## Lesson 8: The Difference Between Theoretical Probabilities and

## Estimated Probabilities

## Classwork

Did you ever watch the beginning of a Super Bowl game? After the traditional handshakes, a coin is tossed to determine which team gets to kick-off first. Whether or not you are a football fan, the toss of a fair coin is often used to make decisions between two groups.

## Examples 1-9: Why a Coin?

Coins were discussed in previous lessons of this module. What is special about a coin? In most cases, a coin has two different sides: a head side (heads) and a tail side (tails). The sample space for tossing a coin is \{heads, tails\}. If each outcome has an equal chance of occurring when the coin is tossed, then the probability of getting heads is $\frac{1}{2}$, or 0.5 . The probability of getting tails is also 0.5 . Note that the sum of these probabilities is 1 .

The probabilities formed using the sample space and what we know about coins are called the theoretical probabilities. Using observed relative frequencies is another method to estimate the probabilities of heads or tails. A relative frequency is the proportion derived from the number of the observed outcomes of an event divided by the total number of outcomes. Recall from earlier lessons that a relative frequency can be expressed as a fraction, a decimal, or a percent. Is the estimate of a probability from this method close to the theoretical probability? The following example investigates how relative frequencies can be used to estimate probabilities.

Beth tosses a coin 10 times and records her results. Here are the results from the 10 tosses:

| Toss | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Result | H | H | T | H | H | H | T | T | T | H |

The total number of heads divided by the total number of tosses is the relative frequency of heads. It is the proportion of the time that heads occurred on these tosses. The total number of tails divided by the total number of tosses is the relative frequency of tails.

1. Beth started to complete the following table as a way to investigate the relative frequencies. For each outcome, the total number of tosses increased. The total number of heads or tails observed so far depends on the outcome of the current toss. Complete this table for the 10 tosses recorded above.

| Toss | Outcome | Total number of heads so far | Relative frequency of heads so far (to the nearest hundredth) | Total number of tails so far | Relative frequency of tails so far (to the nearest hundredth) |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | H | 1 | $\frac{1}{1}=1$ | 0 | $\frac{0}{1}=0$ |
| 2 | H | 2 | $\frac{2}{2}=1$ | 0 | $\frac{0}{2}=0$ |
| 3 | T | 2 | $\frac{2}{3}=0.67$ | 1 | $\frac{1}{3}=0.33$ |
| 4 |  |  |  |  |  |
| 5 |  |  |  |  |  |
| 6 |  |  |  |  |  |
| 7 |  |  |  |  |  |
| 8 |  |  |  |  |  |
| 9 |  |  |  |  |  |
| 10 |  |  |  |  |  |

2. What is the sum of the relative frequency of heads and the relative frequency of tails for each row of the table?
3. Beth's results can also be displayed using a graph. From the table above, complete the graph below using the values of relative frequency of heads so far.

4. Beth continued tossing the coin and recording results for a total of 40 tosses. Here are the results of the next 30 tosses:

| Toss | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Result | T | H | T | H | T | H | H | T | H | T |


| Toss | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Result | H | T | T | H | T | T | T | T | H | T |


| Toss | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Result | H | T | H | T | H | T | H | H | T | T |

As the number of tosses increases, the relative frequency of heads changes. Complete the following table for the 40 coin tosses:

| Number of tosses | Total number of heads so far | Relative frequency of heads so <br> far (to the nearest hundredth) |
| :---: | :--- | :--- |
| 1 |  |  |
| 5 |  |  |
| 10 |  |  |
| 15 |  |  |
| 20 |  |  |
| 25 |  |  |
| 30 |  |  |
| 35 |  |  |
| 40 |  |  |

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5. From the table above, complete the graph below using the relative frequency of heads so far for the total number of tosses of $1,5,10,15,20,25,30,35$, and 40.

6. What do you notice about the changes in the relative frequency of the number of heads so far as the number of tosses increases?
7. If you tossed the coin 100 times, what do you think the relative frequency of heads would be? Explain your answer.
8. Based on the graph and the relative frequencies, what would you estimate the probability of getting heads to be? Explain your answer.

