Lesson 21: The Hunt for Better Notation

Classwork

Opening Exercise

Suppose that $L_1(x, y) = (2x - 3y, 3x + 2y)$ and $L_2(x, y) = (3x + 4y, -4y + 3x)$.
Find the result of performing L_1 and then L_2 on a point (p, q). That is, find $L_2\big(L_1(p, q)\big)$.

Exercises 1–2

1. Calculate each of the following products.

   a. $\begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

   b. $\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 4 \\ -4 \end{pmatrix}$

   c. $\begin{pmatrix} 2 & -4 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$

2. Find a value of k so that $\begin{pmatrix}1 & 2\\ k & 1\end{pmatrix}\begin{pmatrix}3\\ -1\end{pmatrix} = \begin{pmatrix}1\\ 11\end{pmatrix}$.

Exercises 3–9

3. Find a matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ so that we can represent the transformation $L(x,y) = (2x - 3y, 3x + 2y)$ by $L\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$.

4. If a transformation $L\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$ has the geometric effect of rotation and dilation, do you know about the values a, b, c, and d?

5. Describe the form of a matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ so that the transformation $L\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$ has the geometric effect of only dilation by a scale factor r.

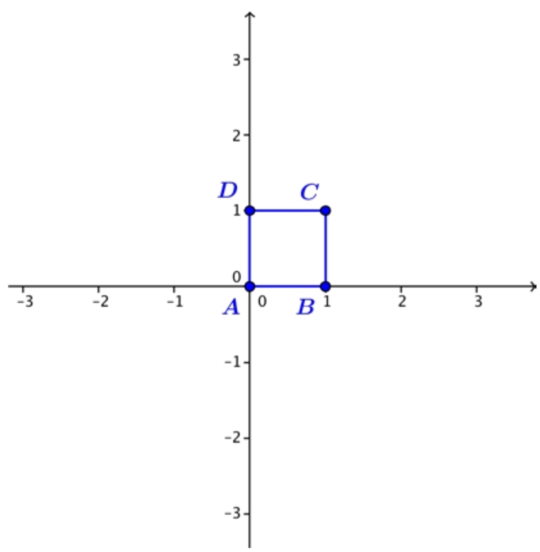
6. Describe the form of a matrix $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ so that the transformation $L\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$ has the geometric effect of only rotation by θ. Describe the matrix in terms of θ.

7. Describe the form of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that the transformation $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ has the geometric effect of rotation by θ and dilation with scale factor r. Describe the matrix in terms of θ and r.

8. Suppose that we have a transformation $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

   a. Does this transformation have the geometric effect of rotation and dilation?

   b. Transform each of the points $A = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, and $D = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and plot the images in the plane shown.



9. Describe the geometric effect of the transformation $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Lesson Summary

For real numbers a, b, c, and d, the transformation $L(x, y) = (ax + by, cx + dy)$ can be represented using matrix multiplication by $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$ and the $\begin{pmatrix} x \\ y \end{pmatrix}$ represents the point (x, y) in the plane.

- The transformation is a counterclockwise rotation by θ if and only if the matrix representation is $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.
- The transformation is a dilation with scale factor k if and only if the matrix representation is $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.
- The transformation is a counterclockwise rotation by $\arg(a + bi)$ and dilation with scale factor $|a + bi|$ if and only if the matrix representation is $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. If we let $r = |a + bi|$ and $\theta = \arg(a + bi)$, then the matrix representation is $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Problem Set

1. Perform the indicated multiplication.
   a. $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$
   b. $\begin{pmatrix} 3 & 5 \\ -2 & -6 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$
   c. $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 6 \\ 8 \end{pmatrix}$
   d. $\begin{pmatrix} 5 & 7 \\ 4 & 9 \end{pmatrix}\begin{pmatrix} 10 \\ 100 \end{pmatrix}$
   e. $\begin{pmatrix} 4 & 2 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} -3 \\ 1 \end{pmatrix}$
   f. $\begin{pmatrix} 6 & 4 \\ 9 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ -3 \end{pmatrix}$
   g. $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   h. $\begin{pmatrix} \pi & 1 \\ 1 & -\pi \end{pmatrix}\begin{pmatrix} 10 \\ 7 \end{pmatrix}$

2. Find a value of k so that $\begin{pmatrix} k & 3 \\ 4 & k \end{pmatrix}\begin{pmatrix} 4 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$.

3. Find values of k and m so that $\begin{pmatrix} k & 3 \\ -2 & m \end{pmatrix}\begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ -10 \end{pmatrix}$.

4. Find values of k and m so that $\begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} k \\ m \end{pmatrix} = \begin{pmatrix} 0 \\ -9 \end{pmatrix}$.

5. Write the following transformations using matrix multiplication.
   a. $L(x, y) = (3x - 2y, 4x - 5y)$
   b. $L(x, y) = (6x + 10y, -2x + y)$
   c. $L(x, y) = (25x + 10y, 8x - 64y)$
   d. $L(x, y) = (\pi x - y, -2x + 3y)$
   e. $L(x, y) = (10x, 100x)$
   f. $L(x, y) = (2y, 7x)$

6. Identify whether or not the following transformations have the geometric effect of rotation only, dilation only, rotation and dilation only, or none of these.
   a. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 4 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   b. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 42 & 0 \\ 0 & 42 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   c. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   d. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   e. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 & 1 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   f. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

7. Create a matrix representation of a linear transformation that has the specified geometric effect.
   a. Dilation by a factor of 4 and no rotation.
   b. Rotation by $180°$ and no dilation.
   c. Rotation by $-\frac{\pi}{2}$ rad and dilation by a scale factor of 3.
   d. Rotation by $30°$ and dilation by a scale factor of 4.

8. Identify the geometric effect of the following transformations. Justify your answer.
   a. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   b. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   c. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -10 & 0 \\ 0 & -10 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$
   d. $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 & 6\sqrt{3} \\ -6\sqrt{3} & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$