## New York State Common Core

## Mathematics Curriculum

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## Precalculus and Advanced Topics • Module 2

## Vectors and Matrices

## OVERVIEW

In Module 1 students learned that throughout the 1800s, mathematicians encountered a number of disparate situations that seemed to call for displaying information via tables and performing arithmetic operations on those tables. One such context arose in Module 1, where students saw the utility of representing linear transformations in the two-dimensional coordinate plane via matrices. Students viewed matrices as representing transformations in the plane and developed an understanding of multiplication of a matrix by a vector as a transformation acting on a point in the plane. This module starts with a second context for matrix representation, networks.

In Topic A, students look at incidence relationships in networks and encode information about them via highdimensional matrices (N-VM.C.6). Questions on counting routes, the results of combining networks, payoffs, and other applications, provide context and use for matrix manipulations: matrix addition and subtraction, matrix product, and multiplication of matrices by scalars (N-VM.C.7, N-VM.C.8).
The question naturally arises as to whether there is a geometric context for higher-dimensional matrices as there is for $2 \times 2$ matrices. Topic $B$ explores this question, extending the concept of a linear transformation from Module 1 to linear transformations in three- (and higher-) dimensional space. The geometric effect of matrix operations-matrix product, matrix sum, and scalar multiplication-are examined, and students come to see, geometrically, that matrix multiplication for square matrices is not a commutative operation, but that it still satisfies the associative and distributive properties (N-VM.C.9). The geometric and arithmetic roles of the zero matrix and identity matrix are discussed, and students see that a multiplicative inverse to a square matrix exists precisely when the determinant of the matrix (given by the area of the image of the unit square in two-dimensional space, or the volume of the image of the unit cube in three-dimensional space) is nonzero (N-VM.C.10). This work is phrased in terms of matrix operations on vectors, seen as matrices with one column (N-VM.C.11).
Topic C provides a third context for the appearance of matrices via the study of systems of linear equations. Students see that a system of linear equations can be represented as a single matrix equation in a vector variable (A-REI.C.8), and that one can solve the system with the aid of the multiplicative inverse to a matrix if it exists (A-REIC.9).

Topic D opens with a formal definition of a vector (the motivation and context for it is well in place at this point) and the arithmetical work for vector addition, subtraction, scalar multiplication, and vector magnitude is explored along with the geometrical frameworks for these operations (N-VM.A.1, N-VM.A.2, N-VM.B.4, N-VM.B.5). Students also solve problems involving velocity and other quantities that can be represented by vectors (N-VM.A.3). Parametric equations are introduced in Topic D allowing students to connect their prior work with functions to vectors.

The module ends with Topic E where students apply their knowledge developed in this module to understand how first-person video games use matrix operations to project three-dimensional objects onto twodimensional screens and animate those images to give the illusion of motion (N-VM.C.8, N-VM.C.9, $\mathrm{N}-\mathrm{VM} . \mathrm{C} .10, \mathrm{~N}-\mathrm{VM} . C .11)$.

## Focus Standards

## Represent and model with vector quantities.

N-VM.A. 1 (+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g. $\mathbf{v},|\mathbf{v}|,||\mathbf{v}||, v)$.
N-VM.A. $2(+)$ Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
N-VM.A. $3 \quad(+)$ Solve problems involving velocity and other quantities that can be represented by vectors.

## Perform operations on vectors.

N-VM.B. $4 \quad(+)$ Add and subtract vectors.
a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that the magnitude of a sum of two vectors is typically not the sum of the magnitudes.
b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
c. Understand vector subtraction $\mathbf{v}-\mathbf{w}$ as $\mathbf{v}+(-\mathbf{w})$, where $-\mathbf{w}$ is the additive inverse of $\mathbf{w}$, with the same magnitude as $\mathbf{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.

N-VM.B. $5 \quad(+)$ Multiply a vector by a scalar.
a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c\left(v_{x}, v_{y}\right)=$ $\left(c v_{x}, c v_{y}\right)$.
b. Compute the magnitude of a scalar multiple $c \mathbf{v}$ using $\|c \mathbf{v}\|=|c| v$. Compute the direction of $c \mathbf{v}$ knowing that when $|c| v \neq 0$, the direction of $c \mathbf{v}$ is either along $\mathbf{v}$ for ( $c>0$ ) or against $\mathbf{v}$ (for $c<0$ ).

## Perform operations on matrices and use matrices in applications

N-VM.C. $6 \quad(+)$ Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network.

N-VM.C. 7 (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled.
N-VM.C. $8 \quad(+)$ Add, subtract, and multiply matrices of appropriate dimensions.
N-VM.C. 9 (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties.

N-VM.C. $10 \quad(+$ Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.
N-VM.C. 11 (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

## Solve systems of equations

A-REI.C. $8 \quad(+)$ Represent a system of linear equations as a single matrix equation in a vector variable.
A-REI.C. 9 (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension $3 \times 3$ or greater.

## Foundational Standards

## Reason quantitatively and use units to solve problems.

N-Q.A. 2 Define appropriate quantities for the purpose of descriptive modeling.*

## Perform arithmetic operations with complex numbers.

N-CN.A. 1 Know there is a complex number $i$ such that $i^{2}=-1$, and every complex number has the form $a+b i$ with $a$ and $b$ real.
N-CN.A. 2 Use the relation $i^{2}=-1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.

## Use complex numbers in polynomial identities and equations.

N-CN.C. 7 Solve quadratic equations with real coefficients that have complex solutions.
N-CN.C. $8 \quad(+)$ Extend polynomial identities to the complex numbers. For example, rewrite $x^{2}+4$ as $(x+2 i)(x-2 i)$.

## Interpret the structure of expressions.

A-SSE.A. 1 Interpret expressions that represent a quantity in terms of its context. ${ }^{\star}$
a. Interpret parts of an expression, such as terms, factors, and coefficients.
b. Interpret complicated expressions by viewing one or more of their parts as a single entity. For example, interpret $P(1+r) n$ as the product of $P$ and a factor not depending on $P$.

## Write expressions in equivalent forms to solve problems.

A-SSE.B. 3 Choose and produce an equivalent form of an expression to reveal, and explain properties of the quantity represented by the expression. ${ }^{\star}$
a. Factor a quadratic expression to reveal the zeros of the function it defines.
b. Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.
c. Use the properties of exponents to transform expressions for exponential functions. For example, the expression $1.15^{t}$ can be rewritten as $\left(1.15^{1 / 12}\right)^{12 t} \approx 1.012^{12 t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is $15 \%$.

## Create equations that describe numbers or relationships.

A-CED.A. 1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

A-CED.A. 2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

A-CED.A. 3 Represent constraints by equations or inequalities, and by systems of equations and/or inequalities, and interpret solutions as viable or non-viable options in a modeling context. For example, represent inequalities describing nutritional and cost constraints on combinations of different foods.

A-CED.A. 4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations. For example, rearrange Ohm's law $V=I R$ to highlight resistance $R$.

## Understand solving equations as a process of reasoning and explain the reasoning.

A-REI.A. 1 Explain each step in solving a simple equation as following from the equality of numbers asserted at the previous step, starting from the assumption that the original equation has a solution. Construct a viable argument to justify a solution method.

## Solve equations and inequalities in one variable.

A-REI.B. 3 Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.

## Solve systems of equations.

A-REI.C. 6 Solve systems of linear equations exactly and approximately (e.g., with graphs), focusing on pairs of linear equations in two variables.

## Extend the domain of trigonometric functions using the unit circle.

F-TF.A. 1 Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
F-TF.A. 2 Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
F-TF.A. 3 (+) Use special triangles to determine geometrically the values of sine, cosine, tangent for $\pi / 3, \pi / 4$ and $\pi / 6$, and use the unit circle to express the values of sine, cosine, and tangent for $\pi-x, \pi+x$, and $2 \pi-x$ in terms of their values for $x$, where $x$ is any real number.

## Prove and apply trigonometric identities.

F-TF.C. $8 \quad$ Prove the Pythagorean identity $\sin ^{2}(\theta)+\cos ^{2}(\theta)=1$ and use it to find $\sin (\theta), \cos (\theta)$, or $\tan (\theta)$ given $\sin (\theta), \cos (\theta)$, or $\tan (\theta)$ and the quadrant of the angle.

## Experiment with transformations in the plane.

G-CO.A. 2 Represent transformations in the plane using, e.g., transparencies and geometry software; describe transformations as functions that take points in the plane as inputs and give other points as outputs. Compare transformations that preserve distance and angle to those that do not (e.g., translation versus horizontal stretch).

G-CO.A. 4 Develop definitions of rotations, reflections, and translations in terms of angles, circles, perpendicular lines, parallel lines, and line segments.

G-CO.A. 5 Given a geometric figure and a rotation, reflection, or translation, draw the transformed figure using, e.g., graph paper, tracing paper, or geometry software. Specify a sequence of transformations that will carry a given figure onto another.

## Translate between the geometric description and the equation for a conic section.

G-GPE.A. 1 Derive the equation of a circle of given center and radius using the Pythagorean Theorem; complete the square to find the center and radius of a circle given by an equation.
G-GPE.A. 2 Derive the equation of a parabola given a focus and directrix.

## Focus Standards for Mathematical Practice

MP. 2 Reason abstractly and quantitatively. Students recognize matrices and justify the transformations that they represent. Students use $3 \times 3$ matrices to solve systems of equations and continue to calculate the determinant of matrices. Students also represent complex numbers as vectors and determine magnitude and direction. Students reason to determine the effect of scalar multiplication and the result of a zero vector.
MP. 4 Model with mathematics. Students initially study matrix multiplication as a tool for modeling networks and create a model of a bus route. Later, students look at matrix transformations and their role in developing video games and create their own video game. The focus of the mathematics in the computer animation is such that the students come to see rotating and translating as dependent on matrix operations and the addition vectors.

MP. 5 Use appropriate tools strategically. As students study $3 \times 3$ matrices, they begin to view matrices as a tool that can solve problems including networks, payoffs, velocity, and force. Students use calculators and computer software to solve systems of three equations and three unknowns using matrices. Computer software is also used to help students visualize three-dimensional changes on a two-dimensional screen and in the creation of their video games.

## Terminology

## New or Recently Introduced Terms

- Argument (The argument of the complex number $z$ is the radian (or degree) measure of the counterclockwise rotation of the complex plane about the origin that maps the initial ray (i.e., the ray corresponding to the positive real axis) to the ray from the origin through the complex number $z$ in the complex plane. The argument of $z$ is denoted $\arg (z)$.)
- Complex Number (A complex number is a number that can be represented by a point in the complex plane. A complex number can be expressed in two forms:

1. The rectangular form of a complex number $z$ is $a+b i$ where $z$ corresponds to the point $(a, b)$ in the complex plane, and $i$ is the imaginary unit. The number $a$ is called the real part of $a+b i$, and the number $b$ is called the imaginary part of $a+b i$. Note that both the real and imaginary parts of a complex number are themselves real numbers.
2. For $z \neq 0$, the polar form of a complex number $z$ is $r(\cos (\theta)+i \sin (\theta))$ where $r=|z|$ and $\theta=\arg (z)$, and $i$ is the imaginary unit.)

- Complex Plane (The complex plane is a Cartesian plane equipped with addition and multiplication operators defined on ordered pairs by
- Addition: $(a, b)+(c, d)=(a+c, b+d)$ When expressed in rectangular form, if $z=a+b i$ and $w=c+d i$, then $z+w=(a+c)+(b+d) i$.
- Multiplication: $(a, b) \cdot(c, d)=(a c-b d, a d+b c)$

When expressed in rectangular form, if $z=a+b i$ and $w=c+d i$, then
$z \cdot w=(a c-b d)+(a d+b c) i$. The horizontal axis corresponding to points of the form $(x, 0)$ is called the real axis, and a vertical axis corresponding to points of the form $(0, y)$ is called the imaginary axis.)

- Conjugate (The conjugate of a complex number of the form $a+b i$ is $a-b i$. The conjugate of $z$ is denoted $\bar{z}$.)
- Determinant of $2 \times 2$ Matrix (The determinant of the $2 \times 2$ matrix $\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$ is the number computed by evaluating $a d-b c$ and is denoted by $\operatorname{det}\left(\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]\right)$.)
- Determinant of $3 \times 3$ Matrix (The determinant of the $3 \times 3$ matrix $\left[\begin{array}{lll}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{array}\right]$ is the number computed by evaluating the expression

$$
a_{11} \operatorname{det}\left(\left[\begin{array}{ll}
a_{22} & a_{23} \\
a_{32} & a_{33}
\end{array}\right]\right)-a_{12} \operatorname{det}\left(\left[\begin{array}{ll}
a_{21} & a_{23} \\
a_{31} & a_{33}
\end{array}\right]\right)+a_{13} \operatorname{det}\left(\left[\begin{array}{ll}
a_{21} & a_{22} \\
a_{31} & a_{32}
\end{array}\right]\right)
$$

and is denoted by det $\left(\left[\begin{array}{lll}a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{array}\right]\right)$.)

- Directed Graph (A directed graph is an ordered pair $D=(V, E)$ with
- $V$ a set whose elements are called vertices or nodes, and
- $E$ a set of ordered pairs of vertices, called arcs or directed edges.)
- Directed Segment (A directed segment $\overrightarrow{A B}$ is the line segment $\overline{A B}$ together with a direction given by connecting an initial point $A$ to a terminal point $B$.)
- Identity Matrix (The $n \times n$ identity matrix is the matrix whose entry in row $i$ and column $i$ for $1 \leq i \leq n$ is 1 , and whose entries in row $i$ and column $j$ for $1 \leq i, j \leq n$ and $i \neq j$ are all zero. The identity matrix is denoted by I.)
- Imaginary Axis (See complex plane.)
- Imaginary Number (An imaginary number is a complex number that can be expressed in the form bi where $b$ is a real number.)
- Imaginary Part (See complex number.)
- Imaginary Unit (The imaginary unit, denoted by $i$, is the number corresponding to the point $(0,1)$ in the complex plane.)
- Incidence Matrix (The incidence matrix of a network diagram is the $n \times n$ matrix such that the entry in row $i$ and column $j$ is the number of edges that start at node $i$ and end at node $j$.)
- Inverse Matrix (An $n \times n$ matrix $A$ is invertible if there exists an $n \times n$ matrix $B$ so that $A B=B A=I$, where $I$ is the $n \times n$ identity matrix. The matrix $B$, when it exists, is unique and is called the inverse of $A$ and is denoted by $A^{-1}$.)
- Linear Function (A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is called a linear function if it is a polynomial function of degree one, that is, a function with real number domain and range that can be put into the form $f(x)=m x+b$ for real numbers $m$ and $b$. A linear function of the form $f(x)=m x+b$ is a linear transformation only if $b=0$.)
- Linear Transformation (A function $L: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ for a positive integer $n$ is a linear transformation if the following two properties hold
- $L(\mathbf{x}+\mathbf{y})=L(\mathbf{x})+L(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{n}$, and
- $L(k \mathbf{x})=k \cdot L(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^{n}$ and $k \in \mathbb{R}$,
where $\mathbf{x} \in \mathbb{R}^{n}$ means that $\mathbf{x}$ is a point in $\mathbb{R}^{n}$.)
- Linear Transformation Induced by Matrix $\boldsymbol{A}$ (Given a $2 \times 2$ matrix $A$, the linear transformation induced by matrix $A$ is the linear transformation $L$ given by the formula $L\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=A \cdot\left[\begin{array}{l}x \\ y\end{array}\right]$. Given a $3 \times 3$ matrix $A$, the linear transformation induced by matrix $A$ is the linear transformation $L$ given by the formula $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=A \cdot\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$.)
- Matrix (An $m \times n$ matrix is an ordered list of $n m$ real numbers, $a_{11}, a_{12}, \ldots, a_{1 n}, a_{21}, a_{22}, \ldots a_{2 n}, \ldots, a_{m 1}, a_{m 2}, \ldots, a_{m n}$, organized in a rectangular array of $m$ rows and $n$ columns: $\left[\begin{array}{cccc}a_{11} & a_{12} & \cdots & a_{1 n} \\ a_{21} & a_{22} & \cdots & a_{2 n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m 1} & a_{m 2} & \cdots & a_{m n}\end{array}\right]$. The number $a_{i j}$ is called the entry in row $i$ and column $j$.)
- Matrix Difference (Let $A$ be an $m \times n$ matrix whose entry in row $i$ and column $j$ is $a_{i j}$, and let $B$ be an $m \times n$ matrix whose entry in row $i$ and column $j$ is $b_{i j}$. Then the matrix difference $A-B$ is the $m \times n$ matrix whose entry in row $i$ and column $j$ is $a_{i j}-b_{i j}$.)
- Matrix Product (Let $A$ be an $m \times n$ matrix whose entry in row $i$ and column $j$ is $a_{i j}$, and let $B$ be an $n \times p$ matrix whose entry in row $i$ and column $j$ is $b_{i j}$. Then the matrix product $\mathrm{A} B$ is the $m \times p$ matrix whose entry in row $i$ and column $j$ is $a_{i 1} b_{1 j}+a_{i 2} b_{2 j}+\cdots+a_{i n} b_{n j}$.)
- Matrix Scalar Multiplication (Let $k$ be a real number, and let $A$ be an $m \times n$ matrix whose entry in row $i$ and column $j$ is $a_{i j}$. Then the scalar product $k \cdot A$ is the $m \times n$ matrix whose entry in row $i$ and column $j$ is $k \cdot a_{i j}$.)
- Matrix Sum (Let $A$ be an $m \times n$ matrix whose entry in row $i$ and column $j$ is $a_{i j}$, and let $B$ be an $m \times n$ matrix whose entry in row $i$ and column $j$ is $b_{i j}$. Then the matrix sum $A+B$ is the $m \times n$ matrix whose entry in row $i$ and column $j$ is $a_{i j}+b_{i j}$.)
- Modulus (The modulus of a complex number $z$, denoted $|z|$, is the distance from the origin to the point corresponding to $z$ in the complex plane. If $z=a+b i$, then $|z|=\sqrt{a^{2}+b^{2}}$.)
- Network Diagram (A network diagram is a graphical representation of a directed graph where the $n$ vertices are drawn as circles with each circle labeled by a number 1 through $n$, and the directed edges are drawn as segments or arcs with arrow pointing from the tail vertex to the head vertex.)
- Polar Form of a Complex Number (The polar form of a complex number $z$ is $r(\cos (\theta)+i \sin (\theta))$ where $r=|z|$ and $\theta=\arg (z)$.)
- Rectangular Form of a Complex Number (The rectangular form of a complex number $z$ is $a+b i$ where $z$ corresponds to the point $(a, b)$ in the complex plane, and $i$ is the imaginary unit. The number $a$ is called the real part of $a+b i$, and the number $b$ is called the imaginary part of $a+b i$.)
- Vector (A vector is described as either a bound or free vector depending on the context. We refer to both bound and free vectors as vectors throughout this module.)
- Bound Vector (A bound vector is a directed line segment (an arrow). For example, the directed line segment $\overrightarrow{A B}$ is a bound vector whose initial point (or tail) is $A$ and terminal point (or tip) is $B$.
Bound vectors are bound to a particular location in space. A bound vector $\overrightarrow{A B}$ has a magnitude given by the length of segment $\overline{A B}$ and direction given by the ray $\overrightarrow{A B}$. Many times, only the magnitude and direction of a bound vector matters, not its position in space. In that case, we consider any translation of that bound vector to represent the same free vector.
- A bound vector or a free vector (see below) is often referred to as just a vector throughout this module. The context in which the word vector is being used determines if the vector is free or bound.)
- Free Vector (A free vector is the equivalence class of all directed line segments (arrows) that are equivalent to each other by translation. For example, scientists often use free vectors to describe physical quantities that have magnitude and direction only, freely placing an arrow with the given magnitude and direction anywhere in a diagram where it is needed. For any directed line segment in the equivalence class defining a free vector, the directed line segment is said to be a representation of the free vector or is said to represent the free vector.
- A bound vector (see above) or a free vector is often referred to as just a vector throughout this module. The context in which the word vector is being used determines if the vector is free or bound.)
- Real Coordinate Space (For a positive integer $n$, the $n$-dimensional real coordinate space, denoted $\mathbb{R}^{n}$, is the set of all $n$-tuple of real numbers equipped with a distance function $d$ that satisfies

$$
d\left[\left(x_{1}, x_{2}, \ldots, x_{n}\right),\left(y_{1}, y_{2}, \ldots, y_{n}\right)\right]=\sqrt{\left(y_{1}-x_{1}\right)^{2}+\left(y_{2}-x_{1}\right)^{2}+\cdots+\left(y_{n}-x_{n}\right)^{2}}
$$

for any two points in the space. One-dimensional real coordinate space is called a number line, and the two-dimensional real coordinate space is called the Cartesian plane.)

- Position Vector (For a point $P\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ in $\mathbb{R}^{n}$, the position vector $\mathbf{v}$, denoted by $\left[\begin{array}{c}v_{1} \\ v_{2} \\ \vdots \\ v_{n}\end{array}\right]$ or $\left\langle v_{1}, v_{2}, \ldots, v_{n}\right\rangle$, is a free vector $\mathbf{v}$ that is represented by the directed line segment $\overrightarrow{O P}$ from the origin $O(0,0,0, \ldots, 0)$ to the point $P$. The real number $v_{i}$ is called the $i^{\text {th }}$ component of the vector $\mathbf{v}$.)
- Vector Addition (For vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^{n}$, the $\operatorname{sum} \mathbf{v}+\mathbf{w}$ is the vector whose $i^{\text {th }}$ component is the sum of the $i^{\text {th }}$ components of $\mathbf{v}$ and $\mathbf{w}$ for $1 \leq i \leq n$. If $\mathbf{v}=\left[\begin{array}{c}v_{1} \\ v_{2} \\ \vdots \\ v_{n}\end{array}\right]$ and $\mathbf{w}=\left[\begin{array}{c}w_{1} \\ w_{2} \\ \vdots \\ w_{n}\end{array}\right]$ in $\mathbb{R}^{n}$, then $\left.\mathbf{v}+\mathbf{w}=\left[\begin{array}{c}v_{1}+w_{1} \\ v_{2}+w_{2} \\ \vdots \\ v_{n}+w_{n}\end{array}\right].\right)$
- Opposite Vector (For a vector $\vec{v}$ represented by the directed line segment $\overrightarrow{A B}$, the opposite vector, denoted $-\mathbf{v}$, is the vector represented by the directed line segment $\overrightarrow{B A}$. If $\mathbf{v}=\left[\begin{array}{c}v_{1} \\ v_{2} \\ \vdots \\ v_{n}\end{array}\right]$ in $\mathbb{R}^{n}$, then $-\mathbf{v}=\left[\begin{array}{c}-v_{1} \\ -v_{2} \\ \vdots \\ -v_{n}\end{array}\right]$.
- Vector Subtraction (For vectors $\mathbf{v}$ and $\mathbf{w}$, the difference $\mathbf{v}-\mathbf{w}$ is the sum of $\mathbf{v}$ and the opposite of $\mathbf{w}$; that is, $\mathbf{v}-\mathbf{w}=\mathbf{v}+(-\mathbf{w})$. If $\mathbf{v}=\left[\begin{array}{c}v_{1} \\ v_{2} \\ \vdots \\ v_{n}\end{array}\right]$ and $\mathbf{w}=\left[\begin{array}{c}w_{1} \\ w_{2} \\ \vdots \\ w_{n}\end{array}\right]$ in $\mathbb{R}^{n}$, then $\mathbf{v}-\mathbf{w}=\left[\begin{array}{c}v_{1}-w_{1} \\ v_{2}-w_{2} \\ \vdots \\ v_{n}-w_{n}\end{array}\right]$.)
- Vector Magnitude (The magnitude or length of a vector $\mathbf{v}$, denoted $\|\mathbf{v}\|$, is the length of any directed line segment that represents the vector. If $\mathbf{v}=\left[\begin{array}{c}v_{1} \\ v_{2} \\ \vdots \\ v_{n}\end{array}\right]$ in $\mathbb{R}^{n}$, then $\|\mathbf{v}\|=\sqrt{v_{1}^{2}+v_{2}^{2}+\cdots+v_{n}^{2}}$, which is the distance from the origin to the associated point $P\left(v_{1}, v_{2}, \ldots, v_{n}\right)$.)
- Vector Scalar Multiplication (For a vector $\mathbf{v}$ in $\mathbb{R}^{n}$ and a real number $k$, the scalar product $k \cdot \mathbf{v}$ is the vector whose $i^{\text {th }}$ component is the product of $k$ and the $i^{\text {th }}$ component of $\vec{v}$ for $1 \leq i \leq n$. If $k$ is a real number and $\mathbf{v}=\left[\begin{array}{c}v_{1} \\ v_{2} \\ \vdots \\ v_{n}\end{array}\right]$ in $\mathbb{R}^{n}$, then $k \cdot \mathbf{v}=\left[\begin{array}{c}k v_{1} \\ k v_{2} \\ \vdots \\ k v_{n}\end{array}\right]$.)
- Vector Representation of a Complex Number (The vector representation of a complex number $z$ is the position vector $\mathbf{z}$ associated to the point $z$ in the complex plane. If $z=a+b i$ for two real numbers $a$ and $b$, then $\mathbf{z}=\left[\begin{array}{l}a \\ b\end{array}\right]$.)
- Translation by a Vector in Real Coordinate Space (A translation by a vector $\mathbf{v}$ in $\mathbb{R}^{n}$ is the translation transformation $T_{\mathbf{v}}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ given by the map that takes $\mathbf{x} \mapsto \mathbf{x}+\mathbf{v}$ for all $\mathbf{x}$ in $\mathbb{R}^{n}$. If $\mathbf{v}=\left[\begin{array}{c}v_{1} \\ v_{2} \\ \vdots \\ v_{n}\end{array}\right]$ in $\mathbb{R}^{n}$, then $T_{\mathbf{v}}\left(\left[\begin{array}{c}x_{1} \\ x_{2} \\ \vdots \\ x_{n}\end{array}\right]\right)=\left[\begin{array}{c}x_{1}+v_{1} \\ x_{2}+v_{2} \\ \vdots \\ x_{n}+v_{n}\end{array}\right]$ for all $\mathbf{x}$ in $\mathbb{R}^{n}$.)
- Zero Matrix (The $m \times n$ zero matrix is the $m \times n$ matrix in which all entries are equal to zero. For example, the $2 \times 2$ zero matrix is $\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$, and the $3 \times 3$ zero matrix is $\left[\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$.)
- Zero Vector (The zero vector in $\mathbb{R}^{n}$ is the vector in which each component is equal to zero. For example, the zero vector in $\mathbb{R}^{2}$ is $\left[\begin{array}{l}0 \\ 0\end{array}\right]$, and the zero vector in $\mathbb{R}^{3}$ is $\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]$.)


## Familiar Terms and Symbols ${ }^{2}$

- Rotation
- Dilation
- Translation
- Rectangular Form


## Suggested Tools and Representations

- Graphing Calculator
- Wolfram Alpha Software
- Geometer's Sketchpad Software
- ALICE 3.1
- Geogebra Software


## Assessment Summary

| Assessment Type | Administered | Format | Standards Addressed |
| :--- | :--- | :--- | :--- |
| Mid-Module |  |  | N-VM.C.6, N-VM.C.7, |
| Assessment Task | After Topic B | Constructed response with rubric | N-VM.C.8, N-VM.C.9, <br> N-VM.C.10, N-VM.C.11 |
|  |  |  | N-VM.A.1, N-VM.A.2, |
| End-of-Module | After Topic E | Constructed response with rubric | N-VM.A.3, N-VM.B.4, <br> Assessment Task |
|  |  |  | N-VM.B.5, N-VM.C.8, |

[^1]
## New York State Common Core

## Mathematics Curriculum

PRECALCULUS AND ADVANCED TOPICS • MODULE 2

## Topic A:

## Networks and Matrices

N-VM.C.6, N-VM.C.7, N-VM.C. 8

| Focus Standards: | N-VM.C. 6 | (+) Use matrices to represent and manipulate data, e.g., to represent payoffs or incidence relationships in a network. |
| :---: | :---: | :---: |
|  | N-VM.C. 7 | (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled. |
|  | N-VM.C. 8 | (+) Add, subtract, and multiply matrices of appropriate dimensions. |
| Instructional Days: | 3 |  |
| Lesson 1: | Introduction | Networks (E) ${ }^{1}$ |
| Lesson 2: | Networks | Matrix Arithmetic (E) |
| Lesson 3: | Matrix Ari | tic in Its Own Right (E) |

In Topic A, students are introduced to a second application of matrices, networks, and use public transportation routes to study the usefulness of matrices. In Lesson 1, students discover the value of matrices in counting routes (N-VM.C.6). Students see that the arithmetic and properties of matrices is the same regardless of the application. Lesson 2 builds on the work of networks as students study a network of subway lines between four cities and a social network. This work allows students to further explore multiplication by a scalar (N-VM.C.7), and matrix addition and subtraction (N-VM.C.8). In Lesson 3, students continue their study of public transportation networks. Matrix addition, subtraction, and multiplication as well as multiplication by a scalar is revisited as students work with square and rectangular matrices (N-VM.C.8). They begin to see that matrix multiplication is not commutative.

In this topic, students make sense of problems involving transportation networks and matrices (MP.1). Students are asked to relate and explain the connection between real-world situations, such as networks, and their matrix representations (MP.2). Matrices are used as a tool to organize and represent transportation and social network systems (MP.5), and show links between these systems with precise and careful calculations (MP.6).

[^2]
## (8) Lesson 1: Introduction to Networks

## Student Outcomes

- Students use matrices to represent and manipulate data from network diagrams.


## Lesson Notes

In Module 1, students used matrix multiplication to perform a linear transformation in the plane or in space. In this module, we use matrices to model new phenomena, starting with networks and graphs, and we develop arithmetic operations on matrices in this context. Students will extend their understanding of the usefulness of matrices from representing functions to representing connections between people and places as they represent and solve a wide variety of problems using matrices.

The emphasis in these lessons is on understanding matrix arithmetic in authentic situations and then performing appropriate calculations to solve problems. Technology can be incorporated to perform arithmetic operations on large matrices. These lessons will address representing and manipulating data using matrices (N-VM.C.6), understanding multiplication by a scalar (N-VM.C.7), and performing arithmetic operations on matrices of appropriate dimensions (N-VM.C.7).

Network diagram: A network diagram is a graph in which the vertices are represented by circles and the vertices are connected by segments, edges, or arcs. A directed graph, or digraph, is a network diagram in which the edges have arrows indicating the direction being traversed.

Note: The term "arc" is being used differently in this definition than the way it is used in geometry. Here it means any curve that connects one vertex to another vertex (or a vertex to itself). Also, sometimes the vertices are represented by circles with the number of the node written inside the circle. The directed edges can intersect each other.

The figure at left is a specific type of network diagram that has arrows indicating direction. This is the more specific network diagram, which can also be called a directed graph. The figure at right is a network diagram.


In these lessons, a network is any system that can be described by a network diagram, which represents the connections between the parts of the system. Matrices are used as a tool that allows us to organize information from these networks and to perform calculations (MP.5).

## Classwork

## Opening (5 minutes)

Open this lesson by asking students to share what they think of when they hear the word "network." This will activate their prior knowledge about every day usage of this word, which can then be used to transition to the mathematical meaning of a network. Give students a minute of silent thinking time, and then have students turn and talk to a partner about their responses. Ask one or two students to share their responses with the entire class.

- What do you think of when you hear the word network?
- I think of social networks, a computer network, a television network, networking with professionals in a particular career field or group.
- For our purposes, a network will be a system of interrelated objects (such as people or places) that we can represent using a network diagram, as shown below.


## Scaffolding:

- Use a word wall to support English Language Learners in your classroom when you introduce new vocabulary.
- The word "network" can be posted on your word wall after you complete the opening.
- Create a Frayer Diagram for Networks - see Precalculus and Advanced Topics, Module 1, Lesson 5 for an example.


## Exploratory Challenges 1-3 (15 minutes)

Introduce the network diagram representing bus routes shown in the student materials. Have students read silently and then summarize with a partner the information on the student materials. Ask them to point out the four cities and the bus routes connecting them. Use the terminology of "vertices" for the cities and "edges" for the routes. Organize students into small groups, and have them respond to Exercises 1-7. These exercises provide a fairly simple context to explore the number of bus lines that connect four cities. However, students are being presented with an entirely new representation of information so they will have to make sense and persevere when working on this with their group members.

## Exploratory Challenge 1

A network diagram depicts interrelated objects by circles that represent the objects and directed edges drawn as segments or arcs between related objects with arrows to denote direction. The network diagram below shows the bus routes that run between four cities, forming a network. The arrows indicate the direction the buses travel.


Figure 1
How many ways can you travel from City 1 to City 4? Explain how you know.
There are three ways to travel from City 1 to City 4. According to the arrows, you can travel from City 1 to City 2 to City 4, from City 1 to City 3 to City 4, or from City 1 to City 3 to City 2 to City 4.

What about these bus routes doesn't make sense?
It is not possible to leave City 4. The direction of the arrows show that there are no bus routes that lead from City 4 to any other city in this network. It is not possible to travel to City 1. The direction of the arrows show that there are no bus routes that lead to City 1.

It turns out there was an error in printing the first route map. An updated network diagram showing the bus routes that connect the four cities is shown below in Figure 2. Arrows on both ends of an edge indicate that buses travel in both directions.


How many ways can you reasonably travel from City 4 to City 1 using the route map in Figure 2? Explain how you know.
There is only one reasonable way. You must go from City 4 to City 2 to City 3 to City 1. The arrows indicate that there is only one route to City 1, which comes from City 3. However, it is possible to travel from City 4 to City 2 to City 3 as many times as desired before traveling from City 4 to City 2 to City 1.

## Exploratory Challenge 2

A rival bus company offers more routes connecting these four cities as shown in the network diagram in Figure 3.


Figure 3
What might the loop at City 1 represent?
This loop could represent a tour bus that takes visitors around City 1 but does not leave the city limits.

How many ways can you travel from City 1 to City 4 if you want to stop in City 2 and make no other stops?
There are three bus routes from City 1 to City 2 and two bus routes from City 2 to City 4, so there are 6 possible ways to travel from City 1 to City 4.

How many possible ways are there to travel from City 1 to City 4 without repeating a city?
City 1 to City 4 with no stops: No routes.
City 1 to City 4 with a stop in City 2: 6 routes.
City 1 to City 4 with a stop in City 3: 1 route.
City 1 to City 4 via City 3 then City 2: 2 routes.
City 1 to City 4 via City 2 then City 3: 6 routes.
Total ways: $0+6+1+2+6=15$ possible ways to travel from City 1 to City 4 without visiting a city more than once.

Pause here to debrief student work so far. Have different groups present their findings, and use the discussion questions that follow to make sure students begin to understand that as networks become more complicated, an organized, numeric representation may be beneficial. The next discussion focuses students on using a table as a tool to organize information.

- As a transportation network grows, these diagrams become more complicated, and keeping track of all of the information can be challenging. People that work with complicated networks use computers to manage and manipulate this information. Provide students with an example of a more complex network diagram by displaying a visual representation of the New York City subway system, an airline flight map, or show a computer networking diagram.
- What challenges did you encounter as you tried to answer Exercises 1-6?
- We had to keep track to the different routes in an organized manner to make sure our answers were accurate.
- How might we represent the possible routes in a more organized manner?
- You could make a list or a table.
- Organizing information in a table can make things easier, especially if computers are used to count and keep track of routes and schedules. What if we wanted to represent a count of ALL the routes that connect ALL the cities? How could we best organize that information?
- We need a grid or table to show connections from city to city.

Have students return to their groups to complete this exercise to conclude the Exploratory Challenge 3 using the table below.

## Exploratory Challenge 3

We will consider a "direct route" to be a route from one city to another without going through any other city. Organize the number of direct routes from each city into the table shown below. The first row showing the direct routes between City 1 and the other cities is complete for you.

## Discussion (5 minutes)

After students have had a few minutes to complete this exercise, copy the table onto a whiteboard, and lead this next discussion to connect the tabular representation of the network to a matrix.

Circulate to check for understanding. In particular, ensure that your students understand what is meant by a direct route: a route from one city to another without going through any other city.

- If we agree that the rows represent the cities of origin and the columns represent the destination cities, then we do not really need the additional labels.

At this point, erase or display the same table with the row and column labels removed. The new table is shown below.

|  | 1 | 2 | 3 | 4 |
| :--- | :--- | :--- | :--- | :--- |
| 1 | 1 | 3 | 1 | 0 |
| 2 | 2 | 0 | 2 | 2 |
| 3 | 2 | 1 | 0 | 1 |
| 4 | 0 | 2 | 1 | 0 |

- Next, if we agree that we will list the cities in numerical order from top to bottom and left to right, then we don't need the individual row or column labels.

At this point, erase the city number labels or display the table shown below.

| 1 | 3 | 1 | 0 |
| :--- | :--- | :--- | :--- |
| 2 | 0 | 2 | 2 |
| 2 | 1 | 0 | 1 |
| 0 | 2 | 1 | 0 |

- Does this array of numbers written like this look familiar to you? What about if we erase the lines bordering the table?
- It looks like a matrix.

Now erase the lines and just show the numbers.

| 1 | 3 | 1 | 0 |
| :--- | :--- | :--- | :--- |
| 2 | 0 | 2 | 2 |
| 2 | 1 | 0 | 1 |
| 0 | 2 | 1 | 0 |
| 1 | 3 | 1 | 0 |
| 2 | 0 | 2 | 2 |
| 2 | 1 | 0 | 1 |
| 0 | 2 | 1 | 0 |

- In the last module, we used matrices to represent transformations, but they are useful in a wide variety of mathematical situations. What do you recall about matrices from Module 1?
- A matrix is an array of numbers organized into $m$ rows and $n$ columns. A matrix containing $m$ rows and $n$ columns has dimensions $m \times n$. The entry in the first row and first column is referred to as $a_{1,1}$. In general, the entry in the $i^{\text {th }}$ row and $j^{\text {th }}$ column would be denoted $a_{i, j}$.


## Scaffolding:

For struggling students, emphasize that the entry in row $i$ and column $j$ of the matrix counts the number of direct routes from the City $i$ to City $j$. Illustrate with a concrete example.

## Exercises 1-7 (12 minutes)

These exercises can be done as a whole class or within small groups depending on how much students recall about matrix terminology from the previous module.

## Exercises 1-7

1. Use the network diagram in Figure 3 to represent the number of direct routes between the four cities in a matrix $\boldsymbol{R}$.

$$
R=\left[\begin{array}{llll}
1 & 3 & 1 & 0 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
0 & 2 & 1 & 0
\end{array}\right]
$$

2. What is the value of $r_{2,3}$ ? What does it represent in this situation?

The value is 2. It is the number of direct routes from City 2 to City 3.
3. What is the value of $r_{2,3} \cdot r_{3,1}$, and what does it represent in this situation?

The value is 4. It represents the number of one-stop routes between City 2 and City 1 that pass through City 3.
4. Write an expression for the total number of one-stop routes from City 4 and City 1 , and determine the number of routes stopping in one city.

$$
r_{4,2} \cdot r_{2,1}+r_{4,3} \cdot r_{3,1}=2 \cdot 2+1 \cdot 2=6
$$

5. Do you notice any patterns in the expression for the total number of one-stop routes from City 4 and City 1?

The middle indices in each expression are the same in each term and represent the cities where a stop was made.

## 6. Create a network diagram for the matrices shown below. Each matrix represents the number of transportation

 routes that connect four cities. The rows are the cities you travel from, and the columns are the cities you travel to.a. $\quad R=\left[\begin{array}{llll}0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0\end{array}\right]$

b. $\quad R=\left[\begin{array}{llll}0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0\end{array}\right]$


This last exercise introduces a new type of network diagram. This exercise will help students understand that networks can be used to represent a wide variety of situations where tracking connections between a number of entities is important. If time is running short, this exercise could be used as an additional Problem Set exercise.

Here is a type of network diagram called an arc diagram.


Suppose the points represent eleven students in your mathematics class, numbered 1 through 11 . Suppose the arcs above and below the line of vertices 1-11 are the people who are friends on a social network.
7. Complete the matrix that shows which students are friends with each other on this social network. The first row has been completed for you.

$$
\left[\begin{array}{ccccccccccc}
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
- & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & - \\
- & - & - & - & - & - & - & - & - & - & -
\end{array}\right]
$$

$$
\left[\begin{array}{lllllllllll}
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0
\end{array}\right]
$$

Number 1 is not friends with Number 10. How many ways could Number 1 get a message to 10 by only going through one other friend?
a. Who has the most friends in this network? Explain how you know.

Number 1 is friends with 7 people, and that's more than anyone else, so Number 1 has the most friends in this network.
b. Is everyone in this network connected at least as a friend of a friend? Explain how you know.

No. Number 2 is not connected to Number 4 as a friend of a friend because Number 2 is only friends with Number 7, and Number 7 and Number 4 are not friends.
c. What is entry $A_{2,3}$ ? Explain its meaning in this context.

The entry is 0 . Number 3 is not friends with Number 2.

## Closing (3 minutes)

Ask students to summarize their work from this lesson with a partner or on their own in writing.

- How can you find the total number of possible routes between two locations in a network?
- Multiply the number of routes from each city stopping at common cities and add the products.
- How does a matrix help you to organize and represent information in a network?
- A matrix allows for succinct numerical representations which are advantageous when network diagrams are complex and contain many nodes and edges.
- How does a network diagram help you to organize and represent information?
- It shows connections between points to help you see the data before organizing it.


## Lesson Summary

Students organize data and use matrices to represent data in an organized way.
A network diagram is a graphical representation of a directed graph where the $n$ vertices are drawn as circles with each circle labeled by a number 1 through $n$, and the directed edges are drawn as segments or arcs with an arrow pointing from the tail vertex to the head vertex.

A matrix is an array of numbers organized into $\boldsymbol{m}$ rows and $\boldsymbol{n}$ columns. A matrix containing $\boldsymbol{m}$ rows and $\boldsymbol{n}$ columns has dimensions $\boldsymbol{m} \times \boldsymbol{n}$. The entry in the first row and first column is referred to as $\boldsymbol{a}_{1,1}$. In general, the entry in the $i^{\text {th }}$ row and $j^{\text {th }}$ column would be denoted $a_{i, j}$.

## Exit Ticket (5 minutes)

| Lesson 1: | Introduction to Networks |
| :--- | :--- |
| Date: | $1 / 24 / 15$ |

Name $\qquad$ Date $\qquad$

## Lesson 1: Introduction to Networks

## Exit Ticket

The following directed graph shows the major roads that connect four cities.


1. Create a matrix $C$ that shows the direct routes connecting the four cities.
2. Use the matrix to determine how many ways are there to travel from City 1 to City 4 with one stop in City 2.
3. What is the meaning of $c_{2,3}$ ?
4. Write an expression that represents the total number of ways to travel between City 2 and City 3 without passing through the same city twice (you can travel through another city on the way from City 2 to City 3).

## Exit Ticket Sample Solutions

The following directed graph shows the major roads that connect four cities.


1. Create a matrix $C$ that shows the direct routes connecting the four cities.

$$
C=\left[\begin{array}{llll}
0 & 3 & 2 & 0 \\
1 & 0 & 1 & 2 \\
1 & 1 & 0 & 2 \\
0 & 2 & 2 & 0
\end{array}\right]
$$

2. Use the matrix to determine how many ways are there to travel from City 1 to City 4 with one stop in City 2.

$$
c_{1,2} \cdot c_{2,4}=3 \cdot 2=6
$$

3. What is the meaning of $c_{2,3}$ ?

Since $c_{2,3}=1$, it means that there is just one road that goes to City 3 from City 2 without going through the other cities.
4. Write an expression that represents the total number of ways to travel between City 2 and City $\mathbf{3}$ without passing through the same city twice (you can travel through another city on the way from City 2 to City 3).

We will count the routes by the number of stops in intermediate cities.
Direct routes between City 2 and City 3: $c_{2,3}=1$
One-stop routes through City 1: $c_{2,1} \cdot c_{1,3}=1 \cdot 2=2$
One-stop routes through City 4: $c_{2,4} \cdot c_{4,3}=2 \cdot 2=4$
Two-stop routes through City 1 then City 4: $c_{2,1} \cdot c_{1,4} \cdot c_{4,3}=1 \cdot 0 \cdot 2=0$
Two-stop routes through City 4 then City 1: $c_{2,4} \cdot c_{4,1} \cdot c_{1,3}=2 \cdot 0 \cdot 2=0$
So there are $c_{2,3}+c_{2,1} \cdot c_{1,3}+c_{2,4} \cdot c_{4,3}+c_{2,1} \cdot c_{1,4} \cdot c_{4,3}+c_{2,4} \cdot c_{4,1} \cdot c_{1,3}=1+2+4+0+0=7$ ways to travel between City 2 and City 3 without passing through the same city twice.

## Problem Set Sample Solutions

1. Consider the railroad map between Cities 1,2 , and 3 , as shown.
a. Create a matrix $R$ to represent the railroad map between Cities 1, 2 and 3.

$$
R=\left[\begin{array}{lll}
0 & 2 & 1 \\
1 & 0 & 0 \\
2 & 3 & 0
\end{array}\right]
$$

b. How many different ways can you travel from City 1 to City 3 without passing through the same city twice?

$r_{1,2} \cdot r_{2,3}+r_{1,1} \cdot r_{1,3}+r_{1,3}=2 \cdot 0+0 \cdot 1+1=1$
c. How many different ways can you travel from City 2 to City 3 without passing through the same city twice?
$r_{2,3}+r_{2,1} \cdot r_{1,3}=0+1 \cdot 1=1$
d. How many different ways can you travel from City 1 to City 2 with exactly one connecting stop?
$r_{1,3} \cdot r_{3,2}=1 \cdot 3=3$
e. Why is this not a reasonable network diagram for a railroad?

More trains arrive in City 2 than leave, and more trains leave City 3 than arrive.
2. Consider the subway map between stations 1,2 , and 3 , as shown.
a. Create a matrix $S$ to represent the subway map between stations 1,2 , and 3.

$$
S=\left[\begin{array}{lll}
0 & 2 & 1 \\
1 & 0 & 2 \\
2 & 1 & 0
\end{array}\right]
$$

b. How many different ways can you travel from station 1 to station 3 without passing through the same station twice?

$s_{1,3}+s_{1,2} \cdot s_{2,3}=1+2 \cdot 2=5$
c. How many different ways can you travel directly from station 1 to station 3 with no stops?
$s_{1,3}=1$
d. How many different ways can you travel from station 1 to station 3 with exactly one stop?
$s_{1,2} \cdot s_{2,3}=2 \cdot 2=4$
e. How many different ways can you travel from station 1 to station 3 with exactly two stops? Allow for stops at repeated stations.
$s_{1,2} \cdot s_{2,1} \cdot s_{1,3}+s_{1,3} \cdot s_{3,2} \cdot s_{2,3}+s_{1,3} \cdot s_{3,1} \cdot s_{1,3}=2 \cdot 1 \cdot 1+1 \cdot 1 \cdot 2+1 \cdot 2 \cdot 1=6$
3. Suppose the matrix $R$ represents a railroad map between cities $1,2,3,4$, and 5 .

$$
R=\left[\begin{array}{lllll}
0 & 1 & 2 & 1 & 1 \\
2 & 0 & 1 & 1 & 0 \\
1 & 1 & 0 & 2 & 2 \\
1 & 1 & 0 & 0 & 2 \\
1 & 1 & 3 & 0 & 0
\end{array}\right]
$$

a. How many different ways can you travel from City 1 to City $\mathbf{3}$ with exactly one connection?

$$
r_{1,2} \cdot r_{2,3}+r_{1,4} \cdot r_{4,3}+r_{1,5} \cdot r_{5,3}=1 \cdot 1+1 \cdot 0+1 \cdot 3=4
$$

b. How many different ways can you travel from City 1 to City 5 with exactly one connection?
$r_{1,2} \cdot r_{2,5}+r_{1,3} \cdot r_{3,5}+r_{1,4} \cdot r_{4,5}=1 \cdot 0+2 \cdot 2+1 \cdot 2=6$
c. How many different ways can you travel from City $\mathbf{2}$ to City 5 with exactly one connection?
$r_{2,1} \cdot r_{1,5}+r_{2,3} \cdot r_{3,5}+r_{2,4} \cdot r_{4,5}=2 \cdot 1+1 \cdot 2+1 \cdot 2=6$
4. Let $B=\left[\begin{array}{lll}0 & 2 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 1\end{array}\right]$ represent the bus routes between 3 cities.
a. Draw an example of a network diagram represented by this matrix.

b. Calculate the matrix $B^{2}=B B$.

$$
B^{2}=\left[\begin{array}{lll}
4 & 3 & 5 \\
5 & 5 & 5 \\
3 & 6 & 5
\end{array}\right]
$$

c. How many routes are there between City 1 and City 2 with one stop in between?
$b_{1,1} b_{1,2}+b_{1,2} b_{2,2}+b_{1,3} b_{3,2}=0 \cdot 1+2 \cdot 1+1 \cdot 1=3$
d. How many routes are there between City $\mathbf{2}$ and City $\mathbf{2}$ with one stop in between?
$b_{2,1} b_{1,2}+b_{2,2} b_{2,2}+b_{2,3} b_{3,2}=1 \cdot 2+1 \cdot 1+2 \cdot 1=5$
e. How many routes are there between City 3 and City 2 with one stop in between?
$b_{3,1} b_{1,2}+b_{3,2} b_{2,2}+b_{3,3} b_{3,2}=2 \cdot 2+1 \cdot 1+1 \cdot 1=6$
f. What is the relationship between your answers to parts (b)-(e)? Formulate a conjecture.

The numbers 3,5, 6 appear in the second column of the matrix $B^{2}$. It seems that the entry in row $i$ and column $j$ of matrix $B^{2}$ is the number of ways to get from city $i$ to city $j$ with one stop.
5. Consider the airline flight routes between Cities $1,2,3$, and 4 , as shown.

a. Create a matrix $F$ to represent the flight map between Cities 1, 2, 3, and 4.

$$
F=\left[\begin{array}{llll}
0 & 2 & 1 & 2 \\
1 & 0 & 1 & 2 \\
1 & 1 & 0 & 1 \\
2 & 2 & 1 & 0
\end{array}\right]
$$

b. How many different routes can you take from City 1 to City 4 with no stops?
$f_{1,4}=2$
c. How many different routes can you take from City 1 to City 4 with exactly one stop?
$f_{1,2} f_{2,4}+f_{1,3} f_{3,4}=2 \cdot 2+1 \cdot 1=5$
d. How many different routes can you take from City 3 to City 4 with exactly one stop?
$f_{3,1} f_{1,4}+f_{3,2} f_{2,4}=1 \cdot 2+1 \cdot 2=4$
e. How many different routes can you take from City 1 to City 4 with exactly two stops? Allow for routes that include repeated cities.
$f_{1,2} f_{2,1} f_{1,4}+f_{1,2} f_{2,3} f_{3,4}+f_{1,3} f_{3,1} f_{1,4}+f_{1,3} f_{3,2} f_{2,4}+f_{1,4} f_{4,1} f_{1,4}+f_{1,4} f_{4,2} f_{2,4}+f_{1,4} f_{4,3} f_{3,4}=$ $(\mathbf{2} \cdot \mathbf{1} \cdot \mathbf{2})+(\mathbf{2} \cdot \mathbf{1} \cdot \mathbf{1})+(\mathbf{1} \cdot \mathbf{1} \cdot \mathbf{2})+(\mathbf{1} \cdot \mathbf{1} \cdot \mathbf{2})+(2 \cdot \mathbf{2} \cdot \mathbf{2})+(\mathbf{2} \cdot \mathbf{2} \cdot \mathbf{2})+(\mathbf{2} \cdot \mathbf{1} \cdot \mathbf{1})=28$
f. How many different routes can you take from City 2 to City 4 with exactly two stops? Allow for routes that include repeated cities.
$f_{2,1} f_{1,2} f_{2,4}+f_{2,1} f_{1,3} f_{3,4}+f_{2,3} f_{3,1} f_{1,4}+f_{2,3} f_{3,2} f_{2,4}+f_{2,4} f_{4,1} f_{1,4}+f_{2,4} f_{4,2} f_{2,4}+f_{2,4} f_{4,3} f_{3,4}=$
$(\mathbf{1} \cdot \mathbf{2} \cdot \mathbf{2})+(\mathbf{1} \cdot \mathbf{1} \cdot \mathbf{1})+(\mathbf{1} \cdot \mathbf{1} \cdot \mathbf{2})+(\mathbf{1} \cdot \mathbf{1} \cdot \mathbf{2})+(\mathbf{2} \cdot \mathbf{2} \cdot \mathbf{2})+(\mathbf{2} \cdot \mathbf{2} \cdot \mathbf{2})+(\mathbf{2} \cdot \mathbf{1} \cdot \mathbf{1})=27$
6. Consider the following directed graph representing the number of ways Trenton can get dressed in the morning (only visible options are shown):

a. What reasons could there be for there to be three choices for shirts after "traveling" to shorts but only two after traveling to pants?

It could be that Trenton has shirts that only make sense to wear with one or the other. For instance, maybe he does not want to wear a button-up shirt with a pair of shorts.
b. What could the order of the vertices mean in this situation?

The order of the vertices is probably the order Trenton gets dressed in.
c. Write a matrix $A$ representing this directed graph.

$$
A=\left[\begin{array}{lllll}
0 & 2 & 3 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{array}\right]
$$

d. Delete any rows of zeros in matrix $A$, and write the new matrix as matrix $B$. Does deleting this row change the meaning of any of the entries of $B$ ? If you had deleted the first column, would the meaning of the entries change? Explain.

$$
B=\left[\begin{array}{lllll}
0 & 2 & 3 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

Deleting the row did not change the meaning of any of the other entries. Each entry $b_{i, j}$ still says how to get from article of clothing $i$ to article of clothing $j$. If we had deleted the first column, then each entry $b_{i, j}$ would represent how to get from article of clothing $i-1$ to article of clothing $j$.
e. Calculate $b_{1,2} \cdot b_{2,4} \cdot b_{4,5}$. What does this product represent?
$2 \cdot 2 \cdot 1=4$
The number of outfits that Trenton can wear assuming he wears pants instead of shorts.
f. How many different outfits can Trenton wear assuming he always wears a watch?

$$
b_{1,2} \cdot b_{2,4} \cdot b_{4,5}+b_{1,3} \cdot b_{3,4} \cdot b_{4,5}=4+9=13
$$

7. Recall the network representing bus routes used at the start of the lesson:


Faced with competition from rival companies, you have been tasked with considering the option of building a toll road going directly from City 1 to City 4. Once built, the road will provide income in the form of tolls and also enable the implementation of a non-stop bus route to and from City 1 and City 4.

Analysts have provided you with the following information (values are in millions of dollars):

|  | Start-up costs <br> (expressed as profit) | Projected minimum <br> profit per year | Projected maximum <br> profit per year |
| :--- | :---: | :---: | :---: |
| Road | $-\$ 63$ | $\$ 65$ | $\$ 100$ |
| New bus route | $-\$ 5$ | $\$ 0.75$ | $\$ 1.25$ |

a. Express this information in a matrix $P$.
$\left[\begin{array}{ccc}-63 & 65 & 100 \\ -5 & 0.75 & 1.25\end{array}\right]$
b. What are the dimensions of the matrix?
$2 \times 3$
c. Evaluate $p_{1,1}+p_{1,2}$. What does this sum represent?
$-63+65=2$
$\$ 2,000$ is the worst-case profit of the road after one year.
d. Solve $p_{1,1}+t \cdot p_{1,2}=0$ for $t$. What does the solution represent?

$$
\begin{aligned}
-63+65 t & =0 \\
65 t & =63 \\
t & =\frac{63}{65} \approx 0.9692
\end{aligned}
$$

It will take about 1 year for the road to break even if we assume the worst-case profit.
e. Solve $p_{1,1}+t \cdot p_{1,3}=0$ for $t$. What does the solution represent?

$$
\begin{aligned}
-63+100 t & =0 \\
100 t & =63 \\
t & =0.63
\end{aligned}
$$

It will take about 7.5 months for the road to break even if we assume the best-case every year.
f. Summarize your results to part (d) and (e).

It will take between 7.5 months and 1 year for the road to break even.
g. Evaluate $p_{1,1}+p_{2,1}$. What does this sum represent?
$-63+-5=-68$
The total cost of the new road and new bus route is $\$ 68,000$.
h. Solve $p_{1,1}+p_{2,1}+t\left(p_{1,2}+p_{2,2}\right)=0$ for $t$. What does the solution represent?

$$
\begin{aligned}
-68+65.75 t & =0 \\
65.75 t & =68 \\
t & =\frac{68}{65.75} \approx 1.034
\end{aligned}
$$

It will take about 1 year for the road and new bus route to break even assuming the worst-case scenario for profit.
i. Make your recommendation. Should the company invest in building the toll road or not? If they build the road, should they also put in a new bus route? Explain your answer.

Answers will vary but should include the length of time it will take for the company to be profitable after the initial investment. Other factors can include considering the positive press for the company from building and maintaining a non-stop route between the cities as well as how this would affect the other routes of their buses.

## (8) Lesson 2: Networks and Matrix Arithmetic

## Student Outcomes

- Students use matrices to represent data based on transportation networks.
- Students multiply a matrix by a scalar, add and subtract matrices of appropriate dimensions, and interpret the meaning of this arithmetic in terms of transportation networks.


## Lesson Notes

This lesson builds on the work in the previous lesson in which students modeled transportation networks with matrices. The primary example used in Lesson 1 was a set of bus routes that connect four cities. This situation will be used to help students discover and define multiplication by a scalar (N-VM.C.7) and matrix addition and subtraction (N-VM.C.8). This lesson helps students understand the meaning of this matrix arithmetic. Matrix multiplication and the properties of matrix arithmetic will be explored further in Lesson 3. Throughout this lesson, students make sense of transportation network diagrams and matrices (MP.1), reason about contextual and abstract situations (MP.2), and use matrices as tools to represent network diagrams (MP.5) with care and precision (MP.6).

## Classwork

## Opening Exercise (5 minutes)

Have students turn and talk to a partner about the following questions to activate prior knowledge about different types of transportation networks and to remind them of yesterday's scenario.

- In yesterday's lesson, you looked at bus routes and roads that connected four cities. What other types of transportation might connect cities?
- Other types of transportation could include trains, airplanes, boats, walking, or biking routes.


## Scaffolding:

Teachers can offer a simplified task for the Opening Exercise by displaying the network diagram and the matrix representation side-by-side and asking:

- Explain why $b_{2,3}=1$.
- Explain why $s_{4,4}=0$.

Use this exercise to check for student understanding about how to create a matrix from a network diagram. Students should work independently on this exercise and then confirm their answers with a partner. If time is a factor, you could have half of the class create the matrix for the subway line and the other half for the bus line. Ask one or two students to share their answers with the class. Take time to review the meaning of the arrows and to make sure students are distinguishing between bus routes (solid) and subway routes (dashed).

## Opening Exercise

Suppose a subway line also connects the four cities. Here is the subway and bus line network. The bus routes connecting the cities are represented by solid lines, and the subway routes are represented by dashed arcs.


Write a matrix $B$ to represent the bus routes and a matrix $S$ to represent the subway lines connecting the four cities.

$$
B=\left[\begin{array}{llll}
1 & 3 & 1 & 0 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
0 & 2 & 1 & 0
\end{array}\right] \text { and } S=\left[\begin{array}{llll}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 2 \\
1 & 2 & 0 & 1 \\
1 & 1 & 2 & 0
\end{array}\right]
$$

## Exploratory Challenge/Exercises 1-6 (15 minutes): Matrix Arithmetic

Organize students into small groups, and invite them to work through the exercises in this Exploratory Challenge. Students will create new matrices and interpret them in terms of the situation. They will consider the product of a scalar and a matrix and the sum of two matrices. At the end of this challenge, you will lead a discussion debriefing the results and then give students time to work with group members to start populating a graphic organizer for these matrix arithmetic operations.

## Scaffolding:

Teachers can offer a simplified diagram if students are struggling with the one presented.


## Exploratory Challenge/Exercises 1-6: Matrix Arithmetic

Use the network diagram from the Opening Exercise and your answers to help you complete this challenge with your group.

1. Suppose the number of bus routes between each city were doubled.
a. What would the new bus route matrix be?

$$
\left[\begin{array}{llll}
2 & 6 & 2 & 0 \\
4 & 0 & 4 & 4 \\
4 & 2 & 0 & 2 \\
0 & 4 & 2 & 0
\end{array}\right]
$$

b. Mathematicians call this matrix $2 B$. Why do you think they call it that?

Each entry is multiplied by 2 , so the values are twice what they were before.
2. What would be the meaning of $10 B$ in this situation?

It would mean each entry in matrix B is multiplied by 10, so the city now has 10 times as many bus routes as before connecting the cities.
3. Write the matrix $10 B$.

$$
10 B=\left[\begin{array}{cccc}
10 & 30 & 10 & 0 \\
20 & 0 & 20 & 20 \\
20 & 10 & 0 & 10 \\
0 & 20 & 10 & 0
\end{array}\right]
$$

4. Ignore whether or not a line connecting cities represents a bus or subway route.
a. Create one matrix that represents all the routes between the cities in this transportation network.
$\left[\begin{array}{llll}1 & 4 & 2 & 1 \\ 3 & 0 & 3 & 4 \\ 3 & 3 & 0 & 2 \\ 1 & 3 & 3 & 0\end{array}\right]$
b. Why would it be appropriate to call this matrix $B+S$ ? Explain your reasoning.