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9. How close is your estimate in Exercise 8 to the theoretical probability of 0.5 ? Would the estimate of this probability have been as good if Beth had only tossed the coin a few times instead of 40 ?

The value you gave in Exercise 8 is an estimate of the theoretical probability and is called an experimental or estimated probability.

## Exercises 1-8

Beth received nine more pennies. She securely taped them together to form a small stack. The top penny of her stack showed heads, and the bottom penny showed tails. If Beth tosses the stack, what outcomes could she observe?

1. Beth wanted to determine the probability of getting heads when she tosses the stack. Do you think this probability is the same as the probability of getting heads with just one coin? Explain your answer.
2. Make a sturdy stack of 10 pennies in which one end of the stack has a penny showing heads and the other end tails. Make sure the pennies are taped securely, or you may have a mess when you toss the stack. Toss the stack to observe possible outcomes. What is the sample space for tossing a stack of 10 pennies taped together? Do you think the probability of each outcome of the sample space is equal? Explain your answer.
3. Record the results of 10 tosses. Complete the following table of the relative frequencies of heads for your 10 tosses:

| Toss | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| Result |  |  |  |  |  |  |  |  |  |  |
| Relative <br> frequency <br> of heads <br> so far |  |  |  |  |  |  |  |  |  |  |

4. Based on the value of the relative frequencies of heads so far, what would you estimate the probability of getting heads to be?
5. Toss the stack of 10 pennies another 20 times. Complete the following table:

| Toss | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Result |  |  |  |  |  |  |  |  |  |  |


| Toss | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Result |  |  |  |  |  |  |  |  |  |  |

6. Summarize the relative frequency of heads so far by completing the following table:

| Number of <br> tosses | Total number <br> of heads so far | Relative frequency <br> of heads so far <br> (to the nearest hundredth) |
| :---: | :---: | :---: |
| 1 |  |  |
| 5 |  |  |
| 10 |  |  |
| 15 |  |  |
| 20 |  |  |
| 25 |  |  |
| 30 |  |  |

7. Based on the relative frequencies for the 30 tosses, what is your estimate of the probability of getting heads? Can you compare this estimate to a theoretical probability like you did in the first example? Explain your answer.
8. Create another stack of pennies. Consider creating a stack using 5 pennies, 15 pennies, or 20 pennies taped together in the same way you taped the pennies to form a stack of 10 pennies. Again, make sure the pennies are taped securely, or you might have a mess!

Toss the stack you made 30 times. Record the outcome for each toss:

| Toss | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Result |  |  |  |  |  |  |  |  |  |  |


| Toss | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Result |  |  |  |  |  |  |  |  |  |  |


| Toss | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Result |  |  |  |  |  |  |  |  |  |  |

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## Lesson Summary

- Observing the long-run relative frequency of an event from a chance experiment (or the proportion of an event derived from a long sequence of observations) approximates the theoretical probability of the event.
- After a long sequence of observations, the observed relative frequencies get close to the probability of the event occurring.
- When it is not possible to compute the theoretical probabilities of chance experiments, then the long-run relative frequencies (or the proportion of events derived from a long sequence of observations) can be used as estimated probabilities of events.


## Problem Set

1. If you created a stack of 15 pennies taped together, do you think the probability of getting a heads on a toss of the stack would be different than for a stack of 10 pennies? Explain your answer.
2. If you created a stack of 20 pennies taped together, what do you think the probability of getting a heads on a toss of the stack would be? Explain your answer.
3. Based on your work in this lesson, complete the following table of the relative frequencies of heads for the stack you created:

| Number <br> of tosses | Total number of <br> heads so far | Relative frequency of heads so <br> far (to the nearest hundredth) |
| :---: | :---: | :---: |
| 1 |  |  |
| 5 |  |  |
| 10 |  |  |
| 15 |  |  |
| 20 |  |  |
| 25 |  |  |
| 30 |  |  |

4. What is your estimate of the probability that your stack of pennies will land heads up when tossed? Explain your answer.
5. Is there a theoretical probability you could use to compare to the estimated probability? Explain you answer.
