Name ____ Date____________________

Lesson 1: Wishful Thinking—Does Linearity Hold?

Exit Ticket

1. Xavier says that $(a+b)^2 \neq a^2 + b^2$ but that $(a+b)^3 = a^3 + b^3$. He says that he can prove it by using the values $a = 2$ and $b = -2$. Shaundra says that both $(a+b)^2 = a^2 + b^2$ and $(a+b)^3 = a^3 + b^3$ are true and that she can prove it by using the values of $a = 7$ and $b = 0$ and also $a = 0$ and $b = 3$. Who is correct? Explain.

2. Does $f(x) = 3x + 1$ display ideal linear properties? Explain.

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Lesson 2: Wishful Thinking—Does Linearity Hold?

Exit Ticket

1. Koshi says that he knows that $\sin(x + y) = \sin(x) + \sin(y)$ because he has plugged in multiple values for x and y and they all work. He has tried $x = 0°$ and $y = 0°$, but he says that usually works, so he also tried $x = 45°$ and $y = 180°$, $x = 90°$ and $y = 270°$, and several others. Is Koshi correct? Explain your answer.

2. Is $f(x) = \sin x$ a linear transformation? Why or why not?

Name ____ Date ____________________

Lesson 3: Which Real Number Functions Define a Linear Transformation?

Exit Ticket

Suppose you have a linear transformation $f: \mathbb{R} \to \mathbb{R}$, where $f(3) = 9$ and $f(5) = 15$.

1. Use the addition property to compute $f(8)$ and $f(13)$.

2. Find $f(12)$ and $f(10)$. Show your work.

3. Find $f(-3)$ and $f(-5)$. Show your work.

4. Find $f(0)$. Show your work.

5. Find a formula for $f(x)$.

6. Draw the graph of the function $y = f(x)$.

Name ____ Date ____________________

Lesson 4: An Appearance of Complex Numbers

Exit Ticket

1. Solve the equation below.

$$2x^2 - 3x + 9 = 4$$

2. What is the geometric effect of multiplying a number by i^4? Explain your answer using words or pictures, and then confirm your answer algebraically.

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Lesson 5: An Appearance of Complex Numbers

Exit Ticket

In Problems 1–4, perform the indicated operations. Write each answer as a complex number $a + bi$.