The total number of routes regardless of transportation mode is found by counting the total routes. We can also find that by adding the number of bus and subway lines between each pair of cities.
5. What would be the meaning of $4 B+2 S$ in this situation?

It would mean the total routes connecting the cities if the number of bus lines were increased by a factor of 4 and the number of subway lines were doubled.
6. Write the matrix $4 B+2 S$. Show work and explain how you found your answer.

First multiply each entry in B by 4, and then multiply each entry in $S$ by 2.

$$
\begin{aligned}
& 4 B=4\left[\begin{array}{llll}
1 & 3 & 1 & 0 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
0 & 2 & 1 & 0
\end{array}\right]=\left[\begin{array}{cccc}
4 & 12 & 4 & 0 \\
8 & 0 & 8 & 8 \\
8 & 4 & 0 & 4 \\
0 & 8 & 4 & 0
\end{array}\right] \\
& 2 S=2\left[\begin{array}{llll}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 2 \\
1 & 2 & 0 & 1 \\
1 & 1 & 2 & 0
\end{array}\right]=\left[\begin{array}{llll}
0 & 2 & 2 & 2 \\
2 & 0 & 2 & 4 \\
2 & 4 & 0 & 2 \\
2 & 2 & 4 & 0
\end{array}\right]
\end{aligned}
$$

Finally, add these two new matrices together by adding corresponding entries.

$$
4 B+2 S=\left[\begin{array}{cccc}
4 & 12 & 4 & 0 \\
8 & 0 & 8 & 8 \\
8 & 4 & 0 & 4 \\
0 & 8 & 4 & 0
\end{array}\right]+\left[\begin{array}{llll}
0 & 2 & 2 & 2 \\
2 & 0 & 2 & 4 \\
2 & 4 & 0 & 2 \\
2 & 2 & 4 & 0
\end{array}\right]=\left[\begin{array}{cccc}
4+0 & 12+2 & 4+2 & 0+2 \\
8+2 & 0+0 & 8+2 & 8+4 \\
8+2 & 4+4 & 0+0 & 4+2 \\
0+2 & 8+2 & 4+4 & 0+0
\end{array}\right]=\left[\begin{array}{cccc}
4 & 14 & 6 & 2 \\
10 & 0 & 10 & 12 \\
10 & 8 & 0 & 6 \\
2 & 10 & 8 & 0
\end{array}\right]
$$

To debrief these exercises, have groups present their solutions to the class. As they present, encourage them to explain how they created the new matrices and what they mean in this situation. Assign each group in your class to present a portion of these exercises. For example, have one or two groups present their solutions to Exercise 3, have another group present Exercises 4 and 5, and then have one or two groups present Exercise 6.

## Discussion (10 minutes)

Here you will introduce matrix arithmetic vocabulary and discuss how we might define matrix subtraction.

- How would you describe the process you used to create the matrix $10 B$ in Exercise 4?
- To create this matrix, multiply each element that represents the number of bus lines by 10.
- This operation is called scalar multiplication. Describe in words how to create the matrix $k A$ where $k$ is a real number and $A$ is a matrix.
- You would multiply each element in matrix $A$ by the real number $k$.
- How would you describe the process you used to create the matrix $B+S$ in Exercise 5?
- To create this matrix, you would count (or add) the corresponding entries for bus and subway routes to find the total between each pair of cities and then record these in a new matrix.
- This operation is called matrix addition. Describe in words how to create a matrix $A+B$ where $A$ and $B$ are matrices with equal dimensions.
- Add each entry in matrix $A$ to its corresponding entry in matrix $B$.
- Why would $A$ and $B$ need to have the same dimensions in order to find their sum?
- Because we have to add corresponding elements, the matrices $A$ and $B$ must have the same dimensions or the operation will not make sense.
- How would we use addition to represent the difference between 5 and 3?

> You add the opposite of the second number. For example, $5-3=5+(-3)=2$.

- How could we create the opposite of a matrix?
- Multiply it by the scalar -1.
- How could you produce a matrix $A-B$ if $A$ and $B$ are matrices with equal dimensions?
- You would simply subtract each element in matrix $B$ from its corresponding element in matrix $A$. More formally, you would compute the sum $A+(-1) B$.


## Scaffolding:

If students are having trouble describing the abstract operations (e.g. $k A$ ), you can provide them with concrete examples like the ones shown below.

- If $A=\left[\begin{array}{ll}2 & 2 \\ 3 & 1\end{array}\right]$, describe how to create $2 A, 3 A$, and $k A$.
- If $A=\left[\begin{array}{ll}2 & 2 \\ 3 & 1\end{array}\right]$ and $B=\left[\begin{array}{ll}2 & 1 \\ 5 & 0\end{array}\right]$, describe how to create $A+B$ and $-B$ and $A-B$.
You can also model using technology (graphing calculators or software) so students have a means to check their work or to help them to accurately calculate when working with larger matrices.


## Exercise 7 (5 minutes)

Now give students time to work with their groups to complete the first two rows of the graphic organizer in Exercise 7. If you would like, you can provide students with sample matrices to work from in the example column.

## Exercise 7

7. Complete this graphic organizer.

> Matrix Operations Graphic Organizer

| Operation | Symbols | Describe How to Calculate | Example Using $3 \times 3$ Matrices |
| :---: | :---: | :---: | :---: |
| Scalar <br> Multiplication | $k A$ | Multiply each element of matrix $A$ by the real number $k$. | $2\left[\begin{array}{lll}1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 0 & 0\end{array}\right]=\left[\begin{array}{lll}2 & 2 & 4 \\ 4 & 4 & 8 \\ 6 & 0 & 0\end{array}\right]$ |
| The Sum of Two Matrices | $A+B$ | Add corresponding elements in each row and column of $A$ and $B$. <br> Matrices $A$ and $B$ must have the same dimensions. | $\left[\begin{array}{lll}1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 0 & 0\end{array}\right]+\left[\begin{array}{lll}2 & 2 & 4 \\ 4 & 4 & 8 \\ 6 & 0 & 0\end{array}\right]=\left[\begin{array}{ccc}3 & 3 & 6 \\ 6 & 6 & 12 \\ 9 & 0 & 0\end{array}\right]$ |
| The Difference of Two Matrices | $\begin{gathered} A-B \\ =A+(-\mathbf{1}) B \end{gathered}$ | Subtract corresponding elements in each row and column of $A$ and $B$. The matrices must have the same dimensions. | $\begin{gathered} {\left[\begin{array}{lll} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 0 & 0 \end{array}\right]-\left[\begin{array}{lll} 2 & 2 & 4 \\ 4 & 4 & 8 \\ 6 & 0 & 0 \end{array}\right]=\left[\begin{array}{ccc} -1 & -1 & -2 \\ -2 & -2 & -4 \\ -3 & 0 & 0 \end{array}\right]} \\ \text { Or } \\ {\left[\begin{array}{lll} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 0 & 0 \end{array}\right]+(-1)\left[\begin{array}{lll} 2 & 2 & 4 \\ 4 & 4 & 8 \\ 6 & 0 & 0 \end{array}\right]=\left[\begin{array}{ccc} -1 & -1 & -2 \\ -2 & -2 & -4 \\ -3 & 0 & 0 \end{array}\right]} \end{gathered}$ |

## Closing ( 5 minutes)

Have students review their entries in Exercise 7 with a partner, and then ask one or two students to share their responses with the entire class. Take a minute to clarify any questions students have about the notation used in the Lesson Summary shown below. Be sure to emphasize that the processes of matrix addition, subtraction, and scalar multiplication apply to matrices that are not square, even though the examples in this section used only square matrices. We can add or subtract any two matrices that have the same dimensions, and we can multiply any matrix by a real number.

## Lesson Summary

Matrix Scalar Multiplication: Let $\boldsymbol{k}$ be a real number, and let $\boldsymbol{A}$ be an $\boldsymbol{m} \times \boldsymbol{n}$ matrix whose entry in row $\boldsymbol{i}$ and column $j$ is $a_{i, j}$. Then the scalar product $\boldsymbol{k} \cdot \boldsymbol{A}$ is the $\boldsymbol{m} \times \boldsymbol{n}$ matrix whose entry in row $i$ and column $j$ is $\boldsymbol{k} \cdot \boldsymbol{a}_{i, j}$.

Matrix Sum: Let $\boldsymbol{A}$ be an $\boldsymbol{m} \times \boldsymbol{n}$ matrix whose entry in row $\boldsymbol{i}$ and column $\boldsymbol{j}$ is $\boldsymbol{a}_{i, j}$, and let $\boldsymbol{B}$ be an $\boldsymbol{m} \times \boldsymbol{n}$ matrix whose entry in row $i$ and column $j$ is $\boldsymbol{b}_{i, j}$. Then the matrix sum $A+B$ is the $\boldsymbol{m} \times \boldsymbol{n}$ matrix whose entry in row $i$ and column $j$ is $a_{i, j}+b_{i, j}$.

Matrix Difference: Let $\boldsymbol{A}$ be an $\boldsymbol{m} \times \boldsymbol{n}$ matrix whose entry in row $\boldsymbol{i}$ and column $\boldsymbol{j}$ is $\boldsymbol{a}_{i, j}$, and let $\boldsymbol{B}$ be an $\boldsymbol{m} \times \boldsymbol{n}$ matrix whose entry in row $i$ and column $j$ is $b_{i, j}$. Then the matrix difference $A-B$ is the $\boldsymbol{m} \times \boldsymbol{n}$ matrix whose entry in row $i$ and column $j$ is $a_{i, j}-b_{i, j}$.

## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 2: Networks and Matrix Arithmetic

## Exit Ticket

The diagram below represents a network of highways and railways between three cities. Highways are represented by black lines, and railways are represented by red lines.


1. Create matrix $A$ that represents the number of major highways connecting the three cities and matrix $B$ that represents the number of railways connecting the three cities.
2. Calculate and interpret the meaning of each matrix in this situation.
a. $A+B$
b. $3 B$
3. Find $A-B$. Does the matrix $A-B$ have any meaning in this situation? Explain your reasoning.

## Exit Ticket Sample Solutions

The diagram below represents a network of highways and railways between three cities. Highways are represented by black lines, and railways are represented by red lines.


1. Create matrix $A$ that represents the number of major highways connecting the three cities and matrix $B$ that represents the number of railways connecting the three cities.

$$
\begin{aligned}
A & =\left[\begin{array}{lll}
0 & 1 & 2 \\
2 & 0 & 2 \\
3 & 1 & 0
\end{array}\right] \\
B & =\left[\begin{array}{lll}
0 & 0 & 2 \\
2 & 0 & 1 \\
1 & 1 & 0
\end{array}\right]
\end{aligned}
$$

2. Calculate and interpret the meaning of each matrix in this situation
a. $\quad A+B$
$A+B=\left[\begin{array}{lll}0 & 1 & 4 \\ 4 & 0 & 3 \\ 4 & 2 & 0\end{array}\right]$. The resulting matrix represents the numbers of ways that a person can travel between
the three cities by taking any of the major highway or railways.
b. $3 B$
$3 B=\left[\begin{array}{lll}0 & 0 & 6 \\ 6 & 0 & 3 \\ 3 & 3 & 0\end{array}\right]$. The resulting matrix means the number of trains between the three cities has tripled.
3. Find $-\boldsymbol{B}$. Does the matrix $\boldsymbol{A}-\boldsymbol{B}$ have any meaning in this situation? Explain your reasoning.
$A-B=\left[\begin{array}{lll}0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0\end{array}\right]$. The entries in matrix $A-B$ represent how many more routes there are by highway than there are by train between two cities. For example, the $\mathbf{2}$ in the third row and first column indicates that there are two more ways to get from City 3 to City 1 by highway than there are by rail.

## Problem Set Sample Solutions

1. For the matrices given below, perform each of the following calculations or explain why the calculation is not possible.

$$
\begin{array}{ll}
A=\left[\begin{array}{ll}
1 & 2 \\
0 & 1
\end{array}\right] & B=\left[\begin{array}{cc}
2 & 1 \\
-1 & 4
\end{array}\right] \\
C=\left[\begin{array}{ccc}
5 & 2 & 9 \\
6 & 1 & 3 \\
-1 & 1 & 0
\end{array}\right] & D=\left[\begin{array}{ccc}
1 & 6 & 0 \\
3 & 0 & 2 \\
1 & 3 & -2
\end{array}\right]
\end{array}
$$

a. $A+B$

$$
\left[\begin{array}{cc}
3 & 3 \\
-1 & 5
\end{array}\right]
$$

b. $2 A-B$

$$
\left[\begin{array}{cc}
0 & 3 \\
1 & -2
\end{array}\right]
$$

c. $A+C$

Matrices $A$ and $C$ cannot be added together because they do not have the same dimensions.
d. $-2 C$

$$
\left[\begin{array}{ccc}
-10 & -4 & -18 \\
-12 & -2 & -6 \\
2 & -2 & 0
\end{array}\right]
$$

e. $4 D-2 C$

$$
\left[\begin{array}{ccc}
-6 & 20 & -18 \\
0 & -2 & 2 \\
6 & 10 & -8
\end{array}\right]
$$

f. $3 B-3 B$

$$
\left[\begin{array}{ll}
0 & 0 \\
0 & 0
\end{array}\right]
$$

g. $5 B-C$

Matrix C cannot be subtracted from 5B because they do not have the same dimensions.
h. $B-3 A$

$$
\left[\begin{array}{cc}
-1 & -5 \\
-1 & 1
\end{array}\right]
$$

i. $C+10 D$

$$
\left[\begin{array}{ccc}
15 & 62 & 9 \\
36 & 1 & 23 \\
9 & 31 & -20
\end{array}\right]
$$

j. $\quad \frac{1}{2} C+D$

$$
\left[\begin{array}{ccc}
\frac{7}{2} & 7 & \frac{9}{2} \\
6 & \frac{1}{2} & \frac{7}{2} \\
\frac{1}{2} & \frac{7}{2} & -2
\end{array}\right]
$$

k. $\quad \frac{1}{4} B$

$$
\left[\begin{array}{cc}
\frac{1}{2} & \frac{1}{4} \\
-\frac{1}{4} & 1
\end{array}\right]
$$

I. $3 D-4 A$

Matrix 4A cannot be subtracted from 3D because they do not have the same dimensions.
m. $\frac{1}{3} B-\frac{2}{3} A$

$$
\left[\begin{array}{cc}
0 & -1 \\
-\frac{1}{3} & \frac{2}{3}
\end{array}\right]
$$

2. For the matrices given below, perform each of the following calculations or explain why the calculation is not possible.

$$
\begin{array}{ll}
A=\left[\begin{array}{lll}
1 & 2 & 1 \\
3 & 0 & 2
\end{array}\right] & B=\left[\begin{array}{ll}
2 & 1 \\
3 & 6 \\
1 & 0
\end{array}\right] \\
C=\left[\begin{array}{ccc}
1 & -2 & 3 \\
1 & 1 & 4
\end{array}\right] & D=\left[\begin{array}{cc}
2 & -1 \\
-1 & 0 \\
4 & 1
\end{array}\right]
\end{array}
$$

a. $\quad A+2 B$

Matrices $A$ and 2B cannot be added together because they do not have the same dimensions.
b. $2 A-C$

$$
\left[\begin{array}{ccc}
1 & 6 & -1 \\
5 & -1 & 0
\end{array}\right]
$$

c. $\quad A+C$

$$
\left[\begin{array}{lll}
2 & 0 & 4 \\
4 & 1 & 6
\end{array}\right]
$$

d. $-2 C$

$$
\left[\begin{array}{ccc}
-2 & 4 & -6 \\
-2 & -2 & -8
\end{array}\right]
$$

e. $4 D-2 C$

Matrix 2C cannot be subtracted from 4D because the matrices do not have the same dimensions.
f. $3 D-3 D$

$$
\left[\begin{array}{ll}
0 & 0 \\
0 & 0 \\
0 & 0
\end{array}\right]
$$

g. $5 B-D$

$$
\left[\begin{array}{cc}
8 & 6 \\
16 & 30 \\
1 & -1
\end{array}\right]
$$

h. $\quad C-3 A$

$$
\left[\begin{array}{ccc}
-2 & -8 & 0 \\
-8 & 1 & -2
\end{array}\right]
$$

i. $B+10 D$

$$
\left[\begin{array}{cc}
22 & -9 \\
-7 & 6 \\
41 & 10
\end{array}\right]
$$

j. $\quad \frac{1}{2} C+A$

$$
\left[\begin{array}{ccc}
\frac{3}{2} & 1 & \frac{5}{2} \\
\frac{7}{2} & \frac{1}{2} & 4
\end{array}\right]
$$

k. $\quad \frac{1}{4} B$
$\left[\begin{array}{cc}\frac{1}{2} & \frac{1}{4} \\ \frac{3}{4} & \frac{3}{2} \\ \frac{1}{4} & 0\end{array}\right]$
I. $3 A+3 B$

Matrices $3 A$ and $3 B$ cannot be added together because they do not have the same dimension.
m. $\frac{1}{3} B-\frac{2}{3} D$

$$
\left[\begin{array}{cc}
-\frac{2}{3} & 1 \\
\frac{5}{3} & 2 \\
-\frac{7}{3} & -\frac{2}{3}
\end{array}\right]
$$

3. Let

$$
A=\left[\begin{array}{cc}
3 & \frac{2}{3} \\
-1 & 5
\end{array}\right]
$$

and

$$
B=\left[\begin{array}{ll}
\frac{1}{2} & \frac{3}{2} \\
4 & 1
\end{array}\right]
$$

a. Let $C=6 A+6 B$. Find matrix $C$.

$$
C=\left[\begin{array}{ll}
21 & 13 \\
18 & 36
\end{array}\right]
$$

b. Let $\boldsymbol{D}=\mathbf{6}(\boldsymbol{A}+\boldsymbol{B})$. Find matrix $\boldsymbol{D}$.

$$
D=\left[\begin{array}{ll}
21 & 13 \\
18 & 36
\end{array}\right]
$$

c. What is the relationship between matrices $C$ and $D$ ? Why do you think that is?

The matrices are the same. Multiplying by a scalar appears to be distributive with matrices.
4. Let $A=\left[\begin{array}{cc}3 & 2 \\ -1 & 5 \\ 3 & -4\end{array}\right]$ and $X$ be a $3 \times 2$ matrix. If $A+X=\left[\begin{array}{cc}-2 & 3 \\ 4 & 1 \\ 1 & -5\end{array}\right]$, then find $X$.

$$
X=\left[\begin{array}{cc}
-5 & 1 \\
5 & -4 \\
-2 & -1
\end{array}\right]
$$

5. Let $A=\left[\begin{array}{lll}1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2\end{array}\right]$ and $B=\left[\begin{array}{lll}2 & 1 & 3 \\ 2 & 2 & 1 \\ 1 & 3 & 1\end{array}\right]$ represent the bus routes of two companies between three cities.
a. Let $C=A+B$. Find matrix $C$. Explain what the resulting matrix and entry $c_{1,3}$ mean in this context.
$C=\left[\begin{array}{lll}3 & 4 & 5 \\ 5 & 3 & 3 \\ 5 & 6 & 3\end{array}\right]$. Entry $c_{1,3}=5$ means that there are 5 ways to get from City 1 to City 3 using either bus company.
b. Let $D=B+A$. Find matrix $D$. Explain what the resulting matrix and entry $d_{1,3}$ mean in this context.
$D=\left[\begin{array}{lll}3 & 4 & 5 \\ 5 & 3 & 3 \\ 5 & 6 & 3\end{array}\right]$. Entry $d_{1,3}=5$ means that there are 5 ways to get from City 1 to City 3 using either bus company.
c. What is the relationship between matrices $C$ and $D$ ? Why do you think that is?

Matrices C and D are equal. When we add two matrices, we add the real numbers in the corresponding spots. When two real numbers are added, the order doesn't matter, so it makes sense that it doesn't matter the order in which we add matrices.
6. Suppose that April's Pet Supply has three stores in Cities 1,2, and 3. Ben's Pet Mart has two stores in Cities 1 and 2. Each shop sells the same type of dog crates in size 1 (small), 2 (medium), 3 (large), and 4 (extra large).
April's and Ben's inventory in each city are stored in the tables below.

|  | April's Pet Supply |  |  |
| :---: | :---: | :---: | :---: |
|  | City 1 | City 2 | City 3 |
| Size 1 | 3 | 5 | 1 |
| Size 2 | 4 | 2 | 9 |
| Size 3 | 1 | 4 | 2 |
| Size 4 | 0 | 0 | 1 |


|  | Ben's Pet Mart |  |
| :---: | :---: | :---: |
|  | City 1 | City 2 |
| Size 1 | 2 | 3 |
| Size 2 | 0 | 2 |
| Size 3 | 4 | 1 |
| Size 4 | 0 | 0 |

a. Create a matrix $A$ so that $\boldsymbol{a}_{i, j}$ represents the number of crates of size $\boldsymbol{i}$ available in April's store $\boldsymbol{j}$.

$$
A=\left[\begin{array}{lll}
3 & 5 & 1 \\
4 & 2 & 9 \\
1 & 4 & 2 \\
0 & 0 & 1
\end{array}\right]
$$

b. Explain how the matrix $B=\left[\begin{array}{lll}2 & 3 & 0 \\ 0 & 2 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]$ can represent the dog crate inventory at Ben's Pet Mart.

The first two columns represent the inventory in Ben's stores in Cities 1 and 2. Since there is no store in City 3. Ben has no inventory of dog crates in this city, which is represented by the zeros in the third column of matrix B.
c. Suppose that April and Ben merge their inventories. Find a matrix that represents their combined inventory in each of the three cities.
Matrix $A+B=\left[\begin{array}{lll}5 & 8 & 1 \\ 4 & 4 & 9 \\ 5 & 5 & 2 \\ 0 & 0 & 1\end{array}\right]$ represents the combined inventory in each of the three cities. For example, since
the entry in row 4 , column 1 is 0 , there are no extra large dog crates in either store in City 1. And, since the entry in row 1, column 2 is 8, there are a combined 8 small crates between the stores in City 2.
7. Jackie has two businesses she is considering buying and a business plan that could work for both. Consider the tables below, and answer the questions following.

|  | Horus's One-Stop Warehouse Supply |  |  | Re's 24-Hour Distributions |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | If business stays the same | If business improves as projected |  | If business stays the same | If business improves as projected |
| Expand to Multiple States | -\$75,000, 000 | \$45, 000, 000 | Expand to Multiple States | -\$99, 000,000 | \$62,500, 000 |
| Invest in Drone Delivery | -\$33, 000, 000 | \$30,000, 000 | Invest in Drone Delivery | -\$49,000,000 | \$29,000, 000 |
| Close and Sell Out | \$20,000, 000 | \$20,000, 000 | Close and Sell Out | \$35, 000, 000 | \$35, 000, 000 |

a. Create matrices $H$ and $R$ representing the values in the tables above such that the rows represent the different options and the columns represent the different outcomes of each option.

$$
\begin{aligned}
H & =\left[\begin{array}{cc}
-75 & 45 \\
-33 & 30 \\
20 & 20
\end{array}\right] \\
R & =\left[\begin{array}{cc}
-99 & 62.5 \\
-49 & 29 \\
35 & 35
\end{array}\right]
\end{aligned}
$$

b. Calculate $R-H$. What does $R-H$ represent?

$$
\left[\begin{array}{cc}
-99 & 62.5 \\
-49 & 29 \\
35 & 35
\end{array}\right]-\left[\begin{array}{cc}
-75 & 45 \\
-33 & 30 \\
20 & 20
\end{array}\right]=\left[\begin{array}{cc}
-24 & 17.5 \\
-16 & -1 \\
15 & 15
\end{array}\right]
$$

$R-H$ represents how much more money Jackie will make buying the second business instead of the first in each situation. Note that negative values mean that Jackie will lose more money with those choices.
c. Calculate $H+R$. What does $H+R$ represent?

$$
\left[\begin{array}{cc}
-75 & 45 \\
-33 & 30 \\
20 & 20
\end{array}\right]+\left[\begin{array}{cc}
-99 & 62.5 \\
-49 & 29 \\
35 & 35
\end{array}\right]=\left[\begin{array}{cc}
-174 & 107.5 \\
-82 & 59 \\
55 & 55
\end{array}\right]
$$

$H+R$ represents what could happen if Jackie buys both businesses.
d. Jackie estimates that the economy could cause fluctuations in her numbers by as much as $5 \%$ both ways. Find matrices to represent the best and worst case scenarios for Jackie.

$$
\begin{aligned}
& 0.95 H=\left[\begin{array}{cc}
-71.25 & 42.75 \\
-31.25 & 28.5 \\
19 & 19
\end{array}\right] \\
& 1.05 H=\left[\begin{array}{cc}
-78.75 & 47.25 \\
-34.65 & 31.5 \\
21 & 21
\end{array}\right] \\
& 0.95 R=\left[\begin{array}{cc}
-94.05 & 59.375 \\
-46.55 & 27.55 \\
33.25 & 33.25
\end{array}\right] \\
& 1.05 R=\left[\begin{array}{cc}
-103.95 & 65.625 \\
-51.45 & 30.45 \\
36.75 & 36.75
\end{array}\right]
\end{aligned}
$$

e. Which business should Jackie buy? Which of the three options should she choose? Explain your choices.

Answers will vary depending on how students feel about risk and income. There are two poor choices. If Jackie is planning to sell out the business she buys, then Re's business results in a guaranteed increase of $\$ 15$ million. Also, if Jackie is planning to invest in drones, then she should buy Horus's business since it is both less risky and results in a greater return.

# (Q. Lesson 3: Matrix Arithmetic in its Own Right 

## Student Outcomes

- Students will use matrices to represent data based on transportation networks.
- Students will multiply matrices of appropriate dimensions and interpret the meaning of this arithmetic in terms of transportation networks.


## Lesson Notes

This lesson builds on the work in the previous lesson by considering a transportation network with two different types of public transportation. This situation will be used to help students discover and define multiplication of two matrices (N-VM.C.8). Students will understand the meaning of this matrix arithmetic. They will begin to discover that matrix multiplication is not commutative. Matrix arithmetic and the properties of matrix arithmetic will be explored further in this lesson as well. Throughout this lesson, students will be making sense of transportation network diagrams and matrices (MP.1), reasoning about contextual and abstract situations (MP.2), and using matrices as tools to represent network diagrams (MP.5) with care and precision (MP.6).

## Classwork

## Opening Exercise (5 minutes)

Students will begin to explore the concept of matrix multiplication by calculating the total number of ways to travel from City 2 to City 1 if the first leg of the trip is by bus and the second leg is by subway. This activity will help students understand how to determine the number of available routes that use two modes of transportation in a particular order. The table will help students understand how to calculate the elements in the product of two matrices and to understand the meaning of that product.

- What do the rows in this table represent?
- Each row is the number of ways to travel from City 2 to City 1 via one of the four cities.
- Why are the total ways to travel between City 2 to City 1 through City 1 equal to 0 ?
- The exercise asks us to find the routes from City 2 to City 1 with the first leg being by bus and the second leg being by subway. There is one way to travel from City 2 to City 1 via bus, but there are no ways to travel from City 1 to City 1 by subway. That would require the diagram to have a dashed loop from City 1 to City 1. Therefore, there are no routes from City 2 to City 1 that use a bus for the first leg of the trip and the subway for the second leg.
- How did you complete the table below?
- We counted the routes between each city and then multiplied the number of bus routes by the number of subway routes. Finally, we had to add all the numbers in the last column to determine the total possible routes.
- How does this table ensure that you counted all the possible routes?
- We counted all the bus routes from City 2 to another city (including itself), and then we counted all the subway routes from one of the four cities to City 1. We considered all possible paths.


## Opening Exercise

The subway and bus line network connecting four cities that we used in Lesson $\mathbf{2}$ is shown below. The bus routes connecting the cities are represented by solid lines, and the subway routes are represented by dashed lines.


## Scaffolding:

- Challenge advanced students to create a matrix to represent this network.
- Struggling students can use the diagram below that contains fewer bus routes and subway lines.


Suppose we want to travel from City 2 to City $\mathbf{1}$, first by bus and then by subway, with no more than one connecting stop.
a. Complete the chart below showing the number of ways to travel from City 2 to City 1 using first a bus and then the subway. The first row has been completed for you.

| First Leg (BUS) |  | Second Leg (SUBWAY) |  | Total Ways to Travel |
| :--- | :--- | :--- | :---: | :---: |
| City 2 to City 1: | 2 | City 1 to City 1: | 0 | $2 \cdot 0=0$ |
| City 2 to City 2: | 0 | City 2 to City 1: | 1 | $0 \cdot 1=0$ |
| City 2 to City 3: | 2 | City 3 to City 1: | 1 | $2 \cdot 1=2$ |
| City 2 to City 4: | 2 | City 4 to City 1: | 1 | $2 \cdot 1=2$ |

b. How many ways are there to travel from City 2 to City 1, first on a bus and then on a subway? How do you know?

The total number of ways is the sum numbers in the last column. There are 4 ways.

## Exploratory Challenge/Exercises 1-12 ( $\mathbf{2 0}$ minutes): The Meaning of Matrix Multiplication

This challenge continues to introduce the notion of matrix multiplication in a meaningful situation. Students will consider travel routes if we use two different means of transportation in a particular order and have at most one connecting city for all pairs of cities.

## Exploratory Challenge/Exercises 1-12: The Meaning of Matrix Multiplication

Suppose we want to travel between all cities, traveling first by bus and then by subway, with no more than one connecting stop.

1. Use a chart like the one in the Opening Exercise to help you determine the total number of ways to travel from City 1 to City 4 using first a bus and then the subway.

| First Leg (BUS) |  | Second Leg (SUBWAY) |  | Total Ways to Travel |
| :--- | :--- | :--- | :--- | :---: |
| City 1 to City 1: | 1 | City 1 to City 4: | 1 | $1 \cdot 1=1$ |
| City 1 to City 2: | 3 | City 2 to City 4: | 2 | $3 \cdot 2=6$ |
| City 1 to City 3: | 1 | City 3 to City 4: | 1 | $1 \cdot 1=1$ |
| City 1 to City 4: | 0 | City 4 to City 4: | 0 | $0 \cdot 0=0$ |

Summing the entries in the last column shows that there are eight different ways to travel from City 1 to City 4 using first a bus and then the subway.
2. Suppose we create a new matrix $P$ to show the number of ways to travel between the cities, first by bus and then by subway, with no more than one connecting stop. Record your answers to Opening Exercise, part (b) and Exercise 1 in this matrix in the appropriate row and column location. We do not yet have enough information to complete the entire matrix. Explain how you decided where to record these numbers in the matrix shown below.

$$
\begin{aligned}
& P=\left[\begin{array}{llll}
- & - & - & - \\
- & - & - & - \\
- & - & - & - \\
- & - & - & -
\end{array}\right] \\
& P=\left[\begin{array}{llll}
-\frac{1}{4} & - & - & 8 \\
- & - & - & - \\
- & - & - & -
\end{array}\right]
\end{aligned}
$$

If the rows represent the starting city and the columns represent the destination city, then the ways to travel from City 2 to City 1 would be $p_{2,1}$, and the ways to travel from City 1 to City 4 would be $p_{1,4}$.

At this point, circulate around the room to check for understanding by making sure that groups are recording the correct entries in the proper locations in the matrix. Debrief as a whole class if a majority of students are struggling. You may want to review matrix notation, especially since students are going to use it in upcoming exercises. For example, if $P$ is a matrix then $p_{i, j}$ is the element in the $i^{\text {th }}$ row and $j^{\text {th }}$ column of the matrix. In Exercise 5 , some groups may continue to use a table, while some may simply create a list. Make sure that students are recording a written explanation of how they calculated their answers. If students can describe the process of multiplying and summing verbally or in writing, then they will be able to calculate the product of two matrices without difficulty.
3. What is the total number of ways to travel from City 3 to City 2 first by bus and then by subway with no more than one connecting stop? Explain how you got your answer and where you would record it in matrix $P$ above.

Find the ways to travel by bus from City 3 to each of the other cities, including City 3 itself. Then for each city, find the ways to travel by subway from that city to City 2. Multiply the number of ways from City 2 to the connecting city by the number of ways from the connecting city to City 2. Finally, add the result of the number of ways through each connecting city.

$$
\begin{array}{ll}
\text { City } 3 \text { to City } 2 \text { via City 1: } & 2 \cdot 1=2 \\
\text { City } 3 \text { to City } 2 \text { via City 2: } & 1 \cdot 0=0 \\
\text { City } 3 \text { to City } 2 \text { via City 3: } & 0 \cdot 2=0 \\
\text { City } 3 \text { to City } 2 \text { via City 4: } & 1 \cdot 1=1
\end{array}
$$

There are $2+0+0+1=3$ ways to travel from City 3 to City 2, using first a bus and then the subway. This would be entered in the $3^{\text {rd }}$ row and $2^{\text {nd }}$ column of matrix $P$.

Recall matrix $B$, which shows the number of bus lines connecting the cities in this transportation network, and matrix $S$, which represents the number of subway lines connecting the cities in this transportation network.

$$
B=\left[\begin{array}{llll}
1 & 3 & 1 & 0 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
0 & 2 & 1 & 0
\end{array}\right] \text { and } S=\left[\begin{array}{llll}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 2 \\
1 & 2 & 0 & 1 \\
1 & 1 & 2 & 0
\end{array}\right]
$$

4. What does the product $b_{1,2} s_{2,4}$ represent in this situation? What is the value of this product?

The product $b_{1,2} s_{2,4}$ represents the number of ways to travel from City 1 to City 4 via City 2, first by bus and then by subway.

$$
b_{1,2} s_{2,4}=3 \cdot 2=6
$$

5. What does $b_{1,4} s_{4,4}$ represent in this situation? What is the value of this product? Does this make sense?

The product $b_{1,4} s_{4,4}$ represents the number of ways to travel from City 1 to City 4 via City 4, first by bus and then by subway.

$$
b_{1,4} s_{4,4}=0 \cdot 0=0
$$

Since there is no bus route from City 1 to City 4 and no subway route from City 4 to City 4, it is impossible to travel from City 1 to City 4 using first a bus and then a subway. Thus, there are no possible routes.
6. Calculate the value of the expression $b_{1,1} s_{1,4}+b_{1,2} s_{2,4}+b_{1,3} s_{3,4}+b_{1,4} s_{4,4}$. What is the meaning of this expression in this situation?

$$
b_{1,1} s_{1,4}+b_{1,2} s_{2,4}+b_{1,3} s_{3,4}+b_{1,4} s_{4,4}=1 \cdot 1+3 \cdot 2+1 \cdot 1+0 \cdot 0=8
$$

Thus, there are 8 total ways to travel from City 1 to City 4 via any one of the cities, first by bus and then by subway.
7. Circle the first row of $B$ and the second column of $S$. How are these entries related to the expression above and your work in Exercise 1?

We multiplied the first element in the row by the first element in the column, the second by the second, third by third and so on, and then we found the sum of those products. This is the same as the sum of the entries in the last column of the table in Exercise 1.
8. Write an expression that represents the total number of ways you can travel between City 2 and City 1 , first by bus and then by subway, with no more than one connecting stop. What is the value of this expression? What is the meaning of the result?

$$
b_{2,1} s_{1,1}+b_{2,2} s_{2,1}+b_{2,3} s_{3,1}+b_{2,4} s_{4,1}=2 \cdot 0+0 \cdot 1+2 \cdot 1+2 \cdot 1=4
$$

Thus, there are 4 total ways to travel from City 2 to City 1 via any one of the cities, first by bus and then by subway.
9. Write an expression that represents the total number of ways you can travel between City 4 and City 1, first by bus and then by subway, with no more than one connecting stop. What is the value of this expression? What is the meaning of the result?

$$
b_{4,1} s_{1,1}+b_{4,2} s_{2,1}+b_{4,3} s_{3,1}+b_{4,4} s_{4,1}=0 \cdot 0+2 \cdot 1+1 \cdot 1+0 \cdot 1=3
$$

Thus, there are 3 total ways to travel from City 4 to City 1 via any one of the cities, first by bus and then by subway.

At this point, you should stop and debrief. Have several students share their results with the whole class. Have different groups present different portions of the Exploratory Challenge on the board or on a document camera. Use these questions that follow to focus the discussion. Have students answer them with a partner or in their small groups before asking for volunteers to share the answer with the entire class.

- What does each element of matrix $P$ represent?
- Entry $p_{i, j}$ represents the number of ways to travel from the $i^{\text {th }}$ city to the $j^{\text {th }}$ city, changing from a bus to the subway one time.
- What patterns do you notice in the expression in Exercise 6 and the expressions you wrote for Exercises 8 and 9 ?
- In the indices, there is a pattern where the first number is the starting city and the last number is the ending city, and the middle indices are the same.
- Complete this sentence: To calculate the element of matrix $P$ in the $2^{\text {nd }}$ row and $4^{\text {th }}$ column, you would...
- Multiply each element in the $2^{\text {nd }}$ row of $B$ by each corresponding element in the $4^{\text {th }}$ column of $S$, and then add the resulting products.
- Complete the sentence: The element of matrix $P$ in the $4^{\text {th }}$ row and $2^{\text {nd }}$ column represents the number of ways to travel...
- From City 4 to City 2 first via bus and then changing to a subway in the connecting city.
- Describe how to calculate any element in matrix $P$.
- You multiply each element in the $i^{\text {th }}$ row of the first matrix by the corresponding element in the $j^{\text {th }}$ column of the second matrix, and then add the resulting products.

The last two exercises have students construct the complete matrix. Allow them time to work through the process. There are many opportunities for careless errors, so taking the time to be accurate and precise is essential (MP.6).
10. Complete matrix $P$ that represents the routes connecting the four cities if you travel first by bus and then by subway.

$$
P=\left[\begin{array}{llll}
4 & 3 & 4 & 8 \\
4 & 8 & 6 & 4 \\
2 & 3 & 5 & 4 \\
3 & 2 & 2 & 5
\end{array}\right]
$$

11. Construct a matrix $M$ that represents the routes connecting the four cities if you travel first by subway and then by bus.

$$
M=\left[\begin{array}{llll}
4 & 3 & 3 & 3 \\
3 & 8 & 3 & 1 \\
5 & 5 & 6 & 4 \\
7 & 5 & 3 & 4
\end{array}\right]
$$

12. Should these two matrices be the same? Explain your reasoning.

This transportation network has different numbers of routes connecting cities in each direction. Traveling from City 1 to City 4 will not have the same number of options as traveling from City 4 to City 1. Therefore, the order in which you select your method of travel makes a difference, which is why the matrices $M$ and $P$ are not equal.

## Discussion (5 minutes)

Use this time to finish debriefing the exercises in the Exploratory Challenge. Define and describe the product of two matrices more formally, and give students an opportunity to take notes on the process.

- During this Exploratory Challenge, you have been calculating what mathematicians define to be the product of two matrices. Does this matrix arithmetic operation seem to relate back to real number multiplication in any way?
- We are working with an array and doing some multiplying when working with real number multiplication. We do not really find products and then add the results.
- Our examples used square $4 \times 4$ matrices, but matrix multiplication can also be applied to matrices of different dimensions. Consider matrices $A=\left[\begin{array}{ll}1 & 2 \\ 3 & 1\end{array}\right]$ and $B=\left[\begin{array}{ccc}2 & 1 & 0 \\ 1 & -2 & 1\end{array}\right]$. Would it make sense to compute $A \cdot B$ ? Why or why not?
- We could compute $A \cdot B$, and we would get $A \cdot B=\left[\begin{array}{lll}1 \cdot 2+2 \cdot 1 & 1 \cdot 1+2 \cdot(-2) & 1 \cdot 0+2 \cdot 1 \\ 3 \cdot 2+1 \cdot 1 & 3 \cdot 1+1 \cdot(-2) & 3 \cdot 0+1 \cdot 1\end{array}\right]$, so

$$
A \cdot B=\left[\begin{array}{ccc}
4 & -3 & 2 \\
7 & 1 & 1
\end{array}\right]
$$

- What about the product $B \cdot A$ ?
- Our multiplication process fails if we try to compute $B \cdot A$ because there are a different number of entries in the columns of $B$ than in the rows of $A$.
- You described how to calculate each element of the product matrix in words. Using symbols, if $P=A \cdot B$, then element $p_{i, j}=a_{i, 1} b_{1, j}+a_{i, 2} b_{2, j}+a_{i, 3} b_{3, j}+\cdots+a_{i, n} b_{m, j}$, where $n$ is the number of columns in $A$ and $m$ is the number of rows in $B$. What must be true about the dimensions of two matrices if we wish to find their product?
- This implies that you can only multiply two matrices if the number of columns in the first matrix is equal to the number of rows in the second matrix.

Have students record this information in their notes to show examples of the product of two matrices. Have them color code a few entries as shown below to highlight the process.

$$
B \cdot S=P
$$

$$
\begin{aligned}
& {\left[\begin{array}{llll}
1 & 3 & 1 & 0 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
0 & 2 & 1 & 0
\end{array}\right] \cdot\left[\begin{array}{llll}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 2 \\
1 & 2 & 0 & 1 \\
1 & 2 & 0
\end{array}\right]=\left[\begin{array}{llll}
4 & 3 & 4 & 8 \\
4 & 8 & 6 & 4 \\
2 & 3 & 5 & 4 \\
3 & 2 & 2 & 5
\end{array}\right]} \\
& p_{1,2}=1 \cdot 1+3 \cdot 0+1 \cdot 2+0 \cdot 1=3 \\
& S \cdot B=M \\
& {\left[\begin{array}{llll}
0 & 1 & 1 & 1 \\
1 & 0 & 1 & 2 \\
1 & 2 & 0 & 1 \\
1 & 1 & 2 & 0
\end{array}\right]\left[\begin{array}{llll}
1 & 3 & 1 & 0 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
0 & 2 & 1 & 0
\end{array}\right]=\left[\begin{array}{llll}
4 & 3 & 3 & 3 \\
3 & 8 & 3 & 1 \\
5 & 5 & 6 & 4 \\
7 & 5 & 3 & 4
\end{array}\right]}
\end{aligned}
$$

- When we multiply two matrices together, such as an $m \times n$ matrix by an $n \times p$ matrix, what is the size of the resulting matrix?
- The resulting matrix has size $m \times p$.

- In Exercises 10 and 11, you showed that matrix $P$ and matrix $M$ were not the same. In real number arithmetic, the operation of multiplication is commutative. What does that mean?
- It means that $a \cdot b=b \cdot a$ when $a$ and $b$ are real numbers.
- Is matrix multiplication commutative? Explain your reasoning.
- The results of Exercises 10-12 and the meaning of the product matrix in this situation implies that even if the two matrices being multiplied are square matrices (equal number of rows and columns), matrix multiplication is not commutative.


## Exercises 13-16 (5 minutes)

These exercises talk about arithmetic operations with matrices. Students are beginning to think abstractly about the properties of matrices and how they compare to properties of real numbers. Three by three matrices are used in the example, and square matrices are used in these exercises, but advanced students could be asked to show that these properties hold regardless of the size of the matrices, as long as added matrices are of equal dimensions and multiplied matrices have appropriate dimensions.

Exercises 13-16
13. Let $A=\left[\begin{array}{ccc}2 & 3 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0\end{array}\right]$
a. Construct a matrix $Z$ such that $A+Z=A$. Explain how you got your answer.

$$
Z=\left[\begin{array}{lll}
0 & 0 & 0 \\
\mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & 0
\end{array}\right]
$$

We would need to add 0 to each element of $A$ to get the same matrix, so each element of $Z$ must be equal to 0 . Also, $Z$ must have the same dimensions as $A$ so we can add corresponding elements.
b. Explain why $\boldsymbol{k} \cdot \boldsymbol{Z}=\boldsymbol{Z}$ for any real number $\boldsymbol{k}$.

Any number multiplied by 0 is equal to $\mathbf{0}$. Scalar multiplication multiplies each element of $Z$ by $\boldsymbol{k}$.
c. The real number 0 has the properties that $a+0=0$ and $a \cdot 0=0$ for all real numbers $a$. Why would mathematicians call $Z$ a zero matrix?

The matrix $Z$ has the same properties in matrix addition as the real number 0 has with real number addition.
14. Suppose each city had a trolley car that ran a route between tourist destinations. The blue loops represent the trolley car routes. Remember that straight lines indicate bus routes, and dotted lines indicate subway routes.

Lesson 3:
Date:
15. In this lesson you learned that the commutative property does not hold for matrix multiplication. This exercise asks you to consider other properties of real numbers applied to matrix arithmetic.
a. Is matrix addition associative? That is, does $(A+B)+C=A+(B+C)$ for matrices $A, B$, and $C$ that have the same dimensions? Explain your reasoning.

Yes. Because all three matrices have the same dimensions, we will add corresponding entries. The entries are real numbers. Since the associative property holds for adding real numbers, it would make sense to hold for the addition of matrices.
b. Is matrix multiplication associative? That is, does $(A \cdot B) \cdot C \cdot A \cdot(B \cdot C)$ for matrices $A, B$, and $C$ for which the multiplication is defined? Explain your reasoning.

Yes. As long as the dimensions of the matrices are such that the multiplications are all defined, computing the products requires that we add and multiply real numbers. Since these operations are associative for real numbers, multiplication will be associative for matrices. It would be like finding the number of routes if we had three modes of transportation and changed modes twice. We can count those totals in different groupings as long as we maintain the order of the transportation modes (e.g. bus to subway to train).
c. Is matrix addition commutative? That is, does $A+B=B+A$ for matrices $A$ and $B$ with the same dimensions?

Yes, because addition of the individual elements is commutative.

| Lesson 3: | Matrix Arithmetic in its Own Right |
| :--- | :--- |
| Date: | $1 / 24 / 15$ |

16. Complete the graphic organizer to summarize your understanding of the product of two matrices.

| Operation | Symbols | Describe How to Calculate | Example Using $3 \times 3$ Matrices |
| :--- | :--- | :--- | :--- |
|  |  | To find the element in the $i^{\text {th }}$ <br> row and $j^{\text {th }}$ column of the |  |
| product matrix, multiply |  |  |  |
| corresponding elements from |  |  |  |
| Matrix |  |  |  |
| Multiplication | $\boldsymbol{A} \cdot \boldsymbol{B}$ | the $i^{\text {th }}$ row of the first matrix by <br> the $j^{\text {th }}$ column of the second <br> matrix, and then add the <br> results. Repeat this process for <br> each element in the product <br> matrix. | $\left[\begin{array}{ccc}2 & 0 & 2 \\ 2 & 2 & -1 \\ 1 & -3 & 0\end{array}\right]\left[\begin{array}{ccc}1 & 1 & 2 \\ 2 & 2 & 4 \\ 3 & 0 & 0\end{array}\right]=\left[\begin{array}{ccc}8 & 2 & 4 \\ 3 & 6 & 12 \\ -5 & -5 & -10\end{array}\right]$ |

## Closing ( 5 minutes)

Have students review their graphic organizer entries in Exercise 16 with a partner, and then ask one or two students to share their responses with the entire class. Take a minute to clarify any questions students have about the notation used in the Lesson Summary shown below.

## Lesson Summary

MATRIX Product: Let $\boldsymbol{A}$ be an $\boldsymbol{m} \times \boldsymbol{n}$ matrix whose entry in row $i$ and column $\boldsymbol{j}$ is $\boldsymbol{a}_{i, j}$, and let $B$ be an $\boldsymbol{n} \times \boldsymbol{p}$ matrix whose entry in row $i$ and column $j$ is $b_{i, j}$. Then the matrix product $A B$ is the $m \times p$ matrix whose entry in row $i$ and column $j$ is $a_{i, 1} b_{1, j}+a_{i, 2} b_{2, j}+\cdots+a_{i, n} b_{n, j}$.

Identity Matrix: The $\boldsymbol{n} \times \boldsymbol{n}$ identity matrix is the matrix whose entry in row $\boldsymbol{i}$ and column $\boldsymbol{i}$ for $\mathbf{1} \leq \boldsymbol{i} \leq \boldsymbol{n}$ is $\mathbf{1}$, and whose entries in row $\boldsymbol{i}$ and column $\boldsymbol{j}$ for $\mathbf{1} \leq \boldsymbol{i}, \boldsymbol{j} \leq \boldsymbol{n}$ and $\boldsymbol{i} \neq \boldsymbol{j}$ are all zero. The identity matrix is denoted by $I$. The $2 \times 2$ identity matrix is $\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$, and the $3 \times 3$ identity matrix is $\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$. If the size of the identity matrix is not explicitly stated, then the size is implied by context.

Zero Matrix: The $\boldsymbol{m} \times \boldsymbol{n}$ zero matrix is the $\boldsymbol{m} \times \boldsymbol{n}$ matrix in which all entries are equal to zero.
For example, the $2 \times 2$ zero matrix is $\left[\begin{array}{ll}\mathbf{0} & \mathbf{0} \\ 0 & 0\end{array}\right]$, and the $3 \times 3$ zero matrix is $\left[\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$. If the size of the zero matrix is not specified explicitly, then the size is implied by context.

## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 3: Matrix Arithmetic in its Own Right

## Exit Ticket

Matrix $A$ represents the number of major highways connecting three cities. Matrix $B$ represents the number of railways connecting the same three cities.

$$
A=\left[\begin{array}{lll}
0 & 3 & 0 \\
2 & 0 & 2 \\
1 & 1 & 2
\end{array}\right] \text { and } B=\left[\begin{array}{lll}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{array}\right]
$$

1. Draw a network diagram for the transportation network of highways and railways between these cities. Use solid lines for highways and dotted lines for railways.


2

3
2. Calculate and interpret the meaning of each matrix in this situation.
a. $A \cdot B$
b. $B \cdot A$
3. In this situation, why does it make sense that $A \cdot B \neq B \cdot A$ ?

## Exit Ticket Sample Solutions

Matrix $A$ represents the number of major highways connecting three cities. Matrix $B$ represents the number of railways connecting the same three cities.

$$
A=\left[\begin{array}{lll}
0 & 3 & 0 \\
2 & 0 & 2 \\
1 & 1 & 2
\end{array}\right] \text { and } B=\left[\begin{array}{lll}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{array}\right]
$$

1. Draw a network diagram for the transportation network of highways and railways between these cities. Use solid lines for highways and dotted lines for railways.

2. Calculate and interpret the meaning of each matrix in this situation.
a. $\quad \boldsymbol{A} \cdot \boldsymbol{B}$
$A B=\left[\begin{array}{lll}3 & 0 & 3 \\ 2 & 4 & 2 \\ 3 & 3 & 2\end{array}\right]$. It indicates the number of ways to get around between three cities by taking highways first and then railways second.
b. $\quad B \cdot A$
$B A=\left[\begin{array}{lll}3 & 1 & 4 \\ 1 & 4 & 2 \\ 2 & 3 & 2\end{array}\right]$. It indicates the number of ways to get around between three cities by taking railways first and then highways second.
3. In this situation, why does it make sense that $A \cdot B \neq B \cdot A$ ?

It makes sense because both transportation networks have a different number of routes connecting cities in each direction. Not every route goes both directions. For example, there are a different number of possible routes from City 1 to City 2 than from City 2 to City 1.

## Problem Set Sample Solutions

1. Let $A=\left[\begin{array}{ll}1 & 3 \\ 2 & 0\end{array}\right]$ and $B=\left[\begin{array}{ll}1 & 2 \\ 4 & 3\end{array}\right]$ represent the bus routes of two companies between 2 cities. Find the product $A \cdot B$, and explain the meaning of the entry in row 1 , column 2 of $A \cdot B$ in the context of this scenario.

The product is $A \cdot B=\left[\begin{array}{ll}1 & 3 \\ 2 & 0\end{array}\right] \cdot\left[\begin{array}{ll}1 & 2 \\ 4 & 3\end{array}\right]=\left[\begin{array}{cc}13 & 11 \\ 2 & 4\end{array}\right]$. The entry in row 1, column 2 to 11, which means that there are 11 possible routes from the first city to the second city, taking first a bus from Company $A$ and then a bus from Company B.
2. Let $A=\left[\begin{array}{lll}1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2\end{array}\right]$ and $B=\left[\begin{array}{lll}2 & 1 & 3 \\ 2 & 2 & 1 \\ 1 & 3 & 1\end{array}\right]$ represent the bus routes of two companies between three cities.
a. Let $C=A \cdot B$. Find matrix $C$, and explain the meaning of entry $c_{1,3}$.

The product is $C=\left[\begin{array}{ccc}10 & 13 & 8 \\ 10 & 11 & 12 \\ 16 & 16 & 17\end{array}\right]$, and $c_{1,3}=8$ means that there are 8 different ways to travel to City 3 from
City 1 by taking a bus from Company $A$ and then a bus from Company B.
b. Nina wants to travel from City 3 to City 1 and back home to City $\mathbf{3}$ by taking a direct bus from Company $\mathbf{A}$ on the way to City 1 and a bus from Company B on the way back home to City 3. How many different ways are there for her to make this trip?

Since $C=A \cdot B=\left[\begin{array}{ccc}10 & 13 & 8 \\ 10 & 11 & 12 \\ 16 & 16 & 17\end{array}\right]$, and Nina wants to travel from City 3 back to City 3, entry $c_{3,3}=17$ means that she has 17 ways to make the trip.
c. Oliver wants to travel from City 2 to City $\mathbf{3}$ by taking first a bus from Company $\mathbf{A}$ and then taking a bus from Company B. How many different ways can he do this?

Since $C=A \cdot B=\left[\begin{array}{ccc}10 & 13 & 8 \\ 10 & 11 & 12 \\ 16 & 16 & 17\end{array}\right]$, and Oliver wants to travel from City 2 to City 3 , he has $c_{2,3}=12$ ways to make the trip.
d. How many routes can Oliver choose from if travels from City 2 to City $\mathbf{3}$ by first taking a bus from Company B and then taking a bus from Company A?

Since $B \cdot A=\left[\begin{array}{ccc}17 & 16 & 12 \\ 12 & 11 & 10 \\ 14 & 9 & 10\end{array}\right]$, and the entry in row 2 , column 3 is 10 , which means that there are 10 ways to get from City 2 to City 3 by taking a bus from Company B first and then a bus from Company A.
3. Recall the bus and trolley matrices from the lesson:

$$
B=\left[\begin{array}{llll}
1 & 3 & 1 & 0 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
0 & 2 & 1 & 0
\end{array}\right] \text { and } I=\left[\begin{array}{llll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
$$

a. Explain why it makes sense that $B I=I B$ in the context of the problem.

Since there is only one way to take a trolley in each city, no matter whether you take the trolley before you take the bus or if you take the bus first, the result is the same number of ways as there are bus routes.
b. Multiply out $B I$ and $I B$ to show $B I=I B$.

$$
B I=\left[\begin{array}{llll}
1 & 3 & 1 & 0 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
0 & 2 & 1 & 0
\end{array}\right], I B=\left[\begin{array}{llll}
1 & 3 & 1 & 0 \\
2 & 0 & 2 & 2 \\
2 & 1 & 0 & 1 \\
0 & 2 & 1 & 0
\end{array}\right]
$$

c. Consider the multiplication that you did in part (b). What about the arrangement of the entries in the identity matrix causes $B I=B$ ?

When you multiply BI, you are multiplying the rows of $B$ by the columns of $I$; since every entry in the columns of I but the row that you are currently multiplying by is 0 , you only get a single value of $B$ to carry over, and it is carried over in the same position. A similar thing happens with IB, but this time it is because the rows of the identity are zero everywhere except at the position you want to carry over.
4. Consider the matrices

$$
A=\left[\begin{array}{ccc}
3 & 1 & -\frac{1}{2} \\
2 & \frac{2}{3} & 4
\end{array}\right] \text { and } B=\left[\begin{array}{cc}
1 & 0 \\
0 & 1 \\
0 & 0
\end{array}\right]
$$

a. Multiply $A B$ and $B A$ or explain why you cannot.

$$
A B=\left[\begin{array}{cc}
3 & \frac{1}{2} \\
2 & \frac{2}{3}
\end{array}\right], B A=\left[\begin{array}{ccc}
3 & 1 & -\frac{1}{2} \\
2 & \frac{2}{3} & 4 \\
0 & 0 & 0
\end{array}\right]
$$

b. Would you consider $B$ to be an identity matrix for $A$ ? Why or why not?

No. $A B$ has all of the same entries as $A$ for those entries that $A B$ has, but $A$ possesses an additional column that $A B$ does not. $B A$ possesses all of the same entries as $A$ for those that exist in $A$, but it also possesses a row of zeros that did not exist in $A$; all three matrices have different dimensions.
c. Would you consider $I_{2}=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$ or $I_{3}=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$ an identity matrix for $A$ ? Why or why not?

Answers may vary. It is true that $I_{2} A=A$ and $A I_{3}=A$, but neither of these can commute with $A$ based on their dimensions. We can say that $I_{2}$ is an identity for $A$ on the left, and $I_{3}$ is an identity for $A$ on the right.
5. We've shown that matrix multiplication is generally not commutative, meaning that as a general rule for two matrices $A$ and $B, A \cdot B \neq B \cdot A$. Explain why $F \cdot G=G \cdot F$ in each of the following examples.
a. $\quad F=\left[\begin{array}{ll}1 & 3 \\ 2 & 0\end{array}\right], \quad G=\left[\begin{array}{ll}2 & 6 \\ 4 & 0\end{array}\right]$.

We see that $F \cdot G=\left[\begin{array}{cc}14 & 6 \\ 4 & 12\end{array}\right]$ and $G \cdot F=\left[\begin{array}{cc}14 & 6 \\ 4 & 12\end{array}\right]$. Because $G=2 F$, we have
$\boldsymbol{F} \cdot \boldsymbol{G}=\boldsymbol{F} \cdot(2 \boldsymbol{F})=2(\boldsymbol{F} \cdot \boldsymbol{F})=(2 \boldsymbol{F}) \cdot \boldsymbol{F}=\boldsymbol{G} \cdot \boldsymbol{F}$.
b. $\quad F=\left[\begin{array}{lll}1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2\end{array}\right], \quad G=\left[\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$.

We see that $F \cdot G=\left[\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$ and $G \cdot F=\left[\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$. The matrix $G$ is the zero matrix, and any matrix multiplied by a zero matrix will result in the zero matrix.
c. $\quad F=\left[\begin{array}{lll}1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2\end{array}\right], \quad G=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$.

We see that $F \cdot G=\left[\begin{array}{lll}1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2\end{array}\right]$ and $G \cdot F=\left[\begin{array}{lll}1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2\end{array}\right]$. The matrix $G$ is an identity matrix. Any square
matrix multiplied by an identity matrix will result in the original matrix.
d. $\quad F=\left[\begin{array}{lll}1 & 3 & 2 \\ 3 & 1 & 2 \\ 4 & 3 & 2\end{array}\right], \quad G=\left[\begin{array}{lll}3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3\end{array}\right]$.

