Lesson 3: Which Real Number Functions Define a Linear Transformation?

Classwork

Opening Exercises

Recall from the previous two lessons that a linear transformation is a function f that satisfies two conditions: (1) $f(x+y) = f(x) + f(y)$ and (2) $f(kx) = kf(x)$. Here, k refers to any real number, and x and y represent arbitrary elements in the domain of f.

1. Let $f(x) = x^2$. Is f a linear transformation? Explain why or why not.

2. Let $g(x) = \sqrt{x}$. Is g a linear transformation? Explain why or why not.

Problem Set

1. Suppose you have a linear transformation $f: \mathbb{R} \to \mathbb{R}$, where $f(2) = 1$ and $f(4) = 2$.
   a. Use the addition property to compute $f(6)$, $f(8)$, $f(10)$, and $f(12)$.
   b. Find $f(20)$, $f(24)$, and $f(30)$. Show your work.
   c. Find $f(-2)$, $f(-4)$, and $f(-8)$. Show your work.
   d. Find a formula for $f(x)$.
   e. Draw the graph of the function $f(x)$.

2. The symbol $\mathbb{Z}$ represents the set of integers, and so $g: \mathbb{Z} \to \mathbb{Z}$ represents a function that takes integers as inputs and produces integers as outputs. Suppose that a function $g: \mathbb{Z} \to \mathbb{Z}$ satisfies $g(a + b) = g(a) + g(b)$ for all integers a and b. Is there necessarily an integer k such that $g(n) = kn$ for all integer inputs n?
   a. Let $k = g(1)$. Compute $g(2)$ and $g(3)$.
   b. Let n be any positive integer. Compute $g(n)$.
   c. Now consider $g(0)$. Since $g(0) = g(0 + 0)$, what can you conclude about $g(0)$?
   d. Lastly, use the fact that $g(n + -n) = g(0)$ to learn something about $g(-n)$, where n is any positive integer.
   e. Use your work above to prove that $g(n) = kn$ for every integer n. Be sure to consider the fact that n could be positive, negative, or 0.

3. In the following problems, be sure to consider all kinds of functions: polynomial, rational, trigonometric, exponential, logarithmic, etc.
   a. Give an example of a function $f: \mathbb{R} \to \mathbb{R}$ that satisfies $f(x \cdot y) = f(x) + f(y)$.
   b. Give an example of a function $g: \mathbb{R} \to \mathbb{R}$ that satisfies $g(x + y) = g(x) \cdot g(y)$.
   c. Give an example of a function $h: \mathbb{R} \to \mathbb{R}$ that satisfies $h(x \cdot y) = h(x) \cdot h(y)$.