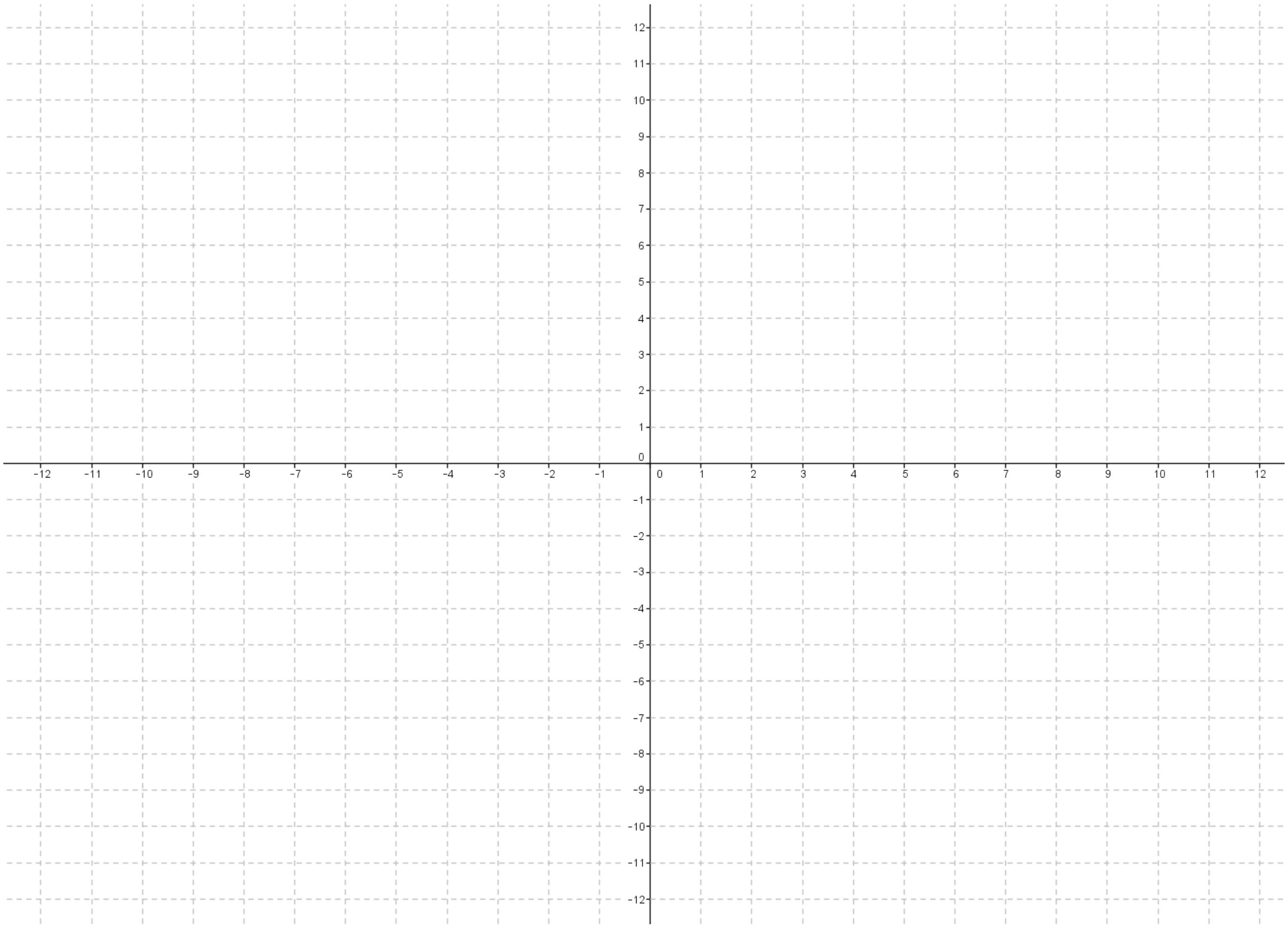
1. Let $z_1 = -2 + i$, $z_2 = 3 - 2i$, and $w = z_1 + z_2$. Find w, and graph z_1, z_2, and w in the complex plane.

2. Let $z_1 = -1 - i$, $z_2 = 2 + 2i$, and $w = z_1 - z_2$. Find w, and graph z_1, z_2, and w in the complex plane.

3. Let $z = -2 + i$ and $w = -2z$. Find w, and graph z and w in the complex plane.

4. Let $z_1 = 1 + 2i$, $z_2 = 2 - i$, and $w = z_1 \cdot z_2$. Find w, and graph z_1, z_2, and w in the complex plane.

Complex Plane Reproducible



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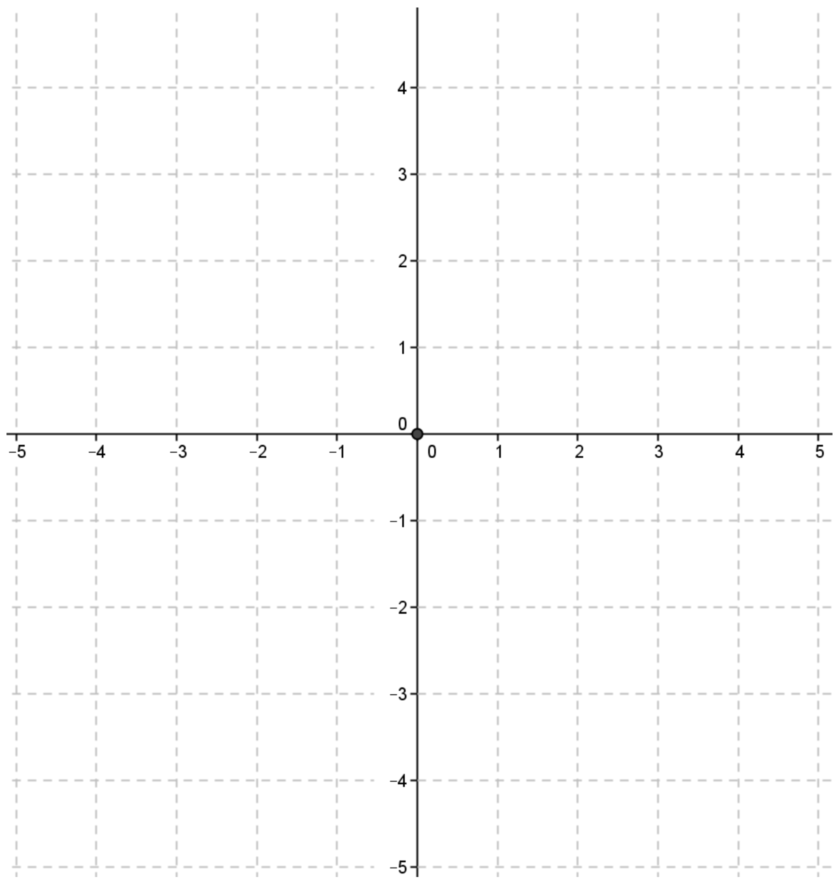
Lesson 6: Complex Numbers as Vectors