We see that $F \cdot G=\left[\begin{array}{ccc}3 & 9 & 6 \\ 9 & 3 & 6 \\ 12 & 9 & 6\end{array}\right]$ and $G \cdot F=\left[\begin{array}{ccc}3 & 9 & 6 \\ 9 & 3 & 6 \\ 12 & 9 & 6\end{array}\right]$. Because $G=3 I$, we have
$F \cdot G=F \cdot(3 I)=3(F \cdot I)=3 I F=3(I \cdot F)=(3 I) \cdot F=G \cdot F$.
6. Let $I_{\boldsymbol{n}}$ be the $\boldsymbol{n} \times \boldsymbol{n}$ identity matrix. For the matrices given below, perform each of the following calculations or explain why the calculation is not possible:

$$
\begin{array}{ll}
A=\left[\begin{array}{ll}
\frac{1}{2} & 3 \\
2 & \frac{2}{3}
\end{array}\right] & B=\left[\begin{array}{ccc}
9 & -1 & 2 \\
-3 & 4 & 1
\end{array}\right] \\
C=\left[\begin{array}{lll}
3 & 1 & 3 \\
1 & 0 & 1 \\
3 & 1 & 3
\end{array}\right] & D=\left[\begin{array}{cccc}
2 & \sqrt{2} & -2 & \frac{1}{2} \\
3 & 2 & 1 & 0
\end{array}\right]
\end{array}
$$

a. $A B$

$$
\left[\begin{array}{ccc}
-\frac{9}{2} & \frac{23}{2} & 4 \\
16 & \frac{2}{3} & \frac{14}{3}
\end{array}\right]
$$

b. $B A$

These matrices have incompatible dimensions multiplied this way.
c. $A C$

A has 2 columns, and $C$ has 3 rows, which means we cannot multiply them.
d. $A B C$

$$
\left[\begin{array}{ccc}
10 & -\frac{1}{2} & 10 \\
\frac{188}{3} & \frac{62}{3} & \frac{188}{3}
\end{array}\right]
$$

e. $A B C D$

Matrix $A B C$ is a $2 \times 3$ matrix, and matrix $D$ is a $2 \times 4$ matrix, which means we cannot multiply them.
f. $A D$

$$
\left[\begin{array}{cccc}
10 & \frac{\sqrt{2}}{2}+6 & 2 & \frac{1}{4} \\
6 & 2 \sqrt{2}+\frac{4}{3} & -\frac{10}{3} & 1
\end{array}\right]
$$

g. $\quad A^{2}$

$$
\left[\begin{array}{cc}
\frac{25}{4} & \frac{7}{2} \\
\frac{7}{3} & \frac{58}{9}
\end{array}\right]
$$

h. $\quad C^{2}$

$$
\left[\begin{array}{ccc}
19 & 6 & 19 \\
6 & 2 & 6 \\
19 & 6 & 19
\end{array}\right]
$$

i. $\quad B C^{2}$

$$
\left[\begin{array}{ccc}
203 & 64 & 203 \\
-14 & -4 & -14
\end{array}\right]
$$

j. $\quad A B C+A D$

ABC and AD have different dimensions, so they cannot be added.
k. $A B I_{2}$
$B$ is $2 \times 3$ and $I_{2}$ is $2 \times 2$, so they cannot be multiplied in this order.
I. $A I_{2} B$

This is the same as

$$
A B=\left[\begin{array}{ccc}
-\frac{9}{2} & \frac{23}{2} & 4 \\
16 & \frac{2}{3} & \frac{14}{3}
\end{array}\right]
$$

m. $\quad C I_{3} B$
$C I_{3}=C$, so this is $C B$, but these have incompatible dimensions and cannot be multiplied.
n. $I_{2} B C$

This is the same as

$$
B C=\left[\begin{array}{ccc}
32 & 11 & 32 \\
-2 & -2 & -2
\end{array}\right]
$$

o. $2 A+B$
$2 A$ and $B$ have different dimensions, so they cannot be added.
p. $B\left(I_{3}+C\right)$

$$
\begin{aligned}
I_{3}+C & =\left[\begin{array}{lll}
4 & 1 & 3 \\
1 & 1 & 1 \\
3 & 1 & 4
\end{array}\right] \\
B\left(I_{3}+C\right) & =\left[\begin{array}{ccc}
41 & 10 & 34 \\
-5 & 2 & -1
\end{array}\right]
\end{aligned}
$$

q. $B+B C$

$$
\left[\begin{array}{ccc}
41 & 10 & 34 \\
-5 & 2 & -1
\end{array}\right]
$$

r. $\quad 4 D I_{4}$

$$
\left[\begin{array}{cccc}
8 & 4 \sqrt{2} & -8 & 2 \\
12 & 8 & 4 & 0
\end{array}\right]
$$

7. Let $\boldsymbol{F}$ be an $\boldsymbol{m} \times \boldsymbol{n}$ matrix. Then what do you know about the dimensions of matrix $G$ in the problems below if each expression has a value?
a. $\boldsymbol{F}+\boldsymbol{G}$
$G$ must have the exact same dimensions as $F$ in order to add them together. That is, the dimensions of $G$ are $m \times n$.
b. $F G$

Here we know that the rows of $G$ must be the same as the columns of $F$; that is, $G$ has $n$ rows.
c. $\boldsymbol{G F}$

We know that $G$ has $m$ columns.
d. $F H G$ for some matrix $H$.

We know nothing about the dimensions of matrix $G$ based on the dimensions of $F$.
8. Consider an $\boldsymbol{m} \times \boldsymbol{n}$ matrix $A$ such that $\boldsymbol{m} \neq \boldsymbol{n}$. Explain why you cannot evaluate $\boldsymbol{A}^{2}$.

The only way to evaluate $A^{2}$ is to multiply $A A$, which implies that $A$ has the same number of rows as it does columns. Since $m \neq n$, then $A^{2}$ does not exist.
9. Let $A=\left[\begin{array}{lll}0 & 1 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 0\end{array}\right], B=\left[\begin{array}{lll}0 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 2 & 0\end{array}\right], C=\left[\begin{array}{lll}0 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & 0\end{array}\right]$ represent the routes of three airlines $A, B$, and $C$ between three cities.
a. Zane wants to fly from City 1 to City $\mathbf{3}$ by taking Airline $\boldsymbol{A}$ first and then Airline $B$ second. How many different ways are there for him to travel?

Since $A \cdot B=\left[\begin{array}{lll}5 & 4 & 1 \\ 2 & 6 & 2 \\ 2 & 2 & 3\end{array}\right]$, and the entry in row 1, column 3 is 1, there is only 1 way for Zane to travel.
b. Zane did not like Airline $A$ after the trip to City 3, so on the way home, Zane decides to fly Airline $C$ first and then Airline $\boldsymbol{B}$ second. How many different ways are there for him to travel?
Since $C \cdot B=\left[\begin{array}{lll}4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 2 & 3\end{array}\right]$, and the entry in row 3, column 1 is 2, there are two ways for Zane to travel.
10. Let $A=\left[\begin{array}{llll}0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 2 & 1 & 0 & 2 \\ 1 & 2 & 1 & 0\end{array}\right]$ represent airline flights of one airline between 4 cities.
a. We use the notation $A^{2}$ to represent the product $A \cdot A$. Calculate $A^{2}$. What do the entries in matrix $A^{2}$ represent?

We see that $A^{2}=\left[\begin{array}{llll}5 & 5 & 4 & 3 \\ 5 & 5 & 2 & 6 \\ 3 & 6 & 6 & 5 \\ 4 & 2 & 5 & 6\end{array}\right]$. The entry in row $i$, column $j$ of matrix $A^{2}$ is the number of ways to get from
City $i$ to City $j$ with one stop in between at one of the other cities.
b. Jade wants to fly from City 1 to City 4 with exactly one stop. How many different ways are there for her to travel?

Since $\left(A^{2}\right)_{1,4}=3$, Jade can choose between three different ways to travel.
c. Now Jade wants to fly from City 1 to City 4 with exactly two stops. How many different ways are there for her to choose?

Since $A^{3}=A^{2} \cdot A=\left[\begin{array}{llll}16 & 15 & 18 & 23 \\ 15 & 19 & 21 & 19 \\ 23 & 19 & 20 & 24 \\ 18 & 21 & 14 & 20\end{array}\right]$, and $\left(A^{3}\right)_{1,4}=23$, there are 23 different ways for Jade to travel

| Lesson 3: | Matrix Arithmetic in its Own Right |
| :--- | :--- |
| Date: | $1 / 24 / 15$ | 1/24/15

## Topic B:

## Linear Transformations of Planes and Space

N-VM.C.7, N-VM.C.8, N-VM.C.9, N-VM.C.10, N-VM.C. 11

| Focus Standards: | N-VM.C. 7 | (+) Multiply matrices by scalars to produce new matrices, e.g., as when all of the payoffs in a game are doubled. |
| :---: | :---: | :---: |
|  | N-VM.C. 8 | (+) Add, subtract, and multiply matrices of appropriate dimensions. |
|  | N-VM.C. 9 | (+) Understand that, unlike multiplication of numbers, matrix multiplication for square matrices is not a commutative operation, but still satisfies the associative and distributive properties. |
|  | N-VM.C. 10 | (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse. |
|  | N-VM.C. 11 | (+) Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors. |
| Instructional Days: | 10 |  |
| Lesson 4: | Linear Trans | rmations Review (S) ${ }^{1}$ |
| Lesson 5: | Coordinates | Points in Space (P) |
| Lesson 6: | Linear Trans | rmations as Matrices (E) |
| Lesson 7: | Linear Trans | rmations Applied to Cubes (E) |
| Lessons 8-9: | Composition | f Linear Transformations (E, E) |
| Lesson 10: | Matrix Mult | ication Is Not Commutative (P) |
| Lesson 11: | Matrix Addi | $n$ Is Commutative (P) |
| Lesson 12: | Matrix Mult | ication Is Distributive and Associative (P) |
| Lesson 13: | Using Matri | perations for Encryption (M) |

Topic B explores the usefulness of matrices with dimensions higher than $2 \times 2$. The concept of a linear transformations from Module 1 is extended to linear transformations in three- (and higher-) dimensional space. In Lessons 4-6, students use what they know about linear transformations performed on real and complex numbers in two dimension and extend that to three dimensional space. They verify the conditions

[^3]of linearity in two dimensional space and make conjectures about linear transformations in three dimensional space. Students add matrices and perform scalar multiplication (N-VM.C.7), exploring the geometric interpretations for these operations in three dimensions. In Lesson 7, students examine the geometric effects of linear transformations in $\mathbb{R}^{3}$ induced by various $3 \times 3$ matrices on the unit cube. Students explore these transformations and discover the connections between a $3 \times 3$ matrix and the geometric effect of the transformation produced by the matrix. The materials support the use of geometry software, such as the freely available GeoGebra, but software is not required. Students extend their knowledge of the multiplicative inverse and that it exists precisely when the determinant of the matrix is non-zero from the area of a unit square in two dimensions to the volume of the unit cube in three-dimensions ( $\mathbf{N}$-VM.C.10).
In Lesson 8, students explore a sequence of transformations in two dimensions and this is extended in Lesson 9 to three dimensions. Students see a sequence of transformations as represented by multiplication of several matrices and relate this to a composition. In Lessons 8-9, students practice scalar and matrix multiplication extensively, setting the stage for properties of matrices studied in Lessons 10 and 11. In Lesson 10 , students discover that matrix multiplication is not commutative and verify this finding algebraically for $2 \times 2$ and $3 \times 3$ matrices (N-VM.C.9). In Lesson 11, students translate points by matrix addition and see that while matrix multiplication is not commutative, matrix addition is commutative. They also write points in two and three dimensions as single column matrices (vectors) and multiply matrices by vectors (N-VM.C.11).

The study of matrices continues in Lesson 12 as students discover that matrix multiplication is associative and distributive (N-VM.C.9). In Lesson 13, students recap their understanding of matrix operations-matrix product, matrix sum, and scalar multiplication-and properties of matrices by using matrices and matrix operations to discover encrypted codes. The geometric and arithmetic roles of the zero matrix and the identity matrix are explored in Lessons 12 and 13. Students understand that the zero matrix is similar to the role of 0 in the real number system and the identity matrix is similar to 1 (N-VM.C.10).
Throughout Topic B, students study matrix operations in two- and three-dimensional space and relate these abstract representations to the transformations they represent (MP.2). Students have opportunities to use computer programs as tools for examining and understanding the geometric effects of transformations produced by matrices on the unit circle (MP.5).

## Lesson 4: Linear Transformations Review

## Student Outcome

- Students will review their understanding of linear transformations on real and complex numbers as well as linear transformations in two-dimensional space. They will use their understanding to form conjectures about linear transformations in three-dimensional space.


## Lesson Notes

In this lesson, students will respond to questions to demonstrate what they understand about linear transformations performed on real and complex numbers, as well as those performed in two-dimensional space and three-dimensional space. They will describe special cases of linear transformations for real and complex numbers. They will also verify the conditions for linear transformations in two-dimensional space and make conjectures about how to define and represent linear transformations in three-dimensional space.

## Classwork

## Discussion (5 minutes): Linear Transformations $\mathbb{R} \rightarrow \mathbb{R}$

During this discussion, call on a variety of students to answer the questions as they are posed.

- Recall that when we write $L: \mathbb{R} \rightarrow \mathbb{R}$, we mean that $L$ is a function that takes real numbers as inputs and produces real numbers as outputs.
- Let's recall the conditions a function $L: \mathbb{R} \rightarrow \mathbb{R}$ must meet to be a called a linear transformation. What are they?
- First, we need $L(x+y)=L(x)+L(y)$.
- Second, we need $L(a \cdot x)=a \cdot L(x)$.
- Good. And what exactly are $a, x$, and $y$ in this case?
- Each of these symbols represents a real number.
- Okay. Now give three examples of formulas that you know represent linear transformations from $\mathbb{R}$ to $\mathbb{R}$.
- $L(x)=2 \cdot x$
- $\quad L(x)=3 \cdot x$
- $\quad L(x)=4 \cdot x$
- Good. So we see that a linear transformation $L: \mathbb{R} \rightarrow \mathbb{R}$ must have the form $L(x)=a \cdot x$ for some real number $a$. Now let's explore the geometric effect that linear transformations have on the real number line. The value of $a$ could be positive, negative, or zero, and the size of $a$ could be big or small. How are these cases different geometrically? Let's explore together.


## Scaffolding:

- Show the following examples:
$f(x)=2 x$
$g(x)=2 x+1$
$h(x)=\frac{1}{2} x$
$j(x)=\frac{1}{2} x-1$
Which are linear transformations? Explain your answer.
- Advanced students can be challenged to explore the questions about the geometric implications of different values of $a$ without any additional cueing.
- Consider the interval $[-2,2]$. What is the image of the interval under the rule $L(x)=3 \cdot x$ ?
- We have $L(-2)=3 \cdot-2=-6$, and $L(2)=3 \cdot 2=6$. It looks as though the image of $[-2,2]$ is $[-6,6]$.

- When we use a scale factor of 3 , the interval expands to 3 times its original size. Now let's try a role reversal: can you think of a mapping would take the interval at the bottom of the figure and map it to the one at the top? Explain your answer.
- The inverse map is $L(x)=\left(\frac{1}{3}\right) \cdot x$. This makes sense because we are contracting the interval so that it ends up being $\frac{1}{3}$ of its original size.
- In some cases, the transformation $L(x)=a \cdot x$ expands an interval, and in other cases it contracts the interval. Can you think of a mapping that leaves the size of the interval unchanged?
- $\quad L(x)=1 \cdot x$ does not change the size of the interval.
- Can you see why mathematicians refer to $L(x)=1 \cdot x$ as the identity mapping? Try to make sense of this phrase.
- Under the identity mapping, each point on the number line is mapped to a location that is identical to its original location.
- Now let's consider some cases where $a$ is a negative number. What is the image of $[-2,2]$ under the rule $L(x)=-1 \cdot x$ ?

- In this case, each point on the number line is reflected across the origin. The result is the same interval we started with, but each point has been taken to the opposite side of the number 0 .
- Describe what happens after the application of $L(x)=-2 \cdot x$ or $L(x)=\left(-\frac{1}{2}\right) \cdot x$.
- First, each interval gets reflected across the origin, and then a dilation gets applied. In the first case, each interval dilates to twice its original size, and in the second case, each interval dilates to half its original size.
- Now let's consider the only remaining case. What happens when we apply the zero map, $L(x)=0 \cdot x$ ?
- Since $L(x)=0 \cdot x=0$ for every real number $x$, the entire number line gets mapped to a single point, namely 0 !
- Perhaps we should call this kind of transformation a collapse, since what used to be a line has now collapsed to a single point.
- Write a quick summary of this discussion in your notebook, and then share what you wrote with a partner.
- Every linear transformation $L: \mathbb{R} \rightarrow \mathbb{R}$ has the form $L(x)=a \cdot x$ for some real number $a$. Each of these transformations is essentially a dilation with scale factor $a$. Special cases include $a=0$, which causes the whole number line to collapse to the origin; $a=1$, which leaves the whole number line unchanged; and $a=-1$, which causes each point on the number line to undergo a reflection through the origin.


## Scaffolding:

Students could create a graphic organizer listing the matrix in one column and transformation represented in the next column.

## Exercises 1-2 (3 minutes)

The students should complete the exercises in pairs. One pair should be asked to share each solution. The presenting pair could represent the mapping geometrically on the board to aid students who are strong visual learners.

Exercises 1-2

1. Describe the geometric effect of each mapping.
a. $\quad L(x)=9 \cdot x$

Dilates the interval by a factor of 9
b. $\quad L(x)=-\frac{1}{2} \cdot x$

Reflects the interval over the origin and then applies a dilation with a scale factor of $\frac{1}{2}$
2. Write the formula for the mappings described.
a. A dilation that expands each interval to $\mathbf{5}$ times its original size.

$$
L(x)=5 \cdot x
$$

b. A collapse of the interval to the number 0 .

$$
L(x)=0 \cdot x
$$

## Discussion ( 10 minutes): Linear Transformations $\mathbb{C} \rightarrow \mathbb{C}$

- In addition to transformations from $\mathbb{R}$ to $\mathbb{R}$, in the previous module we also studied transformations that take complex numbers as inputs and produce complex numbers as outputs, and we used the symbol $L$ : $\mathbb{C} \rightarrow \mathbb{C}$ to denote this.
- As we did for functions from $\mathbb{R}$ to $\mathbb{R}$, let's recall the conditions a function $L$ : $\mathbb{C} \rightarrow \mathbb{C}$ must meet to be a called a linear transformation. What are they?
- Just as before, we need $L(x+y)=L(x)+L(y)$ and $L(a \cdot x)=a \cdot L(x)$.
- What exactly do $a, x$, and $y$ represent in this case?
- Each of these symbols represents a complex number.
- Give three examples of formulas that you know represent linear transformations from $\mathbb{C}$ to $\mathbb{C}$.
- $\quad L(x)=2 \cdot x$
- $\quad L(x)=i \cdot x$
- $L(x)=(3+5 i) \cdot x$
- So we see that a linear transformation $L: \mathbb{C} \rightarrow \mathbb{C}$ must have the form $L(x)=a \cdot x$ for some complex number $a$. As we did above with functions from $\mathbb{R}$ to $\mathbb{R}$, let's explore the geometric effect of some special linear transformations from $\mathbb{C}$ to $\mathbb{C}$.
- What do you remember about the result of multiplying $a+b i$ by $c+d i$ ?
- We know that $(a+b i) \cdot(c+d i)=a c-b d+(a d+b c) \cdot i$.
- Recall that we visualize complex numbers as points in the complex plane. What do you remember about the geometric effect of multiplying one complex number by another?
- Multiplication by a complex number induces a rotation and a dilation.
- Now let's get back to the subject of linear transformations. Discuss the following questions with another student. Analyze each question both algebraically and geometrically. Be prepared to share your analysis with the whole class in a few minutes.
- Is there a linear transformation $L: \mathbb{C} \rightarrow \mathbb{C}$ that leaves each point in the complex plane unchanged? That is, is there an identity map from $\mathbb{C} \rightarrow \mathbb{C}$ ?
- Yes. If we take $a=1+0 i$ and $x=c+d i$, we get this:
- $L(x)=a x=(1+0 i)(c+d i)=1 \cdot c-0 \cdot d+(1 \cdot d+0 \cdot c) i=c+d i=x$.
- Geometrically, multiplying by the complex number $1+0 i$ is the same thing as multiplying by the real number 1, so it makes sense that this action leaves points unchanged.
- Describe the set of linear transformations $L: \mathbb{C} \rightarrow \mathbb{C}$ that induce a pure rotation of points in the complex plane.

- We know that multiplication by a complex number a induces a rotation through the argument of a as well as a dilation by the modulus of $a$. So if we want a pure rotation, then we need to have $|a|=1$. Thus, if $a$ is any point on the unit circle surrounding the origin, then $L(x)=a x$ will induce a pure rotation of the complex plane.

- Describe the set of linear transformations $L: \mathbb{C} \rightarrow \mathbb{C}$ that induce a pure dilation of points in the complex plane.
- If $L(x)=$ ax induces a pure dilation, then the argument of a must be 0 . This means that $a$ is, in fact, a real number.

- Is there a linear transformation $L: \mathbb{C} \rightarrow \mathbb{C}$ that collapses the entire complex plane into a single point?
- Yes. If we choose $a=0+0 i$, then we get $L(x)=a x=(0+0 i) x=0+0 i$ for every $x$. Thus, every point in the complex plane gets mapped to the origin.

- Is there a linear transformation $L: \mathbb{C} \rightarrow \mathbb{C}$ that induces a reflection across the real axis? In particular, where would such a transformation take $5+2 i$ ? Where would it take $1+0 i$ ? Where would it take $0+1 i$ ?
- Since $(5,2)$ maps to $(5,-2)$, the image of $5+2 i$ must be $5-2 i$.
- Since $(1,0)$ maps to $(1,0)$, the image of $1+0 i$ must be $1+0 i$.
- Since $(0,1)$ maps to $(0,-1)$, the image of $0+1 i$ must be $0-1 i$.

- If there is a complex number $a$ such that $L(x)=a x$ induces a reflection across the real axis, then what would its argument have to be? Analyze your work above carefully.
- Since points on the real axis map to themselves, it would appear that the argument of a would have to be 0 . In other words, there must be no rotational component.
- On the other hand, points on the imaginary axis get rotated through $180^{\circ}$. But this is irreconcilable with the statement above. Thus, it appears that there is no complex number a that meets the necessary requirements, so we conclude that reflection across the real axis is not a linear transformation from $\mathbb{C}$ to $\mathbb{C}$.


## Discussion (7 minutes): Linear Transformations $\mathbb{R}^{\mathbf{2}} \rightarrow \mathbb{R}^{\mathbf{2}}$

- Recall that we can visualize complex numbers as points in the complex plane. Let's briefly review some arithmetic with complex numbers.
- Let $z_{1}=3+4 i$, and let $z_{2}=5-2 i$. What is $z_{1}+z_{2}$ ?
- $z_{1}+z_{2}=(3+5)+(4-2) i=8+2 i$
- Thinking of these two complex numbers as points in the coordinate plane, we can write $z_{1}=\binom{3}{4}$ and $z_{2}=\binom{5}{-2}$. Thus, we could just as well write $z_{1}+z_{2}=\binom{3}{4}+\binom{5}{-2}=\binom{8}{2}$. In fact, we can even abandon the context of complex numbers and let addition of ordered pairs take on a life of its own.
- With this in mind, what is $\binom{9}{-1}+\binom{3}{6}$ ?

ㅁ $\binom{9}{-1}+\binom{3}{6}=\binom{9+3}{-1+6}=\binom{12}{5}$

- Now let's review scalar multiplication. With $z=3+5 i$, what is $10 z$ ?
- $10 z=10(3+5 i)=30+50 i$
- Thinking of $z$ as a point in the coordinate plane, we have $z=\binom{3}{5}$, and $10 z=10\binom{3}{5}=\binom{30}{50}$. Once again we can set complex numbers to the side and let scalar multiplication of ordered pairs be an operation in its own right.
- With this in mind, what is $5\binom{4}{-2}$ ?
- $5\binom{4}{-2}=\binom{5 \cdot 4}{5 \cdot-2}=\binom{20}{-10}$
- We saw in Module 1 that multiplication by a complex number is a linear transformation that can be modeled using matrices. For instance, to model the product $(3+4 i) \cdot(x+y i)$, we can write $L\binom{x}{y}=\left(\begin{array}{cc}3 & -4 \\ 4 & 3\end{array}\right)\binom{x}{y}$. Describe the geometric effect of this transformation on points in the plane.
- Each point $\binom{x}{y}$ is dilated by a factor of $\sqrt{3^{2}+4^{2}}=5$ and rotated through an angle that is given by $\arctan \frac{4}{3}$.
- When a matrix is used to model complex-number multiplication, it will always have the form $\left(\begin{array}{cc}a & -b \\ b & a\end{array}\right)$, and we already know that these matrices represent linear transformations. Now let's consider a more general matrix transformation $A\binom{x}{y}=\left(\begin{array}{cc}-3 & 2 \\ 1 & 4\end{array}\right)\binom{x}{y}$. Does this also represent a linear transformation? Let's begin by checking the addition requirement.
- $A\left(\binom{x_{1}}{y_{1}}+\binom{x_{2}}{y_{2}}\right)=A\binom{x_{1}+x_{2}}{y_{1}+y_{2}}=\left(\begin{array}{cc}-3 & 2 \\ 1 & 4\end{array}\right)\binom{x_{1}+x_{2}}{y_{1}+y_{2}}=\binom{-3\left(x_{1}+x_{2}\right)+2\left(y_{1}+y_{2}\right)}{\left(x_{1}+x_{2}\right)+4\left(y_{1}+y_{2}\right)}$
- $A\left(\binom{x_{1}}{y_{1}}+\binom{x_{2}}{y_{2}}\right)=\binom{-3 x_{1}-3 x_{2}+2 y_{1}+2 y_{2}}{x_{1}+x_{2}+4 y_{1}+4 y_{2}}$

ㅁ $\quad A\binom{x_{1}}{y_{1}}=\left(\begin{array}{cc}-3 & 2 \\ 1 & 4\end{array}\right)\binom{x_{1}}{y_{1}}=\binom{-3 x_{1}+2 y_{1}}{x_{1}+4 y_{1}}$

- $A\binom{x_{2}}{y_{2}}=\left(\begin{array}{cc}-3 & 2 \\ 1 & 4\end{array}\right)\binom{x_{2}}{y_{2}}=\binom{-3 x_{2}+2 y_{2}}{x_{2}+4 y_{2}}$
- $A\binom{x_{1}}{y_{1}}+A\binom{x_{2}}{y_{2}}=\binom{-3 x_{1}+2 y_{1}}{x_{1}+4 y_{1}}+\binom{-3 x_{2}+2 y_{2}}{x_{2}+4 y_{2}}=\binom{-3 x_{1}+2 y_{1}-3 x_{2}+2 y_{2}}{x_{1}+4 y_{1}+x_{2}+4 y_{2}}$
- When we compare $A\left(\binom{x_{1}}{y_{1}}+\binom{x_{2}}{y_{2}}\right)$ with $A\binom{x_{1}}{y_{1}}+A\binom{x_{2}}{y_{2}}$, we see that the results are the same.
- Now let's check the scalar multiplication requirement.

$$
\begin{aligned}
& \quad A\left(k\binom{x}{y}\right)=A\binom{k x}{k y}=\left(\begin{array}{cc}
-3 & 2 \\
1 & 4
\end{array}\right)\binom{k x}{k y}=\binom{-3 \cdot k x+2 \cdot k y}{1 \cdot k x+4 \cdot k y} \\
&
\end{aligned}
$$

- When we compare $A\left(k\binom{x}{y}\right)$ with $k \cdot A\binom{x}{y}$, we see that they are the same.
- So we see that the matrix mapping $A\binom{x}{y}=\left(\begin{array}{cc}-3 & 2 \\ 1 & 4\end{array}\right)\binom{x}{y}$ is indeed a linear transformation. In fact, we could use the same technique to prove in general that $A\binom{x}{y}=\left(\begin{array}{ll}a & c \\ b & d\end{array}\right)\binom{x}{y}$ is a linear transformation also.
- So if $x=\binom{x_{1}}{x_{2}}$ and $y=\binom{y_{1}}{y_{2}}$, then we can summarize the linearity requirements as follows:

1. $A(x+y)=A x+A y$
2. $A(k \cdot x)=k \cdot A x$

- This is much cleaner and easier to handle!
- Suppose that the determinant of $\left(\begin{array}{ll}a & c \\ b & d\end{array}\right)$ is zero. What can we say about the mapping $\binom{x}{y} \mapsto\left(\begin{array}{ll}a & c \\ b & d\end{array}\right)\binom{x}{y}$ in this case?
- In Module 1, we learned that the determinant of a matrix represents the area of the image of the unit square. If this area is 0 , then the transformation represents a collapse of some kind. This implies that the transformation is not invertible.


## Discussion (5 minutes): Linear Transformations $\mathbb{R}^{3} \rightarrow \mathbb{R}^{\mathbf{3}}$

- We saw that points in $\mathbb{R}^{2}$ can be represented as ordered pairs $\binom{x_{1}}{x_{2}}$. How do you suppose we might represent points in three-dimensional space?
- Points in $\mathbb{R}^{3}$ can be represented as ordered triples $\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)$.
- How would you guess to define addition and scalar multiplication of points in a three-dimensional setting?
- $\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)+\left(\begin{array}{l}y_{1} \\ y_{2} \\ y_{3}\end{array}\right)=\left(\begin{array}{l}x_{1}+y_{1} \\ x_{2}+y_{2} \\ x_{3}+y_{3}\end{array}\right)$
- $k \cdot\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)=\left(\begin{array}{l}k \cdot x_{1} \\ k \cdot x_{2} \\ k \cdot x_{3}\end{array}\right)$
- What might it mean for a function $L: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ to be a linear transformation?
- If $x$ and $y$ are points in $\mathbb{R}^{3}$ and $k$ is any real number, then we should have $L(x+y)=L(x)+L(y)$ and $L(k \cdot x)=k \cdot L(x)$.
- Any ideas about how to represent such transformations?
- Perhaps a $3 \times 3$ matrix will come into play!
- One last point to ponder: we've explored the geometric effects of linear transformations in one and two dimensions. Give some thought to what effects linear transformations might have in a three-dimensional setting. We'll take up this issue further in the next lesson!


## Closing (10 minutes)

The students should complete a graphic organizer that summarizes their understanding of linear transformations. They can work for a few minutes independently, and then compare their results with a partner. During the last few minutes, volunteers can share the information they included in their graphic organizers. They can share this information aloud or display it on the board so that students can make revisions to their organizers. A suggested format for the organizer is shown.

|  | $L: \mathbb{R} \rightarrow \mathbb{R}$ | $L: \mathbb{C} \rightarrow \mathbb{C}$ | $L: \mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$ | $L: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ |
| :--- | :--- | :--- | :--- | :--- |
| Conditions for $L$ |  |  |  |  |
| General form of $L$ |  |  |  |  |
| What $L$ represents |  |  |  |  |

Note: Students will form conjectures about linear transformations in three-dimensional space, which will be discussed further in later lessons.

## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 4: Linear Transformations Review

## Exit Ticket

1. In Module 1, we learned about linear transformations for any real-number functions. What are the conditions of a linear transformation? If a real-number function is a linear transformation, what is its form? What are the two characteristics of the function?
2. Describe the geometric effect of each mapping:
a. $\quad L(x)=3 x$
b. $\quad L(z)=(\sqrt{2}+\sqrt{2} i) \cdot z$
c. $\quad L(z)=\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)\binom{x}{y}$, where $z$ is a complex number
d. $\quad L(z)=\left(\begin{array}{ll}2 & 0 \\ 0 & 2\end{array}\right)\binom{x}{y}$, where $z$ is a complex number

## Exit Ticket Sample Solutions

1. In Module 1, we learned about linear transformations for any real-number functions. What are the conditions of a linear transformation? If a real-number function is a linear transformation, what is its form? What are the two characteristics of the function?
$L(x+y)=L(x)+L(y), L(k x)=k L(x)$, where $x, y$, and $k$ are real numbers.
It is in the form of $L(x)=m x$, and its graph is a straight line going through the origin. It is an odd function.
2. Describe the geometric effect of each mapping:
a. $\quad L(x)=3 x$

Dilate the interval by a factor of 3 .
b. $\quad L(z)=(\sqrt{2}+\sqrt{2} i) \cdot z$

Rotate $\frac{\pi}{4}$ radians counterclockwise about the origin.
c. $\quad L(z)=\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)\binom{x}{y}$, where $z$ is a complex number

Rotate $\pi$ radians counterclockwise about the origin.
d. $\quad L(z)=\left(\begin{array}{ll}\mathbf{2} & \mathbf{0} \\ \mathbf{0} & 2\end{array}\right)\binom{\boldsymbol{x}}{\boldsymbol{y}}$, where $\boldsymbol{z}$ is a complex number

Dilate z by a factor of 2 .

## Problem Set Sample Solutions

1. Suppose you have a linear transformation $L: \mathbb{R} \rightarrow \mathbb{R}$, where $L(3)=6, L(5)=10$.
a. Use the addition property to find $L(6), L(8), L(10)$, and $L(13)$.

$$
\begin{aligned}
& L(6)=L(3+3)=L(3)+L(3)=6+6=12 \\
& L(8)=L(3+5)=L(3)+L(5)=6+10=16 \\
& L(10)=L(5)+L(5)=10+10=20 \\
& L(13)=L(10+3)=L(10)+L(3)=20+6=26
\end{aligned}
$$

b. Use the multiplication property to find $L(15), L(18)$, and $L(30)$.

$$
\begin{aligned}
& L(15)=L(3 \cdot 5)=3 \cdot L(5)=3 \cdot 10=30 \\
& L(18)=L(3 \cdot 6)=3 \cdot L(6)=3 \cdot 12=36 \\
& L(30)=L(5 \cdot 6)=5 \cdot L(6)=5 \cdot 12=60
\end{aligned}
$$

c. Find $L(-3), L(-8)$, and $L(-15)$
$L(-3)=L(-1 \cdot 3)=-1 \cdot L(3)=-6$
$L(-8)=L(-1 \cdot 8)=-1 \cdot L(8)=-16$
$L(-15)=L(-3 \cdot 5)=-3 \cdot L(5)=-3 \cdot 10=-30$
d. Find the formula for $L(x)$.

Given $L(x)$ is a linear transformation; therefore, it must have a form of $L(x)=m x$, where $m$ is real number.
Given $L(3)=6$; therefore, $3 m=6, m=2 . L(x)=2 x$.
e. Draw the graph of the function $L(x)$.

2. A linear transformation $L: \mathbb{R} \rightarrow \mathbb{R}$ must have the form of $L(x)=a x$ for some real number $a$. Consider the interval $[-5,2]$. Describe the geometric effect of the following, and find the new interval.
a. $\quad L(x)=5 x$

It dilates the interval by a scale factor of $5 ;[-25,10]$.
b. $\quad L(x)=-2 x$

It reflects the interval over the origin and then dilates it with a scale factor of $2 ;[-4,10]$.
3. A linear transformation $L: \mathbb{R} \rightarrow \mathbb{R}$ must have the form of $L(x)=a x$ for some real number $a$. Consider the interval $[-2,6]$. Write the formula for the mapping described, and find the new interval.
a. A reflection over the origin.

$$
L(x)=-x, \quad[-6,2]
$$

b. A dilation with a scale of $\sqrt{2}$.
$L(x)=\sqrt{2} \cdot x, \quad[-2 \sqrt{2}, 6 \sqrt{2}]$
c. A reflection over the origin and a dilation with a scale of $\frac{1}{2}$.

$$
L(x)=-\frac{1}{2} x, \quad[-3,1]
$$

d. A collapse of the interval to the number 0 .
$L(x)=0 x, \quad[0,0]$
4. In Module 1, we used $2 \times 2$ matrices to do transformations on a square, such as a pure rotation, a pure reflection, a pure dilation, and a rotation with a dilation. Now use those matrices to do transformations on this complex number: $z=2+i$. For each transformation below, graph your answers.
a. A pure dilation with a factor of 2 .


$$
\left(\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right), L(z)=\left(\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right)\binom{2}{1}=\binom{4}{2}
$$

b. A pure $\frac{\pi}{2}$ radians counterclockwise rotation about the origin.

$\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right), L(z)=\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)\binom{2}{1}=\binom{-1}{2}$
c. A pure $\pi$ radians counterclockwise rotation about the origin.

$\left(\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right), L(z)=\left(\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right)\binom{2}{1}=\binom{-2}{-1}$
d. A pure reflection about the real axis.

$\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right), L(z)=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)\binom{2}{1}=\binom{2}{-1}$
e. A pure reflection about the imaginary axis.

$\left(\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right), L(z)=\left(\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right)\binom{2}{1}=\binom{-2}{1}$
f. A pure reflection about the line $y=x$.


$$
\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right), L(z)=\left(\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right)\binom{2}{1}=\binom{1}{2}
$$

g. A pure reflection about the line $y=-x$.


$$
\left(\begin{array}{cc}
0 & -1 \\
-1 & 0
\end{array}\right), L(z)=\left(\begin{array}{cc}
0 & -1 \\
-1 & 0
\end{array}\right)\binom{2}{1}=\binom{-1}{-2}
$$

5. Wesley noticed that by multiplying the matrix $\left(\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right)$ by a complex number $z$ produces a pure $\frac{\pi}{2}$ radians counterclockwise rotation, and multiplying by $\left(\begin{array}{ll}2 & 0 \\ 0 & 2\end{array}\right)$ produces a pure dilation with a factor of 2 . So, he thinks he can add these two matrices, which will produce $\left(\begin{array}{cc}2 & -1 \\ 1 & 2\end{array}\right)$ and will rotate $z$ by $\frac{\pi}{2}$ radians counterclockwise and dilate $z$ with a factor of 2 . Is he correct? Explain your reason.

No, he is not correct. For the general transformation of complex numbers, the form is $L(Z)=\left(\begin{array}{cc}a & -b \\ b & a\end{array}\right)\binom{x}{y}$.
By multiplying the matrix $\left(\begin{array}{cc}a & -b \\ b & a\end{array}\right)$ to $z$, it rotates $z$ an angle $\arctan \left(\frac{b}{a}\right)$, and dilates $z$ with a factor of $\sqrt{a^{2}+b^{2}}$. $\arctan \left(\frac{1}{2}\right)=26.565^{\circ}$, and $\sqrt{(1)^{2}+(2)^{2}}=\sqrt{5}$, which is not $\frac{\pi}{2}$ radians or a dilation of a factor of 2 .
6. In Module 1, we learned that there is not any real number that will satisfy $\frac{1}{a+b}=\frac{1}{a}+\frac{1}{b}$, which is the addtition property of linear transformation. However, we discussed that some fixed complex numbers might work. Can you find two pairs of complex numbers that will work? Show you work.
Given $\frac{1}{a+b}=\frac{1}{a}+\frac{1}{b}, a, b \neq 0, \frac{1}{a+b}=\frac{b+a}{a b}, a b=(a+b)^{2}, a b=a^{2}+2 a b+b^{2}$, $a^{2}+a b+b^{2}=0, a^{2}+a d+\frac{1}{4} b^{2}=-\frac{3}{4} b^{2},\left(a+\frac{1}{2} b\right)^{2}=-\frac{3}{4} b^{2}, a+\frac{1}{2} b= \pm \frac{\sqrt{3}}{2} b \cdot i$
$a=\frac{-1 \pm \sqrt{3} \cdot i}{2} b$, for example, $b=2$ and $a=-1+\sqrt{3} \cdot i, b=4$ and $a=-2+2 \sqrt{3} \cdot i$
7. Suppose $L$ is a complex-number function that satisfies the dream conditions: $L(z+w)=L(z)+L(w)$ and $L(k z)=k(z)$ for all complex numbers $z, w$, and $k$. Show $L(z)=m z$ for a fixed complex-number $m$, the only type of complex-number function that satisfies these conditions?

Let $\mathrm{z}=a+b i, w=c+d i$, for addition property:
$L(z+w)=L(z)+L(w)=m(a+b i)+m(c+d i)=m[(a+c)+(b+d) i]$
$L(z+w)=L(a+b i+c+d i)=L((a+c)+(b+d) i)=m((a+c)+(b+d) i) ;$ they are the same.
For multiplication property:

$$
L(k z)=k L(z)=k m z=k m(a+b i)
$$

$L(k z)=L(k(a+b i))=m k(a+b i) ;$ they are the same.
8. For complex numbers, the linear transformation requires $L(x+y)=L(x)+L(y), L(a \cdot x)=a \cdot x$. Prove that in general $L\binom{x}{y}=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)\binom{x}{y}$ is a linear transformation, where $\binom{x}{y}$ represents $z=x+y i$.
For the addition property: $L\left(z_{1}+z_{2}\right)=L\left(\binom{x_{1}}{y_{1}}+\binom{x_{2}}{y_{2}}\right)=L\binom{x_{1}+x_{2}}{y_{1}+y_{2}}=\left(\begin{array}{ll}a & c \\ b & d\end{array}\right)\binom{x_{1}+x_{2}}{y_{1}+y_{2}}=$ $\binom{a\left(x_{1}+x_{2}\right)+c\left(y_{1}+y_{2}\right)}{b\left(x_{1}+x_{2}\right)+d\left(y_{1}+y_{2}\right)}$
$L\left(z_{1}+z_{2}\right)=L\left(z_{1}\right)+L\left(z_{2}\right)=\left(\begin{array}{ll}a & c \\ b & d\end{array}\right)\binom{x_{1}}{y_{1}}+\left(\begin{array}{ll}a & c \\ b & d\end{array}\right)\binom{x_{2}}{y_{2}}=\binom{a x_{1}+c y_{1}}{b x_{1}+d y_{1}}+\binom{a x_{2}+c y_{2}}{b x_{2}+d y_{2}}=$ $\binom{a x_{1}+c y_{1}+a x_{2}+c y_{2}}{b x_{1}+d y_{1}+b x_{2}+d y_{2}}=\binom{a\left(x_{1}+x_{2}\right)+c\left(y_{1}+y_{2}\right)}{b\left(x_{1}+x_{2}\right)+d\left(y_{1}+y_{2}\right)}$, the result is the same.

Now we need to prove the multiplication property.
$L(k z)=L\left(k\binom{x}{y}\right)=L\binom{k x}{k y}=\left(\begin{array}{ll}a & c \\ b & d\end{array}\right)\binom{k x}{k y}=\binom{a k x+c k y}{b k x+d k y}$
$L(k z)=k L(z)=k\left(\begin{array}{ll}a & c \\ b & d\end{array}\right)\binom{x}{y}=k\binom{a x+c y}{b x+d y}=\binom{a k x+c k y}{b k x+d k y} ;$ the result is the same.

## Student Outcomes

- Students use parallelograms to interpret the sum of two points in $\mathbb{R}^{2}$ geometrically. Students make a similar interpretation for the sum of two points in $\mathbb{R}^{3}$.
- Students recognize that scalar multiplication of points in $\mathbb{R}^{2}$ corresponds to a dilation. Students make a similar interpretation for scaling points in $\mathbb{R}^{3}$.


## Lesson Notes

In the Opening Exercise, students perform addition and scalar multiplication, first on complex numbers and then on ordered pairs. This leads into the discussion portion of the lesson, in which students explore geometric interpretations for these operations, first for points in $\mathbb{R}^{2}$ and then for points in $\mathbb{R}^{3}$. This lesson focuses on MP. 3 and MP. 7 as students construct arguments and make use of structure while studying points in space.

This is the first of several lessons that will have students graph in three-dimensional space. This may be difficult for some students to visualize, but you can make it easier by using a corner of your room. Allow the wall seam to be the $z$-axis, and one seam between the wall and the floor to be the $x$-axis while the other is the $y$-axis. You can even set up a tape with a number line in each direction. Have students experiment with plotting different points to get the feel of what graphing in 3-D looks like. It is important for students to understand that since we live in a 3-D world, we must have a way to explain points, objects, etc., in this 3-D space. There are many computer graphics programs that are available free of charge that can support student learning on 3-D graphing such as GeoGebra. Additionally, blank 3-D coordinate axes are supplied at the end of this lesson for use throughout this module.

The study of vectors will form a vital part of this course; notation for vectors varies across different contexts and curricula. These materials will refer to a vector as $\mathbf{V}$ (lowercase, bold, non-italicized) or $\langle 4,5\rangle$ or in column format, $\binom{4}{5}$ or $\left[\begin{array}{l}4 \\ 5\end{array}\right]$.
We will use "let $\mathbf{v}=\langle 4,5\rangle$ " to establish a name for the vector $\langle 4,5\rangle$.
This curriculum will avoid stating $\mathbf{v}=\langle 4,5\rangle$ without the word "let" preceding the equation when naming a vector unless it is absolutely clear from the context that we are naming a vector. However, we will continue to use the " $=$ " to describe vector equations, like $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$, as we have done with equations throughout all other grades.

We will refer to the vector from $A$ to $B$ as "vector $\overrightarrow{A B}$ " - notice, this is a ray with a full arrow. This notation is consistent with the way vectors are introduced in Grade 8 and is also widely used in post-secondary textbooks to describe both a ray and a vector depending on the context. To avoid confusion in this curriculum, the context will be provided or strongly implied, so it will be clear whether the full arrow indicates a vector or a ray. For example, when referring to a ray from $A$ passing through $B$, we will say "ray $\overrightarrow{A B}$ " and when referring to a vector from $A$ to $B$, we will say "vector $\overrightarrow{A B}$ ". Students should be encouraged to think about the context of the problem and not just rely on a hasty inference based on the symbol.

The magnitude of a vector will be signified as $\|\mathbf{v}\|$ (lowercase, bold, non-italicized).

## Classwork

In the Opening Exercise, students practice adding and multiplying complex numbers then ordered pairs displayed as matrices. This allows students to review previously taught skills and then explore the geometric interpretation of these operations in the lesson.

## Opening Exercise (2 minutes)

## Opening Exercise

Compute:
a. $(-\mathbf{1 0}+\mathbf{9 i})+(\mathbf{7}-\mathbf{5 i})$
$-3+4 i$
b. $\quad 5 \cdot(2+3 i)$
$10+15 i$
c. $\quad\binom{5}{-6}+\binom{2}{7}$
$\binom{7}{1}$
d. $\quad-2\binom{3}{-3}$
$\binom{-6}{6}$

## Discussion (8 minutes): Addition Viewed Geometrically

Here are two questions to ponder:

- When we add two points $x$ and $y$, we produce a third point $z=x+y$. How do you think $z$ is related geometrically to $x$ and $y$ ?
- When we multiply a point $x$ by a scalar (say 3 ), we produce another point $y=3 x$. How do you think $y$ is related to $x$ geometrically?
- Take a moment to write down any conjectures you may have about these questions, and then share your thinking with a partner. We'll explore these questions in detail in the upcoming discussion.
- Let $x=\binom{6}{3}$ and let $y=\binom{1}{5}$. Compute $z=x+y$.

$$
z=\binom{6}{3}+\binom{1}{5}=\binom{7}{8}
$$

## Scaffolding:

- To help students visualize the results, have them color code each vector always having the resulting vector a standard color. For example, the first vector could always be blue, the second vector green, and so on, but the resulting vector always red. This will help students visualize.
- When plotting in 3-D, continue color coding. Give students only positive values, and use the corners of the classroom to represent 3-D space having students create the 3 number lines, $x, y$, and $z$.
- For advanced learners, once they have shown understanding of 2-D, allow them to move to 3-D graphing, and try it without scaffolding. Challenge them to create a physical model of 3-D space using a shoe box or something in the classroom - this can be used later for all students.
- To discover the geometric relationships between $x, y$, and $z$, it's helpful to picture the box shown below. Note that the width of this rectangle is 6 , which is the first coordinate of $x$, and the height of the rectangle is 3 , which is the second coordinate of $x$. We can associate $x$ with the diagonal of this box. In fact, we might even go so far as to say that the diagonal is $x$. In some contexts it's useful to think of an ordered pair as a vector, which in this case means an arrow that originates at $\binom{0}{0}$ and terminates at $\binom{6}{3}$.



When we add $\mathbf{y}$ to $\mathbf{x}$, we are effectively performing a translation. Let's draw a picture of this:



- So the question still remains: how exactly are these three arrows related? Take a moment to discuss your thoughts with a partner, and then we'll continue.
- Let's see what happens when we perform the addition the other way around; that is, when we compute $\mathbf{y}+\mathbf{x}$ :


- To really see what's going on, let's look at $\mathbf{x}+\mathbf{y}$ and $\mathbf{y}+\mathbf{x}$ together in the same picture:


- Now a clear picture is starting to emerge. Make a conjecture and share it with a partner. What strikes you about this figure?
- It looks as though the points in the figure form a parallelogram.
- Can you make an argument showing that this is indeed a parallelogram?
- The upper and lower sides both have a rise-to-run ratio of 3: 6, which means they must be parallel.
- The left and right sides both have a rise-to-run ratio of 5: 1, which means they must also be parallel.
- Since both pairs of opposite sides of the figure are parallel, the figure is indeed a parallelogram.


## Exercises 1-3 (3 minutes)

## Exercises

1. Let $\mathrm{x}=\binom{5}{1}, \mathrm{y}=\binom{2}{3}$. Compute $\mathrm{z}=\mathrm{x}+\mathrm{y}$, and draw the associated parallelogram.

$\binom{7}{4}$
2. Let $x=\binom{-4}{2}, y=\binom{1}{3}$. Compute $z=x+y$, and draw the associated parallelogram.

$\binom{-3}{5}$
3. Let $\mathrm{x}=\binom{3}{2}, \mathrm{y}=\binom{-1}{-3}$. Compute $\mathrm{z}=\mathrm{x}+\mathrm{y}$, and draw the associated parallelogram.


$$
\binom{2}{-1}
$$

## Example 1 (3 minutes): Degenerate Parallelograms

- We've seen that adding two points gives a third point which lies at the vertex of a parallelogram. Can you think of a case where no parallelogram is produced? Think for a moment about this.
- Answers will vary. Take two points that lie on a line through the origin. These cases do not produce proper parallelograms.
- We call these "degenerate parallelograms."
- Let's try a few examples. For each problem below, compute the sum of the given points, and draw the associated picture.
- $z=\binom{4}{2}+\binom{2}{1}$

ㅁ $z=\binom{4}{2}+\binom{2}{1}=\binom{6}{3}$



- $\quad z=\binom{4}{2}+\binom{-2}{-1}$
- $z=\binom{4}{2}+\binom{-2}{-1}=\binom{2}{1}$


- $\quad z=\binom{4}{2}+\binom{-4}{-2}$
- $z=\binom{4}{2}+\binom{-4}{-2}=\binom{0}{0}$




## Discussion (6 minutes): Scalar Multiplication

- Now let's turn our attention to scalar multiplication. What is the geometric effect of taking $\mathbf{x}=\binom{2}{1}$ and scaling it by a factor of 3 ? Compute $\mathbf{z}=3\binom{2}{1}$, and then plot $\mathbf{x}$ and $\mathbf{z}$ in the plane.
- We have $\mathbf{z}=3\binom{2}{1}=\binom{6}{3}$.

- Describe the relationship between $\mathbf{x}$ and $3 \mathbf{x}$.
- It appears that $\mathbf{x}$ and $3 \mathbf{x}$ lie on a line through the origin and that the length of the arrow to $3 \mathbf{x}$ is 3 times as long as the arrow to $\mathbf{x}$.
- Does the picture below add to your understanding of the nature of scalar multiplication? Discuss what you see with a partner. Try to make connections between the picture and the underlying arithmetic involved in computing $3\binom{2}{1}$.

- The $x$-coordinate is scaled from 2 to $2 \cdot 3=6$, so the widths of the triangles are in a $3: 1$ ratio.
- The $y$-coordinate is scaled from 1 to $1 \cdot 3=3$, so the heights of the triangles are also in a 3:1 ratio.
- Since both of these triangles have a right angle, they are similar to each other by the SAS principle. This means that the lengths of the arrows to $3 \mathbf{x}$ and to $\mathbf{x}$ are indeed in a $3: 1$ ratio and that $0, \mathbf{x}$, and $3 \mathbf{x}$ do lie on a common line.
- What do you suppose would happen if we scaled $\mathbf{x}=\binom{2}{1}$ by -3 instead? Plot $\mathbf{x}$ and $-3 \mathbf{x}$ in the plane, and then describe the geometric relationship between these two vectors.
- Now we have $\mathbf{z}=-3\binom{2}{1}=\binom{-6}{-3}$.

- The vector $\mathbf{z}=-3 \mathbf{x}$ is 3 times as long as $\mathbf{x}$, but it lies on the opposite side of the origin.
- We call the new vector produced the resultant. Can you state the previous result using the word "resultant"?
- The resultant is 3 times as long as $\mathbf{x}$, but it lies on the opposite side of the origin.


## Exercises 4-6 (2 minutes)

4. Let $\mathrm{x}=\binom{3}{2}$. Compute $\mathrm{z}=2 \mathrm{x}$, and plot x and z in the plane.

$\binom{6}{4}$
5. Let $\mathrm{x}=\binom{-6}{3}$. Compute $\mathrm{z}=\frac{1}{3} \mathrm{x}$, and plot x and z in the plane.

$\binom{2}{-1}$
6. Let $\mathrm{x}=\binom{1}{-1}$. Compute $\mathrm{z}=-3 \mathrm{x}$, and plot x and z in the plane.

$\binom{-3}{3}$

## Example 2 (5 minutes): 3D Addition Viewed Geometrically

- Now that we've discussed the geometric interpretation of adding and scaling points in $\mathbb{R}^{2}$, let's turn our attention to $\mathbb{R}^{3}$. What do you think adding points in $\mathbb{R}^{3}$ will mean from a geometric point of view? What will scaling points mean? Make a conjecture, and then quickly share your thinking with a partner. We'll pursue these questions in detail in the upcoming discussion. If graphing software is available, demonstrate plotting points in 3-D space for the class, and then allow students to try. Being able to visualize points in 3-D space is difficult for students, and they will need lots of practice. Blank 3-D axes are included for teacher and student use at the end of this lesson.
- Let $\mathbf{x}=\left(\begin{array}{l}5 \\ 4 \\ 2\end{array}\right)$ and $\mathbf{y}=\left(\begin{array}{c}-3 \\ 2 \\ 3\end{array}\right)$. Compute $\mathbf{z}=\mathbf{x}+\mathbf{y}$, and then plot each of these three points.


- Can you tell how $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ are related? This picture should help:

- In the resultant, the $x$-value of 5 is translated in the $x$ direction -3 units, the $x$-value of 4 is translated in the $y$ direction 2 units, and the $x$-value of 2 is translated in the $z$ direction 3 units. $\mathbf{z}=\left(\begin{array}{l}2 \\ 6 \\ 5\end{array}\right)$.


## Exercises 7-8 (2 minutes)

7. Let $\mathrm{x}=\left(\begin{array}{l}3 \\ 1 \\ 1\end{array}\right)$ and $\mathrm{y}=\left(\begin{array}{l}1 \\ 3 \\ 1\end{array}\right)$. Compute $\mathrm{z}=\mathrm{x}+\mathrm{y}$, and then plot each of these three points.

$\left(\begin{array}{l}4 \\ 4 \\ 2\end{array}\right)$
8. Let $\mathrm{x}=\left(\begin{array}{l}\mathbf{3} \\ \mathbf{0} \\ \mathbf{0}\end{array}\right)$ and $\mathrm{y}=\left(\begin{array}{l}\mathbf{0} \\ 3 \\ \mathbf{0}\end{array}\right)$. Compute $\mathrm{z}=\mathrm{x}+\mathrm{y}$, and then plot each of these three points.

$\left(\begin{array}{l}3 \\ 3 \\ 0\end{array}\right)$

## Example 3 (5 minutes): Scalar Multiplication in 3D

- What do you suppose it means to perform scalar multiplication in a three-dimensional setting? Make a conjecture and briefly share it with a partner.
- Let $\mathbf{x}=\left(\begin{array}{l}1 \\ 3 \\ 2\end{array}\right)$. Compute $\mathbf{z}=3 \mathbf{x}$, and then plot each of these points. Describe what you see.

- The resultant was dilated by a factor of 3. The resultant lies on the same line through the origin as the vector.
- What do you suppose would happen if we scaled $\mathbf{x}=\left(\begin{array}{l}1 \\ 3 \\ 2\end{array}\right)$ by a factor of -1 ? What if the scale factor were -2 ?
- When we scale by -1 , we get $-1 \cdot \mathbf{x}=-1 \cdot\left(\begin{array}{l}1 \\ 3 \\ 2\end{array}\right)=\left(\begin{array}{l}-1 \\ -3 \\ -2\end{array}\right)$. The resultant is the same length as $\mathbf{x}$, but it points in the opposite direction.
- When we scale by -2 , we get $-2 \cdot \mathbf{x}=-2 \cdot\left(\begin{array}{l}1 \\ 3 \\ 2\end{array}\right)=\left(\begin{array}{l}-2 \\ -6 \\ -4\end{array}\right)$. The resultant is twice as long as $\mathbf{x}$, but again it points in the opposite direction.



## Exercises 9-10 (2 minutes)

9. Let $\mathrm{x}=\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$. Compute $\mathrm{z}=4 \mathrm{x}$, and then plot each of the three points.

$\left(\begin{array}{l}4 \\ 4 \\ 4\end{array}\right)$
10. Let $\mathrm{x}=\left(\begin{array}{l}2 \\ 4 \\ 4\end{array}\right)$. Compute $\mathrm{z}=-\frac{1}{2} \mathrm{x}$, and then plot each of the three points. Describe what you see.


$$
\left(\begin{array}{l}
-1 \\
-2 \\
-2
\end{array}\right)
$$

In the resultant, the direction has reversed, and the length of the ray is half the original.

## Closing (2 minutes)

Write a brief response to the following questions in your notebook, and then share your responses with a partner.

- What did you learn about the geometry of vector addition in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$ ?
- We can use the diagonal of a parallelogram to visualize the resultant vector of the sum of two vectors.
- What did you learn about the geometry of scalar multiplication of vectors in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$ ?
- Multiplying a vector by a scalar produces a resultant that is a dilation of the original vector.


## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 5: Coordinates of Points in Space

## Exit Ticket

1. Find the sum of the following, and plot the points and the resultant. Describe the geometric interpretation.
a. $\binom{3}{1}+\binom{1}{3}$
b. $\binom{2}{1}+\binom{3}{2}$
c. $\binom{-2}{4}+\binom{3}{-2}$
d. $\quad-\binom{3}{1}$
2. Find the sum of the following.
a. $\left(\begin{array}{l}3 \\ 1 \\ 3\end{array}\right)+\left(\begin{array}{l}1 \\ 3 \\ 1\end{array}\right)$
b. $\left(\begin{array}{l}2 \\ 0 \\ 1\end{array}\right)+\left(\begin{array}{l}0 \\ 2 \\ 1\end{array}\right)$

## Exit Ticket Sample Solutions

1. Find the sum of the following, and plot the points and the resultant. Describe the geometric interpretation.
a. $\quad\binom{3}{1}+\binom{1}{3}$

$\binom{4}{4}$ The two points, the resultant, and the origin formed a parallelogram.
b. $\quad\binom{2}{0}+\binom{1}{2}$

$\binom{3}{2}$ The two points, the resultant, and the origin formed a parallelogram.
c. $\quad\binom{-2}{4}+\binom{3}{-2}$

$\binom{1}{2}$ The two points, the resultant, and the origin formed a parallelogram.
d. $\quad-\binom{3}{1}$

$\binom{-3}{-1}$ The resultant is a vector of the same magnitude in the opposite direction.
2. Find the sum of the following.
a. $\quad\left(\begin{array}{l}3 \\ 1 \\ 3\end{array}\right)+\left(\begin{array}{l}1 \\ 3 \\ 1\end{array}\right)$
$\left(\begin{array}{l}4 \\ 4 \\ 4\end{array}\right)$
b. $\quad\left(\begin{array}{l}2 \\ 0 \\ 1\end{array}\right)+\left(\begin{array}{l}0 \\ 2 \\ 1\end{array}\right)$
$\left(\begin{array}{l}2 \\ 2 \\ 2\end{array}\right)$

## Problem Set Sample Solutions

1. Find the sum of the following complex numbers, and graph them on the complex plane. Trace the parallelogram that is formed by those two complex numbers, the resultant, and the origin. Describe the geometric interpretation.
a. $\quad x=\binom{2}{3}, y=\binom{3}{2}$

$\binom{5}{5}$ The two points, the resultant, and the origin formed a parallelogram.
b. $\quad \mathrm{x}=\binom{\mathbf{2}}{\mathbf{4}}, \mathrm{y}=\binom{-4}{2}$.

$\binom{-2}{6}$ The two points, the resultant, and the origin formed a parallelogram.
c. $\quad \mathrm{x}=\binom{\mathbf{2}}{1}, \mathrm{y}=\binom{-\mathbf{4}}{-\mathbf{2}}$.

$\binom{-2}{-1}$ The resultant is double the magnitude of the original vector in the opposite direction.
d. $\quad x=\binom{1}{2}, y=\binom{2}{4}$.

$\binom{3}{6}$ The resultant is triple the length of the original vector in the same direction.
2. Simplify and graph the complex number and the resultant. Describe the geometric effect on the complex number.
a. $\quad \mathrm{x}=\binom{\mathbf{1}}{2}, k=2, k \mathrm{x}=$ ?

$\binom{2}{4}$ The point is dilated by a factor of 2.
b. $\quad \mathrm{x}=\binom{-6}{3}, k=-\frac{1}{3}, k x=$ ?

$\binom{2}{-1}$ The point is dilated by a factor of $\frac{1}{3}$ and mapped to the other side of the origin on the same line.
c. $\quad \mathrm{x}=\binom{3}{-2}, k=0, k x=$ ?

| 0 | $X^{\prime}=(0,0)$ |  |  |
| :---: | :---: | :---: | :---: |
|  | d. 1 |  | 4 |
| -1 | $X=(3,-2)$ |  |  |
| -2 |  |  |  |

$\binom{0}{0}$ The point is mapped to the origin, $(0,0)$.
3. Find the sum of the following points, graph the points and the resultant on a 3-dimensional coordinate plane, and describe the geometric interpretation.
a. $\mathrm{x}=\left(\begin{array}{l}1 \\ 1 \\ 2\end{array}\right), \mathrm{y}=\left(\begin{array}{l}2 \\ 1 \\ 2\end{array}\right)$.

$\left(\begin{array}{l}3 \\ 2 \\ 4\end{array}\right)$ The two points, the resultant, and the origin are on the same plane and formed a parallelogram.
b. $\quad \mathrm{x}=\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right), \mathrm{y}=\left(\begin{array}{l}2 \\ 2 \\ 2\end{array}\right)$.

$\left(\begin{array}{l}3 \\ 3 \\ 3\end{array}\right)$ Three points all collapse onto the same line in space.
c. $\quad \mathrm{x}=\left(\begin{array}{l}\mathbf{2} \\ 0 \\ \mathbf{0}\end{array}\right), \mathrm{y}=\left(\begin{array}{l}\mathbf{0} \\ 2 \\ \mathbf{0}\end{array}\right)$.

$\left(\begin{array}{l}2 \\ 2 \\ 0\end{array}\right)$ The two points, the resultant, and the origin are on the same plane and formed a parallelogram.
4. Simplify the following, graph the point and the resultant on a 3-dimensional coordinate plane, and describe the geometric effect.
a. $\quad \mathrm{x}=\left(\begin{array}{l}2 \\ 1 \\ 1\end{array}\right), k=2, k x=$ ?

