Lesson 30: When Can We Reverse a Transformation?

Classwork

Opening Exercise

a. What is the geometric effect of the following matrices?

i. $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

ii. $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

iii. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

b. Jadavis says that the identity matrix is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Sophie disagrees and states that the identity matrix is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

i. Their teacher, Mr. Kuzy, says they are both correct and asks them to explain their thinking about matrices to each other, but also use a similar example in the real number system. Can you state each of their arguments?

ii. Mr. Kuzy then asks each of them to explain the geometric effect that their matrix would have on the unit square.

c. Given the matrices below, answer the following:

$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 5 & 2 \\ 10 & 4 \end{bmatrix}$

i. Which matrix does not have an inverse? Explain how you know algebraically and geometrically.

ii. If a matrix has an inverse, find it.

Example 1

Given $\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

a. Perform this transformation on the unit square, and sketch the results on graph paper. Label the vertices.

b. Explain the transformation that occurred to the unit square.

c. Find the area of the image.

d. Find the inverse of this transformation.

e. Explain the meaning of the inverse transformation on the unit square.

Exercises 1–8

1. Given $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$.

   a. Perform this transformation on the unit square, and sketch the results on graph paper. Label the vertices.

   b. Explain the transformation that occurred to the unit square.

   c. Find the area of the image.

   d. Find the inverse of this transformation.

   e. Explain the meaning of the inverse transformation on the unit square.

   f. If any matrix produces a dilation with a scale factor of k, what would the inverse matrix produce?

2. Given $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$.

   a. Perform this transformation on the unit square, and sketch the results on graph paper. Label the vertices.

b. Explain the transformation that occurred to the unit square.

c. Find the area of the image.

d. Find the inverse of the transformation.

e. Explain the meaning of the inverse transformation on the unit square.

f. Rewrite the original matrix if it also included a dilation with a scale factor of 2.

g. What is the inverse of this matrix?

3. Find a transformation that would create a 90° counterclockwise rotation about the origin. Set up a system of equations and solve to find the matrix.

4.
   a. Find a transformation that would create a $180°$ counterclockwise rotation about the origin. Set up a system of equations and solve to find the matrix.

   b. Rewrite the matrix to also include a dilation with a scale factor of 5.

5. For which values of a does $\begin{bmatrix} 3 & -100 \\ 900 & a \end{bmatrix}$ have an inverse matrix?

6. For which values of a does $\begin{bmatrix} a & a+4 \\ 2 & a \end{bmatrix}$ have an inverse matrix?

7. For which values of a does $\begin{bmatrix} a+2 & a-4 \\ a-3 & a+3 \end{bmatrix}$ have an inverse matrix?

8. Chethan says that the matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ produces a rotation $\theta°$ counterclockwise. He justifies his work by showing that when $\theta = 60°$, the rotation matrix is $\begin{bmatrix} \cos(60) & -\sin(60) \\ \sin(60) & \cos(60) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. Shayla disagrees and says that the matrix $\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ produces a $60°$ rotation counterclockwise. Tyler says that he has found that the matrix $\begin{bmatrix} 2 & -2\sqrt{3} \\ 2\sqrt{3} & 2 \end{bmatrix}$ produces a $60°$ rotation counterclockwise, too.

a. Who is correct? Explain.

b. Which matrix has the largest scale factor? Explain.

c. Create a matrix with a scale factor less than 1 that would produce the same rotation.

Problem Set

1. Find a transformation that would create a $30°$ counterclockwise rotation about the origin and then its inverse.

2. Find a transformation that would create a $30°$ counterclockwise rotation about the origin, a dilation with a scale factor of 4, and then its inverse.

3. Find a transformation that would create a $270°$ counterclockwise rotation about the origin. Set up a system of equations and solve to find the matrix.

4. Find a transformation that would create a $270°$ counterclockwise rotation about the origin, a dilation with a scale factor of 3, and its inverse.

5. For which values of a does $\begin{bmatrix} 8 & a \\ a & 2 \end{bmatrix}$ have an inverse matrix?

6. For which values of a does $\begin{bmatrix} a & a-4 \\ a+4 & a \end{bmatrix}$ have an inverse matrix?

7. For which values of a does $\begin{bmatrix} 3a & 2a-6 \\ 6a & 4a-12 \end{bmatrix}$ have an inverse matrix?

8. In Lesson 27, we learned the effect of a transformation on a unit square by multiplying a matrix. For example, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$, and $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
   a. Sasha says that we can multiply the inverse of A to those resultants of the square after the transformation to get back to the unit square. Is her conjecture correct? Justify your answer.
   b. From part (a), what would you say about the inverse matrix with regard to the geometric effect of transformations?
   c. A pure rotation matrix is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Prove the inverse matrix for a pure rotation of $\frac{\pi}{4}$ radians counterclockwise is $\begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{bmatrix}$, which is the same as $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{d}{ad-bc} \end{bmatrix}$.
   d. Prove that the inverse matrix of a pure dilation with a factor of 4 is $\begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$, which is the same as $\begin{bmatrix} \frac{d}{ad-bc} & \frac{-c}{ad-bc} \\ \frac{-b}{ad-bc} & \frac{d}{ad-bc} \end{bmatrix}$.

e. Prove that the matrix used to undo a $\frac{\pi}{3}$ radians clockwise rotation and a dilation of a factor of 2 is $\frac{1}{2}\begin{bmatrix}\cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right)\\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right)\end{bmatrix}$, which is the same as $\begin{bmatrix}\frac{d}{ad-bc} & \frac{-c}{ad-bc}\\ \frac{-b}{ad-bc} & \frac{d}{ad-bc}\end{bmatrix}$.

f. Prove that any matrix whose determinant is not 0 will have an inverse matrix to "undo" a transformation. For example, use the matrix $A = \begin{bmatrix}a & c\\ b & d\end{bmatrix}$ and the point $\begin{bmatrix}x\\ y\end{bmatrix}$.

9. Perform the transformation $\begin{bmatrix}2 & 2\\ 2 & 2\end{bmatrix}$ on the unit square.

a. Can you find the inverse matrix that will "undo" the transformation? Explain your reasons arithmetically.

b. When all four vertices of the unit square are transformed and collapsed onto a straight line, what can be said about the inverse?

c. Find the equation of the line that all four vertices of the unit square collapsed onto.

d. Find the equation of the line that all four vertices of the unit square collapsed onto using the matrix $\begin{bmatrix}1 & 3\\ 2 & 6\end{bmatrix}$.

e. A function has an inverse function if and only if it is a one-to-one function. By applying this concept, explain why we do not have an inverse matrix when the transformation is collapsed onto a straight line.

10. The determinants of the following matrices are 0. Describe what pattern you can find among them.

a. $\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}$, $\begin{bmatrix}1 & 2\\ 4 & 8\end{bmatrix}$, and $\begin{bmatrix}1 & -2\\ 2 & -4\end{bmatrix}$

b. $\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$, $\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$, $\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$, and $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$