Exit Ticket

Let $z = -1 + 2i$ and $w = 2 + i$. Find the following, and verify each geometrically by graphing z, w, and each result.

a. $z + w$



b. $z - w$

c. $2z - w$

d. $w - z$

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Lesson 7: Complex Number Division

Exit Ticket

1. Find the multiplicative inverse of $3-2i$. Verify that your solution is correct by confirming that the product of $3-2i$ and its multiplicative inverse is 1.

2. What is the conjugate of $3-2i$?

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Lesson 8: Complex Number Division

Exit Ticket

1. Given $z = 4 - 3i$.
   a. What does $\bar{z}$ mean?

   b. What does $\bar{z}$ do to z geometrically?

   c. What does $|z|$ mean both algebraically and geometrically?

2. Describe how to use the conjugate to divide $2 - i$ by $3 + 2i$, and then find the quotient.

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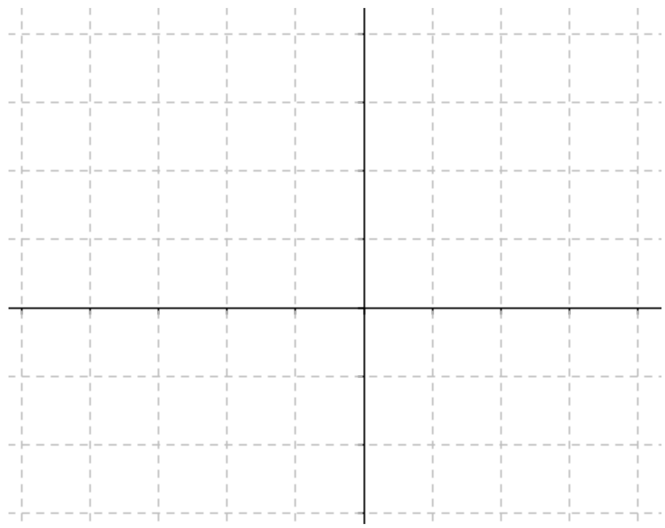
Lesson 9: The Geometric Effect of Some Complex Arithmetic

Exit Ticket

1. Given $z = 3 + 2i$ and $w = -2 - i$, plot the following in the complex plane:
   a. z
   b. w
   c. $z - 2$
   d. $w + 3i$
   e. $w + z$



2. Given $z = a + bi$, what complex number represents the reflection of z about the imaginary axis? Give one example to show why.

3. What is the geometric effect of $T(z) = z + (4 - 2i)$?

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Lesson 10: The Geometric Effect of Some Complex Arithmetic

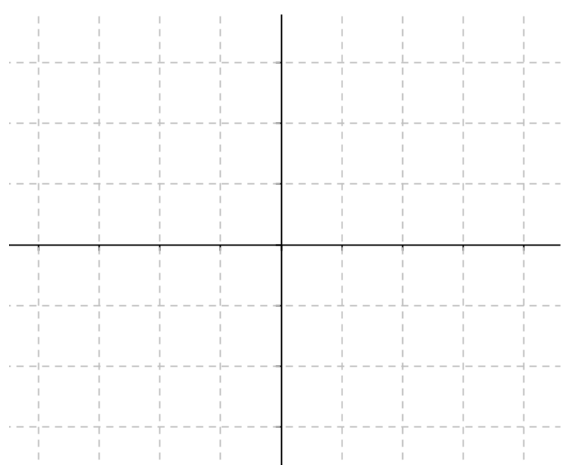
Exit Ticket

1. Given $T(z) = z$, describe the geometric effect of the following:
   a. $T(z) = 5z$

   b. $T(z) = \frac{z}{2}$

   c. $T(z) = i \cdot z$

2. If $z = -2 + 3i$ is the result of a $90°$ counterclockwise rotation about the origin from w, find w. Plot z and w in the complex plane.