$\left(\begin{array}{l}4 \\ 2 \\ 2\end{array}\right)$ The point is dilated by a factor of 2.
b. $\quad \mathrm{x}=\left(\begin{array}{l}2 \\ 2 \\ 2\end{array}\right), k=-1, k \mathrm{x}=$ ?
 $\left(\begin{array}{l}-2 \\ -2 \\ -2\end{array}\right)$ The point is mapped to the other side of the origin on the same line.
5. Find
a. Any two different points whose sum is $\binom{0}{0}$.

Answers vary. $\binom{2}{2},\binom{-2}{-2}$
b. Any two different points in 3 dimensions whose sum is $\left(\begin{array}{l}0 \\ 0 \\ 0\end{array}\right)$.

Answers vary. $\left(\begin{array}{l}2 \\ 2 \\ 2\end{array}\right),\left(\begin{array}{l}-2 \\ -2 \\ -2\end{array}\right)$
c. Any two different complex numbers and their sum will create the degenerate parallelogram.

Answers vary. $\binom{1}{2},\binom{2}{4}$, as long as this answers are in the relation of $\frac{y_{1}}{x_{1}}=\frac{y_{2}}{x_{2}}$
d. Any two different points in 3 dimensions that their sum lie on the same line.

Answers vary. $\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right),\left(\begin{array}{l}2 \\ 2 \\ 2\end{array}\right)$
e. A point that is mapped to $\binom{1}{-3}$ after multiplying - 2 .

$$
\binom{-\frac{1}{2}}{\frac{3}{2}}
$$

f. A point that is mapped to $\left(\begin{array}{c}\frac{1}{2} \\ -2 \\ 4\end{array}\right)$ after multiplying $-\frac{2}{3}$.

$$
\left(\begin{array}{c}
-\frac{3}{4} \\
3 \\
-6
\end{array}\right)
$$

5. $\quad$ Given $x=\binom{2}{1}$ and $y=\binom{-4}{-2}$
a. Find $x+y$ and graph parallelogram that is formed by $x, y, x+y$, and the origin.

$\binom{-2}{-1}$
b. Transform the unit square by multiplying it by the matrix $\left(\begin{array}{ll}2 & -4 \\ 1 & -2\end{array}\right)$, and graph the result.
$\left(\begin{array}{ll}2 & -4 \\ 1 & -2\end{array}\right)\binom{0}{0}=\binom{0}{0},\left(\begin{array}{ll}2 & -4 \\ 1 & -2\end{array}\right)\binom{1}{0}=\binom{2}{1},\left(\begin{array}{ll}2 & -4 \\ 1 & -2\end{array}\right)\binom{1}{1}=\binom{-2}{-1},\left(\begin{array}{ll}2 & -4 \\ 1 & -2\end{array}\right)\binom{0}{1}=\binom{-4}{-2}$

c. What did you find from parts (a) and (b)?

They both have degenerate parallelograms.
d. What is the area of the parallelogram that is formed by part (a)?

The area is $\mathbf{0}$.
e. What is the determinant of the matrix $\left(\begin{array}{ll}2 & -4 \\ 1 & -2\end{array}\right)$ ?
$\left|\begin{array}{ll}2 & -4 \\ 1 & -2\end{array}\right|=2(-2)-1(-4)=-4+4=0$. The determinant is 0.
f. Based on observation, what can you say about the degenerate parallelograms in part (a) and part (b)? For two complex numbers, if $\frac{b_{1}}{a_{1}}=\frac{b_{2}}{a_{2}}$ and the determinant of the matrix is 0 , then they will produce degenerate parallelograms.
7. We learned that when multiplying -1 to a complex number $z$, for example $z=\binom{3}{2}$, the resulting complex number $z_{1}=\binom{-3}{-2}$ will be on the same line but on the opposite side of the origin. What matrix will produce the same effect? Verify your answer.

The matrix that will rotate $\pi$ radians is $\left[\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right]$.
$\left(\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right)\binom{3}{2}=\binom{-3}{-2}$
8. A point $z=\binom{\sqrt{2}}{\sqrt{2}}$ is transformed to $\binom{-2}{0}$. The final step of the transformation is adding the complex number $\binom{0}{-2}$. Describe a possible transformation that can get this result.
$\binom{-2}{0}+(-1)\binom{0}{-2}=\binom{-2}{2}$, from $\binom{\sqrt{2}}{\sqrt{2}}$ to $\binom{-2}{2}$, $a \frac{\pi}{2}$ radians counterclockwise rotation and a dilation with a factor of $\sqrt{2}$ are needed.
$a \frac{\pi}{2}$ radians counterclockwise rotation: $\left[\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right]$; dilation with a factor of $\sqrt{2}:\left[\begin{array}{cc}\sqrt{2} & 0 \\ 0 & \sqrt{2}\end{array}\right]$




## (8) Lesson 6: Linear Transformations as Matrices

## Student Outcomes

- Students verify that $3 \times 3$ matrices represent linear transformations from $\mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$. Students investigate the matrices associated with transformations such as dilations and reflections in a coordinate plane.
- Students explore properties of vector arithmetic, including the commutative, associative, and distributive properties.


## Classwork

## Opening Exercise (3 minutes)

In the Opening Exercise, students are reminded of their work with $2 \times 2$ matrices and prove that the matrix given represents a linear transformation.

## Opening Exercise

Let $A=\left(\begin{array}{ll}7 & -2 \\ 5 & -3\end{array}\right), x=\binom{x_{1}}{x_{2}}$, and $y=\binom{y_{1}}{y_{2}}$. Does this represent a linear transformation? Explain how you know.
A linear transformation satisfies the following conditions: $L(x+y)=L(x)+L(y)$ and $L(k x)=k L(x)$.

$$
\begin{gathered}
A *(x+y)=\left(\begin{array}{ll}
7 & -2 \\
5 & -3
\end{array}\right)\binom{x_{1}+y_{1}}{x_{2}+y_{2}}=\binom{7\left(x_{1}+y_{1}\right)-2\left(x_{2}+y_{2}\right)}{5\left(x_{1}+y_{1}\right)-3\left(x_{2}+y_{2}\right)}=\binom{7 x_{1}+7 y_{1}-2 x_{2}-2 y_{2}}{5 x_{1}+5 y_{1}-3 x_{2}-3 y_{2}} \\
A(x)=\left(\begin{array}{ll}
7 & -2 \\
5 & -3
\end{array}\right)\binom{x_{1}}{x_{2}}=\binom{7\left(x_{1}\right)-2\left(x_{2}\right)}{5\left(x_{1}\right)-3\left(x_{2}\right)} \\
A(y)=\left(\begin{array}{ll}
7 & -2 \\
5 & -3
\end{array}\right)\binom{y_{1}}{y_{2}}=\binom{7\left(y_{1}\right)-2\left(y_{2}\right)}{5\left(y_{1}\right)-3\left(y_{2}\right)} \\
A(x)+A(y)=\binom{7\left(x_{1}\right)-2\left(x_{2}\right)}{5\left(x_{1}\right)-3\left(x_{2}\right)}+\binom{7\left(y_{1}\right)-2\left(y_{2}\right)}{5\left(y_{1}\right)-3\left(y_{2}\right)}=\binom{7\left(x_{1}\right)-2\left(x_{2}\right)+7\left(y_{1}\right)-2\left(y_{2}\right)}{5\left(x_{1}\right)-3\left(x_{2}\right)+5\left(y_{1}\right)-3\left(y_{2}\right)}
\end{gathered}
$$

$A(x+y)=A(x)+A(y)$ by the distributive property.

## Discussion: Linear Transformations $L: \mathbb{R}^{\mathbf{3}} \rightarrow \mathbb{R}^{\mathbf{3}}$ (7 minutes)

- In previous lessons, we have seen that $2 \times 2$ matrices represent linear transformations from $\mathbb{R}^{2} \rightarrow \mathbb{R}^{2}$. Do you think that a $3 \times 3$ matrix would represent a linear transformation from $\mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ ? Can you prove it?
- Let $A=\left(\begin{array}{ccc}7 & -2 & -7 \\ 5 & -3 & -6 \\ -10 & -6 & 6\end{array}\right)$, and let $x=\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)$ and $y=\left(\begin{array}{l}y_{1} \\ y_{2} \\ y_{3}\end{array}\right)$ be points in $\mathbb{R}^{3}$. Show that $A(x+y)=A x+A y$.
- $A \cdot(x+y)=\left(\begin{array}{ccc}7 & -2 & -7 \\ 5 & -3 & -6 \\ -10 & -6 & 6\end{array}\right)\left(\begin{array}{c}x_{1}+y_{1} \\ x_{2}+y_{2} \\ x_{3}+y_{3}\end{array}\right)=\left(\begin{array}{c}7\left(x_{1}+y_{1}\right)-2\left(x_{2}+y_{2}\right)-7\left(x_{3}+y_{3}\right) \\ 5\left(x_{1}+y_{1}\right)-3\left(x_{2}+y_{2}\right)-6\left(x_{3}+y_{3}\right) \\ -10\left(x_{1}+y_{1}\right)-6\left(x_{2}+y_{2}\right)+6\left(x_{3}+y_{3}\right)\end{array}\right)$
- $A x=\left(\begin{array}{ccc}7 & -2 & -7 \\ 5 & -3 & -6 \\ -10 & -6 & 6\end{array}\right)\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)=\left(\begin{array}{c}7 x_{1}-2 x_{2}-7 x_{3} \\ 5 x_{1}-3 x_{2}-6 x_{3} \\ -10 x_{1}-6 x_{2}+6 x_{3}\end{array}\right)$
- $\quad A y=\left(\begin{array}{ccc}7 & -2 & -7 \\ 5 & -3 & -6 \\ -10 & -6 & 6\end{array}\right)\left(\begin{array}{l}y_{1} \\ y_{2} \\ y_{3}\end{array}\right)=\left(\begin{array}{c}7 y_{1}-2 y_{2}-7 y_{3} \\ 5 y_{1}-3 y_{2}-6 y_{3} \\ -10 y_{1}-6 y_{2}+6 y_{3}\end{array}\right)$
- $\quad A x+A y=\left(\begin{array}{c}7 x_{1}-2 x_{2}-7 x_{3}+7 y_{1}-2 y_{2}-7 y_{3} \\ 5 x_{1}-3 x_{2}-6 x_{3}+5 y_{1}-3 y_{2}-6 y_{3} \\ -10 x_{1}-6 x_{2}+6 x_{3}+-10 y_{1}-6 y_{2}+6 y_{3}\end{array}\right)$
- We can see that $A \cdot(x+y)=A x+A y$ by the distributive property.
- Now show that $A(k \cdot x)=k \cdot A x$, where $k$ represents a real number.

$$
\begin{array}{ll} 
& A(k \cdot x)=\left(\begin{array}{ccc}
7 & -2 & -7 \\
5 & -3 & -6 \\
-10 & -6 & 6
\end{array}\right)\left(\begin{array}{l}
k \cdot x_{1} \\
k \cdot x_{2} \\
k \cdot x_{3}
\end{array}\right)=\left(\begin{array}{c}
7 k \cdot x_{1}-2 k \cdot x_{2}-7 k \cdot x_{3} \\
5 k \cdot x_{1}-3 k \cdot x_{2}-6 k \cdot x_{3} \\
-10 k \cdot x_{1}-6 k \cdot x_{2}+6 k \cdot x_{3}
\end{array}\right) \\
& k \cdot A x=k \cdot\left(\begin{array}{ccc}
7 & -2 & -7 \\
5 & -3 & -6 \\
-10 & -6 & 6
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right)=k \cdot\left(\begin{array}{c}
7 x_{1}-2 x_{2}-7 x_{3} \\
5 x_{1}-3 x_{2}-6 x_{3} \\
-10 x_{1}-6 x_{2}+6 x_{3}
\end{array}\right)=\left(\begin{array}{c}
7 k \cdot x_{1}-2 k \cdot x_{2}-7 k \cdot x_{3} \\
5 k \cdot x_{1}-3 k \cdot x_{2}-6 k \cdot x_{3} \\
-10 k \cdot x_{1}-6 k \cdot x_{2}+6 k \cdot x_{3}
\end{array}\right)
\end{array}
$$

- We see that $A(k \cdot x)=k \cdot A x$.
- Let's briefly take a closer look at the reasoning used here. What properties of arithmetic are involved in comparing, for example, $7 \cdot\left(k \cdot x_{1}\right)$ with $k \cdot\left(7 \cdot x_{1}\right)$ ? In other words, how exactly do we know these are the same number?
- We can use the associative property and the commutative property to know these are the same number.
- Okay, now we've proved that the transformation $A(x)$ does indeed represent a linear transformation, at least for this particular $3 \times 3$ matrix. We could use exactly the same reasoning to show that this is also true for any $3 \times 3$ matrix. In fact, mathematicians showed in the 1800s that every linear transformation from $\mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ can be represented by a $3 \times 3$ matrix! This is one reason why the theory of matrices is so powerful.


## Exploratory Challenge 1: The Geometry of 3D Matrix Transformations (10 minutes)

In this activity, students work in groups of four on each of the challenges below. The teacher should monitor the progress of the class as they work independently. For each challenge, the teacher should select a particular group to present their findings to the class at the conclusion of the activity. The teacher should allow approximately 7 minutes for students to work in their groups, and approximately 3 minutes for students to present their findings.

## Scaffolding:

- For students still struggling with matrix multiplication, give them a matrix with blanks to complete, so they can see how many terms they should have in each row such as


Students would fill in the entry for each blank. For example,

$$
\left.\begin{array}{l}
\left(\begin{array}{cc}
1 & 3 \\
-5 & 2 \\
4 & 7
\end{array}\right)\binom{2}{-1}= \\
\left(\frac{1 \cdot 2}{\frac{-5 \cdot 2}{3 \cdot-1}}+\underline{2 \cdot-1}\right. \\
\underline{4 \cdot 2}+\underline{7 \cdot-1}
\end{array}\right) .
$$

- Advanced learners can instead work with a general $3 \times 3$ matrix as linearity conditions are explored. The reasoning is exactly the same.


## Exploratory Challenge 1: The Geometry of 3D Matrix Transformations

a. What matrix in $\mathbb{R}^{2}$ serves the role of $\mathbf{1}$ in the real number system? What is that role?

$$
\left(\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right) \text { serves the role of the multiplicative identity. }
$$

- Find a matrix $A$ such that $A x=x$ for each point $x$ in $\mathbb{R}^{3}$. Describe the geometric effect of this transformation. How might we call such a matrix?
- We can choose $A=\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$.
$\square \quad A x=\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)=\left(\begin{array}{l}1 \cdot x_{1}+0 \cdot x_{2}+0 \cdot x_{3} \\ 0 \cdot x_{1}+1 \cdot x_{2}+0 \cdot x_{3} \\ 0 \cdot x_{1}+0 \cdot x_{2}+1 \cdot x_{3}\end{array}\right)=\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)$
- The name identity matrix makes sense since the output is identical to the input.
b. What matrix in $\mathbb{R}^{2}$ serves the role of 0 in the real number system? What is that role?
$\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right)$ serves the role of the additive identity.
- Find a matrix $A$ such that $A x=0$ for each point $x$ in $\mathbb{R}^{3}$. Describe the geometric effect of this transformation.
- $\left(\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right)\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)=\left(\begin{array}{l}0 \\ 0 \\ 0\end{array}\right)$. This transformation takes every point in space and maps it to the origin.
c. What is the result of scalar multiplication in $\mathbb{R}^{2}$ ?

Multiplying by a scalar, $k$, dilates each point in $\mathbb{R}^{2}$ by a factor of $k$.

- Find a matrix $A$ such that the mapping $x \mapsto A x$ dilates each point in $\mathbb{R}^{3}$ by a factor of 3 .
- $\left(\begin{array}{lll}3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3\end{array}\right)\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)=\left(\begin{array}{l}3 x_{1} \\ 3 x_{2} \\ 3 x_{3}\end{array}\right)=3\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)$. Thus each point in $\mathbb{R}^{3}$ is dilated by a factor of 3 .
d. Given a complex number $a+b i$, what represents the transformation of that point across the real axis?

The conjugate, $a-b i$.

- Investigate the geometric effects of the transformation $\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{array}\right)\left(\begin{array}{l}x_{1} \\ x_{2} \\ x_{3}\end{array}\right)$. Illustrate your findings with one or more specific examples.

$$
\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right)\left(\begin{array}{l}
5 \\
3 \\
4
\end{array}\right)=\left(\begin{array}{c}
5 \\
3 \\
-4
\end{array}\right)
$$

- This transformation induces a reflection across the xy-plane.



## Exploratory Challenge 2: Properties of Vector Arithmetic (18 minutes)

In this activity, students work in groups of four on one of the challenges below. The teacher should assign different challenges to different groups. When a group finishes one of the challenges, they may choose one of the other challenges to work on as time allows.

The teacher should monitor the progress of the class as they work independently. For each challenge, the teacher should select a particular group to present their findings to the class at the conclusion of the activity.

For each challenge, students should explore the question (a) from an algebraic point of view and (b) from a geometric point of view.

The teacher should allow approximately 10 minutes for students to work in their groups and organize their presentations. Allow approximately 9 minutes for students to present their findings.

## Exploratory Challenge 2: Properties of Vector Arithmetic

a. Is vector addition commutative? That is, does $x+y=y+x$ for each pair of points in $\mathbb{R}^{2}$ ? What about points in $\mathbb{R}^{3}$ ?

We want to show that $x+y=y+x$. First let's look at this problem algebraically.
When we compute $x+y$, we get $\binom{x_{1}}{x_{2}}+\binom{y_{1}}{y_{2}}=\binom{x_{1}+y_{1}}{x_{2}+y_{2}}$.
Now let's compute $y+x$. This time we get $\binom{y_{1}}{y_{2}}+\binom{x_{1}}{x_{2}}=\binom{y_{1}+x_{1}}{y_{2}+x_{2}}$.
The commutative property guarantees that the two components of these vectors are equal. For example, $x_{1}+y_{1}=y_{1}+x_{1}$. Since both components are equal, the vectors themselves must be equal. We can make a similar argument to show that vector addition in $\mathbb{R}^{3}$ is commutative.
Next let's see what all of this means geometrically. Let's examine the points $x=\binom{5}{1}$ and $y=\binom{3}{4}$.



| Lesson 6: | Linear Transformations as Matrices |
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Thinking in terms of translations, the sum $x+y=\binom{5}{1}+\binom{3}{4}$ amounts to moving 5 right and 1 up, followed by a movement that takes us 3 right and 4 up.

On the other hand, the sum $y+x=\binom{3}{4}+\binom{5}{1}$ amounts to moving 3 right and 4 up, followed by a movement that takes us 5 right and 1 up.

It should be clear that, in both cases, we've moved a total of 8 units to the right and 5 units up. So it makes sense that, when viewed as translations, these two sums are the same.


Lastly, let's consider vector addition in $\mathbb{R}^{3}$ from a geometric point of view.


Let's consider the vectors $\left(\begin{array}{l}5 \\ 4 \\ 2\end{array}\right)$ and $\left(\begin{array}{c}-3 \\ 2 \\ 3\end{array}\right)$. To show that addition is commutative, let's imagine that Jack and
Jill are moving around a building. We'll send them on two different journeys and see if they reach the same destination.

Jack's movements will model the sum $\left(\begin{array}{l}5 \\ 4 \\ 2\end{array}\right)+\left(\begin{array}{c}-3 \\ 2 \\ 3\end{array}\right)$. He walks 5 units east, 4 units north, and then goes up two flights of stairs. Next, he goes 3 units west, 2 units north, and then goes up 3 flights of stairs.

Jill starts at the same place as Jack. Her movements will model the sum $\left(\begin{array}{c}-3 \\ 2 \\ 3\end{array}\right)+\left(\begin{array}{l}5 \\ 4 \\ 2\end{array}\right)$. She walks 3 units
west, 2 units north, and then goes up 3 flights of stairs. Then she goes 5 units east, 4 units north, and then climbs 2 flights of stairs. It should be clear from this description that Jack and Jill both end up in a location that is 2 units east, 6 units north, and 5 stories above their starting point. In particular, they end up in the same spot!

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b. Is vector addition associative? That is, does $(x+y)+r=x+(y+r)$ for any three points in $\mathbb{R}^{2}$ ? What about points in $\mathbb{R}^{3}$ ?

Let's check to see if $(x+y)+r=x+(y+r)$ for points in $\mathbb{R}^{2}$.

$$
\begin{aligned}
& (x+y)+r=\left(\binom{x_{1}}{x_{2}}+\binom{y_{1}}{y_{2}}\right)+\binom{r_{1}}{r_{2}}=\binom{x_{1}+y_{1}}{x_{2}+y_{2}}+\binom{r_{1}}{r_{2}} \\
& x+(y+r)=\binom{x_{1}}{x_{2}}+\left(\binom{y_{1}}{y_{2}}+\binom{r_{1}}{r_{2}}\right)=\binom{x_{1}}{x_{2}}+\binom{y_{1}+r_{1}}{y_{2}+r_{2}}
\end{aligned}
$$

The associative property guarantees that each of the components are equal. Let's look at the first coordinate. For example, we have $\left(x_{1}+y_{1}\right)+r_{1}$, which is indeed the same as $x_{1}+\left(y_{1}+r_{1}\right)$. We can make a similar argument for vectors in $\mathbb{R}^{3}$.

Next let's examine the problem from a geometric point of view.



These pictures make it clear that these two sums should be the same. In both cases, the overall journey is equivalent to following each of the three paths separately. Now let's look at the 3-dimensional case.



As in the 2-dimsensional case, we are simply reaching the same location in two different ways, both of which are equivalent to following the three individual paths separately.
c. Does the distributive property apply to vector arithmetic? That is, does $k \cdot(x+y)=k x+k y$ for each pair of points in $\mathbb{R}^{2}$ ? What about points in $\mathbb{R}^{3}$ ?

We want to show that $k \cdot(x+y)=k x+k y$.
We have $k \cdot(x+y)=k \cdot\left(\binom{x_{1}}{x_{2}}+\binom{y_{1}}{y_{2}}\right)=k \cdot\binom{x_{1}+y_{1}}{x_{2}+y_{2}}=\binom{k \cdot\left(x_{1}+y_{1}\right)}{k \cdot\left(x_{2}+y_{2}\right)}$.
Next we have $k x+k y=k\binom{x_{1}}{x_{2}}+k\binom{y_{1}}{y_{2}}=\binom{k x_{1}}{k x_{2}}+\binom{k y_{1}}{k y_{2}}=\binom{k x_{1}+k y_{1}}{k x_{2}+k y_{2}}$.
The distributive property guarantees that each of the components are equal. For example,
$\boldsymbol{k} \cdot\left(x_{1}+y_{1}\right)=k x_{1}+k y_{1}$. This means that the vectors themselves are equal. A similar argument can be used to show that this property holds for vectors in $\mathbb{R}^{3}$.

Now let's examine this property geometrically.


Suppose that Jack walks to the point marked $x+y$ and then walks that distance again, ending at the spot marked $2(x+y)$. Now suppose Jill walks to the spot marked $2 x$ and then follows the path labeled $2 y$. The picture makes it clear that Jack and Jill end up at the same spot.

Next let's examine the three-dimensional case. Again, the picture makes it clear that the distributive property holds.

Lesson 6:
Date:
d. Is there an identity element for vector addition? That is, can you find a point $a$ in $\mathbb{R}^{2}$ such that $x+a=x$ for every point $x$ in $\mathbb{R}^{2}$ ? What about for $\mathbb{R}^{3}$ ?

We have $x+a=\binom{x_{1}}{x_{2}}+\binom{a_{1}}{a_{2}}=\binom{x_{1}+a_{1}}{x_{2}+a_{2}}$.
If this sum is equal to $x$, then we have $\binom{x_{1}+a_{1}}{x_{2}+a_{2}}=\binom{x_{1}}{x_{2}}$. This means that $x_{1}+a_{1}=x_{1}$, which implies that $a_{1}=0$. In the same way, we can show that $a_{2}=0$.

Thus the identity element for $\mathbb{R}^{2}$ is $\binom{0}{0}$. Similarly, the identity element for $\mathbb{R}^{3}$ is $\left(\begin{array}{l}0 \\ 0 \\ 0\end{array}\right)$.
Let's think about this geometrically. $\binom{2}{5}+\binom{0}{0}$ means to go to $(2,5)$ and translate by $(0,0)$. That is, we don't translate at all.

e. Does each element in $\mathbb{R}^{2}$ have an additive inverse? That is, if you take a point $a$ in $\mathbb{R}^{2}$, can you find a second point $b$ such that $a+b=0$ ?

We have $a+b=\binom{a_{1}}{a_{2}}+\binom{b_{1}}{b_{2}}=\binom{a_{1}+b_{1}}{a_{2}+b_{2}}$. If this sum is equal to 0 , then $\binom{a_{1}+b_{1}}{a_{2}+b_{2}}=\binom{0}{0}$, which means that $a_{1}+b_{1}=0$. We can conclude that $b_{1}=-a_{1}$. Similarly, $b_{2}=-a_{2}$.

Thus the additive inverse of $\binom{a_{1}}{b_{1}}$ is $\binom{-a_{1}}{-b_{1}}$. For instance, the additive inverse of $\binom{3}{5}$ is $\binom{-3}{-5}$. Geometrically, we see that these vectors have the same length but point in opposite directions. Thus, their sum is $(0,0)$. This makes sense because if Jack walks towards a spot that is 3 miles east and 5 miles north of the origin, and then walks towards the spot that is 3 miles west and 5 miles south of the origin, then he'll end up right back at the origin!


## Closing (2 minutes)

Students should write a brief response to the following questions in their notebooks.

- How can we represent linear transformations from $\mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ ?
- These transformations can be represented using $3 \times 3$ matrices.
- Which properties of real numbers are true for vectors in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$ ?
- Vector addition is commutative and associative. There is a 0 vector, and every vector has an additive inverse. Scalar multiplication distributes over vector addition.


## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 6: Linear Transformations as Matrices

## Exit Ticket

1. Given $x=\binom{1}{2}, y=\binom{4}{2}, z=\binom{3}{1}$, and $k=-2$.
a. Verify the associative property holds: $x+(y+z)=(x+y)+z$.
b. Verify the distributive property holds: $k(x+y)=k x+k y$.
2. Describe the geometric effect of the transformation on the $3 \times 3$ identity matrix given by the following matrices.
a. $\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2\end{array}\right]$
b. $\left[\begin{array}{ccc}-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$
c. $\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1\end{array}\right]$
d. $\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{array}\right]$

## Exit Ticket Sample Solutions

1. Given $x=\binom{1}{2}, y=\binom{4}{2}, z=\binom{3}{1}$, and $k=-2$.
a. Verify the associative property holds: $x+(y+z)=(x+y)+z$.

$$
\begin{aligned}
& x+(y+z)=\binom{1}{2}+\left(\binom{4}{2}+\binom{3}{1}\right)=\binom{1}{2}+\binom{7}{3}=\binom{8}{5} \\
& (x+y)+z=\left(\binom{1}{2}+\binom{4}{2}\right)+\binom{3}{1}=\binom{5}{4}+\binom{3}{1}=\binom{8}{5}
\end{aligned}
$$

b. Verify the distributive property holds: $\boldsymbol{k}(\boldsymbol{x}+\boldsymbol{y})=\boldsymbol{k} \boldsymbol{x}+\boldsymbol{k y}$.

$$
\begin{aligned}
& k(x+y)=-2\left(\binom{1}{2}+\binom{4}{2}\right)=-2\binom{5}{4}=\binom{-10}{-8} \\
& k x+k y=-2\binom{1}{2}+(-2)\binom{4}{2}=\binom{-2}{-4}+\binom{-8}{-4}=\binom{-10}{-8}
\end{aligned}
$$

2. Describe the geometric effect of the transformation on the $3 \times 3$ identity matrix given by the following matrices.
a. $\quad\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2\end{array}\right]$

It dilates the point by a factor of 2 .
b. $\quad\left[\begin{array}{ccc}-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$

It reflects the point about the yz-plane.
c. $\quad\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1\end{array}\right]$

It reflects the point about the $x z$-plane.
d. $\quad\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{array}\right]$.

It reflects the point about the xy-plane.

## Problem Set Sample Solutions

1. Show that the associative property, $x+(y+z)=(x+y)+z$, holds for the following.
a. $\quad x=\binom{3}{-2}, y=\binom{-4}{2}, z=\binom{-1}{5}$

$$
\begin{aligned}
& \binom{3}{-2}+\left(\binom{-4}{2}+\binom{-1}{5}\right)=\binom{3}{-2}+\binom{-5}{7}=\binom{-2}{5} \\
& \left(\binom{3}{-2}+\binom{-4}{2}\right)+\binom{-1}{5}=\binom{-1}{0}+\binom{-1}{5}=\binom{-2}{5}
\end{aligned}
$$

b. $\quad x=\left(\begin{array}{c}2 \\ -2 \\ 1\end{array}\right), y=\left(\begin{array}{c}0 \\ 5 \\ -2\end{array}\right), z=\left(\begin{array}{c}3 \\ 0 \\ -3\end{array}\right)$

$$
\begin{aligned}
& \left(\begin{array}{c}
2 \\
-2 \\
1
\end{array}\right)+\left(\left(\begin{array}{c}
0 \\
5 \\
-2
\end{array}\right)+\left(\begin{array}{c}
3 \\
0 \\
-3
\end{array}\right)\right)=\left(\begin{array}{c}
2 \\
-2 \\
1
\end{array}\right)+\left(\begin{array}{c}
3 \\
5 \\
-5
\end{array}\right)=\left(\begin{array}{c}
5 \\
3 \\
-4
\end{array}\right) \\
& \left(\left(\begin{array}{c}
2 \\
-2 \\
1
\end{array}\right)+\left(\begin{array}{c}
0 \\
5 \\
-2
\end{array}\right)\right)+\left(\begin{array}{c}
3 \\
0 \\
-3
\end{array}\right)=\left(\begin{array}{c}
2 \\
3 \\
-1
\end{array}\right)+\left(\begin{array}{c}
3 \\
0 \\
-3
\end{array}\right)=\left(\begin{array}{c}
5 \\
3 \\
-4
\end{array}\right)
\end{aligned}
$$

2. Show that the distributive property, $k(x+y)=k x+k y$, holds for the following.
a. $\quad x=\binom{5}{-3}, y=\binom{-2}{4}, k=-2$

$$
\begin{aligned}
& -2\left(\binom{5}{-3}+\binom{-2}{4}\right)=-2\binom{3}{1}=\binom{-6}{-2} \\
& -2\binom{5}{-3}+(-2)\binom{-2}{4}=\binom{-10}{6}+\binom{4}{-8}=\binom{-6}{-2}
\end{aligned}
$$

b. $\quad x=\left(\begin{array}{c}3 \\ -2 \\ 5\end{array}\right), y=\left(\begin{array}{c}-4 \\ 6 \\ -7\end{array}\right), k=-3$

$$
\begin{aligned}
& -3\left(\left(\begin{array}{c}
3 \\
-2 \\
5
\end{array}\right)+\left(\begin{array}{c}
-4 \\
6 \\
-7
\end{array}\right)\right)=-3\left(\begin{array}{c}
-1 \\
4 \\
-2
\end{array}\right)=\left(\begin{array}{c}
3 \\
-12 \\
6
\end{array}\right) \\
& -3\left(\begin{array}{c}
3 \\
-2 \\
5
\end{array}\right)+(-3)\left(\begin{array}{c}
-4 \\
6 \\
-7
\end{array}\right)=\left(\begin{array}{c}
-9 \\
6 \\
-15
\end{array}\right)+\left(\begin{array}{c}
12 \\
-18 \\
21
\end{array}\right)=\left(\begin{array}{c}
3 \\
-12 \\
6
\end{array}\right)
\end{aligned}
$$

3. Compute the following.
a. $\quad\left(\begin{array}{lll}1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 2 & 3\end{array}\right)\left(\begin{array}{l}2 \\ 1 \\ 3\end{array}\right)$
$\left(\begin{array}{c}8 \\ 7 \\ 15\end{array}\right)$
b. $\quad\left(\begin{array}{ccc}-1 & 2 & 3 \\ 3 & 1 & -2 \\ 1 & -2 & 3\end{array}\right)\left(\begin{array}{c}1 \\ -2 \\ 1\end{array}\right)$
$\left(\begin{array}{c}-2 \\ -1 \\ 6\end{array}\right)$
c. $\quad\left(\begin{array}{lll}1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 3\end{array}\right)\left(\begin{array}{l}2 \\ 1 \\ 2\end{array}\right)$
$\left(\begin{array}{c}6 \\ 1 \\ 10\end{array}\right)$
4. Let $x=\left(\begin{array}{l}3 \\ 1 \\ 2\end{array}\right)$. Compute $L(x)=\left[\begin{array}{lll}a & d & g \\ b & e & h \\ c & f & i\end{array}\right] \cdot x$, plot the points, and describe the geometric effect to $x$.
a. $\quad\left[\begin{array}{ccc}-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$
$\left(\begin{array}{c}-3 \\ 1 \\ 2\end{array}\right)$. It is reflected about the $y z$-plane.

b. $\quad\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2\end{array}\right]$
$\left(\begin{array}{l}6 \\ 2 \\ 4\end{array}\right)$. It is dilated by a factor of 2 .

c. $\quad\left[\begin{array}{lll}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]$
$\left(\begin{array}{l}1 \\ 3 \\ 2\end{array}\right)$. It is reflected about the vertical plane through
the line $y=x$ on the $x y$-plane.

d. $\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{array}\right]$
$\left(\begin{array}{l}3 \\ 2 \\ 1\end{array}\right)$. It is reflected about the vertical plane through the line
$y=z$ on the $z y$-plane.

5. Let $x=\left(\begin{array}{l}3 \\ 1 \\ 2\end{array}\right)$. Compute $L(x)=\left[\begin{array}{lll}a & d & g \\ b & e & h \\ c & f & i\end{array}\right] \cdot x$. Describe the geometric effect to $x$.
a. $\quad\left[\begin{array}{lll}0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$
$\left(\begin{array}{l}0 \\ 0 \\ 0\end{array}\right)$. It is mapped to the origin.
b. $\quad\left[\begin{array}{lll}3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3\end{array}\right]$
$\left(\begin{array}{l}9 \\ 3 \\ 6\end{array}\right)$. It is dilated by a factor of 3 .
c. $\quad\left[\begin{array}{ccc}-1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{array}\right]$
$\left(\begin{array}{l}-3 \\ -1 \\ -2\end{array}\right)$. It is mapped to the opposite side of the origin on the same line that is equal distance from the origin.
d. $\left[\begin{array}{ccc}-1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$
$\left(\begin{array}{c}-3 \\ 1 \\ 2\end{array}\right)$. It is reflected about the $y z$-plane.
e. $\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1\end{array}\right]$
$\left(\begin{array}{c}3 \\ -1 \\ 2\end{array}\right)$. It is reflected about the $x z$-plane.
f. $\quad\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{array}\right]$
$\left(\begin{array}{c}3 \\ 1 \\ -2\end{array}\right)$. It is reflected about the xy-plane.
g. $\quad\left[\begin{array}{lll}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]$
$\left(\begin{array}{l}1 \\ 3 \\ 2\end{array}\right)$. It is reflected about the vertical plane through the line $y=x$ on the $x y$-plane.
h. $\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{array}\right]$
$\left(\begin{array}{l}3 \\ 2 \\ 1\end{array}\right)$. It is reflected about the vertical plane through the line $y=z$ on the $y z$-plane.
i. $\quad\left[\begin{array}{lll}0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0\end{array}\right]$
$\left(\begin{array}{l}2 \\ 1 \\ 3\end{array}\right)$. It is reflected about the vertical plane through the line $x=z$ on the $x z$-plane.
6. Find the matrix that will transform the point $x=\left(\begin{array}{l}1 \\ 3 \\ 2\end{array}\right)$ to the following point:
a. $\quad\left(\begin{array}{c}-4 \\ -12 \\ -8\end{array}\right)$
$\left[\begin{array}{ccc}-4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4\end{array}\right]$
b. $\quad\left(\begin{array}{l}3 \\ 1 \\ 2\end{array}\right)$
$\left[\begin{array}{lll}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]$
7. Find the matrix/matrices that will transform the point $x=\left(\begin{array}{l}2 \\ 3 \\ 1\end{array}\right)$ to the following point:
a. $\quad x^{\prime}=\left(\begin{array}{l}6 \\ 4 \\ 2\end{array}\right)$
b. $\quad \mathrm{x}^{\prime}=\left(\begin{array}{c}-1 \\ 3 \\ 2\end{array}\right)$
$\left[\begin{array}{lll}0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2\end{array}\right]$
$\left[\begin{array}{ccc}0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0\end{array}\right]$

## (P) Lesson 7: Linear Transformations Applied to Cubes

## Student Outcomes

- Students construct $3 \times 3$ matrices $A$ so that the linear transformation $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=A\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ has a desired geometric effect.
- Students identify the geometric effect of the linear transformation $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=A\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ based on the structure of the $3 \times 3$ matrix $A$.


## Lesson Notes

In this lesson, students will examine the geometric effects of linear transformations in $\mathbb{R}^{3}$ induced by various $3 \times 3$ matrices on the unit cube. This lesson extends work done in Module 1 in which students studied analogous transformations in $\mathbb{R}^{2}$ using $2 \times 2$ matrices. This lesson is written for classes that have access to the GeoGebra demo TransformCubes (http://eureka-math.org/G12M2L7/geogebra-TransformCubes). This GeoGebra demo will allow the students to input values in a $3 \times 3$ matrix between -5 and 5 in increments of 0.1 and visually see the effect of the transformation induced by the matrix on the unit cube. If there aren't enough computers for students to access the demo in pairs or small groups, then the lesson will need to be modified accordingly. If the demo is not available, students can come to the same conclusion by plotting points in 3-dimensional space. In this case, teachers may want to have students plot points and then show visuals from the teacher pages of actual screen shots from the demo to help students visualize the transformations. In addition to the tasks included in the lesson, students are encouraged to explore and play with the demo file to discover the connections between the structure of a $3 \times 3$ matrix and the geometric effect of the transformation induced by the matrix.

## Classwork

## Opening Exercise (10 minutes)

The Opening Exercise allows students to review the geometric effects of linear transformations in the plane induced by various $2 \times 2$ matrices. Students will then extend this idea to transformations in space induced by $3 \times 3$ matrices.

> Opening Exercise
> Consider the following matrices: $A=\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right], B=\left[\begin{array}{ll}2 & 0 \\ 0 & 0\end{array}\right]$, and $C=\left[\begin{array}{cc}2 & -2 \\ 2 & 2\end{array}\right]$
a. Compute the following determinants.
i. $\operatorname{det}(A)$

$$
\operatorname{det}(A)=4-4=0
$$

ii. $\operatorname{det}(B)$
$\operatorname{det}(B)=0-0=0$
iii. $\operatorname{det}(C)$
$\operatorname{det}(C)=4-(-4)=8$
b. Sketch the image of the unit square after being transformed by each transformation.
i. $\quad L_{A}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{ll}1 & 2 \\ 2 & 4\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]$


ii. $\quad L_{B}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{ll}2 & 0 \\ 0 & 0\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]$


iii. $\quad L_{C}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{cc}2 & -2 \\ 2 & 2\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]$



## Scaffolding:

- Pair students so that students who can easily see the transformations are paired with students who struggle.
The first two have no area because each image is a line segment. The third one is a square with sides of length $2 \sqrt{2}$, so its area is $(2 \sqrt{2})^{2}=8$.
d. Explain the connection between the responses to Parts 1 and 3.

The determinant of the matrix $A$ is the area of the image of the unit square under the transformation induced by $A$.

## Discussion (2 minutes)

Use this discussion to have students share the connections they made in part (d) of the Opening Exercise.

- What happens to the unit square under a transformation $L\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=A \cdot\left[\begin{array}{l}x \\ y\end{array}\right]$ when the matrix $A$ has a determinant of 0 ?
- The square is transformed into a line segment.
- Does such a transformation have an inverse?
- No. There is no way to undo this transformation because we don't know where it came from.
- In the same way, we want to explore the geometric effect of a linear transformation induced by matrix multiplication on the unit cube.

| Lesson 7: | Linear Transformations Applied to Cubes |
| :--- | :--- |
| Date: | $1 / 24 / 15$ |

## Example 1 (5 minutes)

If the program is not available, have students plot the original image and the transformation in 3 dimensions and color code them. You can also show the images in the teacher materials below to help students visualize the transformations at first and then have them draw different transformations.

- What do we expect the geometric effect of the transformation $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{array}\right] \cdot\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ will be on the unit cube?
- One way to find out would be to transform the vertices of the cube. If we do that, we find the following transformed points.

- It can be hard to truly see the image when plotting these points in three dimensions. Instead, let's use GeoGebra. Project the GeoGebra demo "TransformCubes.ggb" so that all students can see it. Use the sliders to set $a=1, e=-1, i=-1$, and the remaining entries in the matrix to 0 as shown below. The image above is an exact copy of the file screen that students and teachers can use to make sure they are using the correct settings.
- Using the software, we can see the blue cube before the transformation using matrix $A$, and the green figure is the image of the original cube after the transformation using matrix $A$. What appears to be the geometric effect of multiplication by the matrix $A$ ?
- The green cube is the result of rotating $180^{\circ}$ about the $x$-axis.

It may not be obvious that the green cube is the result of rotating and not from translating, but remind students that a linear transformation never changes the origin. In this case, $L\left(\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]\right)=\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]$. Knowing that the origin doesn't change makes it more obvious that the transformation $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{array}\right] \cdot\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ has the geometric effect of rotation about the $x$-axis.

## Exploratory Challenge 1 (10 minutes)

For this Exploratory Challenge, students use the GeoGebra demo TransformCubes.ggb to explore the geometric effects of a linear transformation defined by matrix multiplication for different matrices $A$. Screen shots of the GeoGebra file is shown in the steps below so students can see transformations and matrices. If possible, allow students to access the software themselves. If there is no way for students to access the software themselves, have them plot the transformed vertices of the unit cube, color coding the original image and its transformation, and draw conclusions from there. Students plotted points in 3 dimensions in the previous lessons. Students may need a quick reminder that if a transformation has an inverse, then each point in the image came from exactly one point on the cube, so that we can undo the function in a well-defined way.

## Exploratory Challenge 1

## For each matrix $\boldsymbol{A}$ given below:

i. Plot the image of the unit cube under the transformation.
ii. Find the volume of the image of the unit cube from part (i).
iii. Does the transformation have an inverse? If so, what is the matrix that induces the inverse transformation?
a. $\quad A=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2\end{array}\right]$

i. The image of the unit cube is a cube with sides of length 2. Thus, the volume of the image of the unit cube is $2 \cdot 2 \cdot 2=8$.
ii. This transformation has an inverse. To undo dilation by 2 , we need to dilate by $\frac{1}{2}$. The matrix that represents the inverse transformation is $B=\left[\begin{array}{ccc}\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2}\end{array}\right]$.
b. $\quad A=\left[\begin{array}{ccc}4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2\end{array}\right]$

i. The image of the unit cube is a box with sides of length 4, 1, and 2. Thus, the volume of the image of the unit cube is $4 \cdot 1 \cdot 2=8$.
ii. This transformation is also invertible. To undo this transformation, we need to scale by $\frac{1}{4}$ in the $x$ direction, by 1 in the $y$-direction, and by $-\frac{1}{2}$ in the $z$-direction. The matrix that will induce this inverse transformation is $B=\left[\begin{array}{ccc}\frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2}\end{array}\right]$.
c. $\quad A=\left[\begin{array}{lll}4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\end{array}\right]$

i. The image of the unit cube is a rectangle with sides of length 2 and 4. Thus, the volume of the image of the unit cube is $4 \cdot 0 \cdot 2=0$.
ii. This transformation is not invertible, since the image was collapsed from a cube onto a rectangle. Many points get sent to the same point on the rectangle, so we cannot undo the transformation in a clear way.
d. Describe the geometric effect of a transformation $L\left(\left[\begin{array}{l}x \\ y \\ Z\end{array}\right]\right)=\left[\begin{array}{lll}a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c\end{array}\right] \cdot\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ for numbers $a, b$, and $c$.

Describe when such a transformation is invertible.
The transformation $L$ scales the side of the cube along the $x$-axis by $a$, the side of the cube along the $y$-axis by $b$, and the side of the cube along the $z$-axis by $c$. If any of the numbers $a, b$, and $c$ are zero, then the image is no longer a three-dimensional figure, so the transformation will not be invertible. Thus, the transformation is invertible only when $a \neq 0, b \neq 0$, and $c \neq 0$.

## Exploratory Challenge 2 ( 10 minutes)

For this challenge, students continue to use the GeoGebra demo TransformCubes.ggb to explore the geometric effects of a linear transformation defined by matrix multiplication. A screen shot of the file is shown below. In this case, students are asked to discover how matrix multiplication can produce rotation about an axis. Encourage the students to play with the software and explore how different geometric effects arise from different matrix structures.

## Exploratory Challenge 2

a. Make a prediction: What would be the geometric effect of the transformation
$L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos \left(45^{\circ}\right) & -\sin \left(45^{\circ}\right) \\ 0 & \sin \left(45^{\circ}\right) & \cos \left(45^{\circ}\right)\end{array}\right] \cdot\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ on the unit cube? Use the GeoGebra demo to test your
conjecture.


Students should recognize that this transformation should have something to do with rotation. However, they will most likely need to see the resulting image before identifying it as rotation by $45^{\circ}$ about the $x$-axis.
b. For each geometric transformation below, find a matrix $A$ so that the geometric effect of $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=A \cdot\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ is the specified transformation.
i. Rotation by $-45^{\circ}$ about the $\boldsymbol{x}$-axis.

$$
A=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \left(-45^{\circ}\right) & \sin \left(-45^{\circ}\right) \\
0 & \sin \left(-45^{\circ}\right) & \cos \left(-45^{\circ}\right)
\end{array}\right]=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{array}\right]
$$

ii. Rotation by $45^{\circ}$ about the $y$-axis.

$$
A=\left[\begin{array}{ccc}
\cos \left(45^{\circ}\right) & 0 & -\sin \left(45^{\circ}\right) \\
0 & 1 & 0 \\
\sin \left(45^{\circ}\right) & 0 & \cos \left(45^{\circ}\right)
\end{array}\right]=\left[\begin{array}{ccc}
\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\
0 & 1 & 0 \\
\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2}
\end{array}\right]
$$

iii. Rotation by $45^{\circ}$ about the $z$-axis.

$$
A=\left[\begin{array}{ccc}
\cos \left(45^{\circ}\right) & -\sin \left(45^{\circ}\right) & 0 \\
\sin \left(45^{\circ}\right) & \cos \left(45^{\circ}\right) & 0 \\
0 & 0 & 1
\end{array}\right]=\left[\begin{array}{ccc}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
0 & 0 & 1
\end{array}\right]
$$

iv. Rotation by $90^{\circ}$ about the $x$-axis.

$$
A=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \left(90^{\circ}\right) & -\sin \left(90^{\circ}\right) \\
0 & \sin \left(90^{\circ}\right) & \cos \left(90^{\circ}\right)
\end{array}\right]=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right]
$$

v. Rotation by $90^{\circ}$ about the $y$-axis.

$$
A=\left[\begin{array}{ccc}
\cos \left(90^{\circ}\right) & 0 & -\sin \left(90^{\circ}\right) \\
0 & 1 & 0 \\
\sin \left(90^{\circ}\right) & 0 & \cos \left(90^{\circ}\right)
\end{array}\right]=\left[\begin{array}{ccc}
0 & 0 & -1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{array}\right]
$$

vi. Rotation by $90^{\circ}$ about the $\mathbf{z}$-axis.

$$
A=\left[\begin{array}{ccc}
\cos \left(90^{\circ}\right) & -\sin \left(90^{\circ}\right) & 0 \\
\sin \left(90^{\circ}\right) & \cos \left(90^{\circ}\right) & 0 \\
0 & 0 & 1
\end{array}\right]=\left[\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

vii. Rotation by $\boldsymbol{\theta}$ about the $\boldsymbol{x}$-axis.

$$
A=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos (\theta) & -\sin (\theta) \\
0 & \sin (\theta) & \cos (\theta)
\end{array}\right]
$$

viii. Rotation by $\theta$ about the $y$-axis.

$$
A=\left[\begin{array}{ccc}
\cos (\theta) & 0 & -\sin (\theta) \\
0 & 1 & 0 \\
\sin (\theta) & 0 & \cos (\theta)
\end{array}\right]
$$

ix. Rotation by $\boldsymbol{\theta}$ about the $\boldsymbol{z}$-axis.

$$
A=\left[\begin{array}{ccc}
\cos (\theta) & -\sin (\theta) & 0 \\
\sin (\theta) & \cos (\theta) & 0 \\
0 & 0 & 1
\end{array}\right]
$$

## Exploratory Challenge 3 (Optional)

This challenge is for students who have completed the other challenges quickly. Allow students to play with the software and try to describe the geometric transformations that arise from $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=A \cdot\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ for the following matrices A. Exact pictures of file screens are shown below to help students and teachers.

Exploratory Challenge 3
Describe the geometric effect of each transformation $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=A \cdot\left[\begin{array}{l}x \\ y \\ Z\end{array}\right]$ for the given matrices $A$. Be as specific as you can.
a. $\quad A=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0\end{array}\right]$


The images of the two cubes coincide; this transformation reflects across the plane through the points $(1,0,0),(1,1,1),(0,1,1)$, and $(0,0,0)$.
b. $\quad A=\left[\begin{array}{lll}2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0\end{array}\right]$


This transformation collapses the cube onto a line segment from $(\mathbf{0}, \mathbf{0}, \mathbf{0})$ to $(3,3,3)$.
c. $\quad A=\left[\begin{array}{ccc}2 & 2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1\end{array}\right]$


This transformation stretches the cube in the $x$ and $y$ directions and then rotates by $-45^{\circ}$.

## Closing (3 minutes)

Ask students to summarize the key points of the lesson in writing or to a partner. Some important summary elements are listed below.

## Lesson Summary

- For a matrix $A$, the transformation $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=A \cdot\left[\begin{array}{l}x \\ y \\ Z\end{array}\right]$ is a function from points in space to points in space.
- Different matrices induce transformations such as rotation, dilation, and reflection.
- The transformation induced by a diagonal matrix $A=\left[\begin{array}{lll}a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c\end{array}\right]$ will scale by $a$ in the direction parallel to the $x$-axis, by $b$ in the direction parallel to the $y$-axis, and by $c$ in the direction parallel to the $z$-axis.
- The matrices $\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos (\theta) & -\sin (\theta) \\ 0 & \sin (\theta) & \cos (\theta)\end{array}\right],\left[\begin{array}{ccc}\cos (\theta) & 0 & -\sin (\theta) \\ 0 & 1 & 0 \\ \sin (\theta) & 0 & \cos (\theta)\end{array}\right]$, and $\left[\begin{array}{ccc}\cos (\theta) & -\sin (\theta) & 0 \\ \sin (\theta) & \cos (\theta) & 0 \\ 0 & 0 & 1\end{array}\right]$ induce rotation by $\theta$ about the $x, y$, and $z$ axes, respectively.


## Exit Ticket ( 5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 7: Linear Transformations Applied to Cubes

## Exit Ticket

1. Sketch the image of the unit cube under the transformation $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=\left[\begin{array}{ccc}3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0\end{array}\right] \cdot\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ on the axes provided.

2. Does the transformation from Question 1 have an inverse? Explain how you know.

## Exit Ticket Sample Solutions

1. Sketch the image of the unit cube under the transformation $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=\left[\begin{array}{ccc}3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0\end{array}\right] \cdot\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ on the axes provided.

2. Does the transformation from Question 1 have an inverse? Explain how you know.

No, this transformation does not have an inverse. The entry in the matrix in the lower right corner is 0 , leading to a collapse of the image of the cube onto a rectangle. Since multiple points on the cube were transformed to the same point on the rectangle, there is no way to undo this transformation.

## Problem Set Sample Solutions

Problems 1 and 2 continue to develop the theory of linear transformations represented by matrix multiplication. Problem 3 can be done with or without access to the GeoGebra demo used in the lesson.

1. Suppose that we have a linear transformation $\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=\left[\begin{array}{lll}a & d & g \\ b & e & h \\ c & f & i\end{array}\right] \cdot\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$, for some matrix $A=\left[\begin{array}{lll}a & d & g \\ b & e & h \\ c & f & i\end{array}\right]$.
a. Evaluate $L\left(\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right]\right)$. How does the result relate to the matrix $A$ ?
$L\left(\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right]\right)=\left[\begin{array}{l}a \\ b \\ c\end{array}\right]$ It is the first column of matrix $A$.
b. Evaluate $L\left(\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]\right)$. How does the result relate to the matrix $A$ ?
$L\left(\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]\right)=\left[\begin{array}{l}d \\ e \\ f\end{array}\right]$ It is the second column of matrix $A$.
c. Evaluate $L\left(\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]\right)$. How does the result relate to the matrix $A$ ?
$L\left(\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]\right)=\left[\begin{array}{l}g \\ h \\ i\end{array}\right]$ It is the third column of matrix $A$.
d. James correctly said that if you know what a linear transformation does to the three points $(\mathbf{1}, \mathbf{0}, \mathbf{0}),(\mathbf{0}, \mathbf{1}, \mathbf{0})$, and $(\mathbf{0}, \mathbf{0}, \mathbf{1})$, you can find the matrix of the transformation. Explain how you can find the matrix of the transformation given the image of these three points.
Create a matrix with $L\left(\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right]\right)$ in the first column, $L\left(\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]\right)$ in the second column, and $L\left(\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]\right)$ in the third column.
2. Use the result from Problem 1(d) to answer the following questions.
a. Suppose a transformation $L: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ satisfies $L\left(\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right]\right)=\left[\begin{array}{l}3 \\ 0 \\ 0\end{array}\right], L\left(\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]\right)=\left[\begin{array}{l}0 \\ 3 \\ 0\end{array}\right]$, and $L\left(\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]\right)=\left[\begin{array}{l}0 \\ 0 \\ 3\end{array}\right]$.
i. What is the matrix $A$ that represents the transformation $L$ ?

$$
A=\left[\begin{array}{lll}
3 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 3
\end{array}\right]
$$

ii. What is the geometric effect of the transformation $L$ ?

The transformation $L$ has the geometric effect of dilation by a factor of 3.
iii. Sketch the image of the unit cube after the transformation by $L$.

b. Suppose a transformation $L: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ satisfies $L\left(\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right]\right)=\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right], L\left(\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]\right)=\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]$, and $L\left(\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]\right)=\left[\begin{array}{c}0 \\ 0 \\ -4\end{array}\right]$.
i. What is the matrix $A$ that represents the transformation $L$ ?

$$
A=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -4
\end{array}\right]
$$

ii. What is the geometric effect of the transformation $L$ ?

The transformation $L$ has the geometric effect of stretching by a factor of $\mathbf{4}$ in the $\mathbf{z}$-direction.
iii. Sketch the image of the unit cube after the transformation by $L$.

c. Suppose a transformation $L: \mathbb{R}^{3} \rightarrow \mathbb{R}^{3}$ satisfies $L\left(\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right]\right)=\left[\begin{array}{c}-2 \\ 0 \\ 0\end{array}\right], L\left(\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]\right)=\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]$, and $L\left(\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]\right)=\left[\begin{array}{c}0 \\ 0 \\ -2\end{array}\right]$.
i. What is the matrix $A$ that represents the transformation $L$ ?

$$
A=\left[\begin{array}{ccc}
-2 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -2
\end{array}\right]
$$

ii. What is the geometric effect of the transformation $L$ ?

The transformation $L$ has the geometric effect of rotating about the $y$-axis by $180^{\circ}$ and stretching by a factor of 2 in the $x$ and $z$ directions.
iii. Sketch the image of the unit cube after transformation by $L$.

3. Find the matrix of the transformation that will produce the following images of the unit cube. Describe the geometric effect of the transformation.
a.


The geometric effect of this transformation is reflection across the xy-plane. The matrix that represents this transformation is $A=\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{array}\right]$.
b.


The geometric effect of this transformation is a stretch in the $y$-direction by a factor of 3 . The matrix that represents this transformation is $A=\left[\begin{array}{lll}\mathbf{1} & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1\end{array}\right]$.
c.


The geometric effect of this transformation is a stretch in the $x$ - and $z$-directions by a factor of 2 and projection onto the $x$ z-plane. The matrix that represents this transformation is $A=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2\end{array}\right]$.
d.


The geometric effect of this transformation is a shear parallel to the $x z$-plane. The matrix that represents this transformation is $A=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1\end{array}\right]$.

# (8. Lesson 8: Composition of Linear Transformations 

## Student Outcomes

- Students compose two linear transformations in the plane by multiplying the associated matrices.
- Students visualize composition of linear transformations in the plane.


## Lesson Notes

In this lesson, students will examine the geometric effects of performing a sequence of two linear transformations in $\mathbb{R}^{2}$ produced by various $2 \times 2$ matrices on the unit square. The GeoGebra demo TransformSquare (http://eureka-math.org/G12M2L8/geogebra-TransformSquare) may be used in this lesson but is not necessary. This GeoGebra demo will allow the students to input values in $2 \times 2$ matrices $A$ between -5 and 5 in increments of 0.1 and visually see the effect of the transformation produced by the matrix $A$ on the unit square. If there aren't enough computers for students to access the demos in pairs or small groups, modify the lesson. The teacher could have students work in groups on transformations, and then as parts of the Exploratory Challenge are finished, the teacher could show specific transformations to the entire class using the teacher computer. Another option would be to have different groups come up and show the transformation that they created by hand and then use the teacher computer to show that transformation using software.

## Classwork

## Opening (3 minutes)

In this lesson, we will consider the transformation that arises from doing two linear transformations in sequence.
Display each transformation matrix below on the board, and ask students what transformation the matrix represents.

| $\left[\begin{array}{ll}2 & 0 \\ 0 & 2\end{array}\right]$ | A dilation with a scale factor of 2. |
| :--- | :--- |
| $\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$ | A reflection across the line $y=x$. |
| $\left[\begin{array}{cc}2 & -1 \\ 1 & 2\end{array}\right]$ | A rotation about the origin by $\theta=\tan ^{-1}\left(\frac{2}{1}\right)$ and a dilation with a scale factor of $\sqrt{2^{2}+1^{2}}=\sqrt{5}$. |

- How could we represent a composition of two of these transformations?


## Opening Exercise (5 minutes)

The focus of this lesson is on students recognizing that a matrix produced by a composition of two linear transformations in the plane is the product of the matrices of each transformation. Thus, we need students to review the process of multiplying $2 \times 2$ matrices. The products used in this Opening Exercise will appear later in the lesson.

## Opening Exercise

Compute the product $A B$ for the following pairs of matrices.
a. $\quad A=\left[\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right], B=\left[\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right]$
$A B=\left[\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right]$
b. $\quad A=\left[\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right], B=\left[\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right]$

$$
A B=\left[\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right]
$$

c. $\quad A=\left[\begin{array}{cc}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{array}\right], B=\left[\begin{array}{ll}4 & 0 \\ 0 & 4\end{array}\right]$
$A B=\left[\begin{array}{cc}2 \sqrt{2} & -2 \sqrt{2} \\ 2 \sqrt{2} & 2 \sqrt{2}\end{array}\right]$
d. $\quad A=\left[\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right], B=\left[\begin{array}{ll}2 & 0 \\ 0 & 2\end{array}\right]$

$$
A B=\left[\begin{array}{ll}
2 & 2 \\
0 & 2
\end{array}\right]
$$

e. $\quad A=\left[\begin{array}{cc}\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{array}\right], B=\left[\begin{array}{cc}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{array}\right]$

$$
A B=\left[\begin{array}{cc}
\frac{\sqrt{3}}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{array}\right]
$$

## Exploratory Challenge ( 25 minutes)

For this challenge, students use the GeoGebra demo TransformSquare.ggb to transform a unit square in the plane in order to explore the geometric effects of performing a sequence of two linear transformations defined by matrix multiplication for different matrices $A$ and $B$. While the software will illustrate a single transformation via matrix multiplication, students will have to reason through the effects of doing two transformations in sequence. If possible, allow students to access the software themselves. If there is no way for students to access the software themselves, have them plot the transformed vertices of the unit square and draw conclusions from there.

## Scaffolding:

- Provide struggling students with colored pencils or markers to use to identify the three figures. For example, use blue for the original square, green for the square transformed by $L_{B}$, and red for the square transformed by $L_{B}$ and then $L_{A}$.
- Ask early finishers to figure out whether or not the geometric effect of performing first $L_{A}$ then $L_{B}$ is the same as performing first $L_{B}$ and then $L_{A}$.


