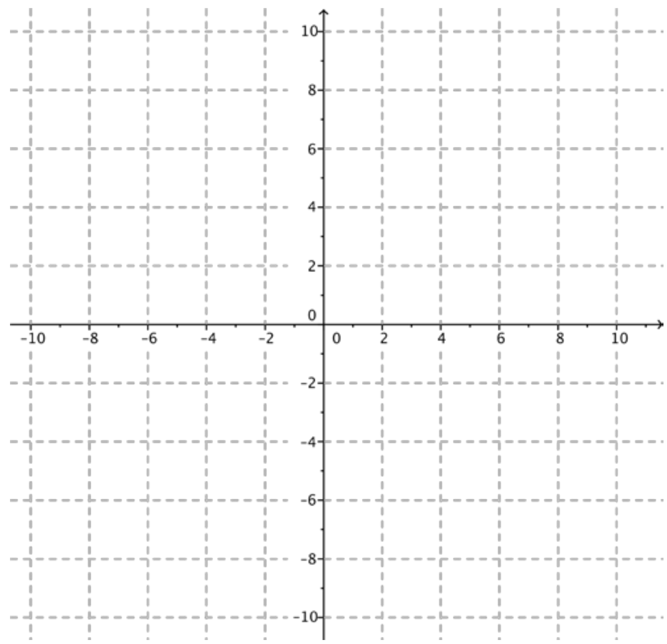
Lesson 15: Justifying the Geometric Effect of Complex Multiplication

Classwork

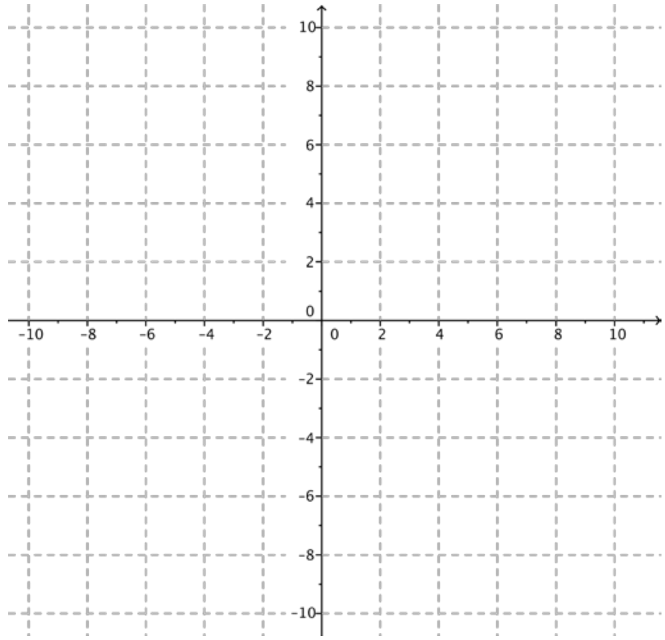
Opening Exercise

For each exercise below, compute the product wz. Then, plot the complex numbers z, w, and wz on the axes provided.

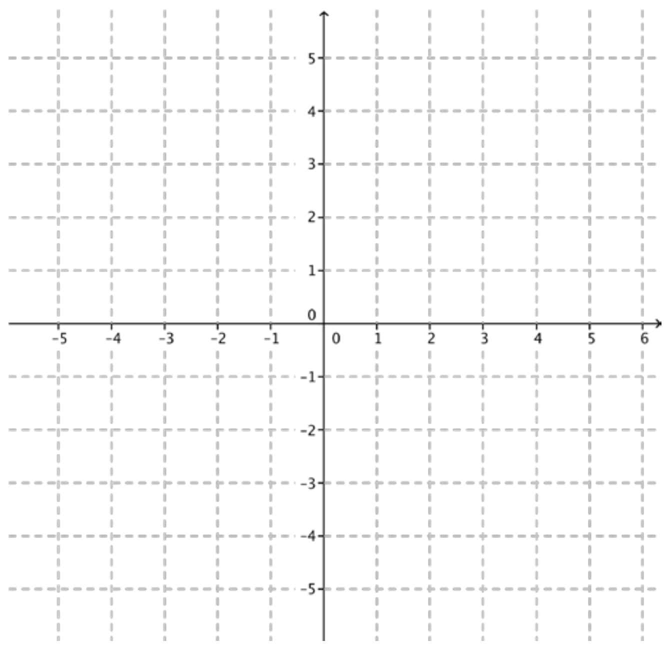
a. $z = 3 + i, w = 1 + 2i$



b. $z = 1 + 2i, w = -1 + 4i$



c. $z = -1 + i, w = -2 - i$



d. For each part (a), (b), and (c), draw line segments connecting each point z, w and wz to the origin. Determine a relationship between the arguments of the complex numbers z, w, and wz.

