Lesson 25: Matrix Multiplication and Addition

Classwork

Opening Exercise

Consider the point $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ that undergoes a series of two transformations: a dilation of scale factor 4 followed by a reflection about the horizontal axis.

a. What matrix produces the dilation of scale factor 4? What is the coordinate of the point after the dilation?

b. What matrix produces the reflection about the horizontal axis? What is the coordinate of the point after the reflection?

c. Could we have produced both the dilation and the reflection using a single matrix? If so, what matrix would both dilate by a scale factor of 4 and produce a reflection about the horizontal axis? Show that the matrix you came up with combines these two matrices.

Example 1: Is Matrix Multiplication Commutative?

a. Take the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ through the following transformations: a rotation of $\frac{\pi}{2}$ and a reflection across the y-axis.

b. Will the resulting point be the same if the order of the transformations is reversed?

c. Are transformations commutative?

d. Let $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Find AB and then BA.

e. Is matrix multiplication commutative?

f. If we apply matrix AB to the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, in what order are the transformations applied.

g. If we apply matrix BA to the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, in what order are the transformations applied.

h. Can we apply $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to matrix BA?

Exercises 1–3

1. Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $M = \begin{pmatrix} 4 & -6 \\ 3 & -2 \end{pmatrix}$.
   a. Find IM.

   b. Find MI.

   c. Do these results make sense based on what you know about the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$?

2. Calculate AB, then BA. Is matrix multiplication commutative?
   a. $A = \begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

   b. $A = \begin{pmatrix} -10 & 1 \\ 3 & 7 \end{pmatrix}$, $B = \begin{pmatrix} -3 & 2 \\ 4 & -1 \end{pmatrix}$

3. Write a matrix that would perform the following transformations in this order: a rotation of $180°$, a dilation by a scale factor of 4, and a reflection across the horizontal axis. Use the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ to illustrate that your matrix is correct.

Example 2: More Operations on Matrices

Find the sum. $\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

Find the difference. $\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

Find the sum. $\begin{pmatrix} 2 & 0 \\ -2 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Exercises 4–5

4. Express each of the following as a single matrix.

   a. $\begin{pmatrix} 6 & -3 \\ 10 & -1 \end{pmatrix} + \begin{pmatrix} -2 & 8 \\ 3 & -12 \end{pmatrix}$

b. $\begin{pmatrix} -2 & 7 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

c. $\begin{pmatrix} 8 & 5 \\ 0 & 15 \end{pmatrix} - \begin{pmatrix} 4 & -6 \\ -3 & 18 \end{pmatrix}$

5. In arithmetic, the additive identity says that for some number a, $a + 0 = 0 + a = 0$. What would be an additive identity in matrix arithmetic?

Lesson Summary

- If L is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and M is given by $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$, then $ML \begin{pmatrix} x \\ y \end{pmatrix}$ is the same as applying the matrix $\begin{pmatrix} pa+rb & pc+rd \\ qa+sb & qc+sd \end{pmatrix}$ to $\begin{pmatrix} x \\ y \end{pmatrix}$.
- If L is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and I is given by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then I acts as a multiplicative identity and $IL = LI = L$.
- If L is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ and O is given by $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then O acts as an additive identity and $O + L = L + O = L$.

Problem Set

1. What type of transformation is shown in the following examples? What is the resulting matrix?
   a. $\begin{pmatrix} \cos\pi & -\sin\pi \\ \sin\pi & \cos\pi \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   b. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   c. $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   d. $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   e. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   f. $\begin{pmatrix} \cos 2\pi & -\sin 2\pi \\ \sin 2\pi & \cos 2\pi \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   g. $\begin{pmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$
   h. $\begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$

2. Calculate each of the following products.
   a. $\begin{pmatrix} 2 & 3 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$
   b. $\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 0 \end{pmatrix}$
   c. $\begin{pmatrix} -1 & -3 \\ -2 & -4 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \end{pmatrix}$
   d. $\begin{pmatrix} -3 & -1 \\ -4 & -2 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 0 & -3 \end{pmatrix}$

e. $\begin{pmatrix} 5 & 0 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix}$

f. $\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & -3 \end{pmatrix}$

3. Calculate each sum or difference.

a. $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$

b. $\begin{pmatrix} -4 & -5 \\ -6 & -7 \end{pmatrix} + \begin{pmatrix} -2 & 3 \\ -1 & 4 \end{pmatrix}$

c. $\begin{pmatrix} -5 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

d. $\begin{pmatrix} 3 \\ -5 \end{pmatrix} - \begin{pmatrix} 7 \\ 9 \end{pmatrix}$

e. $\begin{pmatrix} -4 & -5 \\ -6 & -7 \end{pmatrix} - \begin{pmatrix} -2 & 3 \\ -1 & 4 \end{pmatrix}$

4. In video game programming, Fahad translates a car, whose coordinate is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, 2 units up and 4 units to the right, rotates it $\frac{\pi}{2}$ radians counterclockwise, reflects it about the x-axis, reflects it about the y-axis rotates it $\frac{\pi}{2}$ radians counterclockwise, and finally translates it 4 units down and 2 units to the left. What point represents the final location of the car?