3. Explain the geometric effect of z if you multiply z by w, where $w = 1 + i$.

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Lesson 11: Distance and Complex Numbers

Exit Ticket

1. Kishore said that he can add two points in the coordinate plane like adding complex numbers in the complex plane. For example, for point $A(2, 3)$ and point $B(5, 1)$, he will get $A + B = (7, 4)$. Is he correct? Explain your reasoning.

2. Consider two complex numbers $A = -4 + 5i$ and $B = 4 - 10i$.
   a. Find the midpoint of A and B.

   b. Find the distance between A and B.

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Lesson 12: Distance and Complex Numbers

Exit Ticket

1. Find the distance between the following points.
   a. $(4, -9)$ and $(1, -5)$

   b. $4 - 9i$ and $1 - 5i$

   c. Explain why they have the same answer numerically in parts (a) and (b), but a different perspective in geometric effect.

2. Given point $A = 3 - 2i$ and point $M = -2 + i$, if M is the midpoint of A and another point B, find the coordinates of point B.

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Lesson 13: Trigonometry and Complex Numbers

Exit Ticket

1. State the modulus and argument of each complex number. Explain how you know.
   a. $4 + 0i$

   b. $-2 + 2i$

2. Write each number from Problem 1 in polar form.

3. Explain why $5\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$ and $\frac{5\sqrt{3}}{2} + \frac{5}{2}i$ represent the same complex number.

Trigonometry Review: Additional Resources

1. Evaluate the following.

   a. $\sin(30°)$

   b. $\cos\left(\frac{\pi}{3}\right)$

   c. $\sin(225°)$

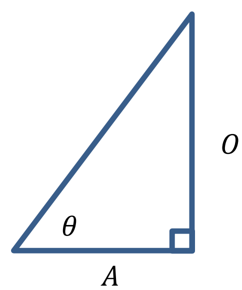
   d. $\cos\left(\frac{5\pi}{6}\right)$

   e. $\sin\left(\frac{5\pi}{3}\right)$

   f. $\cos(330°)$

2. Solve for the acute angle θ, both in radians and degrees, in a right triangle if you are given the opposite side, O, and adjacent side, A. Round to the nearest thousandth.