Exercises

1. Let $w = a + bi$ and $z = c + di$.
   a. Calculate the product wz.

   b. Calculate the moduli $|w|$, $|z|$, and $|wz|$.

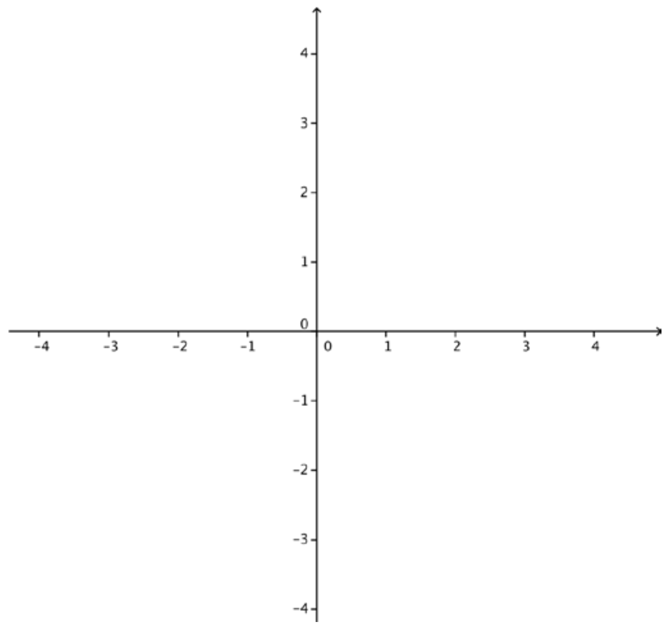
   c. What can you conclude about the quantities $|w|$, $|z|$, and $|wz|$?

2. What does the result of Exercise 1 tell us about the geometric effect of the transformation $L(z) = wz$?

3. If z and w are the complex numbers with the specified arguments and moduli, locate the point that represents the product wz on the provided coordinate axes.

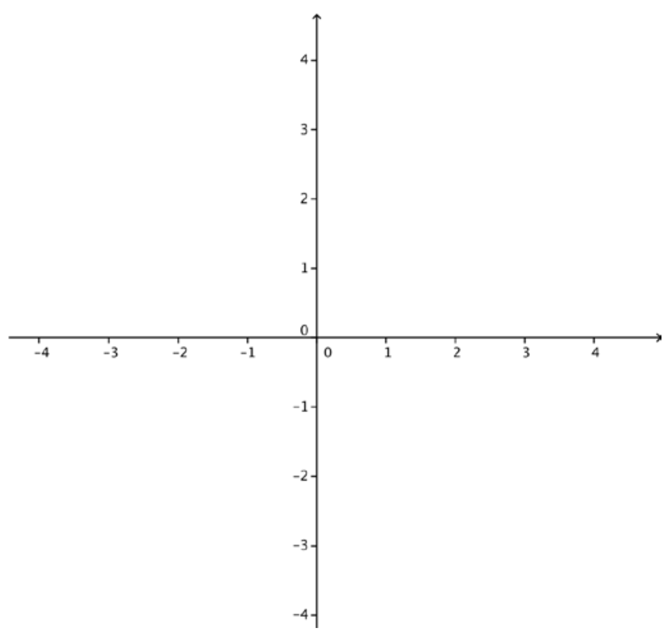
   a. $|w| = 3,\ \arg(w) = \frac{\pi}{4}$