## Exploratory Challenge

1. For each pair of matrices $A$ and $B$ given below:
i. Describe the geometric effect of the transformation $L_{B}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=B \cdot\left[\begin{array}{l}x \\ y\end{array}\right]$.
ii. Describe the geometric effect of the transformation $L_{A}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=A \cdot\left[\begin{array}{l}x \\ y\end{array}\right]$.
iii. Draw the image of the unit square under the transformation $L_{B}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=B \cdot\left[\begin{array}{l}x \\ y\end{array}\right]$.
iv. Draw the image of the transformed square under the transformation $L_{A}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=A \cdot\left[\begin{array}{l}x \\ y\end{array}\right]$.
v. Describe the geometric effect on the unit square of performing first $L_{B}$ then $L_{A}$.
vi. Compute the matrix product $A B$.
vii. Describe the geometric effect of the transformation $L_{A B}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=A B \cdot\left[\begin{array}{l}x \\ y\end{array}\right]$.
a. $\quad A=\left[\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right], B=\left[\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right]$
i. The transformation produced by matrix B has the effect of rotation by $90^{\circ}$ counterclockwise.
ii. The transformation produced by matrix $A$ has the effect of rotation by $90^{\circ}$ counterclockwise.
iii. The green square in the second quadrant is the image of the original square under the transformation produced by $B$.

iv. The red square in the third quadrant is the image of the square after transforming with matrix $A$ then matrix $B$.

v. If we rotate twice by $90^{\circ}$, then the net effect is a rotation by $180^{\circ}$.
vi. The matrix product is $A B=\left[\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right]$.
vii. The transformation produced by matrix $A B$ has the effect of rotation by $180^{\circ}$.
b. $\quad A=\left[\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right], B=\left[\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right]$
i. The transformation produced by matrix $B$ has the effect of reflection across the $y$-axis.
ii. The transformation produced by matrix $A$ has the effect of reflection across the $x$-axis.
iii. The green square in the second quadrant is the image of the original square under the transformation produced by $B$.

iv. The red square in the third quadrant is the image of the square after transforming with matrix $A$ and then matrix B.

v. If we reflect across the $y$-axis and then reflect across the $x$-axis, the net effect is a rotation by $180^{\circ}$.
vi. The matrix product is $A B=\left[\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right]$.
vii. The transformation produced by matrix $A B$ has the effect of rotation by $180^{\circ}$.
c. $\quad A=\left[\begin{array}{cc}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}\end{array}\right], B=\left[\begin{array}{ll}4 & 0 \\ 0 & 4\end{array}\right]$
i. The transformation produced by matrix $B$ has the effect of dilation from the origin with scale factor 4 .
ii. The transformation produced by matrix $A$ has the effect of rotation by $45^{\circ}$.
iii. The large green square is the image of the original unit square under the transformation produced by B.

iv. The tilted red square is the image of the green square under the transformation produced by $A$.

v. If we dilate with factor 4 and then rotate by $45^{\circ}$, the net effect is a rotation by $45^{\circ}$ and dilation with scale factor 4.
vi. The matrix product is $A B=\left[\begin{array}{cc}2 \sqrt{2} & -2 \sqrt{2} \\ 2 \sqrt{2} & 2 \sqrt{2}\end{array}\right]$.
vii. The transformation produced by matrix $A B$ has the effect of rotation by $45^{\circ}$ while scaling by 4 .
d. $\quad A=\left[\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right], B=\left[\begin{array}{ll}2 & 0 \\ 0 & 2\end{array}\right]$
i. The transformation produced by matrix $B$ has the effect of dilation with scale factor 2.
ii. The transformation produced by matrix $A$ has the effect of shearing parallel to the $y$-axis.
iii. The large green square is the image of the original unit square under the transformation produced by B.

iv. The red parallelogram is the image of the green square under the transformation produced by $A$.
v. If we dilate and then shear, the net effect is a dilated shear.
vi. The matrix product is $A B=\left[\begin{array}{ll}2 & 2 \\ 0 & 2\end{array}\right]$.
vii. The transformation produced by matrix $A B$ has the effect of a dilated shear.
e. $\quad A=\left[\begin{array}{cc}\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2}\end{array}\right], B=\left[\begin{array}{cc}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{array}\right]$
i. The transformation produced by matrix $B$ has the effect of rotation by $30^{\circ}$.
ii. The transformation produced by matrix $A$ has the effect of rotation by $-60^{\circ}$.

iii. The tilted green square is the image of the original unit square under the transformation produced by B.

iv. The tilted red square is the image of the green square under the transformation produced by $A$.

v. If we rotate by $30^{\circ}$ counterclockwise and then rotate by $60^{\circ}$ clockwise, the net result is a rotation by $30^{\circ}$ counterclockwise.
vi. The matrix product is $A B=\left[\begin{array}{cc}\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2}\end{array}\right]$
vii. The transformation produced by matrix $A B$ has the effect of rotation by $-30^{\circ}$.
2. Make a conjecture about the geometric effect of the linear transformation produced by the matrix $A B$. Justify your answer.

The linear transformation produced by matrix $A B$ has the same geometric effect of the sequence of the linear transformation produced by matrix $B$ followed by the linear transformation produced by matrix $A$. We have seen in previous work that if we multiply by $A B$, we get the same transformation as when we multiplied by $A$ first and then multiplied the result by $B$.

## Discussion (5 minutes)

In this discussion, we justify why the conjecture the students made at the end of Exploratory Challenge is valid.

- What was the conjecture you made at the end of the Exploratory Challenge?
- The transformation that comes from the matrix $A B$ has the same geometric effect as the sequence of transformations that come from the matrix $B$ first and then $A$.
- Why does matrix $A B$ represent $B$ first and then $A$ ? Doesn't that seem backwards?
- If we look at this as multiplication, we are multiplying

$$
\begin{aligned}
L_{A B}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right) & =(A B) \cdot\left[\begin{array}{l}
x \\
y
\end{array}\right] \\
& =A\left(B \cdot\left[\begin{array}{l}
x \\
y
\end{array}\right]\right) \\
& =L_{A}\left(L_{B}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)\right)
\end{aligned}
$$

so the point $(x, y)$ is transformed first by $L_{B}$, and then the result is transformed by $L_{A}$. It looks backwards, but the transformation on the right happens to our point first. We saw this phenomenon in Geometry when we represented a rotation by $\theta$ about the origin by $R_{0, \theta}$ and a reflection across line $\ell$ by $r_{\ell}$. Then doing the rotation and then the reflection is denoted by $r_{\ell} \circ R_{0, \theta}$, which represents the combined effect of first performing the rotation $R_{O, \theta}$ and then the reflection $r_{\ell}$.

## Closing (3 minutes)

Ask students to summarize the key points of the lesson in writing or to a partner. Some important summary elements are listed below.

## Lesson Summary

The linear transformation produced by matrix $A B$ has the same geometric effect as the sequence of the linear transformation produced by matrix $B$ followed by the linear transformation produced by matrix $A$.

That is, if matrices $A$ and $B$ produce linear transformations $L_{A}$ and $L_{B}$ in the plane, respectively, then the linear transformation $L_{A B}$ produced by the matrix $A B$ satisfies
$L_{A B}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=L_{A}\left(L_{B}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)\right)$.

## Exit Ticket (4 minutes)

Name
Date $\qquad$

## Lesson 8: Composition of Linear Transformations

## Exit Ticket

Let $A$ be the matrix representing a rotation about the origin $135^{\circ}$ and $B$ be the matrix representing a dilation of 6 . Let $x=\left[\begin{array}{c}-1 \\ \frac{1}{2}\end{array}\right]$.
a. Write down $A$ and $B$.
b. Find the matrix representing a dilation of $x$ by 6 , followed by a rotation about the origin of $135^{\circ}$.
c. $\quad$ Graph and label $x, x$ after a dilation of 6 , and $x$ after both transformations have been applied.

## Exit Ticket Sample Solutions

Let $A$ be the matrix representing a rotation about the origin $135^{\circ}$ and $B$ be the matrix representing a dilation of 6 . Let $x=\left[\begin{array}{c}-1 \\ \frac{1}{2}\end{array}\right]$.
a. Write down $A$ and $B$.

$$
A=\left[\begin{array}{cc}
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}
\end{array}\right], B=\left[\begin{array}{ll}
6 & 0 \\
0 & 6
\end{array}\right]
$$

b. Find the matrix representing a dilation $x$ by 6 , followed by a rotation about the origin of $135^{\circ}$.

$$
A B=\left[\begin{array}{cc}
-3 \sqrt{2} & -3 \sqrt{2} \\
3 \sqrt{2} & -3 \sqrt{2}
\end{array}\right]
$$

c. Graph and label $x, x$ after a dilation of 6 , and $x$ after both transformations have been applied.


## Problem Set Sample Solutions

1. Let $A$ be the matrix representing a dilation of $\frac{1}{2}$, and let $B$ be the matrix representing a reflection across the $y$-axis.
a. Write $A$ and $B$.
$A=\left[\begin{array}{ll}\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\end{array}\right], B=\left[\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right]$
b. Evaluate $A B$. What does this matrix represent?
$A B=\left[\begin{array}{cc}-\frac{1}{2} & 0 \\ 0 & \frac{1}{2}\end{array}\right]$
$A B$ is a reflection across the $y$-axis followed by a dilation of $\frac{1}{2}$.
c. Let $x=\left[\begin{array}{l}5 \\ 6\end{array}\right], y=\left[\begin{array}{c}-1 \\ 3\end{array}\right]$, and $z=\left[\begin{array}{c}8 \\ -2\end{array}\right]$. Find $(A B) x,(A B) y$, and $(A B) z$.
$(A B) x=\left[\begin{array}{c}-\frac{5}{2} \\ 3\end{array}\right],(A B) y=\left[\begin{array}{l}\frac{1}{2} \\ \frac{3}{2}\end{array}\right],(A B) z=\left[\begin{array}{c}4 \\ -1\end{array}\right]$
2. Let $A$ be the matrix representing a rotation of $30^{\circ}$, and let $B$ be the matrix representing a dilation of 5 .
a. Write $A$ and $B$.
$A=\left[\begin{array}{cc}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{array}\right], B=\left[\begin{array}{ll}5 & 0 \\ 0 & 5\end{array}\right]$
b. Evaluate $A B$. What does this matrix represent?
$A B=\left[\begin{array}{cc}\frac{5 \sqrt{3}}{2} & -\frac{5}{2} \\ \frac{5}{2} & \frac{5 \sqrt{3}}{2}\end{array}\right]$
$A B$ is a dilation of 5 followed by a rotation of $30^{\circ}$.
c. Let $x=\left[\begin{array}{l}1 \\ 0\end{array}\right]$. Find $(A B) x$.
$(A B) x=\left[\begin{array}{c}\frac{5 \sqrt{3}}{2} \\ \frac{5}{2}\end{array}\right]$
3. Let $A$ be the matrix representing a dilation of 3 , and let $B$ be the matrix representing a reflection across the line $y=x$.
a. Write $A$ and $B$.

$$
A=\left[\begin{array}{ll}
3 & 0 \\
0 & 3
\end{array}\right], B=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]
$$

b. Evaluate $A B$. What does this matrix represent?
$A B=\left[\begin{array}{ll}0 & 3 \\ 3 & 0\end{array}\right]$
$A B$ is a reflection across the line $y=x$ followed by a dilation of 3 .
c. Let $x=\left[\begin{array}{c}-2 \\ 7\end{array}\right]$. Find $(A B) x$.

$$
(A B) x=\left[\begin{array}{l}
21 \\
-6
\end{array}\right]
$$

4. Let $A=\left[\begin{array}{ll}3 & 0 \\ 3 & 3\end{array}\right]$ and $B=\left[\begin{array}{cc}0 & -1 \\ 1 & 0\end{array}\right]$.
a. Evaluate $A B$.

$$
\left[\begin{array}{ll}
0 & -3 \\
3 & -3
\end{array}\right]
$$

b. Let $x=\left[\begin{array}{c}-2 \\ 2\end{array}\right]$. Find $(A B) x$.
$\left[\begin{array}{c}-6 \\ -12\end{array}\right]$
c. Graph $x$ and $(A B) x$.

5. Let $A=\left[\begin{array}{ll}\frac{1}{3} & 0 \\ 2 & \frac{1}{3}\end{array}\right]$ and $B=\left[\begin{array}{cc}3 & 1 \\ 1 & -3\end{array}\right]$.
a. Evaluate $A B$.
$\left[\begin{array}{cc}1 & \frac{1}{3} \\ \frac{19}{3} & 1\end{array}\right]$
b. Let $x=\left[\begin{array}{l}0 \\ 3\end{array}\right]$. Find $(A B) x$.
$\left[\begin{array}{l}1 \\ 3\end{array}\right]$
c. $\quad$ Graph $x$ and ( $A B$ ) $x$.

6. Let $A=\left[\begin{array}{ll}2 & 2 \\ 2 & 0\end{array}\right]$ and $B=\left[\begin{array}{ll}0 & 2 \\ 2 & 2\end{array}\right]$.
a. Evaluate $A B$.
$\left[\begin{array}{ll}4 & 8 \\ 0 & 4\end{array}\right]$
b. Let $x=\left[\begin{array}{c}3 \\ -2\end{array}\right]$. Find $(A B) x$.
$\left[\begin{array}{l}-4 \\ -8\end{array}\right]$

7. Let $A, B, C$ be $2 \times 2$ matrices representing linear transformations.
a. What does $A(B C)$ represent?
$A(B C)$ represents the linear transformation of applying the transformation that $C$ represents followed by the transformation that B represents, followed by the transformation that A represents.
b. Will the pattern established in part (a) be true no matter how many matrices are multiplied on the left?

Yes, in general. When you multiply by a matrix on the left, you are applying a linear transformation after all linear transformations to the right have been applied.
c. Does $(A B) C$ represent something different from $A(B C)$ ? Explain.

No, it does not. This is the linear transformation obtained by applying $C$ and then $A B$, which is $B$ followed by A.
8. Let $A B$ represent any composition of linear transformations in $\mathbb{R}^{2}$. What is the value of $(A B) x$ where $x=\left[\begin{array}{l}0 \\ 0\end{array}\right]$ ? Since a composition of linear transformations in $\mathbb{R}^{2}$ is also a linear transformation, we know that applying it to the origin will result in no change.

# (8. Lesson 9: Composition of Linear Transformations 

## Student Outcomes

- Students use technology to perform compositions of linear transformations in $\mathbb{R}^{3}$.


## Lesson Notes

In Lesson 8, students discovered that if they compose two linear transformations in the plane, $L_{A}$ and $L_{B}$, represented by matrices $A$ and $B$ respectively, then the resulting transformation can be produced using a single matrix $A B$. In this lesson, we extend this result from transformations in the plane to transformations in three-dimensional space, $\mathbb{R}^{3}$. This lesson was designed to be implemented using the GeoGebra applet TransformCubes (http://eureka-math.org/G12M2L9/geogebra-TransformCubes), which allows students to visualize the transformation of the cube.

## Classwork

## Opening Exercise (10 minutes)

The Opening Exercise reviews the discovery from the Problem Set in Lesson 7 that the matrix of a transformation in $\mathbb{R}^{3}$ is determined by the images of the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. This fact will be needed to construct matrices of transformations in this lesson before we compose them. Students may question what is meant by counterclockwise rotation in space; if this happens, let them know that we consider a rotation about the $z$-axis to be a counterclockwise rotation if it rotates the $x y$-plane counterclockwise when placed in its standard orientation as shown to the right.

Students should work these exercises with pencil and paper.


## Opening Exercise

Recall from Problem 1, part (d) of the Problem Set of Lesson 7 that if you know what a linear transformation does to the three points $(\mathbf{1}, \mathbf{0}, \mathbf{0}),(\mathbf{0}, \mathbf{1}, \mathbf{0})$, and $(\mathbf{0}, \mathbf{0}, \mathbf{1})$, you can find the matrix of the transformation. How do the images of these three points lead to the matrix of the transformation?
a. Suppose that a linear transformation $L_{1}$ rotates the unit cube by $90^{\circ}$ counterclockwise about the z-axis. Find the matrix $A_{1}$ of the transformation $L_{1}$.

Since this transformation rotates by $90^{\circ}$ counterclockwise in the $x y$-plane, a vector along the positive $x$-axis will be transformed to lie along the positive $y$-axis, a vector along the positive $y$-axis will be transformed to lie along the negative $y$-axis, and a vector along the z -axis will be left alone. Thus,

## Scaffolding:

- Encourage struggling students to draw the image of a cube before and after the transformation to find the images of the points $(1,0,0),(0,1,0)$, and $(0,0,1)$ in space.

$$
L_{1}\left(\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right], L_{1}\left(\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right)=\left[\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right] \text {, and } L_{1}\left(\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right],
$$

so the matrix of the transformation is $A_{1}=\left[\begin{array}{ccc}0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]$.
b. Suppose that a linear transformation $L_{2}$ rotates the unit cube by $90^{\circ}$ counterclockwise about the $y$-axis. Find the matrix $A_{2}$ of the transformation $L_{2}$.

Since this transformation rotates by $90^{\circ}$ counterclockwise in the $x z$-plane, a vector along the positive $x$-axis will be transformed to lie along the positive $z$-axis, a vector along the $y$-axis will be left alone, and a vector along the positive $z$-axis will be transformed to lie along the negative $x$-axis. Thus,

$$
L_{2}\left(\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right], L_{2}\left(\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right] \text {, and } L_{2}\left(\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]\right)=\left[\begin{array}{c}
-1 \\
0 \\
0
\end{array}\right],
$$

so the matrix of the transformation is $A_{2}=\left[\begin{array}{llc}0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0\end{array}\right]$.
c. $\quad$ Suppose that a linear transformation $L_{3}$ scales by 2 in the $x$-direction, scales by 3 in the $y$-direction, and scales by 4 in the $z$-direction. Find the matrix $A_{3}$ of the transformation $L_{3}$.

Since this transformation scales by 2 in the $x$-direction, by 3 in the $y$-direction, and by 4 in the $z$-direction, a vector along the $x$-axis will be multiplied by 2 , a vector along the $y$-axis will be multiplied by 3 , and $a$ vector along the z -axis will be multiplied by 4 . Thus,

$$
L_{3}\left(\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]\right)=\left[\begin{array}{l}
2 \\
0 \\
0
\end{array}\right], L_{3}\left(\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
3 \\
0
\end{array}\right] \text {, and } L_{3}\left(\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
0 \\
4
\end{array}\right],
$$

so the matrix of the transformation is $A_{3}=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4\end{array}\right]$.
d. Suppose that a linear transformation $L_{4}$ projects onto the $x y$-plane. Find the matrix $A_{4}$ of the transformation $L_{4}$.

Since this transformation projects onto the $x y$-plane, a vector along the $x$-axis will be left alone, a vector along the $y$-axis will be left alone, and a vector along the z-axis will be transformed into the zero vector. Thus,

$$
L_{4}\left(\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]\right)=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right], L_{4}\left(\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right] \text {, and } L_{4}\left(\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right] \text {, }
$$

so the matrix of the transformation is $A_{4}=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]$.
e. Suppose that a linear transformation $L_{5}$ projects onto the $x z$-plane. Find the matrix $A_{5}$ of the transformation $L_{5}$.

Since this transformation projects onto the xz-plane, a vector along the $x$-axis will left alone, a vector along the $y$-axis will be transformed into the zero vector, and a vector along the $z$-axis will be left alone. Thus,

$$
L_{5}\left(\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]\right)=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right], L_{5}\left(\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right] \text {, and } L_{5}\left(\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right],
$$

so the matrix of the transformation is $A_{5}=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]$.
f. Suppose that a linear transformation $L_{6}$ reflects across the plane with equation $y=x$. Find the matrix $A_{6}$ of the transformation $L_{6}$.

Since this transformation reflects across the plane $y=x$, a vector along the positive $x$-axis will be transformed into a vector along the positive $y$-axis with the same length, a vector along the positive $y$-axis will be transformed into a vector along the positive $x$-axis, and a vector along the $z$-axis will be left alone.
Thus,

$$
L_{6}\left(\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right], L_{6}\left(\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right)=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right] \text {, and } L_{6}\left(\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right]\right)=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right] \text {, }
$$

so the matrix of the transformation is $A_{6}=\left[\begin{array}{lll}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]$.
g. Suppose that a linear transformation $L_{7}$ satisfies $L_{7}\left(\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right]\right)=\left[\begin{array}{l}2 \\ 0 \\ 0\end{array}\right], L_{7}\left(\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]\right)=\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]$, and $L_{7}\left(\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]\right)=\left[\begin{array}{l}0 \\ 0 \\ \frac{1}{2}\end{array}\right]$. Find
the matrix $A_{7}$ of the transformation $L_{7}$. What is the geometric effect of this transformation?
The matrix of this transformation is $A_{7}=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\end{array}\right]$. The geometric effect of $L_{7}$ is to stretch by a factor of 2
in the $x$-direction and scale by a factor of $\frac{1}{2}$ in the $z$-direction.
h. Suppose that a linear transformation $L_{8}$ satisfies $L_{8}\left(\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right]\right)=\left[\begin{array}{l}1 \\ 1 \\ 0\end{array}\right], L_{8}\left(\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]\right)=\left[\begin{array}{c}-1 \\ 1 \\ 0\end{array}\right]$, and $L_{8}\left(\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]\right)=\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]$.

Find the matrix of the transformation $L_{8}$. What is the geometric effect of this transformation?
The matrix of this transformation is $A_{8}=\left[\begin{array}{ccc}1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$. The geometric effect of $L_{8}$ is to rotate by $45^{\circ}$ in the $x y$-plane, scale by $\sqrt{2}$ in both the $x$ and $y$ directions, and to not change in the z-direction.

Once students have completed the Opening Exercise with pencil and paper, allow them to use the GeoGebra applet TransformCubes.ggb to check their work before continuing. The remaining exercises rely on the matrices $\mathrm{A}_{1}-\mathrm{A}_{8}$, so ensure that students have found the correct matrices before proceeding with the lesson.

## Discussion (2 minutes)

- In Lesson 8, we saw that for linear transformations in the plane, if $L_{A}$ is a linear transformation represented by a $2 \times 2$ matrix $A$ and $L_{B}$ is a linear transformation represented by a $2 \times 2$ matrix $B$, then the $2 \times 2$ matrix $A B$ is the matrix of the composition of $L_{B}$ followed by $L_{A}$. Today we will explore composition of linear transformations in $\mathbb{R}^{3}$ to see whether the same result extends to the case when $A, B$, and $A B$ are $3 \times 3$ matrices.


## Exploratory Challenge 1 (25 minutes)

In the Exploratory Challenge, students predict the geometric effect of composing pairs of transformations from the Opening Exercise and then check their predictions with the GeoGebra applet TransformCubes.ggb.

## Exploratory Challenge 1

Transformations $L_{1}-L_{8}$ refer to the linear transformations from the Opening Exercise. For each pair,
i. Make a conjecture to predict the geometric effect of performing the two transformations in the order specified.
ii. Find the product of the corresponding matrices, in the order that corresponds to the indicated order of composition. Remember that if we perform a transformation $L_{B}$ with matrix $B$ and then $L_{A}$ with matrix $A$, the matrix that corresponds to the composition $L_{A} \circ L_{B}$ is $A B$. That is, $L_{B}$ is applied first, but matrix $B$ is written second.
iii. Use the GeoGebra applet TransformCubes.ggb to draw the image of the unit cube under the transformation induced by the matrix product in part (ii). Was your conjecture in part (i) correct?
a. Perform $L_{6}$ and then $L_{6}$.
i. Since $L_{6}$ reflects across the plane through $y=x$ that is perpendicular to the $x y$-plane, performing $L_{6}$ twice in succession will result in the identity transformation.
ii. $\quad A_{6} \cdot A_{6}=\left[\begin{array}{lll}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right] \cdot\left[\begin{array}{lll}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$. Since $A_{6} \cdot A_{6}$ is the identity matrix, we know that $L_{6} \circ$ $L_{6}$ is the identity transformation.
iii. The conjecture was correct.

b. Perform $L_{1}$ and then $L_{2}$.
i. Sample student response: Since $L_{1}$ rotates $90^{\circ}$ about the $z$-axis and $L_{2}$ rotates $90^{\circ}$ about the $y$-axis, the composition $L_{2} \circ L_{1}$ should rotate $180^{\circ}$ about the line $y=-x$ in the $x y$-plane.
ii. $\quad A_{2} \cdot A_{1}=\left[\begin{array}{ccc}0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0\end{array}\right] \cdot\left[\begin{array}{ccc}0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]=\left[\begin{array}{ccc}0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0\end{array}\right]$.
iii. The conjecture in part (i) is not correct. While it appears that the composition $L_{2} \circ L_{1}$ is a rotation by $180^{\circ}$ about the $y$-axis, it is not because, for example, point $(0,1,0)$ is transformed to point $(-1,0,0)$ and does not remain on the $y$-axis after the transformation. Thus, this cannot be a rotation around the $y$-axis.

c. Perform $L_{4}$ and then $L_{5}$.
i. Since $L_{4}$ projects onto the $x y$-plane and $L_{5}$ projects onto the $x z$-plane, the composition $L_{5} \circ L_{4}$ will project onto the $x$-axis.
ii. $\quad A_{5} \cdot A_{4}=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{array}\right] \cdot\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$.
iii. The conjecture in part (i) is correct. The cube is first transformed to a square in the $x y$-plane and then transformed onto a segment on the $x$-axis.

d. Perform $L_{4}$ and then $L_{3}$.
i. Since $L_{4}$ projects onto the $x y$-plane and $L_{3}$ scales in the $x, y$, and $z$ directions, the composition $L_{3} \circ L_{4}$ will project onto the $x y$-plane and scale in the $x$ and $y$ directions.
ii. $\quad A_{3} \cdot A_{4}=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4\end{array}\right] \cdot\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0\end{array}\right]$.
iii. The conjecture in part (i) is correct.

e. Perform $L_{3}$ and then $L_{7}$.
i. Since $L_{3}$ scales in the $x, y$, and $z$ directions and so does $L_{7}$, the composition will be a transformation that also scales in all three directions but with different scale factors.
ii. $\quad A_{7} \cdot A_{3}=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\end{array}\right] \cdot\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4\end{array}\right]=\left[\begin{array}{lll}4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2\end{array}\right]$.
iii. The conjecture from (i) is correct.

f. Perform $L_{8}$ and then $L_{4}$.
i. Transformation $L_{8}$ rotates the unit cube by $45^{\circ}$ about the $z$-axis and stretches by a factor of $\sqrt{2}$ in both the $x$ and $y$ directions, while $L_{4}$ projects the image onto the $x y$-plane. The composition will transform the unit cube into a larger square that has been rotated $45^{\circ}$ in the $x y$-plane.
ii.
$A_{4} \cdot A_{8}=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right] \cdot\left[\begin{array}{ccc}1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]=\left[\begin{array}{ccc}1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]$.
iii. The conjecture from part (i) is correct.

g. Perform $L_{4}$ and then $L_{6}$.
i. Transformation $L_{4}$ projects onto the $x y$-plane, and transformation $L_{6}$ reflects across the plane through the line $y=x$ in the $x y$-plane and is perpendicular to the $x y$-plane, so the composition $L_{6} \circ L_{4}$ will appear to be the reflection in the $x y$-plane across the line $y=x$.
ii. $\quad A_{6} \cdot A_{4}=\left[\begin{array}{lll}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right] \cdot\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]=\left[\begin{array}{lll}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$.
iii. The conjecture in part (i) is correct.

h. Perform $L_{2}$ and then $L_{7}$.
i. Since $L_{2}$ rotates $90^{\circ}$ around the $y$-axis and $L_{7}$ scales in the $x$ and $z$ directions, the composition $L_{7} \circ L_{2}$ will rotate and scale simultaneously.
ii. $\quad A_{7} \cdot A_{2}=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\end{array}\right] \cdot\left[\begin{array}{ccc}0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0\end{array}\right]=\left[\begin{array}{ccc}0 & 0 & -2 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0\end{array}\right]$.
iii. The conjecture in part (i) is correct.

i. Perform $L_{8}$ and then $L_{8}$.
i. Since $L_{8}$ rotates by $45^{\circ}$ about the $z$-axis and scales by $\sqrt{2}$ in the $x$ and $y$ directions, performing this transformation twice will rotate by $90^{\circ}$ about the $z$-axis and scale by 2 in the $x$ and $y$ directions.
ii. $\quad A_{8} \cdot A_{8}=\left[\begin{array}{ccc}1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1\end{array}\right] \cdot\left[\begin{array}{ccc}1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]=\left[\begin{array}{ccc}0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]$.
iii. The conjecture in part (i) is correct.


- Do a 30-second Quick Write on what we have discovered in Exploratory Challenge 1, and share with your neighbor.


## Exploratory Challenge 2 (Optional)

This optional challenge is for students who finished Exploratory Challenge 1 early. The challenge below is designed to prompt the question of whether or not order matters when composing two linear transformations, a question that is definitively answered in the next lesson and demonstrates that matrix multiplication is in general not commutative. Students are asked to compose two linear transformations $L_{A}$ and $L_{B}$, with matrices $A$ and $B$ respectively, and to compare $L_{A} \circ L_{B}$ with $L_{B} \circ L_{A}$. The directions for this challenge are left intentionally vague so that students may use either an algebraic or a geometric approach to answer the question.

## Exploratory Challenge 2

Transformations $L_{1}-L_{8}$ refer to the transformations from the Opening Exercise. For each of the following pairs of matrices $A$ and $B$ below, compare the transformations $L_{A} \circ L_{B}$ and $L_{B} \circ L_{A}$.
a. $\quad L_{4}$ and $L_{5}$

Transformation $L_{4} \circ L_{5}$ has matrix representation $A_{4} \cdot A_{5}=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right] \cdot\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$, and transformation $L_{5} \circ L_{4}$ has matrix representation $A_{5} \circ A_{4}=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{array}\right] \cdot\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0\end{array}\right]$.
Since the two transformations have the same matrix representation, they are the same transformation: $L_{5} \circ L_{4}=L_{4} \circ L_{5}$.
b. $\quad L_{2}$ and $L_{5}$

Transformation $L_{2} \circ L_{5}$ has matrix representation $A_{2} \cdot A_{5}=\left[\begin{array}{ccc}0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0\end{array}\right] \cdot\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]=\left[\begin{array}{ccc}0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0\end{array}\right]$, and transformation $L_{5} \circ L_{2}$ has matrix representation $A_{5} \cdot A_{2}=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{array}\right] \cdot\left[\begin{array}{ccc}0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0\end{array}\right]=\left[\begin{array}{ccc}0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0\end{array}\right]$.
Since the two transformations have the same matrix representation, they are the same transformation: $L_{2} \circ L_{5}=L_{5} \circ L_{2}$.
c. $\quad L_{3}$ and $L_{7}$

Transformation $L_{3} \circ L_{7}$ has matrix representation $A_{3} \cdot A_{7}=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4\end{array}\right] \cdot\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\end{array}\right]=\left[\begin{array}{lll}4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2\end{array}\right]$, and transformation $L_{7} \circ L_{3}$ has matrix representation $A_{7} \cdot A_{3}=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\end{array}\right] \cdot\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4\end{array}\right]=\left[\begin{array}{lll}4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2\end{array}\right]$.
Since the two transformations have the same matrix representation, they are the same transformation: $L_{3} \circ L_{7}=L_{7} \circ L_{3}$.
d. $\quad L_{3}$ and $L_{6}$

Transformation $L_{3} \circ L_{6}$ has matrix representation $A_{3} \cdot A_{6}=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4\end{array}\right] \cdot\left[\begin{array}{lll}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]=\left[\begin{array}{lll}0 & 2 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 4\end{array}\right]$, and transformation $L_{6} \circ L_{3}$ has matrix representation $A_{6} \cdot A_{3}=\left[\begin{array}{lll}0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right] \cdot\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4\end{array}\right]=\left[\begin{array}{lll}0 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 4\end{array}\right]$.
Since the two transformations have different matrix representations, they are not the same transformation: $L_{3} \circ L_{6} \neq L_{6} \circ L_{3}$.
e. $\quad L_{7}$ and $L_{1}$

Transformation $L_{7} \circ L_{1}$ has matrix representation $A_{7} \cdot A_{1}=\left[\begin{array}{ccc}2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\end{array}\right] \cdot\left[\begin{array}{ccc}0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]=\left[\begin{array}{ccc}0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\end{array}\right]$, and
transformation $L_{1} \circ L_{7}$ has matrix representation $A_{1} \cdot A_{7}=\left[\begin{array}{ccc}0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right] \cdot\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}\end{array}\right]=\left[\begin{array}{ccc}0 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\end{array}\right]$.
Since the two transformations have different matrix representations, they are not the same transformation: $L_{1} \circ L_{7} \neq L_{7} \circ L_{1}$.
f. What can you conclude about the order in which you compose two linear transformations?

In some cases, the order of composition of two linear transformations matters: for two matrices $A$ and $B$, the transformation $L_{A} \circ L_{B}$ is not always the same transformation as $L_{B} \circ L_{A}$.

## Closing (4 minutes)

Ask students to summarize the key points of the lesson in writing or to a partner. Some important summary elements are listed below.

## Lesson Summary

- The linear transformation induced by a $3 \times 3$ matrix $A B$ has the same geometric effect as the sequence of the linear transformation induced by the $3 \times 3$ matrix $B$ followed by the linear transformation induced by the $3 \times 3$ matrix $A$.
- That is, if matrices $A$ and $B$ induce linear transformations $L_{A}$ and $L_{B}$ in $\mathbb{R}^{3}$, respectively, then the linear transformation $L_{A B}$ induced by the matrix $A B$ satisfies $L_{A B}\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=L_{A}\left(L_{B}\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)\right)$.


## Exit Ticket (4 minutes)

Name
Date $\qquad$

## Lesson 9: Composition of Linear Transformations

## Exit Ticket

Let $A$ be the matrix representing a rotation about the $z$-axis of $45^{\circ}$ and $B$ be the matrix representing a dilation of 2 .
a. Write down $A$ and $B$.
b. Let $x=\left[\begin{array}{c}3 \\ -1 \\ 2\end{array}\right]$. Find the matrix representing a dilation of $x$ by 2 followed by a rotation about the $z$-axis of $45^{\circ}$.
c. Do your best to sketch a picture of $x, x$ after the first transformation, and $x$ after both transformations. You may use technology to help you.

## Exit Ticket Sample Solutions

Let $A$ be the matrix representing a rotation about the $z$-axis $45^{\circ}$ and $B$ be the matrix representing a dilation of 2 .
a. Write down $A$ and $B$.
$A=\left[\begin{array}{ccc}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1\end{array}\right], B=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2\end{array}\right]$
b. Let $x=\left[\begin{array}{c}3 \\ -1 \\ 2\end{array}\right]$. Find the matrix representing a dilation of $x$ by 2 followed by a rotation about the $z$-axis of $45^{\circ}$.
$A B=\left[\begin{array}{ccc}\sqrt{2} & -\sqrt{2} & 0 \\ \sqrt{2} & \sqrt{2} & 0 \\ 0 & 0 & 2\end{array}\right]$
c. Do your best to sketch a picture of $x, x$ after the first transformation, and $x$ after both transformations. You may use technology to help you.


## Problem Set Sample Solutions

1. Let $A$ be the matrix representing a dilation of $\frac{1}{2}$, and let $B$ be the matrix representing a reflection across the $y z$ plane.
a. Write $A$ and $B$.

$$
A=\left[\begin{array}{ccc}
\frac{1}{2} & 0 & 0 \\
0 & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2}
\end{array}\right], B=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

b. Evaluate $A B$. What does this matrix represent?
$A B=\left[\begin{array}{ccc}-\frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2}\end{array}\right]$
$A B$ is a reflection across the yz-plane followed by a dilation of $\frac{1}{2}$.
c. Let $x=\left[\begin{array}{l}5 \\ 6 \\ 4\end{array}\right], y=\left[\begin{array}{c}-1 \\ 3 \\ 2\end{array}\right]$, and $z=\left[\begin{array}{c}8 \\ -2 \\ -4\end{array}\right]$. Find $(A B) x,(A B) y$, and $(A B) z$.
$(A B) x=\left[\begin{array}{c}-\frac{5}{2} \\ 3 \\ 2\end{array}\right],(A B) y=\left[\begin{array}{c}\frac{1}{2} \\ \frac{3}{2} \\ 1\end{array}\right],(A B) z=\left[\begin{array}{c}4 \\ -1 \\ -2\end{array}\right]$
2. Let $A$ be the matrix representing a rotation of $30^{\circ}$ about the $x$-axis, and let $B$ be the matrix representing a dilation of 5 .
a. Write $A$ and $B$.
$A=\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2}\end{array}\right], B=\left[\begin{array}{lll}5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5\end{array}\right]$
b. Evaluate $A B$. What does this matrix represent?
$A B=\left[\begin{array}{ccc}5 & 0 & 0 \\ 0 & \frac{5 \sqrt{3}}{2} & -\frac{5}{2} \\ 0 & \frac{5}{2} & \frac{5 \sqrt{3}}{2}\end{array}\right]$
$A B$ is a dilation of 5 followed by a rotation of $30^{\circ}$ about the $x$-axis.
c. Let $x=\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right], y=\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right], z=\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]$. Find $(A B) x,(A B) y$, and $(A B) z$.

$$
(A B) x=\left[\begin{array}{l}
5 \\
0 \\
0
\end{array}\right]
$$

$$
(A B) y=\left[\begin{array}{c}
0 \\
\frac{5 \sqrt{3}}{2} \\
\frac{5}{2}
\end{array}\right]
$$

$$
(A B) Z=\left[\begin{array}{c}
0 \\
\frac{5}{2} \\
\frac{5 \sqrt{3}}{2}
\end{array}\right]
$$

3. Let $A$ be the matrix representing a dilation of 3 , and let $B$ be the matrix representing a reflection across the plane $y=x$.
a. Write $A$ and $B$.

$$
A=\left[\begin{array}{lll}
3 & 0 & 0 \\
0 & 3 & 0 \\
0 & 0 & 3
\end{array}\right], B=\left[\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

b. Evaluate $A B$. What does this matrix represent?
$A B=\left[\begin{array}{lll}0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 3\end{array}\right]$
$A B$ is a reflection across the $y=x$ plane followed by a dilation of 3.
c. Let $x=\left[\begin{array}{c}-2 \\ 7 \\ 3\end{array}\right]$. Find $(A B) x$.

$$
(A B) x=\left[\begin{array}{c}
21 \\
-6 \\
9
\end{array}\right]
$$

4. Let $A=\left[\begin{array}{lll}3 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 1\end{array}\right], B=\left[\begin{array}{ccc}0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right]$.
a. Evaluate $A B$.
$\left[\begin{array}{ccc}0 & -3 & 0 \\ 3 & -3 & 0 \\ 0 & 0 & 1\end{array}\right]$
b. Let $x=\left[\begin{array}{c}-2 \\ 2 \\ 5\end{array}\right]$. Find $(A B) x$.

$$
\left[\begin{array}{c}
-6 \\
-12 \\
5
\end{array}\right]
$$

c. Graph $x$ and $(A B) x$.

5. Let $A=\left[\begin{array}{lll}\frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & \frac{1}{3}\end{array}\right], B=\left[\begin{array}{ccc}3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1\end{array}\right]$.
a. Evaluate $A B$.
$\left[\begin{array}{ccc}1 & \frac{1}{3} & 0 \\ 1 & -3 & 0 \\ 6 & 2 & \frac{1}{3}\end{array}\right]$
b. Let $x=\left[\begin{array}{l}0 \\ 3 \\ 2\end{array}\right]$. Find $(A B) x$.
$\left[\begin{array}{c}1 \\ -9 \\ 20 \\ \hline 3\end{array}\right]$
c. Graph $x$ and $(A B) x$.

6. Let $A=\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8\end{array}\right], B=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0\end{array}\right]$.
a. Evaluate $A B$.
$\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0\end{array}\right]$
b. Let $x=\left[\begin{array}{c}1 \\ -2 \\ 4\end{array}\right]$. Find $(A B) x$.
$\left[\begin{array}{c}2 \\ -6 \\ 0\end{array}\right]$

d. What does $A B$ represent geometrically?
$A B$ represents a dilation of 2 in the $x$-direction, 3 in the $y$-direction, and a projection onto the $x y$-plane.
7. Let $A, B, C$ be $3 \times 3$ matrices representing linear transformations.
a. What does $A(B C)$ represent?

The linear transformation of applying the linear transformation that $C$ represents followed by the transformation that B represents, followed by the transformation that $A$ represents.
b. Will the pattern established in part (a) be true no matter how many matrices are multiplied on the left?

Yes, in general. When you multiply by a matrix on the left, you are applying a linear transformation after all linear transformations to the right have been applied.
c. Does $(A B) C$ represent something different from $A(B C)$ ? Explain.

No, it does not. This is the linear transformation obtained by applying $C$ then $A B$, which is $B$ followed by $A$.
8. Let $A B$ represent any composition of linear transformations in $\mathbb{R}^{3}$. What is the value of $(A B) x$ where $x=\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]$ ?

Since a composition of linear transformations in $\mathbb{R}^{3}$ is also a linear transformation, we know that applying it to the origin will result in no change.

## Lesson 10: Matrix Multiplication Is Not Commutative

## Student Outcomes

- Students understand that, unlike multiplication of numbers, matrix multiplication is not commutative.


## Lesson Notes

In this lesson, students first demonstrate that linear transformations in the coordinate plane do not commute. Since each linear transformation corresponds to a $2 \times 2$ matrix, students see that the corresponding matrix multiplication must also fail to commute (N-VM.C.9). Students verify this fact algebraically by multiplying matrices in both orders. Work is then extended to coordinates in 3-D space to prove that multiplication of $3 \times 3$ matrices is also not commutative.

## Classwork

## Opening Exercise (7 minutes)

Allow student time to work on the Opening Exercise independently before discussing as a class. Solicit volunteers to demonstrate each transformation in order to illustrate that the result is not the same.

## Consider the vector $\mathbf{v}=\binom{\mathbf{0}}{\mathbf{1}}$.

If v is rotated $45^{\circ}$ counterclockwise and then reflected across the $y$-axis, what is the resulting vector?

$$
\binom{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}
$$

If $v$ is reflected across the $y$-axis and then rotated $45^{\circ}$ counterclockwise about the origin, what is the resulting vector?

$$
\binom{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}
$$

## Scaffolding:

- Consider providing students who struggle with transformations a coordinate grid, a transparency, and a finepoint dry erase marker to perform each transformation.
- Students place the transparency on the grid and draw vector $(0,1)$; then, they perform the rotation with the transparency to help determine the location of the transformed vector.


## Did these linear transformations commute? Explain.

No. $R_{0,45^{\circ}} \circ R_{y} \neq R_{y} \circ R_{0,45^{\circ}}$

- Do you think linear transformations, in general, commute? Explain why or why not.
- No. The resulting vector varies depending on the order in which the transformations are applied.

Write this on the board: $R_{0,45^{\circ}} \circ R_{y} \neq R_{y} \circ R_{0,45^{\circ}}$

- What matrix corresponds to a $45^{\circ}$ counterclockwise rotation?
- $\left(\begin{array}{cc}\cos \left(45^{\circ}\right) & -\sin \left(45^{\circ}\right) \\ \sin \left(45^{\circ}\right) & \cos \left(45^{\circ}\right)\end{array}\right)$
- What matrix corresponds to a reflection across the $y$-axis?
- $\left(\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right)$
- What does the fact that the linear transformations failed to commute tell us about the corresponding matrix multiplication?
- The multiplication of the two transformation matrices must also fail to commute.


## Exercise 1 (5 minutes)

Allow students time to verify algebraically that the matrix multiplication fails to commute. Share results as a class.

Exercises 1-4

1. Let $A$ equal the matrix that corresponds to a $45^{\circ}$ rotation counterclockwise and $B$ equal the matrix that corresponds to a reflection across the $y$-axis. Verify that matrix multiplication does not commute by finding the products $A B$ and $B A$.

$$
\begin{gathered}
A B=\left(\begin{array}{cc}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{array}\right)\left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right)=\left(\begin{array}{cc}
-\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{array}\right) \\
B A=\left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right)\left(\begin{array}{cc}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{array}\right)=\left(\begin{array}{cc}
-\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{array}\right) \\
A B \neq B A
\end{gathered}
$$

- What can be said about multiplication of two $2 \times 2$ matrices?
- Unlike multiplication of two numbers, it is not commutative.
- What do you think about multiplication of two $3 \times 3$ matrices? Why?
- I do not think the multiplication would be commutative either. Each $3 \times 3$ matrix represents a transformation in 3-D space. These transformations will not be commutative; therefore, the corresponding multiplication will not be commutative.


## Exercise $\mathbf{2}$ ( 7 minutes)

Allow students time to verify algebraically that the matrix multiplication fails to commute. Share results as a class.

$$
\text { 2. Let } A=\left(\begin{array}{ccc}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right) \text { and } B=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right) \text {. Verify that matrix multiplication does not commute by finding the }
$$ products $A B$ and $B A$.

$$
\begin{aligned}
& A B=\left(\begin{array}{lll}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right) \\
& B A=\left(\begin{array}{ccc}
0 & -1 & 0 \\
0 & 0 & -1 \\
1 & 0 & 0
\end{array}\right) \\
& A B \neq B A
\end{aligned}
$$

- Summarize to a partner what can be said about linear transformations and, therefore, matrix multiplication.
- Neither of them commutes. The order in which transformations are applied affects the outcome; therefore, the order in which square matrices are multiplied affects the product.


## Exercise 3 (7 minutes)

In this exercise, students will find that the products are equal. We want students to realize that although there are special cases when matrix multiplication appears to be commutative, matrix multiplication is not commutative because as we have seen in previous examples, there are many cases where $A B$ does not equal $B A$ (for square matrices $A$ and $B$ ). Allow students time to work with a partner. Share results as a class.

## Scaffolding:

3. Consider the vector $\mathrm{v}=\left(\begin{array}{l}0 \\ 0 \\ 1\end{array}\right)$.
a. If $v$ is rotated $45^{\circ}$ counterclockwise about the $z$-axis and then reflected across the $x y$ plane, what is the resulting vector?

$$
\left(\begin{array}{c}
0 \\
0 \\
-1
\end{array}\right)
$$

b. If $v$ is reflected across the $x y$-plane and then rotated $45^{\circ}$ counterclockwise about the $z$-axis, what is the resulting vector?

$$
\left(\begin{array}{c}
0 \\
0 \\
-1
\end{array}\right)
$$

c. Verify algebraically that the product of the two corresponding matrices is the same regardless of the order in which they are multiplied.
Let $A=\left(\begin{array}{ccc}\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1\end{array}\right)$ (the matrix representing $45^{\circ}$ rotation about the $Z$-axis) Let $B=\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1\end{array}\right)$. (the matrix representing a reflection across the $x y$-plane)

$$
A B=B A=\left(\begin{array}{ccc}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\
0 & 0 & -1
\end{array}\right)
$$

- What can be said about the order in which the transformations are applied in this particular example?
- The resulting point is the same regardless of the order in which the transformations are applied.
- What can be said about the corresponding matrix multiplication?
- In this case, $A B=B A$, but this is a particular case.
- Summarize with a partner what can be said about matrix multiplication.
- Matrix multiplication is not commutative. However, there are instances where the products are the same. This would correspond geometrically to cases in which the order that transformations are applied does not affect the location of the point.


## Exercise 4 ( 9 minutes)

Allow students time to work with a partner. Have several groups show their results from part (a) in order to establish that multiplication of any two matrices in this form will have the same product. Stress that this is a special case and does not mean that matrix multiplication is commutative.

## Exercise 4

4. Write two matrices in the form $\left(\begin{array}{cc}a & -b \\ b & a\end{array}\right)$.
a. Verify algebraically that the products of these two matrices are equal.

$$
\begin{aligned}
& \text { Let } A=\left(\begin{array}{cc}
2 & -1 \\
1 & 2
\end{array}\right) \text { and } B=\left(\begin{array}{cc}
3 & -2 \\
2 & 3
\end{array}\right) . \\
& A B=\left(\begin{array}{cc}
4 & -7 \\
7 & 4
\end{array}\right) \text { and } B A=\left(\begin{array}{cc}
4 & -7 \\
7 & 4
\end{array}\right) . \\
& A B=B A
\end{aligned}
$$

b. Write each of your matrices as a complex number. Find the product of the two complex numbers.

$$
(2+i)(3+2 i)=4+7 i
$$

c. Why is it the case that any two matrices in the form $\left(\begin{array}{cc}a & -b \\ b & a\end{array}\right)$ have products that are equal regardless of the order in which they are multiplied?

Matrices in this form represent the geometric effect of complex multiplication. Multiplying a complex number $z$ by a complex number $\alpha$ and then by a complex number $\beta$ gives the same answer as multiplying by $\beta$ and then $\alpha$; that is, $\beta(\alpha z)=\alpha(\beta z)$; thus, the corresponding matrix multiplication yields the same product.

- What did you discover about the matrices above? (Allow several groups to share their work.)
- $A B=B A$
- Does this mean matrix multiplication is commutative? Explain.
- No. This is a special case because the matrices are in the form $\left(\begin{array}{cc}a & -b \\ b & a\end{array}\right)$.
- What is the relationship between these matrices and complex numbers?
- Matrices in this form can be used to represent a corresponding complex number. Multiplying these matrices is the same as multiplying two complex numbers.
- Is the multiplication of two complex numbers commutative?
- Yes. Therefore, two matrices in the form $\left(\begin{array}{cc}a & -b \\ b & a\end{array}\right)$ have the same product, but this does not mean that matrix multiplication is commutative.


## Closing ( 5 minutes)

Ask students to summarize in writing what can be said about multiplication of two square matrices and then share with a partner. Discuss key points as a class.

- Is matrix multiplication commutative?
- No.
- How does what we know about linear transformations support this notion?
- The product of the two matrices represents two linear transformations. Since linear transformations do not commute, matrix multiplication must also fail to commute.
- Are there instances when, for two matrices $A$ and $B, A B=B A$ ? Explain.
- Yes, particularly when the two matrices represent complex numbers. The product of the two matrices is also a complex number. Since complex multiplication is commutative, matrix multiplication appears to be commutative in this instance.


## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 10: Matrix Multiplication Is Not Commutative

## Exit Ticket

1. Let $A$ be the matrix representing a rotation about the origin by $60^{\circ}$ and $B$ be the matrix representing a reflection across the $x$-axis.
a. Give two reasons why $A B \neq B A$.
b. Let $x=\binom{1}{-1}$. Evaluate $A(B x)$ and $B(A x)$.
2. Write two matrices, $A, B$, that represent linear transformations where $A(B x)$ and $B(A x)$ where $x=\binom{3}{2}$. Explain why the products are the same.

## Exit Ticket Sample Solutions

1. Let $A$ be the matrix representing a rotation about the origin by $60^{\circ}$ and $B$ be the matrix representing a reflection across the $x$-axis.
a. Give two reasons why $A B \neq B A$.

The linear transformations these matrices represent do not commute, and the matrices themselves do not commute.

$$
A=\left(\begin{array}{cc}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{array}\right) ; B=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) ; A B=\left(\begin{array}{cc}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{array}\right) ; B A=\left(\begin{array}{cc}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{array}\right)
$$

b. Let $x=\binom{1}{-1}$. Evaluate $A(B x)$ and $B(A x)$.

$$
\begin{aligned}
& \boldsymbol{A}(\boldsymbol{B} \boldsymbol{x})=(\boldsymbol{A B}) \boldsymbol{x}=\binom{\frac{1-\sqrt{3}}{2}}{\frac{1+\sqrt{3}}{2}} \\
& \boldsymbol{B}(\boldsymbol{A} \boldsymbol{x})=(\boldsymbol{B} \boldsymbol{A}) \boldsymbol{x}=\binom{\frac{1+\sqrt{3}}{2}}{\frac{1-\sqrt{3}}{2}}
\end{aligned}
$$

2. Write two matrices, $A, B$, that represent linear transformations where $A(B x)$ and $B(A x)$ where $x=\binom{3}{2}$. Explain why the products are the same.
Answers may vary. Examples include any two rotations about the origin, any two matrices representing complex numbers, and a dilation with any other matrix (since scalars commute with all matrices). For example,

$$
\begin{gathered}
A=\left(\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right), B=\left(\begin{array}{cc}
1 & -1 \\
1 & 1
\end{array}\right) \\
A B=B A=\left(\begin{array}{cc}
2 & -2 \\
2 & 2
\end{array}\right) \\
A(B x)=B(A x)=\binom{2}{10}
\end{gathered}
$$

Matrices in the form $\left(\begin{array}{cc}a & -b \\ b & a\end{array}\right)$ represent the geometric effect of complex multiplication. Multiplying a complex number $z$ by a complex number $\alpha$ and then by a complex number $\beta$ gives the same answer as multiplying by $\beta$ and then $\alpha$; that is, $\beta(\alpha z)=\alpha(\beta z)$; thus, the corresponding matrix multiplication yields the same product.

## Problem Set Sample Solutions

1. Let $A$ be the matrix representing a dilation of $2, B$ the matrix representing a rotation of $30^{\circ}$, and $x=\binom{-2}{3}$.
a. Evaluate $A B$.

$$
\left(\begin{array}{cc}
\sqrt{3} & -1 \\
1 & \sqrt{3}
\end{array}\right)
$$

b. Evaluate $B \boldsymbol{A}$.

$$
\left(\begin{array}{cc}
\sqrt{3} & -1 \\
1 & \sqrt{3}
\end{array}\right)
$$

c. Find $A B x$.

$$
\binom{-2 \sqrt{3}-3}{-2+3 \sqrt{3}}
$$

d. Find $B A x$.

$$
\binom{-2 \sqrt{3}-3}{-2+3 \sqrt{3}}
$$

2. Let $A$ be the matrix representing a reflection across the line $y=x, B$ the matrix representing a rotation of $90^{\circ}$, and $x=\binom{1}{\mathbf{0}}$.
a. Evaluate $A B$.

$$
\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

b. Evaluate $\boldsymbol{B A}$.

$$
\left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right)
$$

c. Find $A x$.

$$
\binom{0}{1}
$$

d. Find $B x$.

$$
\binom{0}{1}
$$

e. Find $A B x$.

$$
\binom{1}{0}
$$

f. Find $B A x$.

$$
\binom{-1}{0}
$$

g. Describe the linear transformation represented by $A B$. $A B$ represents $a$ reflection across the $x$-axis.
h. Describe the linear transformation represented by $B A$.
$B A$ represents a reflection across the $y$-axis.
3. Let matrices $A, B$ represent scalars. Answer the following questions.
a. Would you expect $A B=B A$ ? Explain why or why not.

Yes. Scalars are real numbers, and real numbers commute.
b. Let $A=\left(\begin{array}{ll}a & 0 \\ 0 & a\end{array}\right)$ and $B=\left(\begin{array}{ll}b & 0 \\ 0 & b\end{array}\right)$. Show $A B=B A$ through matrix multiplication and explain why.

$$
A B=\left(\begin{array}{cc}
a b & 0 \\
0 & a b
\end{array}\right)=\left(\begin{array}{cc}
b a & 0 \\
0 & b a
\end{array}\right)=B A
$$

These matrices represent complex numbers and are both of the form $\left(\begin{array}{cc}a & -b \\ b & a\end{array}\right)$, so their products will be the same regardless of the order.
4. Let matrices $A, B$ represent complex numbers. Answer the following questions.
a. Let $A=\left(\begin{array}{cc}a & -b \\ b & a\end{array}\right)$ and $B=\left(\begin{array}{cc}c & -d \\ d & c\end{array}\right)$. Show $A B=B A$ through matrix multiplication.

$$
\begin{aligned}
A B & =\left(\begin{array}{ll}
a c-b d & -a d-b c \\
b c+a d & -b d+a c
\end{array}\right) \\
B A & =\left(\begin{array}{ll}
c a-d b & -c b-d a \\
d a+c b & -d b+c a
\end{array}\right)
\end{aligned}
$$

They are equivalent.
b. Would you expect $A C=C A$ for any matrix $C$ ? Explain.

No. Matrix multiplication is not commutative.
c. Let $C$ be any $2 \times 2$ matrix, $\left(\begin{array}{ll}x & y \\ z & w\end{array}\right)$. Show $A C \neq C A$ through matrix multiplication.

$$
\begin{aligned}
A C & =\left(\begin{array}{ll}
a x-b z & a y-b w \\
b x+a z & b y+a w
\end{array}\right) \\
C A & =\left(\begin{array}{lc}
a x+b z & a y+b w \\
a z+b x & -b z+a w
\end{array}\right)
\end{aligned}
$$

d. Summarize your results from Problems 4 and 5.

Matrices representing scalars commute with any other matrix, and matrices representing complex number multiplication only commute when multiplying by other complex numbers (or if $b=0$ ).

| Lesson 10: | Matrix Multiplication Is Not Commutative |
| :--- | :--- |
| Date: | $1 / 24 / 15$ |

5. Quaternions are a number system that extends to complex numbers discovered by William Hamilton in 1843. Multiplication of quaternions is not commutative and is defined as the quotient of two vectors. They are useful in 3-dimensional rotation calculations for computer graphics. Quaternions are formed the following way:

$$
i^{2}=j^{2}=k^{2}=i j k=-1
$$

Multiplication by $\mathbf{- 1}$ and 1 works normally. We can represent all possible multiplications of quaternions through a table:

| $\times$ | $i$ | $j$ | $k$ |
| :---: | :---: | :---: | :---: |
| $i$ | -1 | $k$ | $-j$ |
| $j$ | $-k$ | -1 | $i$ |
| $k$ | $j$ | $-i$ | -1 |

a. Is multiplication of quaternions commutative? Explain why or why not.

No. For example, $\boldsymbol{i j}=\boldsymbol{k}$, but $\boldsymbol{j} \boldsymbol{i}=-\boldsymbol{k}$.
5.
b. Is multiplication of quaternions associative? Explain why or why not.

Yes. $i(j k)=i(-i)=-1$, and $(i j) k=(-k) k=-1$. 1 and -1 work normally.

## Lesson 11: Matrix Addition Is Commutative

## Student Outcomes

- Students prove both geometrically and algebraically that matrix addition is commutative.


## Lesson Notes

In Topic A, we interpreted matrices as representing network diagrams. In that context, an arithmetic system for matrices was natural. Given two matrices $A$ and $B$ of equal dimensions, we defined the matrix product $A B$ and the matrix sum $A+B$. Both of these operations had a meaning within the context of networks. In Topic B , we returned to our interpretation of matrices as representing the geometric effect of linear transformations from Module 1. We have found that the matrix product $A B$ has meaning in this context; it is the composition of transformations. In this lesson, we will explore the question, "Can we give matrix addition meaning in the setting of geometric transformations?"

## Classwork

## Opening Exercise (5 minutes)

Allow students time to work on the Opening Exercise independently before discussing as a class.

## Opening Exercise

Kiamba thinks $A+B=B+A$ for all $2 \times 2$ matrices. Rachel thinks it is not always true. Who is correct? Explain.

Let $A=\left(\begin{array}{ll}a_{11} & a_{12} \\ a_{21} & a_{22}\end{array}\right)$ and $B=\left(\begin{array}{ll}b_{11} & b_{12} \\ b_{21} & b_{22}\end{array}\right)$.
What is the sum of $A+B$ ?
$A+B=\left(\begin{array}{ll}a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22}\end{array}\right)$
$B+A=\left(\begin{array}{cc}b_{11}+a_{11} & b_{12}+a_{12} \\ b_{21}+a_{21} & b_{22}+a_{22}\end{array}\right)$

## Scaffolding:

Provide students with concrete examples if necessary.

$$
\begin{aligned}
& A=\left(\begin{array}{cc}
-2 & 1 \\
3 & 2
\end{array}\right) \text { and } B=\left(\begin{array}{cc}
4 & 0 \\
-1 & 6
\end{array}\right) \\
& A+B=B+A=\left(\begin{array}{ll}
2 & 1 \\
2 & 8
\end{array}\right)
\end{aligned}
$$

The two matrices must be equal because each of the sums must be equal according to the commutative property of addition of real numbers. Kiamba is correct.

- Can we say that matrix addition is commutative?
- Yes. The order in which we add the matrices does not change the sum.
- Will this hold true if we change the dimensions of the matrices being added?
- Yes. Regardless of the size of the matrices, the two sums $(A+B$ and $B+A)$ would still be the same.
[Note: Demonstrate with two $3 \times 3$ matrices if students seem unsure.]
- So we see that matrix addition is commutative, but we still have not determined the geometric meaning of matrix addition.


## Exercise 1 ( 12 minutes)

Allow students time to work in groups on Exercise 1. Optionally, give students colored pencils or a transparency and fine-point dry erase marker to mark their points.

## Exercises 1-6

1. In two-dimensional space, let $A$ be the matrix representing a rotation about the origin through an angle of $45^{\circ}$, and let $B$ be the matrix representing a reflection about the $x$-axis. Let $x$ be the point $\binom{1}{1}$.
a. Write down the matrices $A, B$, and $A+B$.

$$
A=\left(\begin{array}{cc}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{array}\right)
$$

$$
A+B=\left(\begin{array}{cc}
\frac{\sqrt{2}}{2}+1 & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}-1
\end{array}\right)
$$


b. Write down the image points of $A x, B x$, and $(A+B) x$, and plot them on graph paper.
$A x=\binom{0}{\sqrt{2}} \quad B x=\binom{1}{-1} \quad(A+B) x=\binom{1}{\sqrt{2}-1}$
c. What do you notice about $(A+B) x$ compared to $A x$ and $B x$ ?

The point $(A+B) x$ is equal to the sum of the points $A x$ and $B x$ by the distributive property.

- Since $A x+B x=(A+B)(x)$, does it follow that $B x+A x=(B+A)(x)=(A+B)(x)$ ?
- Yes, $A x+B x$ and $B x+A x$ must both map to the same point since we know that the addition of points is commutative. Since $A x+B x$ maps to the point $(A+B)(x)$ by the distributive property, $B x+A x$ must also map to the point $(A+B)(x)$ since $A x+B x=B x+A x$.
- What does this say about matrix addition?
- It confirms what we saw in the opening exercise. Matrix addition is commutative.