   a. $O = 3$ and $A = 4$



   b. $O = 6$ and $A = 1$

   c. $O = 3\sqrt{3}$ and $A = 2$

3. Convert angles in degrees to radians, and convert angles in radians to degrees.

   a. $150°$

   b. $\frac{4\pi}{3}$

   c. $\frac{3\pi}{4}$

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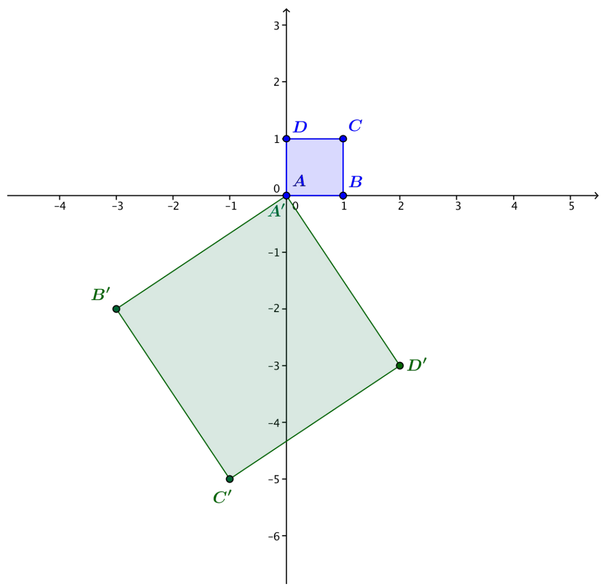
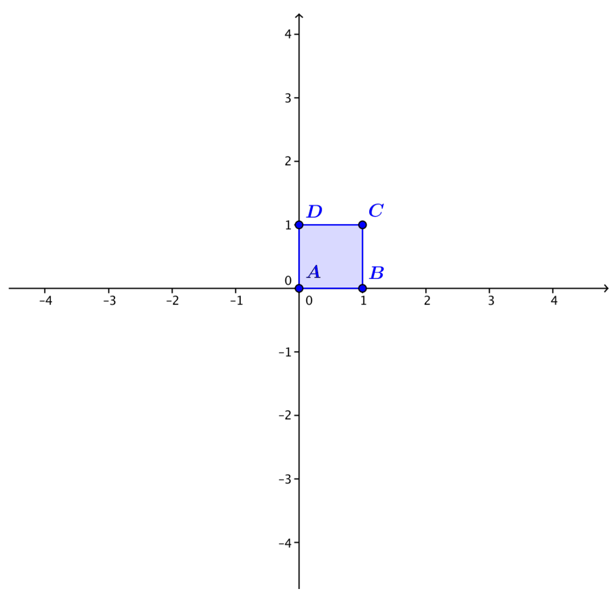
Lesson 14: Discovering the Geometric Effect of Complex Multiplication

Exit Ticket

1. Identify the linear transformation L that takes square $ABCD$ to square $A'B'C'D'$ as shown in the figure on the right.



2. Describe the geometric effect of the transformation $L(z) = (1 - 3i)z$ on the unit square $ABCD$, where $A = 0$, $B = 1$, $C = 1 + i$, and $D = i$. Sketch the unit square transformed by L on the axes at right.



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Lesson 15: Justifying the Geometric Effect of Complex Multiplication

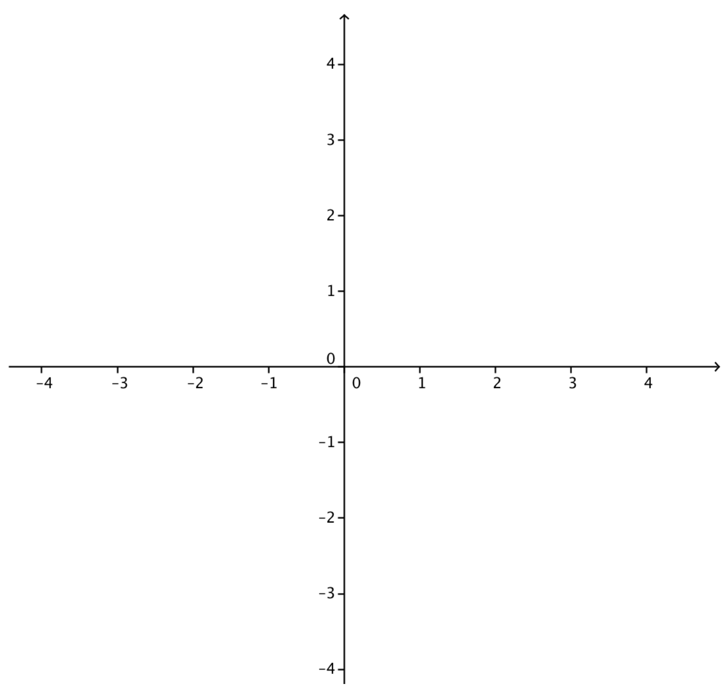
Exit Ticket

1. What is the geometric effect of the transformation $L(z) = (-6 + 8i)z$?

2. Suppose that w is a complex number with $|w| = \frac{3}{2}$ and $\arg(w) = \frac{5\pi}{6}$, and z is a complex number with $|z| = 2$ and $\arg(z) = \frac{\pi}{3}$.

   a. Explain how you can geometrically locate the point that represents the product wz in the coordinate plane.

   b. Plot w, z, and wz on the coordinate grid.



Name ____ Date ____________________

Lesson 16: Representing Reflections with Transformations

Exit Ticket

Explain the process used in the lesson to locate the reflection of a point z across the diagonal line with equation $y = x$. Include figures in your explanation.