   $|z| = \frac{2}{3},\ \arg(z) = -\frac{\pi}{2}$



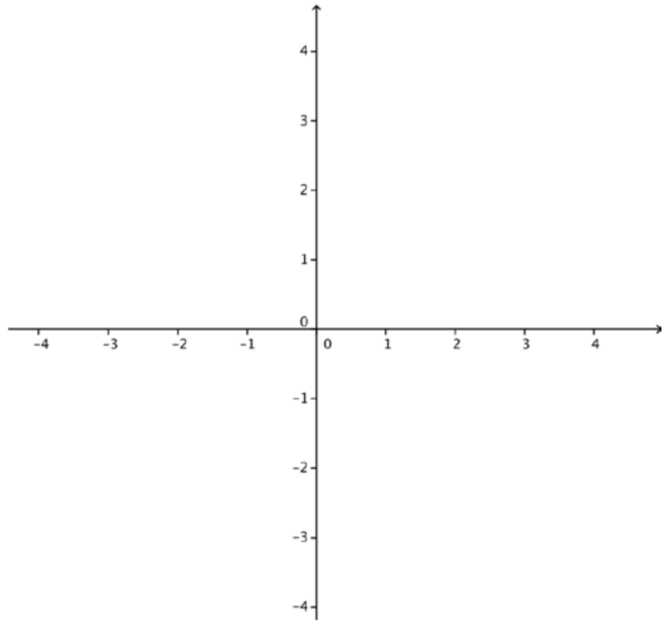
   b. $|w| = 2,\ \arg(w) = \pi$

   $|z| = 1,\ \arg(z) = \frac{\pi}{4}$



c. $|w| = \frac{1}{2},\ \arg(w) = \frac{4\pi}{3}$

$|z| = 4,\ \arg(z) = -\frac{\pi}{6}$



Lesson Summary

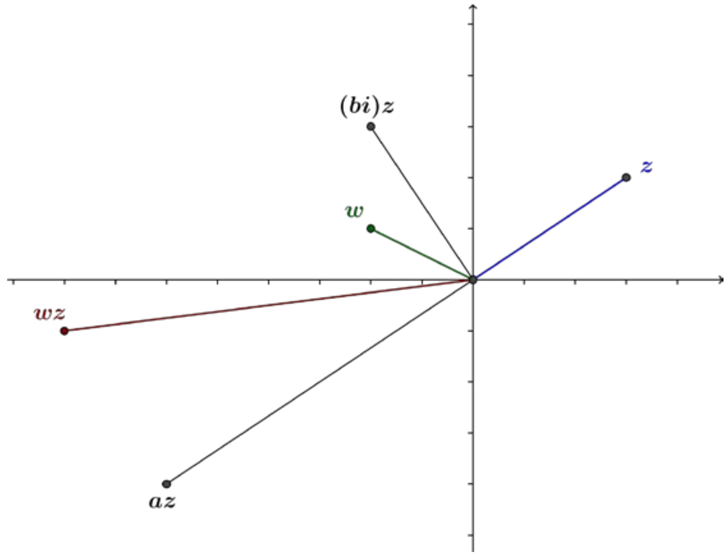
For complex numbers z and w,

- The modulus of the product is the product of the moduli:
$$|wz| = |w| \cdot |z|,$$
- The argument of the product is the sum of the arguments:
$$\arg(wz) = \arg(w) + \arg(z).$$

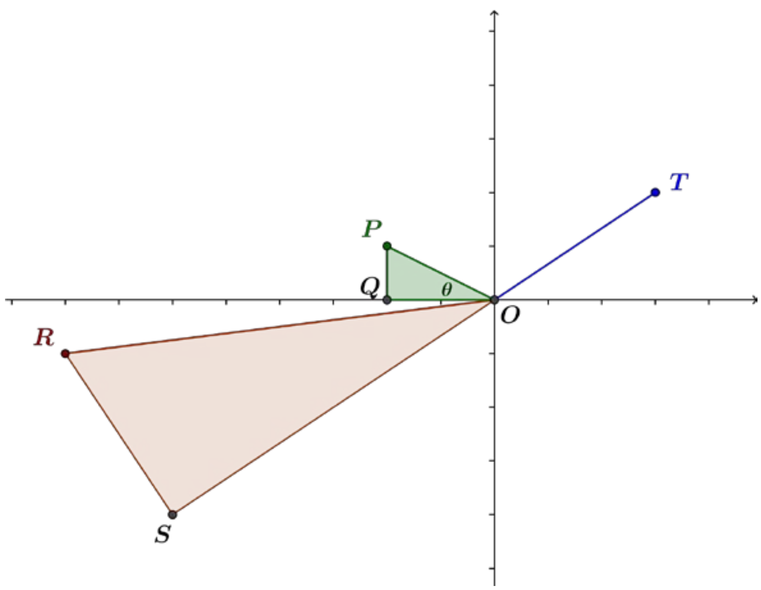
Problem Set

1. In the lesson, we justified our observation that the geometric effect of a transformation $L(z) = wz$ is a rotation by $\arg(w)$ and a dilation by $|w|$ for a complex number w that is represented by a point in the first quadrant of the coordinate plane. In this exercise, we will verify that this observation is valid for any complex number w. For a complex number $w = a + bi$, we only considered the case where $a > 0$ and $b > 0$. There are eight additional possibilities we need to consider.
   a. Case 1: The point representing w is the origin. That is, $a = 0$ and $b = 0$.
   In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.
   b. Case 2: The point representing w lies on the positive real axis. That is, $a > 0$ and $b = 0$.
   In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.
   c. Case 3: The point representing w lies on the negative real axis. That is, $a < 0$ and $b = 0$.
   In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.
   d. Case 4: The point representing w lies on the positive imaginary axis. That is, $a = 0$ and $b > 0$.
   In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.
   e. Case 5: The point representing w lies on the negative imaginary axis. That is, $a = 0$ and $b < 0$.
   In this case, explain why $L(z) = (a + bi)z$ has the geometric effect of rotation by $\arg(a + bi)$ and dilation by $|a + bi|$.

