Lesson 27: Getting a Handle on New Transformations

Classwork

Opening Exercise

Explain the geometric effect of each matrix.

a. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

b. $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

c. $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

d. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

f. $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$

Example 1

Given the transformation $\begin{bmatrix} k & 0 \\ k & 1 \end{bmatrix}$ with $k > 0$:

a. Perform this transformation on the vertices of the unit square. Sketch the image and label the vertices.

b. Calculate the area of the image using the dimensions of the image parallelogram.

c. Confirm the area of the image using the determinant.

d. Perform the transformation on $\begin{bmatrix} x \\ y \end{bmatrix}$.

e. In order for two matrices to be equivalent, each of the corresponding elements must be equivalent. Given that, if the image of this transformation is $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$, find $\begin{bmatrix} x \\ y \end{bmatrix}$.

f. Perform the transformation on $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Write the image matrix.

g. Perform the transformation on the image again, and then repeat until the transformation has been performed four times on the image of the preceding matrix.

Exercise 1

1. Perform the transformation $\begin{bmatrix} k & 0 \\ 1 & k \end{bmatrix}$ with $k > 1$ on the vertices of the unit square.

   a. What are the vertices of the image?

   b. Calculate the area of the image.

   c. If the image of the transformation on $\begin{bmatrix} x \\ y \end{bmatrix}$ is $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$, find $\begin{bmatrix} x \\ y \end{bmatrix}$ in terms of k.

Example 2

Consider the matrix $L = \begin{bmatrix} 2 & 5 \\ -1 & 3 \end{bmatrix}$. For each real number $0 \le t \le 1$ consider the point $(3 + t, 10 + 2t)$.

a. Find point A when $t = 0$.

b. Find point B when $t = 1$.

c. Show that for $t = \frac{1}{2}$, $(3 + t, 10 + 2t)$ is the midpoint of $\overline{AB}$.

d. Show that for each value of t, $(3 + t, 10 + 2t)$ is a point on the line through A and B.

e. Find LA and LB.

f. What is the equation of the line through LA and LB?

g. Show that $L\begin{bmatrix} 3 + t \\ 10 + 2t \end{bmatrix}$ lies on the line through LA and LB.

Problem Set

1. Perform the following transformation on the vertices of the unit square. Sketch the image, label the vertices, and find the area of the image parallelogram.
   a. $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$
   b. $\begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}$
   c. $\begin{bmatrix} 3 & 0 \\ 3 & 1 \end{bmatrix}$
   d. $\begin{bmatrix} 2 & 0 \\ 2 & 2 \end{bmatrix}$
   e. $\begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$
   f. $\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}$
   g. $\begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$
   h. $\begin{bmatrix} 2 & 0 \\ 2 & 3 \end{bmatrix}$
   i. $\begin{bmatrix} 3 & 0 \\ 3 & 5 \end{bmatrix}$

2. Given $\begin{bmatrix} k & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ kx + y \end{bmatrix}$. Find $\begin{bmatrix} x \\ y \end{bmatrix}$ if the image of the transformation is the following:
   a. $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$
   b. $\begin{bmatrix} -3 \\ 2 \end{bmatrix}$
   c. $\begin{bmatrix} 5 \\ -6 \end{bmatrix}$

3. Given $\begin{bmatrix} k & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ kx + y \end{bmatrix}$. Find value of k so that:
   a. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and the image is $\begin{bmatrix} 24 \\ 22 \end{bmatrix}$
   b. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 27 \\ 3 \end{bmatrix}$ and the image is $\begin{bmatrix} 18 \\ 21 \end{bmatrix}$

4. Given $\begin{bmatrix} k & 0 \\ 1 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ x + ky \end{bmatrix}$. Find value of k so that:
   a. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 5 \end{bmatrix}$ and the image is $\begin{bmatrix} -12 \\ 11 \end{bmatrix}$
   b. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{2}{9} \end{bmatrix}$ and image is $\begin{bmatrix} -15 \\ -\frac{1}{3} \end{bmatrix}$

5. Perform the following transformation on the vertices of the unit square. Sketch the image, label the vertices, and find the area of the image parallelogram.

   a. $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$

   b. $\begin{bmatrix} 3 & 0 \\ 1 & 3 \end{bmatrix}$

   c. $\begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}$

   d. $\begin{bmatrix} 2 & 0 \\ 4 & 2 \end{bmatrix}$

   e. $\begin{bmatrix} 4 & 0 \\ 2 & 4 \end{bmatrix}$

   f. $\begin{bmatrix} 3 & 0 \\ 5 & 3 \end{bmatrix}$

6. Consider the matrix $L = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. For each real number $o \leq t \leq 1$, consider the point $(3 + 2t, 12 + 2t)$.

   a. Find the point A when $t = 0$.

   b. Find the point B when $t = 1$.

   c. Show that for $t = \frac{1}{2}$, $(3 + 2t, 12 + 2t)$ is the midpoint of $\overline{AB}$.

   d. Show that for each value of t, $(3 + 2t, 12 + 2t)$ is a point on the line through A and B.

   e. Find LA and LB.

   f. What is the equation of the line through LA and LB?

   g. Show that $L\begin{bmatrix} 3 + 2t \\ 12 + 2t \end{bmatrix}$ lies on the line through LA and LB.