f. Using the result from part (e), determine the exact value of $\sin(75°)$.