## Exercise 2 (13 minutes)

Allow students time to work in groups on Exercise 2 before discussing as a class.
2. For three matrices of equal size, $A, B$, and $C$, does it follow that $A+(B+C)=(A+B)+C$ ?
a. Determine if the statement is true geometrically. Let $A$ be the matrix representing a reflection across the $y$-axis. Let $B$ be the matrix representing a counterclockwise rotation of $30^{\circ}$. Let $C$ be the matrix representing a reflection about the $x$-axis. Let $x$ be the point $\binom{1}{1}$.
$A=\left(\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right) \quad B=\left(\begin{array}{cc}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{array}\right)$

$$
C=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right)
$$

$A+B=\left(\begin{array}{cc}\frac{\sqrt{3}-2}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}+2}{2}\end{array}\right)$

$$
B+C=\left(\begin{array}{cc}
\frac{\sqrt{3}+2}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{\sqrt{3}-2}{2}
\end{array}\right)
$$



From the graph, we see that $A x+(B x+C x)=(A x+B x)+C x$.
b. Confirm your results algebraically.
$A+(B+C)=\left(\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right)+\left(\begin{array}{cc}\frac{\sqrt{3}+2}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}-2}{2}\end{array}\right)=\left(\begin{array}{cc}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{array}\right)$
$(A+B)+C=\left(\begin{array}{cc}\frac{\sqrt{3}-2}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}+2}{2}\end{array}\right)+\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)=\left(\begin{array}{cc}\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2}\end{array}\right)$
c. What do your results say about matrix addition?

Matrix addition is associative.

- What does this exercise tell us about matrix addition?
- It is associative.
- How did the graph support the idea that matrix addition is associative?
- $\quad(A+(B+C)) x$ and $((A+B)+C) x$ mapped to the same point.
- Would this still be true if we used matrices of different (but still equal) dimensions?
- Yes. The addition would still be associative. For example, we could do the same type of operations using $3 \times 3$ matrices, but geometrically it would represent points in 3-D space rather than on a coordinate plane.
- Could we say $A+(B+C)=C+(A+B)$ ?
- Yes. We just demonstrated that $A+(B+C)=(A+B)+C$. We can then say that $(A+B)+C=C+(A+B)$ because we know that matrix addition is commutative.


## Exercises 3-6 (5 minutes)

Allow students time to work in groups on Exercises 3-6 before discussing as a class.
3. If $x=\left(\begin{array}{l}x \\ y \\ z\end{array}\right)$, what are the coordinates of a point $y$ with the property $x+y$ is the origin $O=\left(\begin{array}{l}0 \\ 0 \\ 0\end{array}\right)$ ?

$$
y=\left(\begin{array}{l}
-x \\
-y \\
-z
\end{array}\right)
$$

4. Suppose $A=\left(\begin{array}{ccc}11 & -5 & 2 \\ -34 & 6 & 19 \\ 8 & -542 & 0\end{array}\right)$, and matrix $B$ has the property that $A x+B x$ is the origin. What is the matrix $B$ ?

$$
B=\left(\begin{array}{ccc}
-11 & 5 & -2 \\
34 & -6 & -19 \\
-8 & 542 & 0
\end{array}\right)
$$

5. For three matrices of equal size, $A, B$, and $C$, where $A$ represents a reflection across the line $y=x, B$ represents a counterclockwise rotation of $45^{\circ}, C$ represents a reflection across the $y$-axis, and $x=\binom{1}{2}$ :
a. Show that matrix addition is commutative: $A x+B x=B x+C x$.

$$
\begin{gathered}
A x=\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right)\binom{1}{2}=\binom{-2}{1} \\
B x=\left(\begin{array}{cc}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{array}\right)\binom{1}{2}=\binom{\frac{\sqrt{2}}{2}-\sqrt{2}}{\frac{\sqrt{2}}{2}+\sqrt{2}} \\
A x+B x=\binom{-2+\frac{\sqrt{2}}{2}-\sqrt{2}}{1+\frac{\sqrt{2}}{2}+\sqrt{2}}=B x+A x
\end{gathered}
$$

b. Show that matrix addition is associative: $A x+(B x+C x)=(A x+B x)+C x$.

$$
\begin{gathered}
C x=\left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right)\binom{1}{2}=\binom{-1}{2} \\
A x+(B x+C x)=\binom{-2}{1}+\binom{\frac{\sqrt{2}}{2}-\sqrt{2}-1}{\frac{\sqrt{2}}{2}+\sqrt{2}+2} \\
(A x+B x)+C x=\binom{-2+\frac{\sqrt{2}}{2}-\sqrt{2}}{1+\frac{\sqrt{2}}{2}+\sqrt{2}}+\binom{-1}{2} \\
A x+(B x+C x)=(A x+B x)+C x=\binom{-3+\frac{\sqrt{2}}{2}-\sqrt{2}}{3+\frac{\sqrt{2}}{2}+\sqrt{2}}
\end{gathered}
$$

## Scaffolding:

Provide early finishers with this challenge question.

If $A x=0$ for all $x$, must every entry of $A$ be 0 ?
6. Let $A, B, C$, and $D$ be matrices of the same dimensions. Use the commutative property of addition of two matrices to prove $A+B+C=C+B+A$.

If we treat $A+B$ as one matrix, then

$$
\begin{aligned}
(A+B)+C & =C+(A+B) \\
& =C+(B+\boldsymbol{A}) \\
& =C+B+\boldsymbol{A} .
\end{aligned}
$$

## Closing ( 5 minutes)

Discuss the key points of the lesson as a class.

- What can be said about matrix addition?
- Matrix addition, like addition of numbers, is both commutative and associative.
- What is the geometric meaning of matrix addition? Illustrate with an example.
- $\quad(A+B) x$ maps to the same point as $A x+B x$.
- Answers may vary. $A=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right), B=\left(\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right)$, and $x=\binom{1}{1} . A+B=\left(\begin{array}{cc}2 & 0 \\ 0 & -2\end{array}\right)$,

$$
\begin{aligned}
& (A+B) x=\binom{2}{-2}, A x=\binom{1}{-1}, B x=\binom{1}{-1}, \text { and } A x+B x=\binom{2}{-2} . \text { Therefore, } \\
& (A+B) x=A x+B x
\end{aligned}
$$

## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 11: Matrix Addition Is Commutative

## Exit Ticket

1. Let $x=\binom{3}{-1}, A=\left(\begin{array}{ll}2 & 2 \\ 2 & 0\end{array}\right)$, and $B=\left(\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right)$.
a. Find and plot the points $A x, B x$, and $(A+B) x$ on the axes below.

b. Show algebraically that matrix addition is commutative, $A x+B x=B x+A x$.
2. Let $A=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ and $B=\left(\begin{array}{ll}x & y \\ z & w\end{array}\right)$. Prove $A+B=B+A$.

## Exit Ticket Sample Solutions

1. Let $x=\binom{3}{-1}, A=\left(\begin{array}{ll}2 & 2 \\ 2 & 0\end{array}\right)$, and $B=\left(\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right)$.
a. Find and plot the points $A x, B x$, and $(A+B) x$ on the axes below.

b. Show algebraically that matrix addition is commutative, $A x+B x=B x+A x$.

$$
A x=\binom{4}{6}, B x=\binom{-3}{1}, A x+B x=\binom{1}{7}, \text { and } B x+A x=\binom{1}{7} . A x+B x=B x+A x
$$

2. Let $A=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$ and $B=\left(\begin{array}{ll}x & y \\ z & w\end{array}\right)$. Prove $A+B=B+A$.

$$
\begin{aligned}
A+B & =\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)+\left(\begin{array}{ll}
x & y \\
z & w
\end{array}\right) \\
& =\left(\begin{array}{ll}
a+x & b+y \\
c+d & z+w
\end{array}\right) \\
& =\left(\begin{array}{ll}
x+a & y+b \\
d+c & w+z
\end{array}\right) \\
& =\left(\begin{array}{ll}
x & y \\
z & w
\end{array}\right)+\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right) \\
& =B+A
\end{aligned}
$$

## Problem Set Sample Solutions

1. Let $A$ be matrix transformation representing a rotation of $45^{\circ}$ about the origin and $B$ be a reflection across the $y$-axis. Let $x=(3,4)$.
a. Represent $A$ and $B$ as matrices, and find $A+B$.

$$
A=\left(\begin{array}{cc}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2}
\end{array}\right) ; B=\left(\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right) ; A+B=\left(\begin{array}{cc}
-1+\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & 1+\frac{\sqrt{2}}{2}
\end{array}\right)
$$

b. Represent $A x$ and $B x$ as matrices, and find $(A+B) x$.

$$
A x=\binom{-\frac{\sqrt{2}}{2}}{\frac{7 \sqrt{2}}{2}} ; B x=\binom{-3}{4} ;(A+B) x=\binom{-3-\frac{\sqrt{2}}{2}}{4+\frac{7 \sqrt{2}}{2}}
$$

c. Graph your answer to part (b).

See graph in part (d).
d. Draw the parallelogram containing $A x, B x$, and $(A+B) x$.

2. Let $A$ be matrix transformation representing a rotation of $300^{\circ}$ about the origin and $B$ be a reflection across the $\boldsymbol{x}$-axis. Let $\boldsymbol{x}=(2,-5)$.
a. Represent $A$ and $B$ as matrices, and find $A+B$.

$$
A=\left(\begin{array}{cc}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{array}\right) ; B=\left(\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right) ; A+B=\left(\begin{array}{cc}
\frac{3}{2} & \frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{array}\right)
$$

b. Represent $A x$ and $B x$ as matrices, and find $(A+B) x$.

$$
A x=\binom{1-\frac{5 \sqrt{3}}{2}}{\frac{-5-2 \sqrt{3}}{2}} ; B x=\binom{2}{5} ;(A+B) x=\binom{3-\frac{5 \sqrt{3}}{2}}{\frac{5-2 \sqrt{3}}{2}}
$$

c. Graph your answer to part (b).

See graph in part (d).
d. Draw the parallelogram containing $A x, B x$, and $(A+B) x$.

3. Let $A, B, C$, and $D$ be matrices of the same dimensions.
a. Use the associative property of addition for three matrices to prove $(A+B)+(C+D)=A+(B+C)+D$. If we treat $C+D$ as one matrix, then

$$
\begin{aligned}
& (\boldsymbol{A}+\boldsymbol{B})+(\boldsymbol{C}+\boldsymbol{D})=(\boldsymbol{A}+\boldsymbol{B})+(\boldsymbol{C}+\boldsymbol{D}) \\
& =\boldsymbol{A}+(\boldsymbol{B}+(\boldsymbol{C}+\boldsymbol{D})) \\
& =\boldsymbol{A}+((\boldsymbol{B}+\boldsymbol{C})+\boldsymbol{D}) \\
& =\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})+\boldsymbol{D} .
\end{aligned}
$$

b. Make an argument for the associative and commutative properties of addition of matrices to be true for finitely many matrices being added.

For finitely many matrices, we can always use the formula that has been proven already and break the rest of the problem down into that many pieces, just like we did in part (a). Since this is true for any finite number, we could also use the formula that has been proven for one less than whatever number we are trying to prove the property true for.
4. Let $A$ be an $m \times n$ matrix with element in the $i^{\text {th }}$ row, $j^{\text {th }}$ column $a_{i j}$, and $B$ be an $\boldsymbol{m} \times \boldsymbol{n}$ matrix with element in the $i^{\text {th }}$ row, $j^{\text {th }}$ column $b_{i j}$. Present an argument that $A+B=B+A$.

The entry in the $i^{\text {th }}$ row, $j^{\text {th }}$ column of $A+B$ will be $a_{i j}+b_{i j}$, which is equal to $b_{i j}+a_{i j}$, which is the entry in the $i^{\text {th }}$ row, $j^{\text {th }}$ column of $B+A$. Since this is true for any element of $A+B$, we have that $A+B=B+A$ for any two matrices of equal dimensions.
5. For integers $x, y$, define $x \oplus y=x \cdot y+1$, read " $x$ plus $y$ " where $x \cdot y$ is defined normally.
a. Is this form of addition commutative? Explain why or why not.

Yes. $x \oplus y=x y+1=y x+1=y \oplus x$.
b. Is this form of addition associative? Explain why or why not.

No.
$x \oplus(y \oplus z)=x \oplus(y z+1)=x(y z+1)+1=x y z+x+1$.
$(x \oplus y) \oplus z=(x y+1) \oplus z=(x y+1) z+1=x y z+z+1$.
6. For integers $x, y$, define $x \oplus y=x$.
a. Is this form of addition commutative? Explain why or why not.

No. $x \oplus y=x$, and $y \oplus x=y$.
b. Is this form of addition associative? Explain why or why not.

Yes. $x \oplus(y \oplus z)=x \oplus y=x$, and $(x \oplus y) \oplus z=x \oplus z=x$. CORE'

## Lesson 12: Matrix Multiplication Is Distributive and

## Associative

## Student Outcomes

- Students discover and verify that matrix multiplication is distributive.
- Students discover and verify that matrix multiplication is associative.


## Lesson Notes

In this lesson, students use specific matrix transformations on points to show that matrix multiplication is distributive and associative. They then revisit some of the properties of matrices to prove that these properties hold for all matrices under multiplication.

## Classwork

## Opening Exercise (10 minutes)

Students review transformations represented by matrices studied in previous lessons in preparation for their work in Lesson 13. Students will write the matrix that represents the given transformation. Each transformation should be written on the board, and as the class agrees on the correct matrix that represents that transformation, a student should write the matrix under the transformation heading. Students can do this on paper at their desks, or this can be completed as a Rapid White Board Exchange. This could also be done in pairs or groups, and each group could present one transformation and explain it. The exercise can be modified to a matching activity if students are struggling.

## Opening Exercise

Write the $3 \times 3$ matrix that would represent the transformation listed.
a. No change when multiplying (the multiplicative identity matrix)

$$
\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

b. No change when adding (the additive identity matrix)

$$
\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]
$$

c. A rotation about the $\boldsymbol{x}$-axis of $\boldsymbol{\theta}$ degrees

$$
\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos (\theta) & -\sin (\theta) \\
0 & \sin (\theta) & \cos (\theta)
\end{array}\right]
$$

d. A rotation about the $y$-axis of $\boldsymbol{\theta}$ degrees

$$
\left[\begin{array}{ccc}
\cos (\theta) & 0 & \sin (\theta) \\
0 & 1 & 0 \\
-\sin (\theta) & 0 & \cos (\theta)
\end{array}\right]
$$

e. A rotation about the $z$-axis of $\boldsymbol{\theta}$ degrees

$$
\left[\begin{array}{ccc}
\cos (\theta) & -\sin (\theta) & 0 \\
\sin (\theta) & \cos (\theta & 0 \\
0 & 0 & 1
\end{array}\right]
$$

f. A reflection over the $x y$-plane

$$
\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right]
$$

g. A reflection over the $y z$-plane

$$
\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

h. A reflection over the $x z$-plane

$$
\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

i. A reflection over $y=x$ in the $x y$-plane

$$
\left[\begin{array}{lll}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

## Example 1 (15 minutes)

This example should be completed by students in pairs for parts (a) through (g). Part (h) should be completed as a teacher-led discussion after debriefing together the results of parts (a) through (g). Students look at transformations in two-dimensional space on a point. They predict the transformation, write a matrix that would represent that transformation, and apply the transformation to the point. Through a series of steps, they see that matrix multiplication is distributive and associative, revisit properties of matrices, and prove that these properties hold for all matrices under matrix multiplication.

- What are the dimensions of a matrix that represents two-dimensional space?

$$
\text { 마 } \quad 2 \times 2
$$

- Using your geometric intuition, can you tell me how the point $(1,1)$ would transform if it was rotated $90^{\circ}$ clockwise? What are the new coordinates?
- The $x$-coordinate would stay the same, and the $y$-coordinate would change signs. $(1,-1)$.


## Scaffolding:

Students could create a graphic organizer listing the matrix in one column and transformation represented in the next column.

## Scaffolding:

- Students with spatial issues may not be able to see the transformations in three-dimensions. Consider using a graphing program to help them visualize the transformations or demonstrating each one visually in some other way.
- For advanced learners, have students work in groups to complete this example with no leading questions. They should write up a summary and present their finding during the class discussion.
- What would happen to the point $(1,1)$ if it was rotated $180^{\circ}$ counterclockwise? What are the new coordinates?
- The $x$ - and $y$-coordinates would change signs. $(-1,-1)$.
- If matrix $A$ represents a rotation of $90^{\circ}$ clockwise, write matrix $A$. Show your steps.

$$
\quad A=\left[\begin{array}{cc}
\cos \left(-90^{\circ}\right) & -\sin \left(-90^{\circ}\right) \\
\sin \left(-90^{\circ}\right) & \cos \left(-90^{\circ}\right)
\end{array}\right]=\left[\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right]
$$

- If matrix B represents a reflection about the $x$-axis, write matrix $B$.

$$
B=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]
$$

- If matrix C represents a rotation of $180^{\circ}$ about the $y$-axis, write matrix $\mathbf{C}$. Show your steps.

$$
\therefore \quad C=\left[\begin{array}{cc}
\cos \left(180^{\circ}\right) & -\sin \left(180^{\circ}\right) \\
\sin \left(180^{\circ}\right) & \cos \left(180^{\circ}\right)
\end{array}\right]=\left[\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right]
$$

- $\quad X=\left[\begin{array}{l}1 \\ 1\end{array}\right]$; confirm your answers above by performing the transformations $B X$ and $C X$. Were your instincts correct? If not, analyze your mistakes.
- $B X=\left[\begin{array}{c}1 \\ -1\end{array}\right] ; \quad C X=\left[\begin{array}{c}-1 \\ -1\end{array}\right]$; answers will vary.
- What is $B X+C X$ ? What did you notice about the results of part (e) and part (g)?
- The results are the same matrix.
- Write out the mathematical statement.
- $\quad A(B X+C X)=(A B) X+(A C) X$
- Do you think matrix multiplication is distributive? If so, what would $A(B X+C X)$ be equivalent to?
- $A(B X+C X)=A(B X)+A(C X)$
- What property would be necessary in order for us to prove matrix multiplication is distributive given what we have already proven?
- We have proven for this set of matrices that $A(B X+C X)=(A B) X+(A C) X$, but we need $A(B X+C X)=A(B X)+A(C X)$. That means $(A B) X+(A C) X=A(B X)+A(C X)$, which means the associative property must hold.
- The sum of two matrices is the sum of corresponding elements. What would be the value of $(B+C) X$ ?
- $B X+C X$.
- We also know that applying a matrix $B$ and then matrix $A$ is the same as applying the matrix product $A B$. So, $A(B X)=$ ? And $A(C X)=$ ?
- $\quad(A B) X$
- $\quad(A C) X$
- Let's use what we have just stated to explain why $A(B+C)$ and $A B+A C$ have the same geometric effect on a point or points for any matrices $A, B$, and $C$.
- How do we know that $(A(B+C)) X=A((B+C) X)$ ?
- Applying the product of $A$ and $(B+C)$ has the same effect as applying $(B+C)$ and then $A$.
- Now $A((B+C) X)=A(B X+C X)$. Why?
- The sum of two matrices acts by summing the image.
- We also know that $A(B X+C X)=A(B X)+A(C X)$. Why?
- Each matrix represents a linear transformation.
- Continuing, $A(B X)+A(C X)=(A B) X+(A C) X$. Explain.
- Applying $B$ then $A$ is the same as applying $A B$.
- Applying $C$ then $A$ is the same as applying $A C$.
- And finally, why does $(A B) X+(A C) X=(A B+A C) X$ ?
- The sum of two matrices acts by summing the images.
- Therefore, what must be true?
- $A(B+C)=A B+A C$
- We have just shown that matrix multiplication is distributive and associative.
- Take a few minutes and discuss what we have just proven with your neighbor. Write a summary.


## Example 1

In three-dimensional space, let $A$ represent a rotation of $90^{\circ}$ about the $x$-axis, $B$ represent a reflection about the $y z$-plane, and $C$ represent a rotation of $180^{\circ}$ about the $z$-axis. Let $X=\left[\begin{array}{l}1 \\ 1 \\ 1\end{array}\right]$.
a. As best you can, sketch a three-dimensional set of axes and the location of the point $X$.

b. Using only your geometric intuition, what are the coordinates of $B X$ ? CX ? Explain your thinking.
$B X=\left[\begin{array}{c}-1 \\ 1 \\ 1\end{array}\right] ; C X=\left[\begin{array}{c}-1 \\ -1 \\ 1\end{array}\right]$; answers will vary but could include that when rotating about the $x$-axis $90^{\circ}$, only the
$x$-coordinate would change signs; however, when rotating about the z -axis $180^{\circ}$, the $x$ - and $y$-coordinates would change signs.
c. Write down matrices $B$ and $C$, and verify or disprove your answers to part (b).

$$
B=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right] ; C=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right] ; B X=\left[\begin{array}{c}
-1 \\
1 \\
1
\end{array}\right] ; C X=\left[\begin{array}{c}
-1 \\
-1 \\
1
\end{array}\right]
$$

d. What is the sum of $B X+C X$ ?

$$
B X+C X=\left[\begin{array}{c}
-2 \\
0 \\
2
\end{array}\right]
$$

e. Write down matrix $A$, and compute $A(B X+C X)$.

$$
\begin{aligned}
A & =\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right] \\
A(B X+C X) & =\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right]\left[\begin{array}{c}
-2 \\
0 \\
2
\end{array}\right]=\left[\begin{array}{c}
-2 \\
-2 \\
0
\end{array}\right]
\end{aligned}
$$

f. Compute $A B$ and $A C$.

$$
\begin{aligned}
A B & =\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right]\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right] \\
A C & =\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right]\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right]=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{array}\right]
\end{aligned}
$$

g. Compute $(A B) X,(A C) X$, and their sum. Compare your result to your answer to part (e). What do you notice?

$$
\begin{gathered}
(A B) X=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]=\left[\begin{array}{c}
-1 \\
-1 \\
1
\end{array}\right] \\
(A C) X=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 0 & -1 \\
0 & -1 & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]=\left[\begin{array}{c}
-1 \\
-1 \\
-1
\end{array}\right] \\
(A B) X+(A C) X=\left[\begin{array}{c}
-1 \\
-1 \\
1
\end{array}\right]\left[\begin{array}{l}
-1 \\
-1 \\
-1
\end{array}\right]=\left[\begin{array}{c}
-2 \\
-2 \\
0
\end{array}\right] \\
A(B X+C X)=(A B) X+(A C) X
\end{gathered}
$$

h. In general, must $A(B+C)$ and $A B+A C$ have the same geometric effect on point, no matter what matrices $A, B$, and $C$ are? Explain.

Yes. See full explanation in questions above.

## Exercises 1-2 (10 minutes)

Allow students to complete the exercises in pairs, each doing the work individually and then comparing answers. Some groups may need more help or one-on-one instructions. Exercise 1, part (b) and Exercise 2, part (a) will be fully discussed as a class and are the focus of the lesson summary. Have early finishers write the results of Exercise 1, part (c) and Exercise 2, parts (a)-(b) on large paper, and prepare an explanation to present to the class as part of the lesson summary.

## Exercises 1-2

1. Let $A=\left[\begin{array}{ll}x & z \\ y & w\end{array}\right], B=\left[\begin{array}{ll}a & c \\ b & d\end{array}\right]$, and $C=\left[\begin{array}{ll}e & g \\ f & h\end{array}\right]$.
a. Write down the products $A B, A C$, and $A(B+C)$.

$$
\begin{aligned}
& A B=\left[\begin{array}{cc}
a x+b z & c x+d z \\
a y+b w & c y+d w
\end{array}\right] ; A C=\left[\begin{array}{cc}
e x+f z & g x+h z \\
e y+f w & g y+h w
\end{array}\right] ; \\
& A(B+C)=\left[\begin{array}{ll}
(a+e) x+(b+f) z & (c+g) x+(d+h) z \\
(a+e) y+(b+f) w & (c+g) y+(d+h) w
\end{array}\right]
\end{aligned}
$$

b. Verify that $A(B+C)=A B+A C$.

$$
\begin{aligned}
& A(B+C)=\left[\begin{array}{cc}
(a+e) x+(b+f) z & (c+g) x+(d+h) z \\
(a+e) y+(b+f) w & (c+g) y+(d+h) w
\end{array}\right]= \\
& {\left[\begin{array}{cc}
a x+e x+b z+f z & c x+g x+d z+h z \\
a y+e y+b w+f w & c y+g y+d w+h w
\end{array}\right]} \\
& A B+A C=\left[\begin{array}{cc}
a x+b z & c x+d z \\
a y+b w & c y+d w
\end{array}\right]+\left[\begin{array}{cc}
e x+f z & g x+h z \\
e y+f w & g y+h w
\end{array}\right]= \\
& {\left[\begin{array}{cc}
a x+b z+e x+f z & c x+d z+g x+h z \\
a y+b w+e y+f w & c y+d w+g y+h w
\end{array}\right]}
\end{aligned}
$$

Therefore, $A(B+C)=A B+A C$.
2. Suppose $A, B$, and $C$ are $3 \times 3$ matrices, and $X$ is a point in three-dimensional space.
a. Explain why the point $(A(B C)) X$ must be the same point as $((A B) C) X$.
$(A(B C)) X=(A)(B)(C) X$. Applying $B C$ and then $A$ is the same as applying $C$, then $B$, and then $A$.
$(A)(B)(C) X=((A B) C) X$. Applying $C$, then $B$, and then $A$ is the same as applying $C$ and then $A B$.
b. Explain why matrix multiplication must be associative.

Matrix multiplication is associative because performing the transformation $B$ and then $A$ on a point $X$ is the same as applying the product of $A B$ to point $X$.
c. Verify using the matrices from Exercise 1 that $A(B C)=(A B) C$.

$$
\begin{gathered}
B C=\left[\begin{array}{ll}
a & c \\
b & d
\end{array}\right]\left[\begin{array}{ll}
e & g \\
f & h
\end{array}\right]=\left[\begin{array}{ll}
a e+c f & a g+c h \\
b e+d f & b g+d h
\end{array}\right] \\
A(B C)=\left[\begin{array}{cc}
x & z \\
y & w
\end{array}\right]\left[\begin{array}{ll}
a e+c f & a g+c h \\
b e+d f & b g+d h
\end{array}\right]=\left[\begin{array}{ll}
x(a e+c f)+z(b e+d f) & x(a g+c h)+z(b g+d h) \\
y(a e+c f)+w(b e+d f) & y(a g+c h)+w(b g+d h)
\end{array}\right] \\
A B=\left[\begin{array}{ll}
x & z \\
y & w
\end{array}\right]\left[\begin{array}{ll}
a & c \\
b & d
\end{array}\right]=\left[\begin{array}{cc}
a x+b z & c x+d z \\
a y+b w & c y+d w
\end{array}\right] \\
(A B) C=\left[\begin{array}{ll}
a x+b z & c x+d z \\
a y+b w & c y+d w
\end{array}\right]\left[\begin{array}{ll}
e & g \\
f & h
\end{array}\right]=\left[\begin{array}{cc}
(a x+b z) e+(c x+d z) f & (a x+b z) g+(c x+d z) h \\
(a y+b w) e+(c y+d w) f & (a y+b w) g+(c y+d w) h
\end{array}\right] \\
A(B C)=\left[\begin{array}{cc}
a e x+c f x+b e z+d f z & a g x+c h x+b g z+d h z \\
a e y+d f y+b e w+d f w & a g y+c h y+b g w+d h w
\end{array}\right]=(A B) C
\end{gathered}
$$

## Closing ( 5 minutes)

As a class, discuss the results of Exercises 1 and 2, focusing on the properties of matrix multiplication (distributive and associative) that were discovered/confirmed in this lesson and these exercises. Alternately, revisit the properties used in the example and the exercises of matrix multiplication.

- What matrix multiplication property did you prove in Exercise 1? Explain how you proved it in part (c).
- The distributive property. See exercises to ensure students understand properties.
- In Exercise 2, parts (a) and (b), explain why you knew that matrix multiplication was associative.
- Review steps from exercises to ensure students understand properties.
- Some properties of matrices:
- Each matrix represents a linear transformation, so $A(B X+C X)=A(B X)+A(C X)$.
- The sum of two matrices acts by summing the image points, $(B+C) X=B X+C X$.
- Applying matrix $B$ and then $A$ is the same as applying the matrix product $A B$, meaning $A(B X)=(A B) X$.


## Exit Ticket ( 5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 12: Matrix Multiplication Is Distributive and Associative

## Exit Ticket

In three-dimensional space, matrix $A$ represents a $180^{\circ}$ rotation about the $y$-axis, matrix $B$ represents a reflection about the $x z$-plane, and matrix $C$ represents a reflection about $x y$-plane. Answer the following:
a. Write matrices $A, B$, and $C$.
b. If $X=\left[\begin{array}{l}2 \\ 2 \\ 2\end{array}\right]$, compute $A(B X+C X)$.
c. What matrix operations are equivalent to $A(B X+C X)$ ? What property is shown?
d. Would $(A(B C)) X=((A B) C) X$ ? Why?

## Exit Ticket Sample Solutions

In three-dimensional space, matrix $A$ represents a $180^{\circ}$ rotation about the $y$-axis, matrix $B$ represents a reflection about the $x z$-plane, and matrix $C$ represents a reflection about $x y$-plane. Answer the following:
a. Write matrices $A, B$ and $C$.

$$
A=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right] ; B=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right] ; C=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right]
$$

b. If $X=\left[\begin{array}{l}2 \\ 2 \\ 2\end{array}\right]$, compute $A(B X+C X)$.

$$
\begin{gathered}
B X=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{array}\right]\left[\begin{array}{l}
2 \\
2 \\
2
\end{array}\right]=\left[\begin{array}{c}
2 \\
-2 \\
-2
\end{array}\right] ; C X=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right]\left[\begin{array}{l}
2 \\
2 \\
2
\end{array}\right]=\left[\begin{array}{c}
2 \\
2 \\
-2
\end{array}\right] \\
B X+C X=\left[\begin{array}{c}
4 \\
0 \\
-4
\end{array}\right] ; A(B X+C X)=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right]\left[\begin{array}{c}
4 \\
0 \\
-4
\end{array}\right]=\left[\begin{array}{c}
-4 \\
0 \\
4
\end{array}\right]
\end{gathered}
$$

c. What matrix operations are equivalent to $A(B X+C X)$ ? What property is shown?
$A(B X+C X)=(A B) X+(A C) X ;$ matrix multiplication is distributive.
d. Would $(A(B C)) X=((A B) C) X$ ? Why?

Yes; matrix multiplication is associative.

## Problem Set Sample Solutions

1. Let matrix $A=\left(\begin{array}{cc}3 & -2 \\ -1 & 0\end{array}\right)$, matrix $B=\left(\begin{array}{ll}4 & 4 \\ 3 & 9\end{array}\right)$, and matrix $C=\left(\begin{array}{cc}8 & 2 \\ 7 & -5\end{array}\right)$. Calculate the following:
a. $A B$

$$
\left(\begin{array}{cc}
6 & -6 \\
-4 & -4
\end{array}\right)
$$

b. $A C$

$$
\left(\begin{array}{cc}
10 & 16 \\
-8 & -2
\end{array}\right)
$$

c. $\quad \boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})$

$$
B+C=\left(\begin{array}{ll}
12 & 6 \\
10 & 4
\end{array}\right) ; A(B+C)=\left(\begin{array}{cc}
16 & 10 \\
-12 & -6
\end{array}\right) ; A B+A C=\left(\begin{array}{cc}
16 & 10 \\
-12 & -6
\end{array}\right)
$$

We have that $A(B+C)=A B+A C$.
d. $A B+A C$

$$
\left(\begin{array}{cc}
16 & 10 \\
-12 & -6
\end{array}\right)
$$

e. $\quad(A+B) C$
$(A+B) C=A C+B C$, so $B C$ has not been calculated yet. We get,

$$
B C=\left(\begin{array}{ll}
60 & -12 \\
87 & -39
\end{array}\right) .
$$

So,

$$
(A+B) C=\left(\begin{array}{cc}
70 & 4 \\
79 & -41
\end{array}\right) .
$$

f. $A(B C)$

$$
\begin{aligned}
A(B C) & =A\left(\begin{array}{ll}
60 & -12 \\
87 & -39
\end{array}\right) \\
& =\left(\begin{array}{cc}
6 & 42 \\
-60 & 12
\end{array}\right)
\end{aligned}
$$

2. Apply each of the transformations you found in Problem 1 to the points $x=\binom{1}{1}, y=\binom{-3}{2}$, and $x+y$.
a. $\quad(A B) x=\binom{0}{-8}$
( $A B$ ) $y=\binom{-30}{4}$
$(A B)(x+y)=\binom{-30}{-4}$
b. $\quad(A C) x=\binom{26}{-10}$
$(A C) y=\binom{2}{20}$
$(A C)(x+y)=\binom{28}{10}$
c. $\quad(A(B+C)) x=\binom{26}{-18}$
$(A(B+C)) y=\binom{-28}{24}$
$(A(B+C))(x+y)=\binom{-2}{6}$
d. Same as part (c)
$(A B+A C) x=\binom{26}{-18}$
$(A B+A C) y=\binom{-28}{24}$
$(A B+A C)(x+y)=\binom{-2}{6}$
e. $\quad((A+B) C) x=\binom{74}{38}$
$((A+B) C) y=\binom{-202}{-319}$
$((A+B) C)(x+y)=\binom{-128}{-281}$
f. $\quad(A(B C)) x=\binom{48}{-48}$

$$
(A(B C)) y=\binom{66}{204}
$$

$$
(A(B C))(x+y)=\binom{114}{156}
$$

3. Let $A, B, C$, and $D$ be any four square matrices of the same dimensions. Use the distributive property to evaluate the following:
a. $\quad(A+B)(C+D)$

$$
(A+B) C+(A+B) D=A C+B C+A D+B D
$$

b. $\quad(A+B)(A+B)$

$$
A A+A B+B A+B B
$$

c. What conditions need to be true for part (b) to equal $A A+2 A B+B B$ ?
$A B=B A$ needs to be true.
4. Let $A$ be a $2 \times 2$ matrix and $B, C$ be the scalar matrices $B=\left(\begin{array}{ll}2 & 0 \\ 0 & 2\end{array}\right)$, and $C=\left(\begin{array}{ll}3 & 0 \\ 0 & 3\end{array}\right)$. Answer the following questions.
a. Evaluate the following:
i. $B C$

$$
\left(\begin{array}{ll}
6 & 0 \\
0 & 6
\end{array}\right)
$$

ii. $C B$

$$
\left(\begin{array}{ll}
6 & 0 \\
0 & 6
\end{array}\right)
$$

iii. $\quad B+C$

$$
\left(\begin{array}{ll}
5 & 0 \\
0 & 5
\end{array}\right)
$$

iv. $B-C$

$$
\left(\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right)
$$

b. Are your answers to part (a) what you expected? Why or why not?

Answers may vary. Students should expect that the matrices will behave like real numbers since they represent scalars.
c. Let $A=\left(\begin{array}{ll}x & y \\ z & w\end{array}\right)$; does $A B=B A$ ? Does $A C=C A$ ?

Yes. $A B=\left(\begin{array}{ll}2 x & 2 y \\ 2 z & 2 w\end{array}\right)=B A$, and $A C=\left(\begin{array}{ll}3 x & 3 y \\ 3 z & 3 w\end{array}\right)$.
d. What is $(A+B)(A+C)$ ? Write the matrix $A$ with the letter and not in matrix form. How does this compare to $(x+2)(x+3)$ ?

$$
\begin{aligned}
(A+B)(A+C) & =A A+B A+A C+B C \\
& =A A+2 A+3 A+\left(\begin{array}{ll}
6 & 0 \\
0 & 6
\end{array}\right) \\
& =A A+5 A+\left(\begin{array}{ll}
6 & 0 \\
0 & 6
\end{array}\right)
\end{aligned}
$$

This is identical to $(x+2)(x+3)$, with $x=A$.
e. With $B$ and $C$ given as above, is it possible to factor $A A-A-B C$ ?

Yes. We need factors of -BC that add to -1. It looks like B and - $C$ work, so we get,

$$
(A+B)(A-C)
$$

5. Define the sum of any two functions with the same domain to be the function $f+g$ such that for each $x$ in the domain of $f$ and $g,(f+g)(x)=f(x)+g(x)$. Define the product of any two functions to be the function $f g$, such that for each $\boldsymbol{x}$ in the domain of $\boldsymbol{f}$ and $\boldsymbol{g},(\boldsymbol{f} \boldsymbol{g})(\boldsymbol{x})=(\boldsymbol{f}(\boldsymbol{x}))(\boldsymbol{g}(\boldsymbol{x}))$.
Let $f, g$, and $h$ be real-valued functions defined by the equations $f(x)=3 x+1, g(x)=-\frac{1}{2} x+2$, and $h(x)=x^{2}-4$.
a. Does $f(g+h)=f g+f h$ ?

Yes. If we can show that $(\boldsymbol{f}(\boldsymbol{g}+\boldsymbol{h}))(\boldsymbol{x})=(\boldsymbol{f} \boldsymbol{g})(\boldsymbol{x})+(\boldsymbol{f} \boldsymbol{h})(\boldsymbol{x})$, then we will have shown that the functions are equal to each other.

$$
\begin{aligned}
(f(g+h))(x) & =(f(x))((g+h)(x)) \\
& =(3 x+1)\left(-\frac{1}{2} x+2+x^{2}-4\right) \\
& =(3 x+1)\left(\left(-\frac{1}{2} x+2\right)+\left(x^{2}-4\right)\right) \\
& =(3 x+1)\left(-\frac{1}{2} x+2\right)+(3 x+1)\left(x^{2}-4\right) \\
& =(f(x))(g(x))+(f(x))(h(x)) \\
& =(f g)(x)+(f h)(x)
\end{aligned}
$$

b. Show that this is true for any three functions with the same domains.

$$
\begin{aligned}
(f(g+h))(x) & =(f(x))((\boldsymbol{g}+\boldsymbol{h})(x)) \\
& =(f(x))(\boldsymbol{g}(\boldsymbol{x})+\boldsymbol{h}(\boldsymbol{x})) \\
& =(\boldsymbol{f}(\boldsymbol{x}))(\boldsymbol{g}(\boldsymbol{x}))+(\boldsymbol{f}(\boldsymbol{x}))(\boldsymbol{h}(\boldsymbol{x})) \\
& =(\boldsymbol{f} \boldsymbol{g})(\boldsymbol{x})+(\boldsymbol{f h})(\boldsymbol{x})
\end{aligned}
$$

c. Does $f \circ(g+h)=f \circ g+f \circ h$ for the functions described above?

No.

$$
\begin{aligned}
(f \circ(g+h))(x) & =f\left(-\frac{1}{2} x+2+x^{2}-4\right) \\
& =3 \cdot\left(-\frac{1}{2} x+2+x^{2}-4\right)+1 \\
& =3 \cdot\left(-\frac{1}{2} x+2\right)+3 \cdot\left(x^{2}-4\right)+1
\end{aligned}
$$

The addition by 1 prevented it from working. If $f(x)$ would have been a proportion, then the composition would have worked.

# 田 <br> <br> Lesson 13: Using Matrix Operations for Encryption 

 <br> <br> Lesson 13: Using Matrix Operations for Encryption}

## Student Outcomes

- Students study and practice the properties of matrix multiplication.
- Students understand the role of the multiplicative identity matrix.


## Lesson Notes

Data encryption has become a necessity with the rise of sensitive data being stored and transmitted via computers. The methods included in this section are not secure enough to use for applications such as Internet banking, but they result in codes that are not easy to break and provide a good introduction to the ideas of encryption. Interested students can research RSA public-key encryption, which relies on the fact that factoring extremely large numbers is a very difficult and slow process. Students interested in the history of data encryption can research the Cherokee and Choctaw Code Talkers from World War I and the Navajo Code Talkers from World War II.

This lesson reinforces concepts of matrix multiplication, matrix inverses, and the identity matrix in the context of encoding and decoding strings of characters using multiplication by either an encoding matrix or its inverse decoding matrix. This lesson aligns with N-VM.C. 6 (Use matrices to represent and manipulate data), N-VM.C. 8 (Add, subtract, and multiply matrices of appropriate dimensions), and N-VM.C. 10 (Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse.)

The activity in Exercise 2 requires that six stations be set up in advance around the classroom as the messages have been encoded four times. At each station, post the specified decoding matrix:

Station 1:

$$
D_{1}=\left[\begin{array}{cc}
-1 & -1 \\
1 & \frac{1}{2}
\end{array}\right]
$$

Station 4

$$
D_{4}=\left[\begin{array}{cc}
\frac{1}{2} & -\frac{1}{2} \\
-\frac{3}{2} & \frac{5}{2}
\end{array}\right]
$$

Divide students into at least six groups numbered 1-6, assign each group their coded message, and start them at their numbered station. Groups will apply the decoding matrix to their message and then move to the next station. After applying four decoding matrices, the original message will be revealed. Each group will decode 20 characters of the original message, combining the results into the full quote from the entire class.

Station 2:

$$
D_{2}=\left[\begin{array}{ll}
\frac{1}{6} & 0 \\
0 & \frac{1}{3}
\end{array}\right]
$$

Station 5

$$
D_{5}=\left[\begin{array}{cc}
-1 & 1 \\
0 & 1
\end{array}\right]
$$

Station 3:

$$
D_{3}=\left[\begin{array}{cc}
\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{4}{3}
\end{array}\right]
$$

Station 6:

$$
D_{6}=\left[\begin{array}{cc}
2 & -1 \\
-3 & 2
\end{array}\right]
$$

## Classwork

## Opening (7 minutes)

The phrase "The crow flies at midnight" appears to have first occurred in lan Fleming's James Bond novel From Russia with Love. It has since become a coded message in spy movies and television shows.

## Opening

A common way to send coded messages is to assign each letter of the alphabet to a number 1-26 and send the message as a string of integers. For example, if we encode the message "THE CROW FLIES AT MIDNIGHT" according to the chart below, we get the string of numbers

$$
20,8,5,0,3,18,15,23,0,6,12,9,5,19,0,1,20,0,13,9,4,14,9,7,8,20 .
$$

| A | B | C | D | E | F | G | H | I | J | K | L | M |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |


| N | O | P | Q | R | S | T | U | V | W | X | Y | Z | SPACE |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 0 |

However, codes such as these are easily broken using an analysis of the frequency of numbers that appear in the coded messages.

We can instead encode a message using matrix multiplication. If a matrix $E$ has an inverse, then we can encode a message as follows.

- First, convert the characters of the message to integers between 1 and 26 using the chart above.
- If the encoding matrix $E$ is an $\boldsymbol{n} \times \boldsymbol{n}$ matrix, then break up the numerical message into $\boldsymbol{n}$ rows of the same length. If needed, add extra zeros to make the rows the same length.
- Place the rows into a matrix $M$.
- Compute the product $E M$ to encode the message.
- The message is sent as the numbers in the rows of the matrix EM.


## Example (10 minutes)

- A common way to send coded messages is to assign each letter of the alphabet to a number 1-26 and send the message as a string of integers. For example, if we encode the message "THE CROW FLIES AT MIDNIGHT" according to the chart below, we get the string of numbers

$$
20,8,5,0,3,18,15,23,0,6,12,9,5,19,0,1,20,0,13,9,4,14,9,7,8,20
$$

| A | B | C | D | E | F | G | H | I | J | K | L | M |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |


| N | O | P | Q | R | S | T | U | V | W | X | Y | Z | SPACE |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 0 |

However, codes such as these are easily broken using an analysis of the frequency of numbers that appear in the coded messages. For example if the coded phrase is $20,8,5,0,3,18,15,23,0,6,12,9,5,19,0,1,20,0,13,9,4,14,9,7$, 8,20 , we can see that there are three of the letters assigned to the integer 20 , so we may want to try putting a common letter in for the number 20 like $\mathrm{S}, \mathrm{T}$, or E . If we assume the word the is used to start the phrase since we have three letters then a space, that would lead us to think that maybe the number 5 is E , and so on.

- We can instead encode a message using matrix multiplication. If a matrix $E$ has an inverse, then we can encode a message as follows.
- First, convert the characters of the message to integers between 1 and 26 using the chart above.
- If the encoding matrix $E$ is an $n \times n$ matrix, then break up the numerical message into $n$ rows of the same length. If needed, add extra zeros to make the rows the same length. For example if we want to use an encoding matrix that is $2 \times 2$, we would write the message in two rows of equal length, filling in zero for the last number if the number of letters was odd. If the encoding matrix is $3 \times 3$, the message would be written in three equal rows adding zeros as necessary.
- Place the rows into a matrix $M$.
- Compute the product $E M$ to encode the message.
- The message is sent as the numbers in the rows of the matrix EM.
- Using the matrix $E=\left[\begin{array}{ll}2 & 1 \\ 3 & 1\end{array}\right]$, we encode our message as follows:

$$
4,14,9,7,8,20
$$

- Since $E$ is a $2 \times 2$ matrix, we need to break up our message into two rows.

$$
\begin{aligned}
& 20,8,5,0,3,18,15,23,0,6,12,9,5 \\
& 19,0,1,20,0,13,9,4,14,9,7,8,20
\end{aligned}
$$

## Scaffolding:

- Students who are struggling can be given a simpler phrase such as "Be Happy" or "Dream Big."
- Have advanced learners find their own phrase of 30 characters or more and encode using a $3 \times 3$ matrix.
- Then we place the rows into a matrix $M$.

$$
M=\left[\begin{array}{ccccccccccccc}
20 & 8 & 5 & 0 & 3 & 18 & 15 & 23 & 0 & 6 & 12 & 9 & 5 \\
19 & 0 & 1 & 20 & 0 & 13 & 9 & 4 & 14 & 9 & 7 & 8 & 20
\end{array}\right]
$$

- Explain how matrix $M$ represents "THE CROW FLIES AT MIDNIGHT."
- Each letter and space in the phrase was assigned an integer value, and these numbers represent the letters in the phrase.
- We encode the message into matrix $C$ by multiplying $E \cdot M$.

$$
\begin{aligned}
C=E \cdot M & =\left[\begin{array}{ll}
2 & 1 \\
3 & 1
\end{array}\right]\left[\begin{array}{ccccccccccccc}
20 & 8 & 5 & 0 & 3 & 18 & 15 & 23 & 0 & 6 & 12 & 9 & 5 \\
19 & 0 & 1 & 20 & 0 & 13 & 9 & 4 & 14 & 9 & 7 & 8 & 20
\end{array}\right] \\
C & =\left[\begin{array}{llllllllllll}
59 & 16 & 11 & 20 & 6 & 49 & 39 & 50 & 14 & 21 & 31 & 26 \\
79 & 24 & 16 & 20 & 9 & 67 & 54 & 73 & 14 & 27 & 43 & 35 \\
35
\end{array}\right]
\end{aligned}
$$

- Thus, the coded message that we send is

$$
59,16,11,20,6,49,39,50,14,21,31,26,30,79,24,16,20,9,67,54,73,14,27,43,35,35 .
$$

If this coded message is intercepted, then it cannot easily be decoded unless the recipient knows how it was originally encoded.
Be sure to work through this discussion and emphasize that the way to decode a message is to multiply by the inverse of the encoding message.

- Using what you know about how the message was encoded, as well as matrix multiplication, describe how you would decode this message.
- We need to know a decoding matrix $D$.
- How can we find that matrix?
- The decoding matrix is the inverse of the encoding matrix, so $D=E^{-1}$.
- What is the decoding matrix?

$$
\quad D=E^{-1}=\left[\begin{array}{ll}
2 & 1 \\
3 & 1
\end{array}\right]^{-1}=\frac{1}{2-3}\left[\begin{array}{cc}
1 & -1 \\
-3 & 2
\end{array}\right]=\left[\begin{array}{cc}
-1 & 1 \\
3 & -2
\end{array}\right] .
$$

- Decode this message!

$$
\therefore \quad \begin{aligned}
D \cdot C & =\left[\begin{array}{cc}
-1 & 1 \\
3 & -2
\end{array}\right]\left[\begin{array}{llllllllllllll}
59 & 16 & 11 & 20 & 6 & 49 & 39 & 50 & 14 & 21 & 31 & 26 & 30 \\
79 & 24 & 16 & 20 & 9 & 67 & 54 & 73 & 14 & 27 & 43 & 35 & 35
\end{array}\right] \\
& =\left[\begin{array}{ccccccccccccc}
20 & 8 & 5 & 0 & 3 & 18 & 15 & 23 & 0 & 6 & 12 & 9 & 5 \\
19 & 0 & 1 & 20 & 0 & 13 & 9 & 4 & 14 & 9 & 7 & 8 & 20
\end{array}\right] .
\end{aligned}
$$

As expected, this is the matrix $M$ that stored our original message "THE CROW FLIES AT MIDNIGHT."

- Why does this process work?
- The coded message stored in matrix $C$ is the product of matrices $E$ and $M$, so $C=E \cdot M$. We then decode the message stored in matrix $C$ by multiplying by matrix $D$. Since matrices $D$ and $E$ are inverses, we have

$$
D \cdot(E \cdot M)=(D \cdot E) \cdot M=I \cdot M=M .
$$

So, encoding and then decoding will return the original message in matrix $M$.

- Explain to your neighbor what you learned about how to encode and decode messages. Teachers should use this as an informal way to check for understanding.


## Exercise 1 (7 minutes)

The encoded phrase in this exercise is "ARCHIMEDES." Archimedes (c. 287-212 BCE, Greece) is regarded as the greatest mathematician of his age and one of the greatest of all time. He developed and applied an early form of integral calculus to derive correct formulas for the area of a circle, volume of a sphere, and area under a parabola. He also found accurate approximations of irrational numbers such as $\sqrt{3}$ and $\pi$. However, during his lifetime he was known more for his inventions such as the Archimedean screw, compound pulleys, and weapons such as the Claw of Archimedes used to protect Syracuse in times of war.
The original message is stored in matrix $M=\left[\begin{array}{llll}1 & 18 & 3 & 8 \\ 9 & 13 & 5 & 4 \\ 5 & 19 & 0 & 0\end{array}\right]$, and the matrix used to encode the message is $E=\left[\begin{array}{lll}1 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 1\end{array}\right]$. This exercise introduces students to using a larger matrix to perform the encoding and decoding and requires that students practice matrix multiplication with non-integer matrix entries. Additionally, because students have not learned a method for finding the inverse of a $3 \times 3$ matrix, they must demonstrate understanding of the meaning of a matrix inverse in order to decode this matrix.

## Exercises

1. You have received an encoded message: $34,101,13,16,23,45,10,8,15,50,8,12$. You know that the message was encoded using matrix $E=\left[\begin{array}{lll}1 & 2 & 3 \\ 0 & 2 & 1 \\ 1 & 1 & 1\end{array}\right]$.
a. Store your message in a matrix $C$. What are the dimensions of $C$ ?

There are 12 numbers in the coded message, and it was encoded using a $3 \times 3$ matrix. Thus, the matrix $C$ needs to have three rows. That means $C$ has four columns, so $C$ is a $3 \times 4$ matrix.
b. You have forgotten whether the proper decoding matrix is matrix $X, Y$, or $Z$ as shown below. Determine which of these is the correct matrix to use to decode this message.

$$
X=\left[\begin{array}{ccc}
-\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\
-\frac{2}{3} & \frac{1}{3} & -\frac{2}{3}
\end{array}\right], Y=\left[\begin{array}{ccc}
-\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} \\
-\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\
\frac{2}{3} & -\frac{1}{3} & \frac{2}{3}
\end{array}\right], Z=\left[\begin{array}{ccc}
-\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\
\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3}
\end{array}\right]
$$

Matrices used to encode and decode messages must be inverses of each other. Thus, the correct decoding matrix is the matrix $D$ so that $\boldsymbol{D} \cdot \boldsymbol{E}=I$. We can find the correct decoding matrix by multiplying $\boldsymbol{X} \cdot \boldsymbol{E}, \boldsymbol{Y} \cdot \boldsymbol{E}$, and $\boldsymbol{Z} \cdot \boldsymbol{E}$.

$$
\begin{aligned}
& X \cdot E=\left[\begin{array}{ccc}
-\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\
-\frac{2}{3} & \frac{1}{3} & -\frac{2}{3}
\end{array}\right] \cdot\left[\begin{array}{lll}
1 & 2 & 3 \\
0 & 2 & 1 \\
1 & 1 & 1
\end{array}\right]=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
-\frac{4}{3} & -\frac{4}{3} & -\frac{7}{3}
\end{array}\right] \\
& Y \cdot E=\left[\begin{array}{ccc}
-\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} \\
-\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\
\frac{2}{3} & -\frac{1}{3} & \frac{2}{3}
\end{array}\right] \cdot\left[\begin{array}{lll}
1 & 2 & 3 \\
0 & 2 & 1 \\
1 & 1 & 1
\end{array}\right]=\left[\begin{array}{ccc}
\frac{1}{2} & 0 & 0 \\
-\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \\
\frac{4}{3} & \frac{4}{3} & \frac{7}{3}
\end{array}\right] \\
& Z \cdot E=\left[\begin{array}{ccc}
-\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\
\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3}
\end{array}\right] \cdot\left[\begin{array}{lll}
1 & 2 & 3 \\
0 & 2 & 1 \\
1 & 1 & 1
\end{array}\right]=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
\end{aligned}
$$

Since $Z \cdot E=I$, we know that $Z$ is the decoding matrix we need.
c. Decode the message.

Using matrix $Z$ to decode, we have

$$
\begin{aligned}
M=Z \cdot C & =\left[\begin{array}{ccc}
-\frac{1}{3} & -\frac{1}{3} & \frac{4}{3} \\
-\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\
\frac{2}{3} & -\frac{1}{3} & -\frac{2}{3}
\end{array}\right] \cdot\left[\begin{array}{cccc}
34 & 101 & 13 & 16 \\
23 & 45 & 10 & 8 \\
15 & 50 & 8 & 12
\end{array}\right] \\
& =\left[\begin{array}{cccccc}
-\frac{34}{3}-\frac{23}{3}+\frac{60}{3} & -\frac{101}{3}-\frac{45}{3}+\frac{200}{3} & -\frac{13}{3}-\frac{10}{3}+\frac{32}{3} & -\frac{16}{3}-\frac{8}{3}+\frac{48}{3} \\
-\frac{34}{3}+\frac{46}{3}+\frac{15}{3} & -\frac{101}{3}+\frac{90}{3}+\frac{50}{3} & -\frac{13}{3}+\frac{20}{3}+\frac{8}{3} & -\frac{16}{3}+\frac{16}{3}+\frac{12}{3} \\
\frac{68}{3}-\frac{23}{3}-\frac{30}{3} & \frac{202}{3}-\frac{90}{3}-\frac{100}{3} & \frac{26}{3}-\frac{10}{3}-\frac{16}{3} & \frac{32}{3}-\frac{8}{3}-\frac{24}{3}
\end{array}\right] \\
& =\left[\begin{array}{cccc}
1 & 18 & 3 & 8 \\
9 & 13 & 5 & 4 \\
5 & 4 & 0 & 0
\end{array}\right] .
\end{aligned}
$$

The decoded message is "ARCHIMEDES."

## Exercises 2-3 (15 minutes)

In this exercise, groups of students decode separate parts of a message that have been encoded four times; as groups complete the decoding of their portion of the message, have them record it in a location that all students can see-either on the white board or projected through a document camera, for example. The decoded messages will together spell out a famous quote by Albert Einstein: "Do not worry about your difficulties in mathematics. I can assure you that mine are still greater." You may substitute a different quote if you would like, perhaps a school motto. Carefully encode each of six portions of a quote stored in matrices $M_{1}$ to $M_{6}$ using encoding matrices $E_{1}$ to $E_{6}$ as follows.

$$
\begin{aligned}
& C_{1}=E_{1} \cdot E_{2} \cdot E_{3} \cdot E_{4} \cdot M_{1} \\
& C_{2}=E_{2} \cdot E_{3} \cdot E_{4} \cdot E_{5} \cdot M_{2} \\
& C_{3}=E_{3} \cdot E_{4} \cdot E_{5} \cdot E_{6} \cdot M_{3} \\
& C_{4}=E_{4} \cdot E_{5} \cdot E_{6} \cdot E_{1} \cdot M_{4} \\
& C_{5}=E_{5} \cdot E_{6} \cdot E_{1} \cdot E_{2} \cdot M_{5} \\
& C_{6}=E_{6} \cdot E_{1} \cdot E_{2} \cdot E_{3} \cdot M_{6}
\end{aligned}
$$

## Scaffolding:

For struggling students, select a shorter quote, or encode it in fewer than four steps.
$E_{1}=\left[\begin{array}{cc}1 & 2 \\ -2 & -2\end{array}\right] ; E_{2}=\left[\begin{array}{ll}6 & 0 \\ 0 & 3\end{array}\right] ; E_{3}=\left[\begin{array}{ll}4 & 1 \\ 1 & 1\end{array}\right] ; E_{4}=\left[\begin{array}{ll}5 & 1 \\ 3 & 1\end{array}\right] ; E_{5}=\left[\begin{array}{cc}-1 & 1 \\ 0 & 1\end{array}\right] ; E_{6}=\left[\begin{array}{ll}2 & 1 \\ 3 & 2\end{array}\right]$
Divide the class into at least six groups of two or three students, numbered 1-6, assigning multiple groups to the same number as needed. Set up six stations around the room in a circular arrangement. Have each group start at the station with the same number as the group-Group 1 starts at Station 1, Group 2 starts at Station 2, etc. At each station, the groups apply the posted decoding matrix to their encoded message shown below, and then they progress to the next station, with groups at Station 6 proceeding to Station 1. It will require four decoding steps with different matrices (such as $D_{2}, D_{3}, D_{4}$ and $D_{5}$ ) to uncover a group's portion of the original message.

At each station, post the matrix listed below.
Station 1: $D_{1}=\left[\begin{array}{cc}-1 & -1 \\ 1 & \frac{1}{2}\end{array}\right]$
Station 2: $D_{2}=\left[\begin{array}{ll}\frac{1}{6} & 0 \\ 0 & \frac{1}{3}\end{array}\right]$
Station 3: $D_{3}=\left[\begin{array}{cc}\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{4}{3}\end{array}\right]$
Station 4: $D_{4}=\left[\begin{array}{cc}\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{5}{2}\end{array}\right]$
Station 5: $D_{5}=\left[\begin{array}{cc}-1 & 1 \\ 0 & 1\end{array}\right]$
Station 6: $D_{6}=\left[\begin{array}{cc}2 & -1 \\ -3 & 2\end{array}\right]$
2. You have been assigned a group number. The message your group receives is listed below. This message is TOP SECRET! It is of such importance that it has been encoded four times.
Your group's portion of the coded message is listed below.
Group 1:
$1500,3840,0,3444,3420,4350,0,4824,3672,3474,-2592,-6660,0,-5976,-5940,-7560,0,-8388$, -6372,-6048

Group 2:
$2424,3024,-138,396,-558,-1890,-1752,1512,-2946,1458,438,540,-24,72,-90,-324,-300$, 270, -510, 270

Group 3:
$489,1420,606,355,1151,33,1002,829,99,1121,180,520,222,130,422,12,366,304,36,410$

Group 4:
$-18,10,-18,44,-54,42,-6,-74,-98,-124,0,10,-12,46,-26,42,-4,-36,-60,-82$

Group 5:
$-120,0,-78,-54,-84,-30,0,-6,-108,-30,-120,114,42,0,-12,42,0,36,0,0$

Group 6:
$126,120,60,162,84,120,192,42,84,192,-18,-360,-90,-324,0,-18,-216,-36,-90,-324$
a. Store your message in a matrix $C$ with two rows. How many columns does matrix $C$ have?
(Sample responses are provided for Group 1.) Our message is stored in a matrix with ten columns:
$C=\left[\begin{array}{cccccccccc}1500 & 3840 & 0 & 3444 & 3420 & 4350 & 0 & 4824 & 3672 & 3474 \\ -2592 & -6660 & 0 & -5976 & -5940 & -7560 & 0 & -8388 & -6372 & -6048\end{array}\right]$.
b. Begin at the station of your group number, and apply the decoding matrix at this first station.

$$
\begin{aligned}
D_{1} & =\left[\begin{array}{cc}
-1 & -1 \\
1 & \frac{1}{2}
\end{array}\right] \\
D_{1} \cdot C & =\left[\begin{array}{cc}
-1 & -1 \\
1 & \frac{1}{2}
\end{array}\right] \cdot\left[\begin{array}{cccccccccc}
1500 & 3840 & 0 & 3444 & 3420 & 4350 & 0 & 4824 & 3672 & 3474 \\
-2592 & -6660 & 0 & -5976 & -5940 & -7560 & 0 & -8388 & -6372 & -6048
\end{array}\right] \\
& =\left[\begin{array}{ccccccccc}
1092 & 2820 & 0 & 2352 & 2520 & 3210 & 0 & 3564 & 2700 \\
204 & 510 & 0 & 456 & 450 & 570 & 0 & 630 & 486 \\
450
\end{array}\right]
\end{aligned}
$$

c. Proceed to the next station in numerical order; if you are at Station 6, proceed to Station 1. Apply the decoding matrix at this second station.

$$
\begin{aligned}
D_{2} & =\left[\begin{array}{ll}
\frac{1}{6} & 0 \\
0 & \frac{1}{3}
\end{array}\right] \\
D_{2} \cdot D_{1} \cdot C & =\left[\begin{array}{cc}
\frac{1}{6} & 0 \\
0 & \frac{1}{3}
\end{array}\right] \cdot\left[\begin{array}{ccccccccccc}
1092 & 2820 & 0 & 2352 & 2520 & 3210 & 0 & 3564 & 2700 & 2574 \\
204 & 510 & 0 & 456 & 450 & 570 & 0 & 630 & 486 & 450
\end{array}\right] \\
& =\left[\begin{array}{cccccccccc}
182 & 470 & 0 & 422 & 420 & 535 & 0 & 594 & 450 & 429 \\
68 & 170 & 0 & 152 & 150 & 150 & 0 & 210 & 162 & 150
\end{array}\right]
\end{aligned}
$$

d. Proceed to the next station in numerical order; if you are at Station 6, proceed to Station 1. Apply the decoding matrix at this third station.

$$
\begin{aligned}
D_{3} & =\left[\begin{array}{cc}
\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{4}{3}
\end{array}\right] \\
D_{3} \cdot D_{2} \cdot D_{1} \cdot C & =\left[\begin{array}{cc}
\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{4}{3}
\end{array}\right] \cdot\left[\begin{array}{cccccccccccc}
182 & 470 & 0 & 422 & 420 & 535 & 0 & 594 & 450 & 429 \\
68 & 170 & 0 & 152 & 150 & 150 & 0 & 210 & 162 & 150
\end{array}\right] \\
& =\left[\begin{array}{ccccccccc}
38 & 100 & 0 & 90 & 90 & 115 & 0 & 128 & 96 \\
30 & 70 & 0 & 62 & 60 & 75 & 0 & 82 & 66 \\
57
\end{array}\right]
\end{aligned}
$$

e. Proceed to the next station in numerical order; if you are at Station 6, proceed to Station 1. Apply the decoding matrix at this fourth station.

$$
\begin{aligned}
D_{4} & =\left[\begin{array}{cc}
\frac{1}{2} & -\frac{1}{2} \\
-\frac{3}{2} & \frac{5}{2}
\end{array}\right] \\
D_{4} \cdot D_{3} \cdot D_{2} \cdot D_{1} \cdot C & =\left[\begin{array}{cc}
\frac{1}{2} & -\frac{1}{2} \\
-\frac{3}{2} & \frac{5}{2}
\end{array}\right] \cdot\left[\begin{array}{ccccccccccc}
38 & 100 & 0 & 90 & 90 & 115 & 0 & 128 & 96 & 93 \\
30 & 70 & 0 & 62 & 60 & 75 & 0 & 82 & 66 & 57
\end{array}\right] \\
& =\left[\begin{array}{ccccccccc}
4 & 15 & 0 & 14 & 15 & 20 & 0 & 23 & 15 \\
18 & 25 & 0 & 20 & 15 & 15 & 0 & 13 & 21 \\
3
\end{array}\right]
\end{aligned}
$$

f. Decode your message.