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Lesson 17: The Geometric Effect of Multiplying by a Reciprocal

Exit Ticket

Let $z = 1 + \sqrt{3}i$ and $w = \sqrt{3} - i$. Describe each complex number as a transformation of z, and then write the number in rectangular form and identify its modulus and argument.

1. $\frac{z}{w}$

2. $\frac{1}{wz}$

Name ______________________________ Date ______________

1. Given $z = 3 - 4i$ and $w = -1 + 5i$:

   a. Find the distance between z and w.

   b. Find the midpoint of the segment joining z and w.

2. Let $z_1 = 2 - 2i$ and $z_2 = (1 - i) + \sqrt{3}(1 + i)$.

   a. What is the modulus and argument of z_1?

   b. Write z_1 in polar form. Explain why the polar and rectangular forms of a given complex number represent the same number.

c. Find a complex number w, written in the form $w = a + ib$, such that $wz_1 = z_2$.

d. What is the modulus and argument of w?

e. Write w in polar form.

f. When the points z_1 and z_2 are plotted in the complex plane, explain why the angle between z_1 and z_2 measures $\arg(w)$.

g. What type of triangle is formed by the origin and the two points represented by the complex numbers z_1 and z_2? Explain how you know.

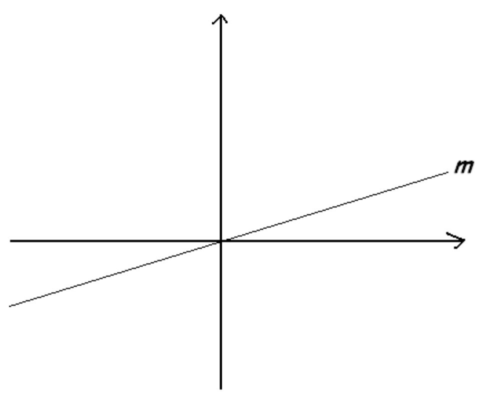
h. Find the complex number, v, closest to the origin that lies on the line segment connecting z_1 and z_2. Write v in rectangular form.

3. Let z be the complex number $2 + 3i$ lying in the complex plane.

   a. What is the conjugate of z? Explain how it is related geometrically to z.

   b. Write down the complex number that is the reflection of z across the vertical axis. Explain how you determined your answer.

Let m be the line through the origin of slope $\frac{1}{2}$ in the complex plane.



   c. Write down a complex number, w, of modulus 1 that lies on m in the first quadrant in rectangular form.

d. What is the modulus of wz?

e. Explain the relationship between wz and z. First, use properties of modulus to answer this question, and then give an explanation involving transformations.

f. When asked,
"What is the argument of $\frac{1}{w}z$?"
Paul gave the answer: $\arctan\left(\frac{3}{2}\right) - \arctan\left(\frac{1}{2}\right)$, which he then computed to two decimal places.

Provide a geometric explanation that yields Paul's answer.

g. When asked,
"What is the argument of $\frac{1}{w}z$?"
Mable did the complex number arithmetic and computed $z \div w$.

She then gave an answer in the form $\arctan\left(\frac{a}{b}\right)$ for some fraction $\frac{a}{b}$. What fraction did Mable find? Up to two decimal places, is Mable's final answer the same as Paul's?

Name ____ Date ____________________

Lesson 18: Exploiting the Connection to Trigonometry

Exit Ticket

1. Write $(2+2i)^8$ as a complex number in the form $a+bi$ where a and b are real numbers.

2. Explain why complex number of the form $(a+ai)^n$ will either be a pure imaginary or a real number when n is an even number.

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Lesson 19: Exploiting the Connection to Trigonometry

Exit Ticket

Find the fourth roots of $-2 - 2\sqrt{3}i$.

Name ____ Date ____________________

Lesson 20: Exploiting the Connection to Cartesian Coordinates

Exit Ticket

1. Find the scale factor and rotation induced by the transformation $L(x, y) = (-6x - 8y, 8x - 6y)$.

2. Explain how the transformation of complex numbers $L(x + iy) = (a + bi)(x + iy)$ leads to the transformation of points in the coordinate plane $L(x, y) = (ax - by, bx + ay)$.