f. Case 6: The point representing $w = a + bi$ lies in the second quadrant. That is, $a < 0$ and $b > 0$. Points representing , z, az, $(bi)z$, and $wz = az + (bi)z$ are shown in the figure below.



For convenience, rename the origin O and let $P = w$, $Q = a$, $R = wz$, $S = az$, and $T = z$, as shown below. Let $m(\angle POQ) = \theta$.



i. Argue that $\triangle OPQ \sim \triangle ORS$.

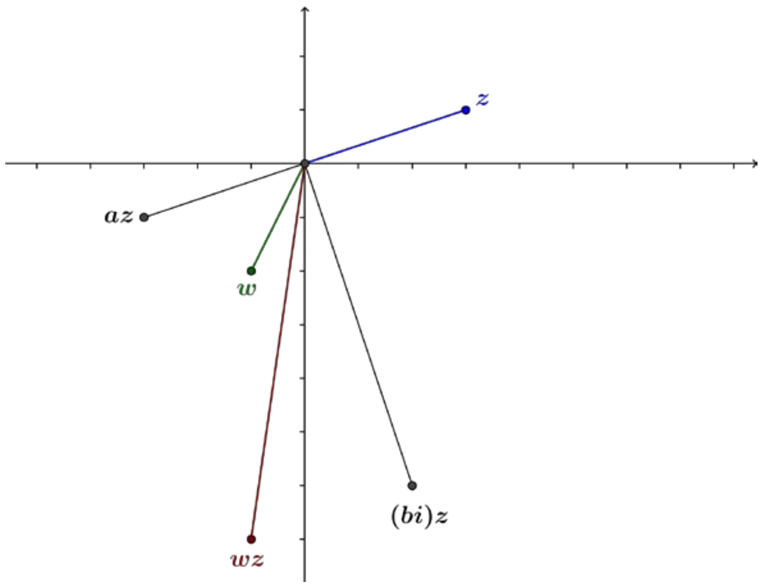
ii. Express the argument of az in terms of $\arg(z)$.

iii. Express $\arg(w)$ in terms of θ, where $\theta = m(\angle POQ)$.

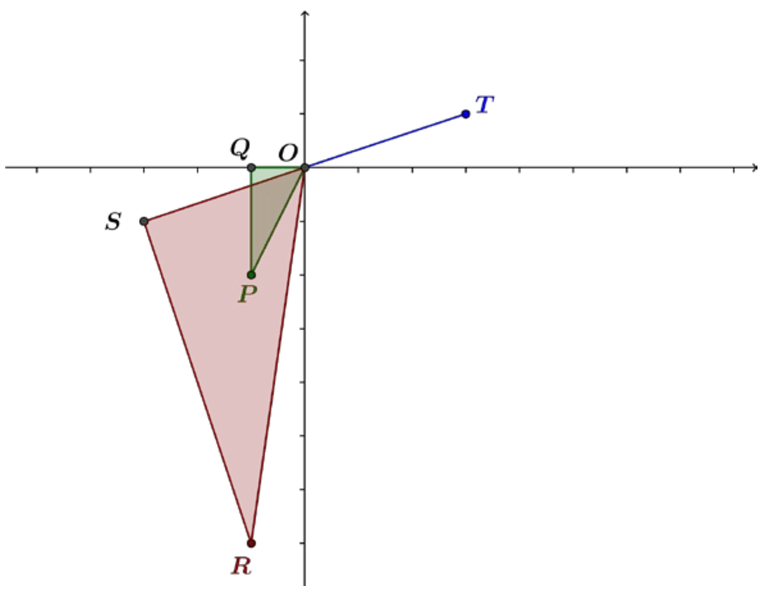
iv. Explain why $\arg(wz) = \arg(az) - \theta$.

v. Combine your responses from parts (ii), (iii) and (iv) to express $\arg(wz)$ in terms of $\arg(z)$ and $\arg(w)$.

g. Case 7: The point representing $w = a + bi$ lies in the third quadrant. That is, $a < 0$ and $b < 0$.
Points representing , z, az, $(bi)z$, and $wz = az + (bi)z$ are shown in the figure below.