The numerical message is $4,15, \mathbf{0}, 14,15,20,0,23,15,18,18,25,0,20,15,15,0,13,21,3$, which represents the characters "DO NOT WORRY TOO MUCH."
3. Sydnie was in Group 1 and tried to decode her message by calculating the matrix ( $\boldsymbol{D}_{1} \cdot \boldsymbol{D}_{2} \cdot \boldsymbol{D}_{3} \cdot \boldsymbol{D}_{4}$ ) and then multiplying $\left(D_{1} \cdot D_{2} \cdot D_{3} \cdot D_{4}\right) \cdot C$. This produced the matrix

$$
M=\left[\begin{array}{cccccccccc}
\frac{10526}{3} & \frac{27020}{3} & 0 & \frac{24242}{3} & 8030 & \frac{30655}{3} & 0 & 11336 & 8616 & 8171 \\
-1455 & -3735 & 0 & -3351 & -3330 & -\frac{8475}{2} & 0 & -4701 & -3573 & -\frac{6177}{2}
\end{array}\right] .
$$

a. How did she know that she made a mistake?

If Sydnie had properly decoded her message, all entries in the matrix $M$ would be integers between 0 and 26.
b. Matrix $C$ was encoded using matrices $E_{1}, E_{2}, E_{3}$ and $E_{4}$, where $D_{1}$ decodes a message encoded by $E_{1}, D_{2}$ decodes a message encoded by $E_{2}$ and so on. What is the relationship between matrices $E_{1}$ and $D_{1}$, between $E_{2}$ and $D_{2}$, etc.?

Matrices $E_{1}$ and $D_{1}$ are inverse matrices, as are $E_{2}$ and $D_{2}, E_{3}$, and $D_{3}$ and so on.
c. The matrix that Sydnie received was encoded by $C=E_{1} \cdot E_{2} \cdot E_{3} \cdot E_{4} \cdot M$. Explain to Sydnie how the decoding process works to recover the original matrix $M$, and devise a correct method for decoding using multiplication by a single decoding matrix.

Since $C=E_{1} \cdot E_{2} \cdot E_{3} \cdot E_{4} \cdot M$, we can recover the original matrix $M$ by multiplying both sides of this equation by the proper decoding matrix at each step, remembering that $D_{1} \cdot E_{1}=I, D_{2} \cdot E_{2}=I$, etc.

$$
\begin{aligned}
C & =E_{1} \cdot E_{2} \cdot E_{3} \cdot E_{4} \cdot M \\
D_{1} \cdot C & =D_{1} \cdot\left(E_{1} \cdot E_{2} \cdot E_{3} \cdot E_{4} \cdot M\right) \\
& =\left(D_{1} \cdot E_{1}\right) \cdot\left(E_{2} \cdot E_{3} \cdot E_{4} \cdot M\right) \\
& =I \cdot\left(E_{2} \cdot E_{3} \cdot E_{4} \cdot M\right) \\
& =\left(E_{2} \cdot E_{3} \cdot E_{4} \cdot M\right) \\
D_{2} \cdot D_{1} \cdot C & =D_{2} \cdot\left(E_{2} \cdot E_{3} \cdot E_{4} \cdot M\right) \\
& =\left(D_{2} \cdot E_{2}\right) \cdot\left(E_{3} \cdot E_{4} \cdot M\right) \\
& =I \cdot\left(E_{3} \cdot E_{4} \cdot M\right) \\
& =\left(E_{2} \cdot E_{3} \cdot E_{4} \cdot M\right) \\
D_{3} \cdot D_{2} \cdot D_{1} \cdot C & =D_{3} \cdot\left(E_{3} \cdot E_{4} \cdot M\right) \\
& =\left(D_{3} \cdot E_{3}\right) \cdot\left(E_{4} \cdot M\right) \\
& =I \cdot\left(E_{4} \cdot M\right) \\
& =E_{4} \cdot M \\
D_{4} \cdot D_{3} \cdot D_{2} \cdot D_{1} \cdot C & =D_{4} \cdot\left(E_{4} \cdot M\right) \\
& =\left(D_{4} \cdot E_{4}\right) \cdot M \\
& =I \cdot M \\
& =M .
\end{aligned}
$$

Since matrix multiplication is associative, this means that $M=\left(D_{4} \cdot D_{3} \cdot D_{2} \cdot D_{1}\right) \cdot C$.
d. Apply the method you devised in part (c) to your group's message to verify that it works.

$$
\begin{aligned}
\left(D_{4} \cdot D_{3} \cdot D_{2} \cdot D_{1}\right) & =\left[\begin{array}{cc}
\frac{1}{2} & -\frac{1}{2} \\
-\frac{3}{2} & \frac{5}{2}
\end{array}\right] \cdot\left[\begin{array}{cc}
\frac{1}{3} & -\frac{1}{3} \\
-\frac{1}{3} & \frac{4}{3}
\end{array}\right] \cdot\left[\begin{array}{cc}
\frac{1}{6} & 0 \\
0 & \frac{1}{3}
\end{array}\right] \cdot\left[\begin{array}{cc}
-1 & -1 \\
1 & \frac{1}{2}
\end{array}\right] \\
& =\left[\begin{array}{cc}
\frac{1}{3} & -\frac{5}{6} \\
-\frac{4}{3} & \frac{23}{6}
\end{array}\right] \cdot\left[\begin{array}{cc}
-\frac{1}{6} & -\frac{1}{6} \\
\frac{1}{3} & \frac{1}{6}
\end{array}\right]=\left[\begin{array}{cc}
-\frac{1}{3} & -\frac{7}{36} \\
\frac{3}{2} & \frac{31}{36}
\end{array}\right]
\end{aligned}
$$

So,

$$
\begin{aligned}
& \left(D_{4} \cdot D_{3} \cdot D_{2} \cdot D_{1}\right) \cdot C \\
& \quad=\left[\begin{array}{cc}
-\frac{1}{3} & -\frac{7}{36} \\
3 & \frac{31}{36}
\end{array}\right] \cdot\left[\begin{array}{cccccccccccc}
1500 & 3840 & 0 & 3444 & 3420 & 4350 & 0 & 4824 & 3672 & 3474 \\
-2592 & -6660 & 0 & -5976 & -5940 & -7560 & 0 & -8388 & -6372 & -6048
\end{array}\right] \\
& \quad=\left[\begin{array}{cccccccc}
4 & 15 & 0 & 14 & 15 & 20 & 0 & 23 \\
18 & 25 & 0 & 20 & 15 & 15 & 0 & 13 \\
21 & 18 \\
18 & 31
\end{array}\right] .
\end{aligned}
$$

This is the same decoded message that we found in Exercise 2, part (f).

## Exercise 4 (optional, 8 minutes)

The encoded phrase in this exercise is "RAMANUJAN." Srinivasa Ramanujan (1887-1920) was a self-taught mathematician from India who made significant contributions to many branches of mathematics, particularly analysis and number theory, compiling thousands of mathematical results. Although he died young, he is widely considered to be one of the greatest mathematicians of his time.

The original message is stored in matrix $M=\left[\begin{array}{ccc}18 & 1 & 13 \\ 1 & 14 & 21 \\ 10 & 1 & 14\end{array}\right]$, and the matrix used to encode the message is $E=\left[\begin{array}{ccc}1 & 2 & 1 \\ -1 & 0 & 2 \\ 1 & 1 & -1\end{array}\right]$. In this optional exercise, students need to reason through the process of encoding and decoding to recover a missing entry in the decoding matrix when the encoding matrix is unknown. Use this exercise as an extension for students who have finished the previous exercises quickly.
4. You received a coded message in the matrix $C=\left[\begin{array}{ccc}30 & 30 & 69 \\ 2 & 1 & 15 \\ 9 & 14 & 20\end{array}\right]$. However, the matrix $D$ that will decode this message has been corrupted, and you do not know the value of entry $d_{12}$. You know that all entries in matrix $D$ are integers. Using $x$ to represent this unknown entry, the decoding matrix $D$ is given by $D=\left[\begin{array}{ccc}2 & x & -4 \\ -1 & 2 & 3 \\ 1 & -1 & -2\end{array}\right]$. Decode the message in matrix $C$.

Decoding the message requires that we multiply D • C:

$$
\begin{aligned}
D \cdot C & =\left[\begin{array}{ccc}
2 & x & -4 \\
-1 & 2 & 3 \\
1 & -1 & -2
\end{array}\right] \cdot\left[\begin{array}{ccc}
30 & 30 & 69 \\
2 & 1 & 15 \\
9 & 14 & 20
\end{array}\right] \\
& =\left[\begin{array}{ccc}
24+2 k & 4+x & 58+15 x \\
1 & 14 & 21 \\
10 & 1 & 14
\end{array}\right] .
\end{aligned}
$$

Since we know all entries are integers and that the entries represent letters, we know that

$$
\begin{aligned}
& 0 \leq 24+2 x \leq 26 \\
& 0 \leq 4+x \leq 26 \\
& 0 \leq 58+15 x \leq 26
\end{aligned}
$$

Solving these inequalities gives

$$
\begin{aligned}
-12 & \leq x \leq 1 \\
-4 & \leq x \leq 22 \\
-\frac{58}{15} & \leq x \leq-\frac{32}{15}
\end{aligned}
$$

Because we know that $x$ is an integer, the third inequality becomes $-3 \leq x \leq-3$, so we know that $x=3$. Then the decoded message is

$$
D \cdot C=\left[\begin{array}{ccc}
24+2(-3) & 4+(-3) & 58+15(-3) \\
1 & 14 & 21 \\
10 & 1 & 14
\end{array}\right]
$$

thus,

$$
D \cdot C=\left[\begin{array}{ccc}
18 & 1 & 13 \\
1 & 14 & 21 \\
10 & 1 & 14
\end{array}\right]
$$

and the decoded message is "RAMANUJAN."

## Closing (3 minutes)

Ask students to write a brief answer to the question, "How do matrix inverses make encoding and decoding messages possible?" Then, have students share responses with a partner before sharing responses as a class.

- How do matrix inverses make encoding and decoding messages possible?
- If an $n \times n$ matrix $E$ is invertible, then it can be used to encode a message. We store the message in a matrix $M$, where $M$ has $n$ rows, and then encode it by multiplying $E \cdot M$. To decode the message, we multiply $E^{-1} \cdot(E \cdot M)=\left(E^{-1} \cdot E\right) \cdot M=I \cdot M=M$.


## Exit Ticket (4 minutes)

The message encoded in the problem in the Exit Ticket is "HYPATIA." Hypatia (Hy-pay-shuh) of Alexandria (born between 350-370 CE, died 415 CE) is one of the earliest known female mathematicians. She was the head of the Neoplatonic School in Alexandria, Egypt, and the head of the Library of Alexandria. She was murdered in a religious conflict. None of her mathematical works have survived.
$\qquad$ Date $\qquad$

## Lesson 13: Using Matrix Operations for Encryption

## Exit Ticket

Morgan used matrix $E=\left[\begin{array}{cc}1 & -2 \\ -1 & 3\end{array}\right]$ to encode the name of her favorite mathematician in the message $-32,7,14,1,52,2,-13,-1$.
a. How can you tell whether or not her message can be decoded?
b. Decode the message, or explain why the original message cannot be recovered.

## Exit Ticket Sample Solutions

Morgan used matrix $E=\left[\begin{array}{cc}1 & -2 \\ -1 & 3\end{array}\right]$ to encode the name of her favorite mathematician in the message

$$
-32,7,14,1,52,2,-13,-1
$$

a. How can you tell whether or not her message can be decoded?

Since the matrix $E$ has determinant $\operatorname{det}(E)=3-2=1$, we know that $\operatorname{det}(E) \neq 0$, so then a decoding matrix $D=E^{-1}$ exists.
b. Decode the message, or explain why the original message cannot be recovered.

First, we place the coded message into a $2 \times 4$ matrix $C$. Using $D=E^{-1}=\left[\begin{array}{ll}3 & 2 \\ 1 & 1\end{array}\right]$, we have

$$
\begin{aligned}
M & =D \cdot C \\
& =\left[\begin{array}{cc}
3 & 2 \\
1 & 1
\end{array}\right] \cdot\left[\begin{array}{cccc}
-32 & 7 & 14 & 1 \\
52 & 2 & -13 & -1
\end{array}\right] \\
& =\left[\begin{array}{cccc}
-96+104 & 21+4 & 42-26 & 3-2 \\
-32+52 & 7+2 & 14-13 & 1-1
\end{array}\right] \\
& =\left[\begin{array}{cccc}
8 & 25 & 16 & 1 \\
20 & 9 & 1 & 0
\end{array}\right] .
\end{aligned}
$$

The decoded message is "HYPATIA."

## Problem Set Sample Solutions

Problems 1-4 are optional as they are practice on skills previously taught and assessed. Problems 6-9 allow students to practice the use of matrix multiplication for coding and decoding messages.

1. Let $A=\left[\begin{array}{ll}1 & 3 \\ 2 & 5\end{array}\right], B=\left[\begin{array}{cc}-2 & 7 \\ 3 & -4\end{array}\right], C=\left[\begin{array}{cc}-5 & 3 \\ 2 & -1\end{array}\right], Z=\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$, and $I=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$. Evaluate the following.
a. $A+B$
$\left[\begin{array}{cc}-1 & 10 \\ 5 & 1\end{array}\right]$
b. $B+A$

$$
\left[\begin{array}{cc}
-1 & 10 \\
5 & 1
\end{array}\right]
$$

. $\quad \boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$
$\left[\begin{array}{cc}-6 & 13 \\ 7 & 0\end{array}\right]$
d. $(A+B)+C$
e. $A+I$
f. $A+Z$
$\left[\begin{array}{ll}2 & 3 \\ 2 & 6\end{array}\right]$
$\left[\begin{array}{ll}1 & 3 \\ 2 & 5\end{array}\right]$
g. $A \cdot Z$
i. $\quad I \cdot A$
$\left[\begin{array}{ll}1 & 3 \\ 2 & 5\end{array}\right]$
k. $B \cdot A$

$$
\left[\begin{array}{cc}
12 & 29 \\
-5 & -11
\end{array}\right]
$$

m. $\quad C \cdot A$
$\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$
o. $\quad A \cdot(B+C)$
$\left[\begin{array}{cc}8 & -5 \\ 11 & -5\end{array}\right]$
q. $\quad C \cdot B \cdot A$

$$
\left[\begin{array}{cc}
-75 & -178 \\
29 & 69
\end{array}\right]
$$

r. $\quad A \cdot C \cdot B$

$$
\left[\begin{array}{cc}
-2 & 7 \\
3 & -4
\end{array}\right]
$$

s. $\operatorname{det}(A)$
$-1$
t. $\operatorname{det}(B)$
$-13$
u. $\operatorname{det}(C)$
$-1$
v. $\operatorname{det}(Z)$

0
x. $\quad \operatorname{ddet}(\boldsymbol{A} \cdot \boldsymbol{B} \cdot \boldsymbol{C})$
y. $\operatorname{det}(\boldsymbol{C} \cdot \boldsymbol{B} \cdot \boldsymbol{A})$
2. For any $2 \times 2$ matrix $A$ and any real number $k$, show that if $k A=\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$, then $k=0$ or $A=\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$. Let $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$; then $k A=\left[\begin{array}{ll}k a & k b \\ k c & k d\end{array}\right]$. Suppose that $k A=\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$.
Case 1: Suppose $k \neq 0$. Then $k a=0, k b=0, k c=0$, and $k d=0$; all imply that $a=b=c=d=0$. Thus, if $k \neq$ 0 , then $A=\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$.

Case 2: Suppose that $A \neq\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$. Then at least one of $a, b, c$ and $d$ is not zero, so $k a=0, k b=0, k c=0$, and $k d=0$ imply that $k=0$.

Thus, if $k A=\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$, then either $k=0$ or $A=\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$.
3. Claire claims that she multiplied $A=\left[\begin{array}{cc}-3 & 2 \\ 0 & 4\end{array}\right]$ by another matrix $X$ and obtained $\left[\begin{array}{cc}-3 & 2 \\ 0 & 4\end{array}\right]$ as her result. What matrix did she multiply by? How do you know?

She multiplied $A$ by the multiplicative identity matrix $I=\left[\begin{array}{ll}1 & 0 \\ 0 & 1\end{array}\right]$. Since the product is a $2 \times 2$ matrix, we know that $X$ is $a \times 2$ matrix of the form $=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$. Multiplying $A \cdot X$ gives $A \cdot X=\left[\begin{array}{cc}-3 a+2 c & -3 b+2 d \\ 0 a+4 c & 0 b+4 d\end{array}\right]$. Since
$A \cdot X=A$, we have the following system of equations:

$$
\begin{aligned}
-3 a+2 c & =-3 \\
-3 b+2 d & =2 \\
0 a+4 c & =0 \\
0 b+4 d & =4
\end{aligned}
$$

The third and fourth equations give $c=0$ and $d=1$, respectively, and substituting into the first two equations gives $-3 a=-3$ and $-3 b+2=2$. Thus, $a=1$ and $b=0$, and the matrix $X$ must be $X=I$.
4. Show that the only matrix $B$ such that $A+B=A$ is the zero matrix.

Let $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$ and $B=\left[\begin{array}{ll}x & y \\ z & w\end{array}\right]$; then we have $A+B=\left[\begin{array}{ll}a+x & b+y \\ c+z & d+w\end{array}\right], a+x=a, b+y=b, c+z=c$, and $d+w=d$. In each case, solving for the elements of $B$, we find that $B=\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$.
5. A $2 \times 2$ matrix of the form $\left[\begin{array}{ll}a & 0 \\ 0 & b\end{array}\right]$ is a diagonal matrix. Daniel calculated

$$
\begin{aligned}
& {\left[\begin{array}{ll}
2 & 0 \\
0 & 2
\end{array}\right] \cdot\left[\begin{array}{cc}
2 & 3 \\
5 & -3
\end{array}\right]=\left[\begin{array}{cc}
4 & 6 \\
10 & -6
\end{array}\right]} \\
& {\left[\begin{array}{cc}
2 & 3 \\
5 & -3
\end{array}\right] \cdot\left[\begin{array}{cc}
2 & 0 \\
0 & 2
\end{array}\right]=\left[\begin{array}{cc}
4 & 6 \\
10 & -6
\end{array}\right]}
\end{aligned}
$$

and concluded that if $X$ is a diagonal matrix and $A$ is any other matrix, then $X \cdot A=A \cdot X$.
a. Is there anything wrong with Daniel's reasoning? Prove or disprove that if $X$ is a diagonal $2 \times 2$ matrix, then $\boldsymbol{X} \cdot \boldsymbol{A}=\boldsymbol{A} \cdot \boldsymbol{X}$ for any other matrix $\boldsymbol{A}$.

Yes, there is something wrong with Daniel's reasoning. A single example does not establish that a statement is true, and the example he calculated used a special case of a diagonal matrix in which the entries on the main diagonal are equal.

If $A=\left[\begin{array}{ll}1 & 2 \\ 3 & 4\end{array}\right]$ and $X=\left[\begin{array}{ll}2 & 0 \\ 0 & 3\end{array}\right]$, then $X \cdot A=\left[\begin{array}{cc}2 & 4 \\ 9 & 12\end{array}\right]$ and $A \cdot X=\left[\begin{array}{cc}2 & 6 \\ 6 & 12\end{array}\right]$. Thus, it is not true that $X \cdot A=A$ $X$ for all diagonal matrices $X$ and all other matrices $A$.
b. For $3 \times 3$ matrices, Elda claims that only diagonal matrices of the form $X=\left[\begin{array}{lll}c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c\end{array}\right]$ satisfy $X \cdot A=A \cdot X$ for any other $3 \times 3$ matrix $A$. Is her claim correct?

Elda is correct since $X=\left[\begin{array}{lll}c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c\end{array}\right]=c I$. Then,
$X \cdot A=c I \cdot A=c(I \cdot A)=c A=A c=(A \cdot I) c=A \cdot(c I)=A \cdot X$
for all matrices $A$.
6. Calvin encoded a message using $E=\left[\begin{array}{cc}2 & 2 \\ -1 & 3\end{array}\right]$, giving the coded message $4,28,42,56,2,-6,-1,52$. Decode the message, or explain why the original message cannot be recovered.

Putting the message in a $2 \times 4$ matrix, we have $C=\left[\begin{array}{cccc}4 & 28 & 42 & 56 \\ 2 & -6 & 1 & 52\end{array}\right]$. We can decode the message with $D=E^{-1}=\frac{1}{6-(-2)}\left[\begin{array}{cc}3 & -2 \\ 1 & 2\end{array}\right]=\left[\begin{array}{cc}\frac{3}{8} & -\frac{2}{8} \\ \frac{1}{8} & \frac{2}{8}\end{array}\right]$. Then the original message is found in message $M$ :

$$
M=D \cdot C=\left[\begin{array}{cc}
\frac{3}{8} & -\frac{2}{8} \\
\frac{1}{8} & \frac{2}{8}
\end{array}\right] \cdot\left[\begin{array}{cccc}
4 & 28 & 42 & 56 \\
2 & -6 & -1 & 52
\end{array}\right]=\left[\begin{array}{cccc}
1 & 12 & 16 & 8 \\
1 & 2 & 5 & 20
\end{array}\right]
$$

The original message is "ALPHABET."
7. Decode the message below using the matrix $D=\left[\begin{array}{ccc}1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 2 & 1\end{array}\right]$ :

$$
22,17,24,9,-1,14,-9,34,44,64,47,77
$$

The decoded message is found by multiplying $\left[\begin{array}{ccc}1 & 1 & -1 \\ -1 & 0 & 2 \\ 1 & 2 & 1\end{array}\right] \cdot\left[\begin{array}{cccc}22 & 17 & 24 & 9 \\ -1 & 14 & -9 & 34 \\ 44 & 64 & 47 & 77\end{array}\right]=\left[\begin{array}{cccc}3 & 18 & 25 & 16 \\ 20 & 15 & 7 & 18 \\ 1 & 16 & 8 & 25\end{array}\right]$. Then the message is "CRYPTOGRAPHY."
8. Brandon encoded his name with the matrix $E=\left[\begin{array}{ll}1 & 1 \\ 2 & 2\end{array}\right]$, producing the matrix $C=\left[\begin{array}{cccc}6 & 33 & 15 & 14 \\ 12 & 66 & 30 & 28\end{array}\right]$. Decode the message, or explain why the original message cannot be recovered.

Brandon used a matrix that is not invertible. The original matrix cannot be recovered.
9. Janelle used the encoding matrix $E=\left[\begin{array}{cc}1 & 2 \\ 1 & -1\end{array}\right]$ to encode the message "FROG" by multiplying
$C=\left[\begin{array}{cc}6 & 18 \\ 15 & 7\end{array}\right] \cdot\left[\begin{array}{cc}1 & 2 \\ 1 & -1\end{array}\right]=\left[\begin{array}{cc}24 & 30 \\ 22 & 37\end{array}\right]$. When Taylor decoded it, she computed
$M=\left[\begin{array}{cc}-1 & 2 \\ 1 & 1\end{array}\right] \cdot\left[\begin{array}{cc}24 & 30 \\ 22 & 37\end{array}\right]=\left[\begin{array}{cc}20 & 44 \\ 2 & -7\end{array}\right]$. What went wrong?
Janelle multiplied her matrices in the wrong order. When Janelle tried to decode the matrix $C=\left[\begin{array}{ll}24 & 30 \\ 22 & 37\end{array}\right]$ using the decoding matrix $D=\left[\begin{array}{cc}1 & 2 \\ 1 & -1\end{array}\right]^{-1}$, she ended up calculating

$$
\begin{aligned}
D \cdot C & =D \cdot M \cdot E \\
& =E^{-1} \cdot M \cdot E .
\end{aligned}
$$

Because matrix multiplication is not commutative, $E^{-1} \cdot M \cdot E \neq M$, Taylor was unable to recover the original message.

Name
Date $\qquad$

1. Kyle wishes to expand his business and is entertaining four possible options. If he builds a new store he expects to make a profit of 9 million dollars if the market remains strong; however, if market growth declines, he could incur a loss of 5 million dollars. If Kyle invests in a franchise, he could profit 4 million dollars in a strong market but lose 3 million dollars in a declining market. If he modernizes his current facilities, he could profit 4 million dollars in a strong market but lose 2 million dollars in a declining one. If he sells his business, he will make a profit of 2 million dollars irrespective of the state of the market.
a. Write down a $4 \times 2$ payoff matrix $P$ summarizing the profits and losses Kyle could expect to see with all possible scenarios. (Record a loss as a profit in a negative amount.) Explain how to interpret your matrix.
b. Kyle realized that all his figures need to be adjusted by $10 \%$ in magnitude due to inflation costs. What is the appropriate value of a real number $\lambda$ so that the matrix $\lambda P$ represents a correctly adjusted payoff matrix? Explain your reasoning. Write down the new payoff matrix $\lambda P$.
c. Kyle is hoping to receive a cash donation of 1 million dollars. If he does, all the figures in his payoff matrix will increase by 1 million dollars.

Write down a matrix $Q$ so that if Kyle does receive this donation, his new payoff matrix is given by $Q+\lambda P$. Explain your thinking.
2. The following diagram shows a map of three land masses, numbered region 1 , region 2 , and region 3, connected via bridges over water. Each bridge can be traversed in either direction.

a. Write down a $3 \times 3$ matrix $A$ with $a_{i j}$, for $i=1$, 2 , or 3 and $j=1$, 2 , or 3 , equal to the number of ways to walk from region $i$ to region $j$ by crossing exactly one bridge. Notice that there are no paths that start and end in the same region crossing exactly one bridge.
b. Compute the matrix product $A^{2}$.
c. Show that there are 10 walking routes that start and end in region 2 , crossing over water exactly twice. Assume each bridge, when crossed, is fully traversed to the next land mass.
d. How many walking routes are there from region 3 to region 2 that cross over water exactly three times? Again, assume each bridge is fully traversed to the next land mass.
e. If the number of bridges between each pair of land masses is doubled, how does the answer to part (d) change? That is, what would be the new count of routes from region 3 to region 2 that cross over water exactly three times?
3. Let $P=\left[\begin{array}{cc}3 & -5 \\ 5 & 3\end{array}\right]$ and $Q=\left[\begin{array}{cc}-2 & 1 \\ -1 & -2\end{array}\right]$.
a. Show the work needed and compute $2 P-3 Q$.
b. Show the work needed and compute $P Q$.
c. Show that $P^{2} Q=P Q P$.
4.
a. Show that if the matrix equation $(A+B)^{2}=A^{2}+2 A B+B^{2}$ holds for two square matrices $A$ and $B$ of the same dimension, then these two matrices commute under multiplication.
b. Give an example of a pair of $2 \times 2$ matrices $A$ and $B$ for which $(A+B)^{2} \neq A^{2}+2 A B+B^{2}$.
c. In general, does $A B=B A$ ? Explain.
d. In general, does $A(B+C)=A B+A C$ ? Explain.
e. In general, does $A(B C)=(A B) C$ ? Explain.
5. Let $I$ be the $3 \times 3$ identity matrix and $A$ the $3 \times 3$ zero matrix. Let the $3 \times 1$ column $x=\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ represent a point in three dimensional space. Also, set $P=\left[\begin{array}{lll}2 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 4\end{array}\right]$.
a. Use examples to illustrate how matrix $A$ plays the same role in matrix addition that the number 0 plays in real number addition. Include an explanation of this role in your response.
b. Use examples to illustrate how matrix I plays the same role in matrix multiplication that the number 1 plays in real number multiplication. Include an explanation of this role in your response.
c. What is the row 3 , column 3 entry of $(A P+I)^{2}$ ? Explain how you obtain your answer.
d. Show that $(P-1)(P+1)$ equals $P^{2}-I$.
e. $\quad$ Show that $P x$ is sure to be a point in the $x z$-plane in three-dimensional space.
f. Is there a $3 \times 3$ matrix $Q$, not necessarily the matrix inverse for $P$, for which $Q P x=x$ for every $3 \times 1$ column $x$ representing a point? Explain your answer.
g. Does the matrix $P$ have a matrix inverse? Explain your answer.
h. What is the determinant of the matrix $P$ ?
6. What is the image of the point given by the $3 \times 1$ column matrix $\left[\begin{array}{c}1 \\ 2 \\ -1\end{array}\right]$ when it is rotated $45^{\circ}$ about the $z$-axis in the counterclockwise direction (as according to the orientation of the $x y$-plane) and then $180^{\circ}$ about the $y$-axis?
$\left.\begin{array}{|c|l|l|l|l|}\hline \text { A Progression Toward Mastery } \\ \text { Assessment } & \begin{array}{l}\text { STEP 1 } \\ \text { Missing or } \\ \text { incorrect answer } \\ \text { and little evidence } \\ \text { of reasoning or } \\ \text { application of } \\ \text { mathematics to } \\ \text { solve the problem. }\end{array} & \begin{array}{l}\text { STEP 2 } \\ \text { Missing or } \\ \text { incorrect answer } \\ \text { but evidence of } \\ \text { some reasoning or } \\ \text { application of } \\ \text { mathematics to } \\ \text { solve the problem. }\end{array} & \begin{array}{l}\text { STEP 3 } \\ \text { A correct answer } \\ \text { with some } \\ \text { evidence of } \\ \text { reasoning or } \\ \text { application of } \\ \text { mathematics to } \\ \text { solve the problem, } \\ \text { or an incorrect }\end{array} & \begin{array}{l}\text { STEP 4 } \\ \text { A correct answer } \\ \text { supported by } \\ \text { substantial }\end{array} \\ \text { evidence of solid } \\ \text { reasoning or } \\ \text { application of } \\ \text { mathematics to } \\ \text { solve the problem. }\end{array}\right\}$

|  | C $\begin{aligned} & \text { N-VM.C. } 6 \\ & \text { N-VM.C. } 8 \end{aligned}$ | Student shows little or no evidence of interpreting matrix entries. | Student answers question using matrix, but uses wrong entry. | Student answers correctly but does not fully explain answer. | Student answers correctly and fully explains answer. |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\begin{gathered} \text { d } \\ \text { N-VM.C. } 6 \\ \text { N-VM.C. } 8 \end{gathered}$ | Student shows little or no evidence of matrix operations needed to answer question. | Student multiplies to find $A^{3}$ but does not explain answer or explains wrong entry as answer. | Student multiplies to find $A^{3}$ but makes mistakes multiplying. The answer is incorrect, but reasoning and justification are correct. | Student multiplies to find $A^{3}$, identifies correct answer, and justifies it correctly. |
|  | $\begin{gathered} \text { e } \\ \text { N-VM.C. } 6 \\ \text { N-VM.C. } 7 \\ \text { N-VM.C. } 8 \end{gathered}$ | Student shows little or no evidence of matrix operations needed to answer question. | Student multiplies to find $8 A^{3}$ but does not explain answer or explains wrong entry as answer. | Student multiplies to find $8 A^{3}$ and explains answer, but not completely. | Student multiplies to find $8 A^{3}$ and explains answer completely. |
| 3 | a $\begin{aligned} & \text { N-VM.C. } 7 \\ & \text { N-VM.C. } 8 \end{aligned}$ | Student shows little or no evidence of matrix operations. | Student shows some knowledge of matrix operations but makes mistakes on two or more entries in the final matrix. | Student shows knowledge of matrix operations but has one entry wrong in final matrix. | Student shows knowledge of matrix operations arriving at correct final matrix. |
|  | b $\text { N-VM.C. } 8$ | Student shows little or no evidence of matrix multiplication. | Student shows some knowledge of matrix multiplication but makes mistakes on two or more entries in the final matrix. | Student shows knowledge of matrix multiplication but has one entry wrong in final matrix. | Student shows knowledge of matrix multiplication arriving at correct final matrix. |
|  | $\begin{gathered} \text { C } \\ \text { N-VM.C. } 8 \end{gathered}$ | Student shows little or no evidence of matrix multiplication. | Student shows some knowledge of matrix multiplication but makes mistakes leading to incorrect answer. | Student shows knowledge of matrix multiplication, finding correct matrices, but does not explain that they are equal. | Student shows knowledge of matrix multiplication, finding correct matrices, and explains that they are equal. |
| 4 | a $\text { N-VM.C. } 9$ | Student makes little or no attempt to answer question. | Student expands the binomial $(A+B)^{2}$ but does not continue with proof, or steps are not correct. | Student expands the binomial $(A+B)^{2}$ and shows that $B A=A B$ but does not explain reasoning for these matrices being commutative under multiplication. | Student expands the binomial $(A+B)^{2}$, shows that $B A=A B$, and explains reasoning that these matrices are commutative under multiplication. |


|  | b $\text { N-VM.C. } 9$ | Student makes little or no attempt to find matrices. | Student lists two $2 \times 2$ matrices but does not support or prove answer. | Student lists two $2 \times 2$ matrices but makes mistakes in calculations or reasoning to support answer. | Student lists two $2 \times 2$ matrices and shows supporting evidence to verify answer. |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\begin{gathered} \text { c } \\ \text { N-VM.C. } 9 \end{gathered}$ | Student states that $A B=B A$ is true for matrices. | Student states that $A B \neq B A$ but does not support answer with reasoning. | Student states that $A B \neq B A$ and attempts to explain reasoning but does not use the term commute or commutative. | Student states that $A B \neq B A$, explains reasoning, and states that matrix multiplication is not generally commutative. |
|  | d $\text { N-VM.C. } 9$ | Student states that $A(B+C) \neq A B+B C .$ | Student states that $A(B+C)=A B+B C$ but does not support answer with reasoning. | Student states that $A(B+C)=A B+A C$ <br> and attempts to explain but makes minor errors in reasoning. | Student states that $A(B+C)=A B+A C$ and explains reasoning correctly. |
|  | e $\text { N-VM.C. } 9$ | Student states that $A(B C) \neq(A B) C .$ | Student states that $A(B C)=(A B) C$ but does not support answer with reasoning. | Student states that $A(B C)=(A B) C$ and attempts to explain but makes minor errors in reasoning. | Student states that $A(B C)=(A B) C$ and explains reasoning correctly. |
| 5 | $\begin{gathered} \text { a } \\ \text { N-VM.C. } 10 \end{gathered}$ | Student shows little of no understanding of the $3 \times 3$ zero matrix. | Student writes the $3 \times 3$ zero matrix but does not explain its role in matrix addition. | Student writes the $3 \times 3$ zero matrix, showing an example of its role in matrix addition but does not explain the connection to the number zero in real number addition. | Student writes the $3 \times 3$ zero matrix, shows and example of its role in matrix addition, and explains the connection to the number zero in real number addition. |
|  | b $\text { N-VM.C. } 10$ | Student shows little of no understanding of the $3 \times 3$ identity matrix. | Student writes the $3 \times 3$ identity matrix but does not explain its role in matrix multiplication. | Student writes the $3 \times 3$ identity matrix, showing an example of its role in matrix multiplication but does not explain the connection to the number one in real number multiplication. | Student writes the $3 \times 3$ identity matrix, shows an example of its role in matrix multiplication, and explains the connection to the number one in real number multiplication. |
|  | c $\begin{gathered} \text { N-VM.C. } 8 \\ \text { N-VM.C. } 10 \end{gathered}$ | Student shows little or no understanding of matrix operations. | Student finds $(A P+I)^{2}$ but does not identify the entry in row 3 column 3. | Student finds $(A P+I)^{2}$ and identifies the entry in row 3 column 3 but does not explain answer. | Student finds $(A P+I)^{2}$, identifies the entry in row 3 column 3, and explains answer. |


|  | d $\begin{gathered} \text { N-VM.C. } 9 \\ \text { N-VM.C. } 10 \end{gathered}$ | Student shows little or no understanding of matrix operations. | Student calculates two of $P-1, P+1$, or $P^{2}-I$ correctly. | Student calculates $(P-1)(P+1)$ and $P^{2}-I$ but does not explain why the expressions are equal. | Student calculates $(P-1)(P+1)$ and $P^{2}-I$ and explains why the expressions are equal. |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | e $\text { N-VM.C. } 11$ | Student shows little or no understanding of matrix operations. | Student sets up $P x$ but does not find the matrix representing the product. | Student sets up and finds the matrix representing $P x$ but does not explain the meaning of the point in 3-dimensional space. | Students sets up and finds the matrix representing $P x$ and explains the meaning of the point in 3dimensional space. |
|  | $\begin{gathered} \mathbf{f} \\ \text { N-VM.C. } 11 \end{gathered}$ | Student shows little or no understanding of matrix operations. | Student finds $P x$ but does not find $Q P x$ or explain reasoning. | Student finds $Q P x$ and attempts to explain, but not clearly, why $Q$ cannot exist. | Student finds $Q P x$ and clearly shows that matrix $Q$ cannot exist. |
|  | $\begin{gathered} \mathbf{g} \\ \text { N-VM.C. } 10 \end{gathered}$ | Student shows little or no understanding of inverse matrices. | Student shows some understanding of inverse matrices but cannot answer or explain question. | Student says that the inverse does not exist and attempts to explain but explanation has minor mistakes. | Student clearly explains why the inverse matrix does not exist. |
|  | h $\text { N-VM.C. } 10$ | Student shows little or no understanding of the determinant of matrix $P$. | Student incorrectly attempts to find the determinant. | Student states that the determinant is zero but with no explanation. | Student explains clearly why the determinant is zero. |
| 6 | N-VM.C. 11 | Student shows little or no understanding of matrices producing rotations. | Student attempts to write the matrices producing rotations but with errors or with only one correct. | Student writes the correct matrices producing the rotations required but makes calculation errors leading to an incorrect final answer. | Student writes the correct matrices producing the rotations required and calculates the correct final image point. |

Name $\qquad$ Date $\qquad$

1. Kyle wishes to expand his business and is entertaining four possible options. If he builds a new store he expects to make a profit of 9 million dollars if the market remains strong; however, if market growth declines, he could incur a loss of 5 million dollars. If Kyle invests in a franchise, he could profit 4 million dollars in a strong market but lose 3 million dollars in a declining market. If he modernizes his current facilities, he could profit 4 million dollars in a strong market but lose 2 million dollars in a declining one. If he sells his business, he will make a profit of 2 million dollars irrespective of the state of the market.
a. Write down a $4 \times 2$ payoff matrix $P$ summarizing the profits and losses Kyle could expect to see with all possible scenarios. (Record a loss as a profit in a negative amount.) Explain how to interpret your matrix.

We have $P=\left[\begin{array}{cc}9 & -5 \\ 4 & -3 \\ 4 & -2 \\ 2 & 2\end{array}\right]$.
Here the four rows correspond to, in turn, the options of building a new store, investing in a franchise, modernizing, and selling. The first column gives the payoffs in a strong market, and the second column gives the payoffs in a declining market.

All entries are in units of millions of dollars.
Note: Other presentations for the matrix $P$ are possible.
b. Kyle realized that all his figures need to be adjusted by $10 \%$ in magnitude due to inflation costs. What is the appropriate value of a real number $\lambda$ so that the matrix $\lambda P$ represents a correctly adjusted payoff matrix? Explain your reasoning. Write down the new payoff matrix $\lambda P$.

Each entry in the matrix needs to increase $5 \%$ in magnitude. This can be accomplished by multiplying each entry by 1.10. If we set $\lambda=1.1$, then $\lambda P=\left[\begin{array}{cc}9.9 & -5.5 \\ 4.4 & -3.3 \\ 4.4 & -2.2 \\ 2.2 & 2.2\end{array}\right]$ is the appropriate new payoff matrix.
c. Kyle is hoping to receive a cash donation of 1 million dollars. If he does, all the figures in his payoff matrix will increase by 1 million dollars.

Write down a matrix $Q$ so that if Kyle does receive this donation, his new payoff matrix is given by $Q+\lambda P$. Explain your thinking.
$\operatorname{Set} Q=\left[\begin{array}{ll}1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1\end{array}\right]$. Then $Q+\lambda P$ is the matrix $\lambda P$ with 1 added to each entry. This is the effect we seek, increasing each expected payoff by 1 million dollars.
2. The following diagram shows a map of three land masses, numbered region 1 , region 2 , and region 3, connected via bridges over water. Each bridge can be traversed in either direction.

a. Write down a $3 \times 3$ matrix $A$ with $a_{i j}$, for $i=1$, 2 , or 3 and $j=1$, 2 , or 3 , equal to the number of ways to walk from region $i$ to region $j$ by crossing exactly one bridge. Notice that there are no paths that start and end in the same region crossing exactly one bridge.

$$
A=\left[\begin{array}{lll}
0 & 1 & 1 \\
1 & 0 & 3 \\
1 & 3 & 0
\end{array}\right]
$$

b. Compute the matrix product $A^{2}$.

$$
A^{2}=\left[\begin{array}{ccc}
2 & 3 & 3 \\
3 & 10 & 1 \\
3 & 1 & 10
\end{array}\right]
$$

c. Show that there are 10 walking routes that start and end in region 2 , crossing over water exactly twice. Assume each bridge, when crossed, is fully traversed to the next land mass.

The entries of $A^{2}$ give the number of paths via two bridges between land regions. As the row 2 , column 2 entry of $A^{2}$ is 10, this is the count of two-bridge journeys that start and end in region 2.
d. How many walking routes are there from region 3 to region 2 that cross over water exactly three times? Again, assume each bridge is fully traversed to the next land mass.

The entries of $A^{3}$ give the counts of three-bridge journeys between land masses. We seek the row 3 , column 2 entry of the product.

$$
\left[\begin{array}{ccc}
2 & 3 & 3 \\
3 & 10 & 1 \\
3 & 1 & 9
\end{array}\right]\left[\begin{array}{lll}
0 & 1 & 1 \\
1 & 0 & 3 \\
1 & 3 & 0
\end{array}\right]
$$

This entry is $(3 \cdot 1)+(1 \cdot 0)+(9 \cdot 3)=30$. Therefore, there are 30 such routes.
e. If the number of bridges between each pair of land masses is doubled, how does the answer to part (d) change? That is, what would be the new count of routes from region 3 to region 2 that cross over water exactly three times?

We are now working with the matrix 2A. The number of routes from region 3 to region 2 via three bridges is the row 3 , column 2 entry of $(2 A)^{3}=8 A^{3}$. As all the entries are multiplied by eight, there are $8 \times 30=240$ routes of the particular type we seek.
3. Let $P=\left[\begin{array}{cc}3 & -5 \\ 5 & 3\end{array}\right]$ and $Q=\left[\begin{array}{cc}-2 & 1 \\ -1 & -2\end{array}\right]$ for some fixed real numbers $a, b, c$, and $d$.
a. Show the work needed and compute $2 P-3 Q$.

$$
2 P-3 Q=\left[\begin{array}{cc}
6 & -10 \\
10 & 6
\end{array}\right]-\left[\begin{array}{cc}
-6 & 3 \\
-3 & -6
\end{array}\right]=\left[\begin{array}{cc}
12 & -13 \\
13 & 12
\end{array}\right]
$$

b. Show the work needed and compute $P Q$.

$$
P Q=\left[\begin{array}{cc}
-6+5 & 3+10 \\
-10-3 & 5-6
\end{array}\right]=\left[\begin{array}{cc}
-1 & 13 \\
-13 & -1
\end{array}\right]
$$

c. Show that $P^{2} Q=P Q P$.
$P Q=\left[\begin{array}{cc}-1 & 13 \\ -13 & -1\end{array}\right]$
$P^{2} Q=P(P Q)=\left[\begin{array}{cc}62 & 44 \\ -44 & 62\end{array}\right]$
$P Q P=(P Q) P=\left[\begin{array}{cc}-1 & 13 \\ -13 & -1\end{array}\right]\left[\begin{array}{cc}3 & -5 \\ 5 & 3\end{array}\right]=\left[\begin{array}{cc}62 & 44 \\ -44 & 62\end{array}\right]$
These are identical matrices.
4.
a. Show that if the matrix equation $(A+B)^{2}=A^{2}+2 A B+B^{2}$ holds for two square matrices $A$ and $B$ of the same dimension, then these two matrices commute under multiplication.

We have $(A+B)^{2}=(A+B)(A+B)$.
By the distributive rule, which does hold for matrices, this equals $A(A+B)+B(A+B)$, which, again by the distributive rule, equals $A^{2}+A B+B A+B^{2}$.

On the other hand, $A^{2}+2 A B+B^{2}$ equals $A^{2}+A B+A B+B^{2}$.
So, if $(A+B)^{2}=A^{2}+2 A B+B^{2}$, then we have $A^{2}+A B+B A+B^{2}=A^{2}+A B+A B+B^{2}$.
Adding $-A^{2}$ and $-A B$ and $-B^{2}$ to each side of this equation gives $B A=A B$.
This shows that $A$ and $B$ commute under multiplication in this special case when $(A+B)^{2}=A^{2}+2 A B+B^{2}$, but in general, matrix multiplication is not commutative.
b. Give an example of a pair of $2 \times 2$ matrices $A$ and $B$ for which $(A+B)^{2} \neq A^{2}+2 A B+B^{2}$.

A pair of matrices that do not commute under multiplication, such as $A=\left[\begin{array}{ll}1 & 0 \\ 1 & 1\end{array}\right]$ and $B=\left[\begin{array}{ll}1 & 1 \\ 0 & 1\end{array}\right]$, should do the trick.
To check: $(A+B)^{2}=\left[\begin{array}{ll}2 & 1 \\ 1 & 2\end{array}\right]^{2}=\left[\begin{array}{ll}5 & 4 \\ 4 & 5\end{array}\right]$

$$
\begin{aligned}
A^{2}+2 A B+B^{2} & =\left[\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right]^{2}+2\left[\begin{array}{ll}
1 & 0 \\
1 & 1
\end{array}\right]\left[\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right]+\left[\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right]^{2} \\
& =\left[\begin{array}{ll}
1 & 0 \\
2 & 1
\end{array}\right]+2\left[\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right]+\left[\begin{array}{ll}
1 & 2 \\
0 & 1
\end{array}\right] \\
& =\left[\begin{array}{ll}
4 & 4 \\
4 & 6
\end{array}\right]
\end{aligned}
$$

These are indeed different.
c. In general, does $A B=B A$ ? Explain.

No. Since matrix multiplication can represent linear transformations, we know that they will not always commute since linear transformations do not always commute.
d. In general, does $A(B+C)=A B+A C$ ? Explain.

Yes. Consider the effect on the point $x$ made by both sides of the equation. On the lefthand side, the transformation $B+C$ is applied to the point $x$, but we know that this is the same as $B x+C x$ from our work with linear transformations. Applying the transformation represented by $A$ to either $B x+C x$ or $(B+C) x$ now is $A B x+A C x$ because they work like linear maps.
e. In general, does $A(B C)=(A B) C$ ? Explain.

Yes. If we consider the effect the matrices on both side make on a point $x$, the matrices are applied in the exact same order, $C$, then $B$, then $A$, regardless of whether $A B$ is computed first or $B C$ is computed first.
5. Let $I$ be the $3 \times 3$ identity matrix and $A$ the $3 \times 3$ zero matrix. Let the $3 \times 1$ column $x=\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ represent a point in three dimensional space. Also, set $P=\left[\begin{array}{lll}2 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 4\end{array}\right]$.
a. Use examples to illustrate how matrix $A$ plays the same role in matrix addition that the number 0 plays in real number addition. Include an explanation of this role in your response.

The sum of two $3 \times 3$ matrices is determined by adding entries in corresponding positions of the two matrices to produce a new $3 \times 3$ matrix. Each and every entry of matrix $A$ is zero, so a sum of the form $A+P$, where $P$ is another $3 \times 3$ matrix, is given by adding zero to each entry of $P$. Thus, $A+P=P$. For example:

$$
\left[\begin{array}{lll}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right]+\left[\begin{array}{lll}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6 & 9
\end{array}\right]=\left[\begin{array}{lll}
0+1 & 0+4 & 0+7 \\
0+2 & 0+5 & 0+8 \\
0+3 & 0+6 & 0+9
\end{array}\right]=\left[\begin{array}{lll}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6 & 9
\end{array}\right] .
$$

This is analogous to the role of zero in the real number system: $0+p=p$ for every real number $p$.

In the same way, $P+A=P$ for all $3 \times 3$ matrices $P$, analogous to $p+0=p$ for all real numbers $p$.
b. Use examples to illustrate how matrix I plays the same role in matrix multiplication that the number 1 plays in real number multiplication. Include an explanation of this role in your response.
We have $I=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$. By the definition of matrix multiplication we see, for example:

$$
\begin{aligned}
{\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right] \times\left[\begin{array}{lll}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6 & 9
\end{array}\right] } & =\left[\begin{array}{lll}
(1 \cdot 1)+(0 \cdot 2)+(0 \cdot 3) & (1 \cdot 4)+(0 \cdot 5)+(0 \cdot 6) & (1 \cdot 7)+(0 \cdot 8)+(0 \cdot 9) \\
(0 \cdot 1)+(1 \cdot 2)+(0 \cdot 3) & (0 \cdot 4)+(1 \cdot 5)+(0 \cdot 6) & (0 \cdot 7)+(1 \cdot 8)+(0 \cdot 9) \\
(0 \cdot 1)+(0 \cdot 2)+(1 \cdot 3) & (0 \cdot 4)+(0 \cdot 5)+(1 \cdot 6) & (0 \cdot 7)+(0 \cdot 8)+(1 \cdot 9)
\end{array}\right] \\
& =\left[\begin{array}{lll}
1 & 4 & 7 \\
2 & 5 & 8 \\
3 & 6 & 9
\end{array}\right]
\end{aligned}
$$

We have, in general, that $1 \times P=P$ for every $3 \times 3$ matrix $P$. We also have $P \times 1=P$.
Thus, the matrix I plays the role of the number 1 in real number arithmetic where $1 \times p=p$ and $p \times 1=p$ for each real number $p$.
c. What is the row 3 , column 3 entry of $(A P+I)^{2}$ ? Explain how you obtain your answer.

Since $A$ is the zero matrix, $A P$ equals the zero matrix. That is, $A P=A$.
Thus, $(A P+1)^{2}=(A+1)^{2}=1^{2}=1$.
So, $(A P+1)^{2}$ is just the $3 \times 3$ identity matrix. The row 3 , column 3 entry is thus 1 .
d. Show that $(P-1)(P+1)$ equals $P^{2}-I$.

We have $(P-I)(P+1)=P^{2}-\mid P+P I-I^{2}$ (using the distributive property which holds for matrices). Since $P I=P, I P=P$, and $1^{2}=1$, this equals $P^{2}-P+P+1=P^{2}-1$.
e. Show that $P x$ is sure to be a point in the $x z$-plane in three-dimensional space.

We have

$$
P x=\left[\begin{array}{lll}
2 & 0 & 5 \\
0 & 0 & 0 \\
0 & 0 & 4
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{c}
2 x+5 z \\
0 \\
4 z
\end{array}\right]
$$

The image is a point with $y$-coordinate zero and so is a point in the xz-plane in three-dimensional space.
f. Is there a $3 \times 3$ matrix $Q$, not necessarily the matrix inverse for $P$, for which $Q P x=x$ for every $3 \times$ 1 column $x$ representing a point? Explain your answer.

If there were such a matrix $Q$, then $Q P x=x$ for $x=\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]$. But $P x=\left[\begin{array}{lll}2 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 4\end{array}\right]\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]=\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]$, and so $Q P x=Q\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]=\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]$, which is not $x=\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]$ after all. There can be no such matrix $Q$.
g. Does the matrix $P$ have a matrix inverse? Explain your answer.

If $P$ had a matrix inverse $P^{-1}$, then we would have $P^{-1} P=1$ and so $P^{-1} P x=x$ for all $3 \times 1$ columns $x$ representing a point. By part $(d)$, there is no such matrix.

OR
By part (c), $P$ takes all points in the three-dimensional space and collapses them to a plane. So there are points that are taken to the same image point by P. Thus, no inverse transformation, $\mathrm{P}^{-1}$, can exist.
h. What is the determinant of the matrix $P$ ?

The unit cube is mapped onto a plane, and so the image of the unit cube under $P$ has zero volume. The determinant of $P$ is thus $O$.

OR
By part (e), $P$ has no multiplicative inverse, and so its determinant must be 0 .
6. What is the image of the point given by the $3 \times 1$ column matrix $\left[\begin{array}{c}1 \\ 2 \\ -1\end{array}\right]$ when it is rotated $45^{\circ}$ about the $z$-axis in the counterclockwise direction (as according to the orientation of the $x y$-plane) and then $180^{\circ}$ about the $y$-axis?

A rotation about the z-axis of $45^{\circ}$ is effected by multiplication by the matrix:

$$
R_{1}=\left[\begin{array}{ccc}
\cos \left(45^{\circ}\right) & -\sin \left(45^{\circ}\right) & 0 \\
\sin \left(45^{\circ}\right) & \cos \left(45^{\circ}\right) & 0 \\
0 & 0 & 1
\end{array}\right]=\left[\begin{array}{ccc}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\
0 & 0 & 1
\end{array}\right]
$$

So, the image of the point under the first rotation is

$$
R_{1}\left[\begin{array}{c}
1 \\
2 \\
-1
\end{array}\right]=\left[\begin{array}{c}
-\frac{1}{\sqrt{2}} \\
\frac{3}{\sqrt{2}} \\
-1
\end{array}\right]
$$

A rotation of $180^{\circ}$ about the $y$-axis is effected by multiplication by

$$
R_{2}=\left[\begin{array}{ccc}
\cos \left(180^{\circ}\right) & 0 & -\sin \left(180^{\circ}\right) \\
0 & 1 & 0 \\
\sin \left(180^{\circ}\right) & 0 & \cos \left(180^{\circ}\right)
\end{array}\right]=\left[\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{array}\right] .
$$

The final image point we seek is thus

$$
R_{2}\left[\begin{array}{c}
-\frac{1}{\sqrt{2}} \\
\frac{3}{\sqrt{2}} \\
-1
\end{array}\right]=\left[\begin{array}{c}
\frac{1}{\sqrt{2}} \\
\frac{3}{\sqrt{2}} \\
1
\end{array}\right] .
$$

## New York State Common Core

PRECALCULUS AND ADVANCED TOPICS•MODULE 2

## Topic C:

## Systems of Linear Equations

N-VM.C.10, A-REI.C.8, A-REI.C. 9

| Focus Standards: | N-VM.C. 10 | (+) Understand that the zero and identity matrices play a role in matrix addition and multiplication similar to the role of 0 and 1 in the real numbers. The determinant of a square matrix is nonzero if and only if the matrix has a multiplicative inverse. |
| :---: | :---: | :---: |
|  | A-REI.C. 8 | (+) Represent a system of linear equation as a single matrix equation in a vector variable. |
|  | A-REI.C. 9 | (+) Find the inverse of a matrix if it exists and use it to solve systems of linear equations (using technology for matrices of dimension $3 \times 3$ or greater). |
| Instructional Days: | 3 |  |
| Lesson 14: | Solving Equa | S Involving Linear Transformations of the Coordinate Plane (P) ${ }^{1}$ |
| Lesson 15: | Solving Equa | ns Involving Linear Transformations of the Coordinate Space (P) |
| Lesson 16: | Solving Gene | Systems of Linear Equations (P) |

Topic C provides a third context for the appearance of matrices via the study of systems of linear equations. Students see that a system of linear equations can be represented as a single matrix equation in a vector variable (A-REI.C.8) and that one can solve the system with the aid of the multiplicative inverse to a matrix, if it exists (A-REI.C.9, N-VM.C.10).

In Lesson 14, students will explore the relationship between linear transformations of points in twodimensional space and systems of equations. They represent systems of equations as linear transformations represented by matrix equations and apply inverse matrix multiplication to find the solutions to systems of equations, establishing a foundation for solving systems of three or more equations using inverse matrix operations (A-REI.C.8, A-REI.C.9, N-VM.C.10). This work is expanded in Lesson 15 to more complicated systems of equations as students use software to calculate inverse matrices for systems of degree 3 and apply the inverse of the coefficient matrix to the linear transformation equation to solve systems (A-REI.C.8, A-REI.C.9, N-VM.C.10). Topic C concludes with Lesson 16 as students discover that, while it is difficult to geometrically describe linear transformations in four- or higher-dimensional space, the mathematics behind representing systems of equations as a linear transformation using matrices is valid for higher-degree space.

[^4]They apply this reasoning to represent complicated systems of equations using matrices and use technology to solve the systems (A-REI.C.8, A-REI.C.9, N-VM.C.10).

Throughout Topic C , students are using calculators to perform matrix operations, find the inverse of matrices, and solve systems of equations with matrices (MP.5). Students also see that the structure of solving systems does not change as systems become more complicated or bigger in size (MP.7).

## Lesson 14: Solving Equations Involving Linear

 Transformations of the Coordinate Plane
## Student Outcomes

- Students will represent systems of equations as linear transformations of the form $L x=b$, using matrix notation.
- Students will discover that the systems of equations written in the form $L x=b$ can be solved by computing $x=L^{-1} b$ for all invertible matrices, $L$ and they will apply this process to find solutions to systems of two linear equations in two variables.


## Lesson Notes

In this lesson, students will explore the relationship between linear transformations of points in two-dimensional space and systems of equations. They will represent systems of equations as linear transformations represented by matrix equations and will apply inverse matrix multiplication to find the solutions to systems of equations, which will establish a foundation for solving systems of three or more equations using inverse matrix operations.

## Classwork

## Opening Exercise (3 minutes)

Students briefly review how to represent linear transformations of points in the coordinate plane as matrices in preparation for representing systems of equations as linear transformations. Students analyze and critique the reasoning in the given problem. Students can perform the matrix multiplication for each problem to verify that the matrices correctly represent the transformations. Debrief with the class after students work individually.

## Opening Exercise

Ahmad says the matrix $\left[\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right]$ applied to the point $\left[\begin{array}{l}4 \\ 1\end{array}\right]$ will reflect the point to $\left[\begin{array}{l}1 \\ 4\end{array}\right]$. Randelle says that applying the matrix to the given point will produce a rotation of $180^{\circ}$ about the origin. Who is correct? Explain your answer, and verify the result.
Randelle is correct. Applying the matrix $\left[\begin{array}{cc}-1 & 0 \\ 0 & -1\end{array}\right]$ to the given point produces a rotation of $180^{\circ}$ about the origin of the point $\left[\begin{array}{l}4 \\ 1\end{array}\right]$ to the image point $\left[\begin{array}{l}-4 \\ -1\end{array}\right]$. Applying the matrix $\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]$ to the given point would produce a reflection to the image point $\left[\begin{array}{l}1 \\ 4\end{array}\right]$.