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Lesson 21: The Hunt for Better Notation

Exit Ticket

1. Evaluate the product $\begin{pmatrix} 10 & 2 \\ -8 & -5 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

2. Find a matrix representation of the transformation $L(x, y) = (3x + 4y, x - 2y)$.

3. Does the transformation $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ represent a rotation and dilation in the plane? Explain how you know.

Name ______________________________ Date ______________

Lesson 22: Modeling Video Game Motion with Matrices

Exit Ticket

1. Consider the function $h(t) = \begin{pmatrix} t+5 \\ t-3 \end{pmatrix}$. Draw the path that the point $P = h(t)$ traces out as t varies within the interval $0 \le t \le 4$.

2. The position of an object is given by the function $f(t) = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix}$, where t is measured in seconds.

   a. Write $f(t)$ in the form $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$.

   b. Find how fast the object is moving in the horizontal direction and in the vertical direction.

3. Write a function $f(x, y)$ which will translate all points in the plane 2 units to the left and 5 units downward.

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Lesson 23: Modeling Video Game Motion with Matrices

Exit Ticket

Write a function $f(t)$ that incorporates the following actions. Make a drawing of the path the point follows during the time interval $0 \leq t \leq 3$.

a. During the time interval $0 \leq t \leq 1$, move the point $(8, 6)$ through $\frac{\pi}{4}$ radians about the origin in a counter-clockwise direction.

b. During the time interval $1 < t \leq 3$, move the image along a straight line to $(6, -8)$.

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Lesson 24: Matrix Notation Encompasses New Transformations!

Exit Ticket

What type of transformation is shown in the following examples? What is the resulting matrix?

1. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

2. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

3. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

5. What is the multiplicative identity matrix? What is it similar to in the set of real numbers? Explain your answer.

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Lesson 25: Matrix Multiplication and Addition

Exit Ticket

1. Carmine has never seen matrices before but must quickly understand how to add, subtract, and multiply matrices. Explain the following problems to Carmine.

   a. $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$

   b. $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -5 & 8 \end{pmatrix}$

   c. $\begin{pmatrix} 2 & 3 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 4 & -3 \end{pmatrix} = \begin{pmatrix} 12 & -7 \\ 20 & -16 \end{pmatrix}$

2. Explain to Carmine the significance of the zero matrix and the multiplicative identity matrix.

Name ____ Date ____________________

Lesson 26: Getting a Handle on New Transformations

Exit Ticket

Perform the transformation $\begin{bmatrix} -2 & 5 \\ 4 & -1 \end{bmatrix}$ on the unit square.

a. Draw the unit square and the image after this transformation.

b. Label the vertices. Explain the effect of this transformation on the unit square.

c. Calculate the area of the image. Show your work.

Name ____ Date ____________________

Lesson 27: Getting a Handle on New Transformations

Exit Ticket

Given the transformation $\begin{bmatrix} 0 & k \\ 1 & k \end{bmatrix}$ with $k > 0$:

a. Find the area of the image of the transformation performed on the unit matrix.

b. The image of the transformation on $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$; find $\begin{bmatrix} x \\ y \end{bmatrix}$ in terms of k. Show your work.

Name ____ Date ____________________

Lesson 28: When Can We Reverse a Transformation?

Exit Ticket

$A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$ $\quad$ $B = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

1. Is matrix A the inverse of matrix B? Show your work and explain your answer.

2. What is the determinant of matrix B? Of matrix A?

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Lesson 29: When Can We Reverse a Transformation?

Exit Ticket

$A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$

1. Find the inverse of A. Show your work and confirm your answer.

2. Explain why the matrix $\begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$ has no inverse.

Name ______________________________ Date ____________

Lesson 30: When Can We Reverse a Transformation?

Exit Ticket

A and B are 2×2 matrices. I is the 2×2 multiplicative identity matrix.