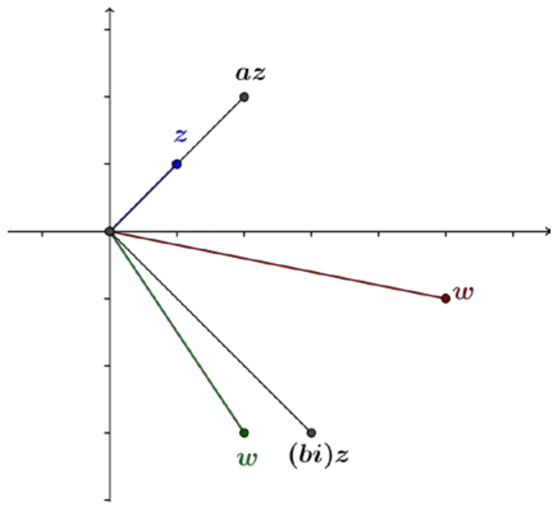


For convenience, rename the origin O and let $P = w$, $Q = a$, $R = wz$, $S = az$, and $T = z$, as shown below. Let $m(\angle POQ) = \theta$.

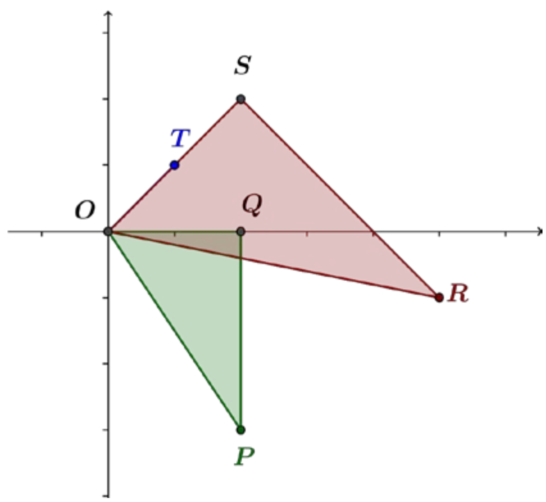


i. Argue that $\triangle OPQ \sim \triangle ORS$.
ii. Express the argument of az in terms of $\arg(z)$.
iii. Express $\arg(w)$ in terms of θ, where $\theta = m(\angle POQ)$.
iv. Explain why $\arg(wz) = \arg(az) + \theta$.
v. Combine your responses from parts (ii), (iii), and (iv) to express $\arg(wz)$ in terms of $\arg(z)$ and $\arg(w)$.

h. Case 8: The point representing $w = a + bi$ lies in the fourth quadrant. That is, $a > 0$ and $b < 0$.
Points representing , z, az, $(bi)z$, and $wz = az + (bi)z$ are shown in the figure below.



For convenience, rename the origin O, and let $P = w$, $Q = a$, $R = wz$, $S = az$, and $T = z$, as shown below.
Let $m(\angle POQ) = \theta$.



i. Argue that $\triangle\, OPQ \sim \triangle\, ORS$.

ii. Express $\arg(w)$ in terms of θ, where $\theta = m(\angle POQ)$.

iii. Explain why $m(\angle QOR) = \theta - \arg(z)$.

iv. Express $\arg(wz)$ in terms of $m(\angle QOR)$

v. Combine your responses from parts (ii), (iii), and (iv) to express $\arg(wz)$ in terms of $\arg(z)$ and $\arg(w)$.

2. Summarize the results of Problem 1, parts (a)–(h) and the lesson.

3. Find a linear transformation L that will have the geometric effect of rotation by the specified amount without dilating.
   a. $45°$ counterclockwise
   b. $60°$ counterclockwise
   c. $180°$ counterclockwise
   d. $120°$ counterclockwise
   e. $30°$ clockwise
   f. $90°$ clockwise
   g. $180°$ clockwise
   h. $135°$ clockwise

4. Suppose that we have linear transformations L_1 and L_2 as specified below. Find a formula for $L_2\big(L_1(z)\big)$ for complex numbers z.
   a. $L_1(z) = (1 + i)z$ and $L_2(z) = (1 - i)z$
   b. $L_1(z) = (3 - 2i)z$ and $L_2(z) = (2 + 3i)z$
   c. $L_1(z) = (-4 + 3i)z$ and $L_2(z) = (-3 - i)z$
   d. $L_1(z) = (a + bi)z$ and $L_2(z) = (c + di)z$ for real numbers a, b, c and d.