1. If $AB = A$, name the matrix represented by B.

2. If $A + B = A$, name the matrix represented by B.

3. If $AB = I$, name the matrix represented by B.

4. Do the matrices have inverses? Justify your answer.

   a. $\begin{bmatrix} -2 & 6 \\ -3 & 9 \end{bmatrix}$

   b. $\begin{bmatrix} -2 & 6 \\ 3 & 9 \end{bmatrix}$

5. Find a value of a, such that the given matrix has an inverse.

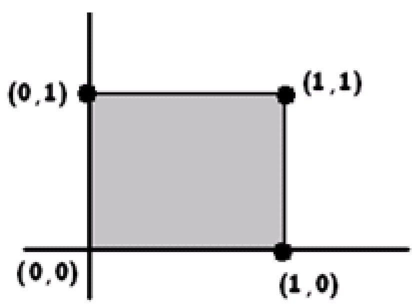
   a. $\begin{bmatrix} -4 & 3a \\ 2 & 9 \end{bmatrix}$

   b. $\begin{bmatrix} 5 & a \\ -a & 5 \end{bmatrix}$

Name ____ Date ____________________

1. Consider the transformation on the plane given by the 2×2 matrix $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ for a fixed positive number $k > 1$.

   a. Draw a sketch of the image of the unit square under this transformation (the unit square has vertices$(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$). Be sure to label all four vertices of the image figure.



The Unit Square

b. What is the area of the image parallelogram?

c. Find the coordinates of a point $\begin{pmatrix} x \\ y \end{pmatrix}$ whose image under the transformation is $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$.

d. The transformation $\begin{pmatrix} 1 & k \\ 0 & k \end{pmatrix}$ is applied once to the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then once to the image point, then once to the image of the image point, and then once to the image of the image of the image point, and so on. What are the coordinates of a tenfold image of the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, that is, the image of the point after the transformation has been applied 10 times?

2. Consider the transformation given by $\begin{pmatrix} \cos(1) & -\sin(1) \\ \sin(1) & \cos(1) \end{pmatrix}$.

a. Describe the geometric effect of applying this transformation to a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane.

b. Describe the geometric effect of applying this transformation to a point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the plane twice: once to the point and then once to its image.

c. Use part (b) to prove $\cos(2) = \cos^2(1) - \sin^2(1)$ and $\sin(2) = 2\sin(1)\cos(1)$.

3.

a. Explain the geometric representation of multiplying a complex number by $1 + i$.

b. Write $(1 + i)^{10}$ as a complex number of the form $a + bi$ for real numbers a and b.

c. Find a complex number $a + bi$, with a and b positive real numbers, such that $(a + bi)^3 = i$.

d. If z is a complex number, is there sure to exist, for any positive integer n, a complex number w such that $w^n = z$? Explain your answer.

e. If z is a complex number, is there sure to exist, for any negative integer n, a complex number w such that $w^n = z$? Explain your answer.

4. Let $P = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

   a. Give an example of a 2×2 matrix A, not with all entries equal to zero, such that $PA = O$.

   b. Give an example of a 2×2 matrix B with $PB \neq O$.

   c. Give an example of a 2×2 matrix C such that $CR = R$ for all 2×2 matrices R.

d. If a 2×2 matrix D has the property that $D + R = R$ for all 2×2 matrices R, must D be the zero matrix O? Explain.

e. Let $E = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix}$. Is there 2×2 matrix F so that $EF = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $FE = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$? If so, find one. If not, explain why no such matrix F can exist.

5. In programming a computer video game, Mavis coded the changing location of a space rocket as follows: At a time t seconds between $t = 0$ seconds and $t = 2$ seconds, the location $\begin{pmatrix} x \\ y \end{pmatrix}$ of the rocket is given by

$$\begin{pmatrix} \cos\left(\frac{\pi}{2}t\right) & -\sin\left(\frac{\pi}{2}t\right) \\ \sin\left(\frac{\pi}{2}t\right) & \cos\left(\frac{\pi}{2}t\right) \end{pmatrix}\begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

At a time of t seconds between $t = 2$ seconds and $t = 4$ seconds, the location of the rocket is given by $\begin{pmatrix} 3-t \\ 3-t \end{pmatrix}$.

a. What is the location of the rocket at time $t = 0$? What is its location at time $t = 4$?

b. Petrich is worried that Mavis may have made a mistake and the location of the rocket is unclear at time $t = 2$ seconds. Explain why there is no inconsistency in the location of the rocket at this time.

c. What is the area of the region enclosed by the path of the rocket from time $t = 0$ to time $t = 4$?

d. Mavis later decided that the moving rocket should be shifted five places farther to the right. How should she adjust her formulations above to accomplish this translation?