## Example 1 (20 minutes)

Students should complete part (a) with a partner. After a few minutes, each pair can share its example with another pair. The students should complete part (b) in small groups, e.g., with the pairs that exchanged examples for part (a). After a few minutes, a different group should describe each example from part (b) and justify why the linear transformation will not produce an image point of $\left[\begin{array}{l}4 \\ 1\end{array}\right]$. Students should be allowed to question the reasoning of other groups. Students should recognize that not all linear transformations will produce a desired image point. In particular, linear transformations represented by matrices with a determinant of 0 will not transform a point $\left[\begin{array}{l}x \\ y\end{array}\right]$ to any desired image point. Part (c) should be completed as a teacher-led discussion. Students will discover that systems of equations in two-dimensional space can be represented as a linear transformation represented by the equation $L x=b$, where $L$ represents the linear transformation of point $x$ resulting in image point $b$. The coordinates of the pre-image can be found by applying the reverse of the linear transformation to the equation $L x=b$, resulting in the equation $x=L^{-1} b$ for all invertible matrices $L$.

- How can you verify that the transformation matrix you found in part (a) accurately represents the transformation you described?
- Multiply the pre-image by the matrix to verify that the result is the column matrix $\left[\begin{array}{l}4 \\ 1\end{array}\right]$.
- If we calculate the determinants of the transformation matrices you found for part (a), what do they have in common?
- They are all nonzero.
- Why can't the transformation matrix $\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$ produce an image point of $\left[\begin{array}{l}4 \\ 1\end{array}\right]$ ?
- It collapses the points to the origin.
- Why can't the transformation matrix $\left[\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right]$ produce an image point of $\left[\begin{array}{l}4 \\ 1\end{array}\right]$ ?
- It produces only ordered pairs that have the same $x$ - and $y$-coordinates.
- Why can't the transformation matrix $\left[\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right]$ produce an image point of $\left[\begin{array}{l}4 \\ 1\end{array}\right]$ ? - It transforms all points to the $y$ - axis.
- What do you notice about the determinants of the examples that cannot produce the image point of $\left[\begin{array}{l}4 \\ 1\end{array}\right]$ ?
- They all have determinants of 0 .
- What have we learned about matrices that have a determinant of 0 ?
- They do not have inverses.
- If matrix $L$ represents a linear transformation, what does $L^{-1}$ represent geometrically?
- It represents undoing the transformation.
- So if $L$ represents a rotation $90^{\circ}$ clockwise about the origin, what would $L^{-1}$ represent?
- Rotation of $90^{\circ}$ counterclockwise or $270^{\circ}$ clockwise about the origin.


## Scaffolding:

- Advanced students can find additional examples of linear transformations that are not invertible.
- Advanced students could be asked to make conjectures about the properties of linear transformations that can/cannot produce the image point without being prompted about the determinant.
- When reviewing Example 2, select several pre-image points and multiply them by the transformation matrices. Ask students if they notice any patterns in the image points and to make a conjecture about whether they could produce the image point $\left[\begin{array}{l}4 \\ 1\end{array}\right]$.
- Then if $L$ does not have an inverse, what does that suggest geometrically?
- The transformation cannot be undone using another single transformation.
- How would this apply to the matrix $\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$ ?
- You cannot perform a single transformation to reverse the transformation.
- Looking at part (c), what does the equation $L\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}4 \\ 1\end{array}\right]$ represent geometrically?
- A linear transformation is applied to the point $\left[\begin{array}{l}x \\ y\end{array}\right]$ to produce the image point $\left[\begin{array}{l}4 \\ 1\end{array}\right]$.
- Explain geometrically how we could find the coordinates of $\left[\begin{array}{l}x \\ y\end{array}\right]$ given the equation $L\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}4 \\ 1\end{array}\right]$.
- You can undo the linear transformation represented by matrix $L$.
- What would this look like algebraically?
- $L^{-1} L\left[\begin{array}{l}x \\ y\end{array}\right]=L^{-1}\left[\begin{array}{l}4 \\ 1\end{array}\right]$.
- And this simplifies to? Explain.
- $\left[\begin{array}{l}x \\ y\end{array}\right]=L^{-1}\left[\begin{array}{l}4 \\ 1\end{array}\right]$ because $L^{-1} L=I$; it represents applying and then undoing the linear transformation.
- Does $L$ have an inverse? How can you tell?
- Yes. It has a determinant of 5, and all square matrices with nonzero determinants are invertible.
- Recall the formula that if $A=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$ where $a d-b c \neq 0$, then $A^{-1}=\frac{1}{a d-b c}\left[\begin{array}{cc}d & -b \\ -c & a\end{array}\right]$. Given this, find the pre-image point $\left[\begin{array}{l}x \\ y\end{array}\right]$.
- $\quad L^{-1}\left[\begin{array}{l}4 \\ 1\end{array}\right]=\frac{1}{2(1)-(-1)(3)}\left[\begin{array}{cc}1 & 1 \\ -3 & 2\end{array}\right]\left[\begin{array}{l}4 \\ 1\end{array}\right]=\frac{1}{5}\left[\begin{array}{c}5 \\ -10\end{array}\right]=\left[\begin{array}{c}1 \\ -2\end{array}\right]$
- How else could you have found the coordinates of $\left[\begin{array}{l}x \\ y\end{array}\right]$ without using inverse matrices?
- Multiply the matrices on the left side of the equation $\left[\begin{array}{cc}2 & -1 \\ 3 & 1\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}4 \\ 1\end{array}\right]$ to create the system of equations:

$$
\begin{aligned}
& 2 x-y=4 \\
& 3 x+y=1
\end{aligned}
$$

- How can we verify that the solution we found using inverse matrices is correct?
- Substitute the ordered pair $\left[\begin{array}{c}1 \\ -2\end{array}\right]$ into both equations, and verify that the resulting number sentences are true.
- What does our solution mean geometrically?
- If you apply the transformation represented by the matrix $\left[\begin{array}{cc}2 & -1 \\ 3 & 1\end{array}\right]$ to the point $\left[\begin{array}{c}1 \\ -2\end{array}\right]$, the image point will have coordinates of $\left[\begin{array}{l}4 \\ 1\end{array}\right]$.
- What do you notice about the relationship between the system of equations and the transformation matrix $L$ ? Explain your observations to your partner.
- The entries in the matrix represent the coefficients of the system when both equations are written in standard form.
- Why might it be useful to learn the technique of using inverse matrix operations to solve systems rather than using algebraic methods like substitution or elimination? Discuss your ideas with your partner.
- Algebraic methods can become messy and cumbersome with increasing numbers of equations and variables.


## Example 1

a. Describe a transformation not already discussed that results in an image point of $\left[\begin{array}{l}4 \\ 1\end{array}\right]$, and represent the transformation using a $2 \times 2$.
Answers will vary. An example of an appropriate response is as follows: A rotation of the point $\left[\begin{array}{c}-1 \\ 4\end{array}\right] 90^{\circ}$ to the point $\left[\begin{array}{l}4 \\ 1\end{array}\right]$ can be represented with the matrix $\left[\begin{array}{cc}0 & 1 \\ -1 & 0\end{array}\right]$.
b. Determine whether any of the matrices listed represent linear transformations that can produce the image point $\left[\begin{array}{l}4 \\ 1\end{array}\right]$. Justify your answers by describing the transformations represented by the matrices.
i. $\quad\left[\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right]$

This matrix represents a collapse to the origin, so it cannot produce the image point $\left[\begin{array}{l}4 \\ 1\end{array}\right]$.
ii. $\quad\left[\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right]$

This matrix represents a transformation to the diagonal defined by $y=x$, so it cannot produce the image point $\left[\begin{array}{l}4 \\ 1\end{array}\right]$.
iii. $\left[\begin{array}{ll}0 & 0 \\ 0 & 1\end{array}\right]$

This matrix represents a transformation to the $y$-axis, so it cannot produce the image point $\left[\begin{array}{l}4 \\ 1\end{array}\right]$.
c. Suppose a linear transformation $L$ is represented by the matrix $\left[\begin{array}{cc}2 & -1 \\ 3 & 1\end{array}\right]$. Find a point $L\left[\begin{array}{l}x \\ y\end{array}\right]$ so that $L\left[\begin{array}{l}x \\ y\end{array}\right]=$ $\left[\begin{array}{l}4 \\ 1\end{array}\right]$.

$$
\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{c}
1 \\
-2
\end{array}\right]
$$

## Exercises 1-4 (11 minutes)

Allow students to complete the work in pairs. They should both complete the work and then compare answers. Some pairs may need more help or additional instruction. After a few minutes, discuss Exercise 1 to ensure that students are able to represent the systems of equations as linear transformations using matrix notation. Exercise 3 will be discussed in detail to summarize the main points of the lesson. Exercise 4 is a challenge exercise.

## Exercises 1-4

1. Given the system of equations

$$
\begin{aligned}
& 2 x+5 y=4 \\
& 3 x-8 y=-25
\end{aligned}
$$

a. Show how this system can be written as a statement about a linear transformation of the form $L \boldsymbol{x}=\boldsymbol{b}$, with $x=\left[\begin{array}{l}x \\ y\end{array}\right]$ and $b=\left[\begin{array}{c}4 \\ -25\end{array}\right]$.

$$
\left[\begin{array}{cc}
2 & 5 \\
3 & -8
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{c}
4 \\
-25
\end{array}\right]
$$

b. Determine whether $L$ has an inverse. If it does, compute $L^{-1} b$, and verify that the coordinates represent the solution to the system of equations.

$$
\begin{aligned}
L^{-1} b & =\frac{1}{2(-8)-(5)(3)}\left[\begin{array}{cc}
-8 & -5 \\
-3 & 2
\end{array}\right] \\
L^{-1} b & =\frac{1}{-31}\left[\begin{array}{cc}
-8 & -5 \\
-3 & 2
\end{array}\right]\left[\begin{array}{c}
4 \\
-25
\end{array}\right]=\frac{1}{-31}\left[\begin{array}{c}
93 \\
-62
\end{array}\right]=\left[\begin{array}{c}
-3 \\
2
\end{array}\right]
\end{aligned}
$$

Verification using back substitution:

$$
\begin{aligned}
& 2(-3)+5(2)=4 \\
& 3(-3)-8(2)=-25
\end{aligned}
$$

2. The path of a piece of paper carried by the wind into a tree can be modeled with a linear transformation, where $L=\left[\begin{array}{cc}3 & -4 \\ 5 & 3\end{array}\right]$ and $b=\left[\begin{array}{c}6 \\ 10\end{array}\right]$.
a. Write an equation that represents the linear transformation of the piece of paper.

$$
\left[\begin{array}{cc}
3 & -4 \\
5 & 3
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{c}
6 \\
10
\end{array}\right]
$$

b. Solve the equation from part (a).

$$
\left[\begin{array}{l}
x \\
y
\end{array}\right]=\frac{1}{29}\left[\begin{array}{cc}
3 & 4 \\
-5 & 3
\end{array}\right]\left[\begin{array}{c}
6 \\
10
\end{array}\right]=\frac{1}{29}\left[\begin{array}{c}
58 \\
0
\end{array}\right]=\left[\begin{array}{l}
2 \\
0
\end{array}\right]
$$

c. Use your solution to provide a reasonable interpretation of the path of the piece of paper under the transformation by the wind.

Answers will vary. An example of an appropriate response would be that the piece of paper started on the ground 2 feet to the right of the location defined as the origin, and it was moved by the wind to a spot 6 feet to the right of the origin and 10 feet above the ground (in the tree).
3. For each system of equations, write the system as a linear transformation represented by a matrix and apply inverse matrix operations to find the solution, or explain why this procedure cannot be performed.
a. $6 x+2 y=1$
$y=3 x+1$

$$
\begin{aligned}
& {\left[\begin{array}{cc}
6 & 2 \\
-3 & 1
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right] }=\left[\begin{array}{l}
1 \\
1
\end{array}\right] \\
& {\left[\begin{array}{l}
x \\
y
\end{array}\right]=\frac{1}{12}\left[\begin{array}{cc}
1 & -2 \\
3 & 6
\end{array}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right]=\frac{1}{12}\left[\begin{array}{c}
-1 \\
9
\end{array}\right]=\left[\begin{array}{c}
\frac{-1}{12} \\
\frac{3}{4}
\end{array}\right] }
\end{aligned}
$$ cori

b. $\quad \begin{aligned} 4 x-6 y & =10 \\ 2 x-3 y & =1\end{aligned}$

This system cannot be represented as a linear transformation because the transformation matrix $L$ has a determinant of 0 . The system represents parallel lines, so there is no solution.
4. In a two-dimensional plane, $A$ represents a rotation of $30^{\circ}$ counterclockwise about the origin, $B$ represents a reflection over the line $y=x$, and $C$ represents a rotation of $60^{\circ}$ counterclockwise about the origin.
a. Write matrices $A, B$, and $C$.

$$
A=\left[\begin{array}{cc}
\frac{\sqrt{3}}{2} & -\frac{1}{2} \\
\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{array}\right] B=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] C=\left[\begin{array}{cc}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{array}\right]
$$

b. Transformations $A, B$, and $C$ are applied to point $\left[\begin{array}{l}x \\ y\end{array}\right]$ successively and produce the image point $\left[\begin{array}{c}1+2 \sqrt{3} \\ 2-\sqrt{3}\end{array}\right]$. Use inverse matrix operations to find $\left[\begin{array}{l}x \\ y\end{array}\right]$.

We must apply the inverse transformations in the reverse order. The inverse of matrix $C$ is

$$
C^{-1}=1\left[\begin{array}{cc}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{array}\right]
$$

Applied to the image

$$
\left[\begin{array}{cc}
\frac{1}{2} & \frac{\sqrt{3}}{2} \\
-\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{array}\right]\left[\begin{array}{c}
1+2 \sqrt{3} \\
2-\sqrt{3}
\end{array}\right]=\left[\begin{array}{c}
-1+2 \sqrt{3} \\
-2-\sqrt{3}
\end{array}\right]
$$

The inverse of matrix $B$ is $B^{-1}=-1\left[\begin{array}{cc}0 & -1 \\ -1 & 0\end{array}\right]$. Applied to the image $\left[\begin{array}{ll}0 & 1 \\ 1 & 0\end{array}\right]\left[\begin{array}{c}-1+2 \sqrt{3} \\ -2-\sqrt{3}\end{array}\right]=\left[\begin{array}{c}-2-\sqrt{3} \\ -1+2 \sqrt{3}\end{array}\right]$.
The inverse of matrix $A$ is

$$
A^{-1}=1\left[\begin{array}{cc}
\frac{\sqrt{3}}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{array}\right]
$$

Applied to the image

$$
\left[\begin{array}{cc}
\frac{\sqrt{3}}{2} & \frac{1}{2} \\
-\frac{1}{2} & \frac{\sqrt{3}}{2}
\end{array}\right]\left[\begin{array}{c}
-2-\sqrt{3} \\
-1+2 \sqrt{3}
\end{array}\right]=\left[\begin{array}{c}
-2 \\
4
\end{array}\right]
$$

## Closing ( 6 minutes)

As a class, discuss the results of Exercises 1-3, focusing on how to represent systems of two linear equations as a linear transformation represented by matrix multiplication.

- How does the format of a linear system affect how it can be written as a linear transformation?
- The equations need to be written in standard form so the coefficients are represented accurately in the transformation matrix $L$.
- What is the geometric interpretation of your conclusion to Exercise 3, part (b)?
- There are no points in the coordinate plane that, when the transformation represented by the matrix $\left[\begin{array}{ll}4 & -6 \\ 2 & -3\end{array}\right]$ is applied, will result in an image point of $\left[\begin{array}{c}10 \\ 1\end{array}\right]$.
- How are problems involving linear transformations solved in the coordinate plane?
- Systems of equations with a single solution can be represented as a linear transformation in the form of the equation $L x=b$, where $L$ is the transformation matrix, $x=\left[\begin{array}{l}x \\ y\end{array}\right]$ and $b=$ the coordinates of the image.
- The coordinates of the pre-image can be found by multiplying both sides of the transformation equation by $L^{-1}$, which effectively undoes the transformation.
- When the determinant of $L$ is 0 , the linear transformation matrix does not have an inverse, and there is no solution to the system of equations whose coefficients are represented by $L$.


## Exit Ticket ( 5 minutes)

Name $\qquad$ Date $\qquad$

# Lesson 14: Solving Equations Involving Linear Transformations of the Coordinate Plane 

## Exit Ticket

In two-dimensional space, point $x$ is rotated $180^{\circ}$ to the point $\left[\begin{array}{l}3 \\ 4\end{array}\right]$.
a. Represent the transformation of point $x$ using an equation in the format $L x=b$.
b. Use inverse matrix operations to find the coordinates of $x$.
c. Verify that this solution makes sense geometrically.

## Exit Ticket Sample Solutions

In two-dimensional space, point $x$ is rotated $180^{\circ}$ to the point $\left[\begin{array}{l}3 \\ 4\end{array}\right]$.
a. Represent the transformation of point $x$ using an equation in the format $L x=b$.

$$
\left[\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
3 \\
4
\end{array}\right]
$$

b. Use inverse matrix operations to find the coordinates of $x$.

$$
\left[\begin{array}{l}
x \\
y
\end{array}\right]=\frac{1}{1}\left[\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right]\left[\begin{array}{l}
3 \\
4
\end{array}\right]=\left[\begin{array}{l}
-3 \\
-4
\end{array}\right]
$$

c. Verify that this solution makes sense geometrically.

When a point is rotated $180^{\circ}$ about the origin in the coordinate plane, the $x$-and $y$-coordinates of the image point are the opposite of those of the pre-image point.

## Problem Set Sample Solutions

1. In a two-dimensional plane, a transformation represented by $L=\left[\begin{array}{cc}1 & 5 \\ 2 & -4\end{array}\right]$ is applied to point $x$, resulting in an image point $\left[\begin{array}{l}0 \\ 5\end{array}\right]$. Find the location of the point before it was transformed.
a. Write an equation to represent the linear transformation of point $x$.

$$
\left[\begin{array}{cc}
1 & 5 \\
2 & -4
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
0 \\
5
\end{array}\right]
$$

b. Solve the equation to find the coordinates of the pre-image point.

$$
\begin{aligned}
& {\left[\begin{array}{l}
x \\
y
\end{array}\right]=\frac{-1}{14}\left[\begin{array}{cc}
-4 & -5 \\
-2 & 1
\end{array}\right]\left[\begin{array}{l}
0 \\
5
\end{array}\right]} \\
& {\left[\begin{array}{l}
x \\
y
\end{array}\right]=\frac{-1}{14}\left[\begin{array}{c}
-25 \\
5
\end{array}\right]=\left[\begin{array}{c}
\frac{25}{14} \\
\frac{-5}{14}
\end{array}\right]}
\end{aligned}
$$

2. Find the location of the point $\left[\begin{array}{l}x \\ y\end{array}\right]$ before it was transformed when given:
a. The transformation $L=\left[\begin{array}{ll}3 & 5 \\ 1 & 2\end{array}\right]$ and the resultant is $\left[\begin{array}{l}1 \\ 2\end{array}\right]$. Verify your answer.

$$
\begin{aligned}
{\left[\begin{array}{ll}
3 & 5 \\
1 & 2
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\left[\begin{array}{l}
1 \\
2
\end{array}\right] \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\frac{1}{1}\left[\begin{array}{cc}
2 & -5 \\
-1 & 3
\end{array}\right]\left[\begin{array}{l}
1 \\
2
\end{array}\right] \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\left[\begin{array}{c}
-8 \\
5
\end{array}\right] \\
{\left[\begin{array}{ll}
3 & 5 \\
1 & 2
\end{array}\right]\left[\begin{array}{c}
-8 \\
5
\end{array}\right] } & =\left[\begin{array}{c}
1 \\
2
\end{array}\right]
\end{aligned}
$$

b. The transformation $L=\left[\begin{array}{cc}4 & 7 \\ -1 & -2\end{array}\right]$ and the resultant is $\left[\begin{array}{c}2 \\ -1\end{array}\right]$. Verify your answer.

$$
\begin{aligned}
{\left[\begin{array}{cc}
4 & 7 \\
-1 & -2
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\left[\begin{array}{c}
2 \\
-1
\end{array}\right] \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\frac{1}{-1}\left[\begin{array}{cc}
-2 & -7 \\
1 & 4
\end{array}\right]\left[\begin{array}{c}
2 \\
-1
\end{array}\right] \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =-1\left[\begin{array}{c}
3 \\
-2
\end{array}\right]=\left[\begin{array}{c}
-3 \\
2
\end{array}\right] \\
{\left[\begin{array}{cc}
4 & 7 \\
-1 & -2
\end{array}\right]\left[\begin{array}{c}
-3 \\
2
\end{array}\right] } & =\left[\begin{array}{c}
2 \\
-1
\end{array}\right]
\end{aligned}
$$

c. The transformation $L=\left[\begin{array}{cc}0 & -1 \\ 2 & 1\end{array}\right]$ and the resultant is $\left[\begin{array}{l}1 \\ 3\end{array}\right]$. Verify your answer.

$$
\begin{aligned}
{\left[\begin{array}{cc}
0 & -1 \\
2 & 1
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\left[\begin{array}{l}
1 \\
3
\end{array}\right] \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\frac{1}{2}\left[\begin{array}{cc}
1 & 1 \\
-2 & 0
\end{array}\right]\left[\begin{array}{c}
1 \\
3
\end{array}\right] \\
{\left[\begin{array}{c}
x \\
y
\end{array}\right] } & =\frac{1}{2}\left[\begin{array}{c}
4 \\
-2
\end{array}\right]=\left[\begin{array}{c}
2 \\
-1
\end{array}\right] \\
{\left[\begin{array}{cc}
0 & -1 \\
2 & 1
\end{array}\right]\left[\begin{array}{c}
2 \\
-1
\end{array}\right] } & =\left[\begin{array}{l}
1 \\
3
\end{array}\right]
\end{aligned}
$$

d. The transformation $L=\left[\begin{array}{cc}2 & 3 \\ 0 & -1\end{array}\right]$ and the resultant is $\left[\begin{array}{l}3 \\ 0\end{array}\right]$. Verify your answer.

$$
\begin{aligned}
{\left[\begin{array}{cc}
2 & 3 \\
0 & -1
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\left[\begin{array}{l}
3 \\
0
\end{array}\right] \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\frac{1}{-2}\left[\begin{array}{cc}
-1 & -3 \\
0 & 2
\end{array}\right]\left[\begin{array}{l}
3 \\
0
\end{array}\right] \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\frac{1}{-2}\left[\begin{array}{c}
-3 \\
0
\end{array}\right]=\left[\begin{array}{l}
3 \\
2 \\
0
\end{array}\right] \\
{\left[\begin{array}{cc}
2 & 3 \\
0 & -1
\end{array}\right]\left[\begin{array}{l}
3 \\
2 \\
0
\end{array}\right] } & =\left[\begin{array}{l}
3 \\
0
\end{array}\right]
\end{aligned}
$$

e. The transformation $L=\left[\begin{array}{cc}2 & -1 \\ 1 & 2\end{array}\right]$ and the resultant is $\left[\begin{array}{l}3 \\ 2\end{array}\right]$. Verify your answer.

$$
\begin{aligned}
{\left[\begin{array}{cc}
2 & -1 \\
1 & 2
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\left[\begin{array}{l}
3 \\
2
\end{array}\right] \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\frac{1}{5}\left[\begin{array}{cc}
2 & 1 \\
-1 & 2
\end{array}\right]\left[\begin{array}{l}
3 \\
2
\end{array}\right] \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right] } & =\frac{1}{5}\left[\begin{array}{l}
8 \\
1
\end{array}\right]=\left[\begin{array}{l}
\frac{8}{5} \\
\frac{1}{5}
\end{array}\right] \\
{\left[\begin{array}{cc}
2 & -1 \\
1 & 2
\end{array}\right]\left[\begin{array}{l}
\frac{8}{5} \\
\frac{1}{5}
\end{array}\right] } & =\left[\begin{array}{l}
3 \\
2
\end{array}\right]
\end{aligned}
$$

3. On a computer assembly line, a robot is placing a CPU onto a motherboard. The robot's arm is carried out by the transformation $L=\left[\begin{array}{ll}2 & 3 \\ 1 & 2\end{array}\right]$.
a. If the CPU is attached to the motherboard at point $\left[\begin{array}{c}-2 \\ 3\end{array}\right]$, at what location does the robot pick up the CPU?

$$
\begin{gathered}
{\left[\begin{array}{ll}
2 & 3 \\
1 & 2
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{c}
-2 \\
3
\end{array}\right]} \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right]=\frac{1}{1}\left[\begin{array}{cc}
2 & -3 \\
-1 & 2
\end{array}\right]\left[\begin{array}{c}
-2 \\
3
\end{array}\right]=\left[\begin{array}{c}
-13 \\
8
\end{array}\right]}
\end{gathered}
$$

b. If the CPU is attached to the motherboard at point $\left[\begin{array}{l}3 \\ 2\end{array}\right]$, at what location does the robot pick up the CPU?

$$
\begin{gathered}
{\left[\begin{array}{ll}
2 & 3 \\
1 & 2
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
3 \\
2
\end{array}\right]} \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right]=\frac{1}{1}\left[\begin{array}{cc}
2 & -3 \\
-1 & 2
\end{array}\right]\left[\begin{array}{l}
3 \\
2
\end{array}\right]=\left[\begin{array}{l}
0 \\
1
\end{array}\right]}
\end{gathered}
$$

c. Find the transformation $L=\left[\begin{array}{cc}-1 & c \\ b & 3\end{array}\right]$ that will place the CPU starting at $\left[\begin{array}{c}2 \\ -3\end{array}\right]$ onto the motherboard at the location $\left[\begin{array}{c}-8 \\ 3\end{array}\right]$.

$$
\begin{gathered}
{\left[\begin{array}{cc}
-1 & c \\
b & 3
\end{array}\right]\left[\begin{array}{c}
2 \\
-3
\end{array}\right]=\left[\begin{array}{c}
-8 \\
3
\end{array}\right]} \\
-2-3 c=-8, c=2 \\
2 b-9=3, b=6 \\
{\left[\begin{array}{cc}
-1 & 2 \\
6 & 3
\end{array}\right]}
\end{gathered}
$$

4. On a construction site, a crane is moving steel beams from a truck bed to workers. The crane is programed to perform the transformation $L=\left[\begin{array}{ll}1 & 1 \\ 2 & 3\end{array}\right]$.
a. If the workers are at location $\left[\begin{array}{l}2 \\ 5\end{array}\right]$, where does the truck driver need to unload the steel beams so that the crane can pick them up and bring them to the workers?

$$
\begin{gathered}
{\left[\begin{array}{ll}
1 & 1 \\
2 & 3
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
2 \\
5
\end{array}\right]} \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right]=\frac{1}{1}\left[\begin{array}{cc}
3 & -1 \\
-2 & 1
\end{array}\right]\left[\begin{array}{l}
2 \\
5
\end{array}\right]=\left[\begin{array}{l}
1 \\
1
\end{array}\right]}
\end{gathered}
$$

b. If the workers move to another location $\left[\begin{array}{c}-3 \\ 1\end{array}\right]$, where does the truck driver need to unload the steel beams so that the crane can pick them up and bring them to the workers?

$$
\begin{gathered}
{\left[\begin{array}{ll}
1 & 1 \\
2 & 3
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{c}
-3 \\
1
\end{array}\right]} \\
{\left[\begin{array}{l}
x \\
y
\end{array}\right]=\frac{1}{1}\left[\begin{array}{cc}
3 & -1 \\
-2 & 1
\end{array}\right]\left[\begin{array}{c}
-3 \\
1
\end{array}\right]=\left[\begin{array}{c}
-10 \\
7
\end{array}\right]}
\end{gathered}
$$

5. A video game soccer player is positioned at $\left[\begin{array}{l}0 \\ 2\end{array}\right]$, where he kicks the ball. The ball goes into the goal, which is at point $\left[\begin{array}{c}10 \\ 0\end{array}\right]$. When the player moves to point $\left[\begin{array}{l}1 \\ 1\end{array}\right]$ and kicks the ball, he misses the goal. The ball lands at point $\left[\begin{array}{l}10 \\ -1\end{array}\right]$. What is the program/transformation $L=\left[\begin{array}{ll}a & c \\ b & d\end{array}\right]$ that this video soccer player uses?

$$
\begin{gathered}
{\left[\begin{array}{ll}
a & c \\
b & d
\end{array}\right]\left[\begin{array}{l}
0 \\
2
\end{array}\right]=\left[\begin{array}{c}
10 \\
0
\end{array}\right], 2 c=10, c=5.2 d=0, d=0} \\
{\left[\begin{array}{ll}
a & 5 \\
b & 0
\end{array}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right]=\left[\begin{array}{l}
10 \\
-1
\end{array}\right], a+5=10, a=5 . b=-1} \\
{\left[\begin{array}{cc}
5 & 5 \\
-1 & 0
\end{array}\right]}
\end{gathered}
$$

6. Tim bought 5 shirts and 3 pair of pants, and it cost him $\$ 250$. Scott bought 3 shirts and 2 pair of pants, and it cost him $\$ 160$. All the shirts have the same cost, and all the pants have the same cost.
a. Write a system of linear equations to find the cost of the shirts and pants.

$$
\left\{\begin{array}{l}
5 S+3 P=250 \\
3 S+2 P=160
\end{array}\right.
$$

b. Show how this system can be written as a statement about a linear transformation of the form $L \boldsymbol{x}=\boldsymbol{b}$ with $x=\left[\begin{array}{l}S \\ P\end{array}\right]$ and $b=\left[\begin{array}{l}250 \\ 160\end{array}\right]$.

$$
\left[\begin{array}{ll}
5 & 3 \\
3 & 2
\end{array}\right]\left[\begin{array}{l}
S \\
P
\end{array}\right]=\left[\begin{array}{l}
250 \\
160
\end{array}\right]
$$

c. Determine whether $L$ has an inverse. If it does, compute $L^{-1} b$, and verify your answer to the system of equations.
The determinant of $\left[\begin{array}{ll}5 & 3 \\ 3 & 2\end{array}\right]$ is 1 . $L^{-1}=\frac{1}{1}\left[\begin{array}{cc}2 & -3 \\ -3 & 5\end{array}\right]=\left[\begin{array}{cc}2 & -3 \\ -3 & 5\end{array}\right]$

$$
\left[\begin{array}{l}
S \\
P
\end{array}\right]=\left[\begin{array}{cc}
2 & -3 \\
-3 & 5
\end{array}\right]\left[\begin{array}{l}
250 \\
160
\end{array}\right]=\left[\begin{array}{l}
20 \\
50
\end{array}\right]
$$

Verification using back substitution: $5(20)+3(50)=250,3(20)+2(50)=160$
7. In a two-dimensional plane, $A$ represents a reflection over the $x$-axis, $B$ represents a reflection over the $y$-axis, and $C$ represents a reflection over the line $y=x$.
a. Write matrices $A, B$, and $C$.

$$
A=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] B=\left[\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right] C=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]
$$

b. Write an equation for each linear transformation, assuming that each one produces an image point of $\left[\begin{array}{l}-2 \\ -3\end{array}\right]$. For transformation A,

$$
\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
-2 \\
-3
\end{array}\right]
$$

For transformation B,

$$
\left[\begin{array}{cc}
-1 & 0 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
-2 \\
-3
\end{array}\right]
$$

For transformation $C$,

$$
\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
-2 \\
-3
\end{array}\right]
$$

c. Use inverse matrix operations to find the pre-image point for each equation. Explain how your solutions make sense based on your understanding of the effect of each geometric transformation on the coordinates of the pre-image points.

For transformation $A,\left[\begin{array}{l}x \\ y\end{array}\right]=-1\left[\begin{array}{cc}-1 & 0 \\ 0 & 1\end{array}\right]\left[\begin{array}{l}-2 \\ -3\end{array}\right]=\left[\begin{array}{c}-2 \\ 3\end{array}\right]$. When a point is reflected over the $x$-axis, the $x$-coordinate remains unchanged, and the $y$-coordinate changes signs.

For transformation $B,\left[\begin{array}{l}x \\ y\end{array}\right]=-1\left[\begin{array}{cc}1 & 0 \\ 0 & -1\end{array}\right]\left[\begin{array}{c}-2 \\ -3\end{array}\right]=\left[\begin{array}{c}2 \\ -3\end{array}\right]$. When a point is reflected over the $y$-axis, the $y$-coordinate remains unchanged, and the $x$-coordinate changes signs.

For transformation $C,\left[\begin{array}{l}x \\ y\end{array}\right]=-1\left[\begin{array}{cc}0 & -1 \\ -1 & 0\end{array}\right]\left[\begin{array}{l}-2 \\ -3\end{array}\right]=\left[\begin{array}{l}-3 \\ -2\end{array}\right]$. When a point is reflected over the line $y=x$, the coordinates of the pre-image point are interchanged ( $x$ and $y$ are switched).
8. A system of equations is shown:

$$
\begin{aligned}
& 2 x+5 y+z=3 \\
& 4 x+y-z=5 \\
& 3 x+2 y+4 z=1
\end{aligned}
$$

a. Represent this system as a linear transformation in three-dimensional space represented by a matrix equation in the form of $L x=b$.

$$
\left[\begin{array}{ccc}
2 & 5 & 1 \\
4 & 1 & -1 \\
3 & 2 & 4
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{l}
3 \\
5 \\
1
\end{array}\right]
$$

b. What assumption(s) need to be made to solve the equation in part (a) for $x$.

To use inverse operations, we need to assume that $L$ has an inverse.
c. Use algebraic methods to solve the system.

Adding equations 1 and 2 gives $6 x+6 y=8$.
Adding 4 times equation 2 and equation 3 gives $19 x+6 y=21$.
Subtracting $(6 x+6 y=8)$ from $(19 x+6 y=21)$ gives $13 x=13$, so $x=1$.
Back substituting into $6 x+6 y=8$ gives $6 y=2$, or $y=\frac{1}{3}$.
Back substituting for $y$ and $x$ into the first equation gives $2(1)+5\left(\frac{1}{3}\right)+z=3$, so $z=-\frac{2}{3}$.

$$
x=\left[\begin{array}{c}
1 \\
\frac{1}{3} \\
\frac{-2}{3}
\end{array}\right]
$$

9. Assume

$$
L^{-1}=\frac{1}{78}\left[\begin{array}{ccc}
-6 & 18 & 6 \\
19 & -5 & -6 \\
-5 & -11 & 18
\end{array}\right]
$$

Use inverse matrix operations to solve the equation from Problem 8, part (a) for $x$. Verify that your solution is the same as the one you found in Problem 8, part (c).

$$
\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\frac{1}{78}\left[\begin{array}{ccc}
-6 & 18 & 6 \\
19 & -5 & -6 \\
-5 & -11 & 18
\end{array}\right]\left[\begin{array}{l}
3 \\
5 \\
1
\end{array}\right]=\left[\begin{array}{c}
1 \\
\frac{1}{3} \\
\frac{-2}{3}
\end{array}\right]
$$

which is the same solution found in part (c).

## Lesson 15: Solving Equations Involving Linear

 Transformations of the Coordinate Space
## Student Outcomes

- Students will represent systems of three or more simultaneous equations as a linear transformation in the form $A x=b$, where $A$ represents the linear transformation, $x$ represents the coordinates of the pre-image, and $b$ represents the coordinates of the image point.
- Students will apply inverse matrix operations to solve systems of linear equations with three equations.


## Lesson Notes

The students will expand on their previous work from Lesson 14 to represent more complicated systems of equations as linear transformations in coordinate space. They will use software to calculate inverse matrices for systems of degree 3 and will apply the inverse of the coefficient matrix to the linear transformation equation to solve systems. Students will model real-world situations that can be written as systems of equations and solve the systems in a specific context including finding the intersection points of lines, designing a card game, and determining the number of coaches needed for a school athletic program (MP.4).

## Classwork

## Opening Exercise (5 minutes)

The students should complete the Opening Exercise in pairs. They should find the answer independently and then verify the solution with a partner. One pair could display the algebraic solution process on the board while another pair displays the matrix method of solving the system.

$$
\begin{aligned}
& \text { Opening Exercise } \\
& \text { Mariah was studying currents in two mountain streams. She determined that five times the } \\
& \text { current in stream A was } 8 \text { feet per second stronger than twice the current in stream B. Another } \\
& \text { day she found that double the current in stream A plus ten times the current in stream B was } 3 \\
& \text { feet per second. She estimated the current in stream A to be } 1.5 \text { feet per second and stream B to } \\
& \text { be almost still ( } 0 \text { feet per second). Was her estimate reasonable? Explain your answer after } \\
& \text { completing parts (a)-(c). } \\
& \text { arite a system of equations to model this problem. } \\
& \qquad \begin{aligned}
5 x=8+2 y \\
\text { a. Wray }
\end{aligned} \\
& \qquad \begin{array}{l}
2 x+10 y
\end{array}
\end{aligned}
$$

## Scaffolding:

Provide a simpler system:

$$
\begin{aligned}
& x+2 y=-1 \\
& 2 x-y=3
\end{aligned}
$$

Use scaffolded questions such as:

- Is $(1,2)$ a solution to this system of equations? Explain how you know.
- Is $\left(\frac{43}{27},-\frac{1}{54}\right)$ a solution to this system of equations? Explain how you know.
b. Solve the system using algebra.

Procedures may vary. An example of an appropriate algebraic procedure is shown:
Multiply the first equation by 5: $5(5 x-2 y=8) \rightarrow 25 x-10 y=40$
Add this equation to the second equation:

$$
\begin{aligned}
25 x-10 y & =40 \\
+2 x+10 y & =3 \\
27 x & =43 \\
x & =\frac{43}{27}
\end{aligned}
$$

Back substitute the $x$-value into the first equation and isolate $y$.

$$
\begin{aligned}
5 \cdot \frac{43}{27}-2 y & =8 \\
-2 y & =\frac{1}{27} \\
y & =-\frac{1}{54}
\end{aligned}
$$

Solution:

$$
\left[\begin{array}{r}
\frac{43}{27} \\
-\frac{1}{54}
\end{array}\right]
$$

c. $\quad$ Solve the system by representing it as a linear transformation of the point $x$ and then applying the inverse of the transformation matrix $L$ to the equation. Verify that the solution is the same as that found in part (b).

$$
\begin{aligned}
& {\left[\begin{array}{ll}
5 & -2 \\
2 & 10
\end{array}\right]\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
8 \\
3
\end{array}\right]} \\
& {\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{ll}
5 & -2 \\
2 & 10
\end{array}\right]^{-1}\left[\begin{array}{l}
8 \\
3
\end{array}\right]} \\
& \qquad\left[\begin{array}{l}
x \\
y
\end{array}\right]=\frac{1}{54}\left[\begin{array}{ll}
10 & 2 \\
-2 & 5
\end{array}\right]\left[\begin{array}{l}
8 \\
3
\end{array}\right]=\left[\begin{array}{r}
\frac{43}{27} \\
-\frac{1}{54}
\end{array}\right]
\end{aligned}
$$

which is the same solution as in part (b).
Mariah's estimate was reasonable because $x=\frac{43}{27} \approx 1.59 \mathrm{ft} / \mathrm{sec}$ and $y=-\frac{1}{54} \approx-0.02 \mathrm{ft} / \mathrm{sec}$. Since the signs are opposite, the currents are moving in opposite directions.

## Discussion (3 minutes)

- Do you prefer the algebraic or matrix method for solving systems of two linear equations? Explain why you prefer one method over the other. Share your thoughts with a partner.
- Answers will vary. An example of an appropriate response would be to state a preference for the algebraic method because it does not require calculating the inverse of the transformation matrix.
- Describe when it might be more efficient to use algebra to solve systems of equations. Discuss your ideas with your partner.
- Answers will vary. An example of an appropriate response would be that it would be more efficient to use algebra to solve systems of equations when there are a few equations or when it is clear that the system is either inconsistent or dependent.
- Describe when it might be more efficient to use matrices to solve systems of equations. Discuss your ideas with your partner.
- Answers will vary. An example of an appropriate response would be that using matrices might be more efficient to solve systems of equations when the system consists of several equations with several variables.


## Scaffolding:

- Students above grade level can complete the entire example with a partner with no leading questions from the teacher.
- Struggling students may need help setting up equations. Consider giving them questions to lead to the system such as:
Look at the first hand. If $x$ is the value of green cards, $y$ the value of yellow cards, and $z$ the value of blue cards, let's write an equation for just the first hand.

Example 1 (12 minutes)
Students should complete parts (a)-(c) in pairs during the first few minutes. One pair could display their procedure and algebraic solution on the board, while another pair displays the linear transformation equation. Alternatively, each pair could display first their algebraic solution and then the transformation equation on white boards for a quick check. For part (d), a free software program or graphing calculator app should be used to demonstrate how to input matrices and calculate their inverses. The program should be projected on the board for student viewing. Alternatively, the screen of a graphing calculator could be projected to demonstrate the use of technology to calculate matrix inverses. Part (e) should be completed as a teacher-led exercise. The calculation could first be performed using technology, and then students could multiply $A^{-1}$ by hand to verify that the software calculation is accurate.

- What features of the system of equations lend it to being solved using algebra?
- Answers will vary but might include that elimination could be used to find the value of the variables quickly because some of the equations in the system have corresponding coefficients that are either identical or opposites, which means that a variable could be eliminated by adding or subtracting two of the equations as they are written. Also, the values of the variables are integers, which facilitate computations when performing back substitution.
- How did you find matrix $L$ ?
- The matrix is constructed of the coefficients of the variables of the system.
- Can you always construct matrix $L$ using the coefficients of the variables in the order in which they appear in the system? Why?
- No. The equations must be written in standard form first.
- When we looked at two-dimensional space, how did we represent a system of equations using matrices?
- We wrote the system using the equation $L x=b$.
- What did the equation represent?
- A linear transformation of the point $x$ to the image point $b$.
- Could all systems be represented as a linear transformation of a point to a desired image point in twodimensional space?
- No
- How can you tell when systems could be solved using the linear transformation matrix equation?
- The determinant of the matrix $L$ must be nonzero.
- How did we solve the equation $L x=b$ when the determinant of $L$ is nonzero?
- We applied $L^{-1}$ to both sides of the equation.
- What is the result?
- $\quad x=L^{-1} b$
- And what does $x$ represent geometrically?
- The pre-image point that, once $L$ is applied to it, results in image point $b$.
- Why can't we solve the equation $L x=b$ if the determinant is 0 ?
- $\quad L$ does not have an inverse when the determinant is 0 .
- What does this imply geometrically?
- The transformation $L$ cannot be reversed to produce point $x$.
- Now, for three-dimensional space, how can we represent a system of equations as a linear transformation of a point?
- The same way that we represent two-dimensional systems, with the equation $L x=b$.
- And what condition do you think should be met to find the point $x$ using this equation?
- The determinant must be nonzero.
- Right. And how can you determine if a $3 \times 3$ matrix has a nonzero determinant?
- Answers may vary but might include using the diagonal method as shown:

$$
\begin{aligned}
& \text { If } L=\left[\begin{array}{lll}
l_{1,1} & l_{1,2} & l_{1,3} \\
l_{2,1} & l_{2,2} & l_{2,3} \\
l_{3,1} & l_{3,2} & l_{3,3}
\end{array}\right],\left|\begin{array}{lll}
l_{1,1} & l_{1,2} & l_{1,3} \\
l_{2,1} & l_{2,2} & l_{2,3} \\
l_{3,1} & l_{3,2} & l_{3,3}
\end{array}\right|=l_{1,1} l_{2,2} l_{3,3}+l_{1,2} l_{2,3} l_{3,1}+l_{1,3} l_{2,1} l_{3,2}-l_{1,2} l_{2,1} l_{3,3}- \\
& l_{1,1} l_{2,3} l_{3,2}-l_{1,3} l_{2,2} l_{3,1}
\end{aligned}
$$

- How can we use technology to determine if a matrix $L$ has an inverse?
- Answers will vary but will probably include entering the matrix into the software program and applying the inverse function to it.
- How do we know matrix $L$ for the system in our example has an inverse?
- We can calculate it directly using the software.
- How can we use technology to solve the equation $L x=b$ ?
- Why does it make sense that our solution to part (e) was the same as the solution we found in part (b)?
- In part (e) we calculated $L^{-1} b$, which is equal to $x$, and $x$ represents the point $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$, which is the solution to the system.
- In this problem, which method of solving did you prefer and why? Discuss your reasoning with a partner.
- Answers will vary but might include that the algebraic method was preferable for this system because the elimination method was easy to use given the coefficients of the variables in the equations.
- What factors would you consider when trying to determine whether to solve a system of equations in threedimensional space using algebraic methods versus using matrices? Share your ideas with your partner.
- Answers may vary but might include the size and relationship between the corresponding coefficients in the equations, e.g., whether elimination could be used easily to find the values of the variables.


## Example 1

Dillon is designing a card game where different colored cards are assigned point values. Kryshna is trying to find the value of each colored card. Dillon gives him the following hints. If I have 3 green cards, 1 yellow card, and 2 blue cards in my hand, my total is 9 . If I discard 1 blue card, my total changes to 7 . If I have $\mathbf{1}$ card of each color (green, yellow, and blue), my cards total 1.
a. Write a system of equations for each hand of cards if $x=$ value of green cards, $y=$ value of yellow cards, and $z=$ value of blue cards.

$$
\begin{array}{r}
3 x+y+2 z=9 \\
3 x+y+z=7 \\
x+y+z=1
\end{array}
$$

b. Solve the system using any method you choose.

Answers will vary. An example of an appropriate response is shown.
Subtract the second equation from the first.

$$
\begin{aligned}
3 x \mp y+2 z & =9 \\
-(3 x \mp y+z & =7) \\
z & =2
\end{aligned}
$$

Subtract the third equation from the second.

$$
\begin{aligned}
3 x \mp y+z & =7 \\
-(x+y+z & =1) \\
2 x & =6 \\
x & =3
\end{aligned}
$$

Back substitute values of $x$ and $z$ into the first equation, and isolate $y$.

$$
\begin{aligned}
3(3) \mp y+2(2) & =9 \\
13-y & =9 \\
y & =-4
\end{aligned}
$$

Solution: $\left[\begin{array}{c}3 \\ -4 \\ 2\end{array}\right]$ Green cards are worth 3 points, yellow cards -4 points, and blue cards 2 points.
c. Let $x=\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ and $b=\left[\begin{array}{l}9 \\ 7 \\ 1\end{array}\right]$. Find a matrix $L$ so that the linear transformation equation $L x=b$ would produce image coordinates that are the same as the solution to the system of equations.

$$
L=\left[\begin{array}{lll}
3 & 1 & 2 \\
3 & 1 & 1 \\
1 & 1 & 1
\end{array}\right]
$$

## Scaffolding:

- Provide students with a flow chart that outlines the steps to solving systems of equations using inverse matrix operations.
- Provide written directions or printed screen shots to students to aid them in entering matrices and calculating their inverses using software.
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d. Enter matrix $L$ into a software program or app, and try to calculate its inverse. Does $L$ have an inverse? If so, what is it?

$$
L^{-1}=\left[\begin{array}{ccc}
0 & 0.50 & -0.50 \\
-1 & 0.50 & 1.50 \\
1 & -1 & 0
\end{array}\right]
$$

e. Calculate $L^{-1}\left[\begin{array}{l}9 \\ 7 \\ 1\end{array}\right]$. Verify that the result is equivalent to the solution to the system you calculated in part (b). Why should the solutions be equivalent?

$$
\begin{aligned}
L^{-1}\left[\begin{array}{l}
9 \\
7 \\
1
\end{array}\right] & =\left[\begin{array}{ccc}
0 & 0.50 & -0.50 \\
-1 & 0.50 & 1.50 \\
1 & -1 & 0
\end{array}\right]\left[\begin{array}{l}
9 \\
7 \\
1
\end{array}\right] \\
{\left[\begin{array}{ccc}
0 & 0.50 & -0.50 \\
-1 & 0.50 & 1.50 \\
1 & -1 & 0
\end{array}\right]\left[\begin{array}{l}
9 \\
7 \\
1
\end{array}\right] } & =\left[\begin{array}{c}
3 \\
-4 \\
2
\end{array}\right]
\end{aligned}
$$

which is equivalent to the solution from part (b). This makes sense because the system of equations can be represented using the linear transformation equation $L x=b$. From what we learned in Lesson $14, x=L^{-1} b$, so the solution found using inverse matrix operations should be the same as the solution set when solving the system of equations using algebra.

## Exercises 1-3 (15 minutes)

The students should be placed into small groups. Each group should solve one of the problems. The group should spend about 5 minutes determining whether to solve the system using algebra or matrices, assisted by technology. They should solve the system using the method they chose and then verify the solution using back substitution. For the next few minutes, they should meet with the other groups assigned the same problem and verify their solutions, as well as discuss their arguments for the solution method they selected. During the last few minutes, each problem should be presented to the entire class. Presenters should display the problem, state the solution method chosen, justify the method selected, and display the solution. Each small group will need access to software that can be used to determine the inverse of matrices.

## Exercises 1-3

1. The system of equations is given:

$$
\begin{aligned}
& 2 x-4 y+6 z=14 \\
& 9 x-3 y+z=10 \\
& 5 x+9 z=1
\end{aligned}
$$

a. Solve the system using algebra or matrix operations. If you use matrix operations, include the matrices you entered into the software and the calculations you performed to solve the system.

Answers will vary. An appropriate response is included.
Matrix method:

## Scaffolding:

Challenge advanced students by having them write a system of equations which cannot be solved using inverse matrix operations, and have them explain the process they used in constructing their system.
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Solving Equations Involving Linear Transformations of the Coordinate
Space

$$
\begin{gathered}
A^{-1}=\left[\begin{array}{ccc}
-\frac{27}{340} & \frac{9}{85} & \frac{7}{170} \\
-\frac{19}{85} & -\frac{3}{85} & \frac{13}{85} \\
\frac{3}{68} & -\frac{1}{17} & \frac{3}{34}
\end{array}\right] \\
x=A^{-1} b=\left[\begin{array}{ccc}
-\frac{27}{340} & \frac{9}{85} & \frac{7}{170} \\
-\frac{19}{85} & -\frac{3}{85} & \frac{13}{85} \\
\frac{3}{68} & -\frac{1}{17} & \frac{3}{34}
\end{array}\right]\left[\begin{array}{c}
14 \\
10 \\
1
\end{array}\right]=\left[\begin{array}{c}
-\frac{1}{85} \\
-\frac{283}{85} \\
\frac{2}{17}
\end{array}\right]
\end{gathered}
$$

b. Verify your solution is correct.

$$
\begin{aligned}
2\left(-\frac{1}{85}\right)-4\left(-\frac{283}{85}\right)+6\left(\frac{2}{17}\right) & =14 \\
9\left(-\frac{1}{85}\right)-3\left(-\frac{283}{85}\right)+1\left(\frac{2}{17}\right) & =10 \\
5\left(-\frac{1}{85}\right)+9\left(\frac{2}{17}\right) & =1
\end{aligned}
$$

c. Justify your decision to use the method you selected to solve the system.

Answers will vary. An example of an appropriate response is shown: The coefficients of the variables in the system did not contain corresponding variables that were identical or opposites, which might indicate that elimination might be a slower method than using the matrix method.
2. An athletic director at an all-boys high school is trying to find out how many coaches to hire for the football, basketball, and soccer teams. To do this, he needs to know the number of boys that play each sport. He does not have names or numbers but finds a note with the following information listed:
The total number of boys on all three teams is $\mathbf{8 6}$.
The number of boys that play football is 7 less than double the total number of boys playing the other two sports. The number of boys that play football is 5 times the number of boys playing basketball.
a. Write a system of equations representing the number of boys playing each sport where $x$ is the number of boys playing football, $y$ basketball, and $z$ soccer.

$$
\begin{aligned}
x+y+z & =86 \\
2(y+z)-7 & =x \\
x & =5 y
\end{aligned}
$$

b. Solve the system using algebra or matrix operations. If you use matrix operations, include the matrices you entered into the software and the calculations you performed to solve the system.

Answers will vary. An appropriate response is shown.
Matrix method:

$$
\begin{aligned}
A & =\left[\begin{array}{ccc}
1 & 1 & 1 \\
-1 & 2 & 2 \\
1 & -5 & 0
\end{array}\right] b=\left[\begin{array}{c}
86 \\
7 \\
0
\end{array}\right] \\
A^{-1} & =\left[\begin{array}{ccc}
\frac{2}{3} & -\frac{1}{3} & 0 \\
\frac{2}{15} & -\frac{1}{15} & -\frac{1}{5} \\
\frac{1}{5} & \frac{2}{5} & \frac{1}{5}
\end{array}\right] \\
x & =A^{-1} b=\left[\begin{array}{ccc}
\frac{2}{3} & -\frac{1}{3} & 0 \\
\frac{2}{15} & -\frac{1}{15} & -\frac{1}{5} \\
\frac{1}{5} & \frac{2}{5} & \frac{1}{5}
\end{array}\right]\left[\begin{array}{c}
86 \\
7 \\
0
\end{array}\right]=\left[\begin{array}{l}
55 \\
11 \\
20
\end{array}\right]
\end{aligned}
$$

c. Verify that your solution is correct.

$$
\begin{aligned}
1(55)+1(11)+1(20) & =86 \\
2(11+20)-7 & =55 \\
55 & =5(11)
\end{aligned}
$$

d. Justify your decision to use the method you selected to solve the system.

Answers will vary. An example of an appropriate response is shown: Using technology to solve the system using matrices generally takes less time than using algebra, especially when the solution set contains fractions.
3. Kyra had $\$ \mathbf{2 0}, 000$ to invest. She decided to put the money into three different accounts earning $\mathbf{3} \%, \mathbf{5} \%$, and $7 \%$ simple interest respectively and earned a total of $\$ \mathbf{9 2 0 . 0 0}$ in interest. She invested half as much money at $7 \%$ as at 3\%. How much did she invest in each account?
a. Write a system of equations that models this situation.

$$
\begin{aligned}
x+y+z & =20000 \\
0.03 x+0.05 y+0.07 z & =920 \\
x-2 z & =0
\end{aligned}
$$

b. Find the amount invested in each account.

She invested $\$ 8,000$ at $3 \%, \$ 8,000$ at $5 \%$, and $\$ 4,000$ at $7 \%$.

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## Closing ( 5 minutes)

Have students summarize in writing the process of solving a problem using systems of equations in three-dimensional space and matrices. Students could individually write a bulleted list including important steps in the procedure along with suggestions for recognizing special cases and recommendations for when representing a system as a linear transformation and applying inverse matrix operations is an efficient method of finding the solution. Have students share their results with a partner, and if time permits, students can display a compilation of their bulleted ideas on chart paper or on the board.

- Create a system of equations.
- Systems of equations in three-dimensional space can be represented as linear transformations using the matrix equation $A x=b$, where $A$ represents the linear transformation of point $x$ and $b$ represents the image point after the linear transformation.
- The entries of matrix $A$ are the coefficients of the system, and $b$ is the column matrix representing the constants for the system.
- By applying inverse matrix operations, the solution to the system can be found by calculating $A^{-1} b$.
- Software programs can be used to calculate the inverse matrices for $3 \times 3$ systems and can be an efficient way to solve systems in three-dimensional space.
- As with systems in two-dimensional space, systems in three-dimensional space can only be represented as linear transformations of $x$ when $A$ has an inverse, e.g., the determinant of $A$ is nonzero.
- If the determinant of $A$ is 0 , there is no single solution to the system.
- When the coefficients of corresponding variables are not identical or opposites, which expedite the process of elimination, it may be quicker to use matrices to solve the system rather than algebraic methods.


## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

# Lesson 15: Solving Equations Involving Linear Transformations of the Coordinate Space 

## Exit Ticket

The lemonade sales at a baseball game were described as follows:
The number of small lemonades purchased was the number of mediums sold plus double the number of larges sold.
The total number of all sizes sold was 70.
One and a half times the number of smalls purchased plus twice the number of mediums sold was 100 .

Use a system of equations and its matrix representation to determine the number of small, medium, and large lemonades sold.

Lesson 15:

## Exit Ticket Sample Solutions

The lemonade sales at a baseball game were described as follows:
The number of small lemonades purchased was the number of mediums sold plus double the number of larges sold.
The total number of all sizes sold was 70 .
One and a half times the number of smalls purchased plus twice the number of mediums sold was 100.

Use a system of equations and its matrix representation to determine the number of small, medium, and large lemonades sold.

$$
\begin{gathered}
2 l+m=s \\
s+m+l=70 \\
1.5 s+2 m=100 \\
{\left[\begin{array}{ccc}
-1 & 1 & 2 \\
1 & 1 & 1 \\
1.5 & 2 & 0
\end{array}\right]\left[\begin{array}{c}
s \\
m \\
l
\end{array}\right]=\left[\begin{array}{c}
0 \\
70 \\
100
\end{array}\right]} \\
\operatorname{det}(A)=6 \\
A^{-1}:\left[\begin{array}{ccc}
\frac{-4}{9} & \frac{8}{9} & \frac{-2}{9} \\
\frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \\
\frac{1}{9} & \frac{7}{9} & \frac{-4}{9}
\end{array}\right] \\
A^{-1} b:\left[\begin{array}{c}
40 \\
20 \\
10
\end{array}\right]
\end{gathered}
$$

Based on the system, projected sales are $\mathbf{4 0}$ small lemonades, 20 medium lemonades, and 10 large lemonades.

## Problem Set Sample Solutions

1. A small town has received funding to design and open a small airport. The airport plans to operate flights from three airlines. The total number of flights scheduled is $\mathbf{1 0 0}$. The airline with the greatest number of flights is planned to have double the sum of the flights of the other two airlines. The plan also states that the airline with the greatest number of flights will have 40 more flights than the airline with the least number of flights.
a. Represent the situation described with a system of equations. Define all variables.

$$
\begin{aligned}
a+b+c & =100 \\
a & =2(b+c) \\
a & =c+40
\end{aligned}
$$

$a=$ number of flights for the airline with the most flights
$b=$ number of flights for the airline with the second greatest number of flights
$c=$ number of flights for the airline with the least number of flights
Note: variables used may differ.

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b. Represent the system as a linear transformation using the matrix equation $A x=b$. Define matrices $A, x$, and b.

Equation:

$$
\left[\begin{array}{ccc}
1 & 1 & 1 \\
1 & -2 & -2 \\
1 & 0 & -1
\end{array}\right]\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right]=\left[\begin{array}{c}
100 \\
0 \\
40
\end{array}\right]
$$

A:

$$
\left[\begin{array}{ccc}
1 & 1 & 1 \\
1 & -2 & -2 \\
1 & 0 & -1
\end{array}\right]
$$

$x:$

$$
\left[\begin{array}{l}
\mathrm{a} \\
\mathrm{~b} \\
\mathrm{c}
\end{array}\right]
$$

$b:$
$\left[\begin{array}{c}100 \\ 0 \\ 40\end{array}\right]$
c. Explain how you can determine if the matrix equation has a solution without solving it.

Answers will vary but should indicate that technology can be used to verify that the matrix A has a nonzero determinant or that it has an inverse.
d. Solve the matrix equation for $x$.

$$
x=A^{-1} b=\left[\begin{array}{c}
66 \frac{2}{3} \\
6 \frac{2}{3} \\
26 \frac{2}{3}
\end{array}\right]
$$

e. Discuss the solution in context.

The solution indicates that the airlines, from greatest to least number of flights, should have $66 \frac{2}{3}, 26 \frac{2}{3}$, and $6 \frac{2}{3}$ flights, respectively. This does not make sense given that the number of flights must be a whole number. Therefore, the number of flights granted should be approximately 67, 27, and 7, which would satisfy the second and third conditions and would results in only 1 more than the total number of flights planned for the airport.
2. A new blockbuster movie opens tonight, and several groups are trying to buy tickets. Three types of tickets are sold: adult, senior (over 65), and youth (under 10). A groups of 3 adults, 2 youths, and 1 senior pays $\$ 54.50$ for their tickets. Another group of 6 adults and 12 youths pays $\$ 151.50$. A final group of 1 adult, 4 youths, and 1 senior pays $\$ 49.00$. What is the price for each type of ticket?
a. Represent the situation described with a system of equations. Define all variables.

$$
\begin{aligned}
3 a+2 y+1 s & =54.50 \\
6 a+12 y & =151.50 \\
1 a+4 y+1 s & =49
\end{aligned}
$$

$a=$ price of an adult ticket
$y=$ price of a youth ticket
$s=$ price of a senior ticket
Note: variables used may differ.

| Lesson 15: | Solving Equations Involving Linear Transformations of the Coordinate |
| :--- | :--- |
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b. Represent the system as a linear transformation using the matrix equation $A x=b$.

$$
A x=b\left[\begin{array}{ccc}
3 & 2 & 1 \\
6 & 12 & 0 \\
1 & 4 & 1
\end{array}\right]\left[\begin{array}{l}
a \\
y \\
s
\end{array}\right]=\left[\begin{array}{c}
54.50 \\
151.50 \\
49
\end{array}\right]
$$

c. Explain how you can determine if the matrix equation has a solution without solving it.

Answers will vary but should indicate that technology can be used to verify that the matrix A has a nonzero determinant or that it has an inverse.
d. Solve the matrix equation for $x$.

$$
x=A^{-1} b=\left[\begin{array}{c}
10.25 \\
7.50 \\
8.75
\end{array}\right]
$$

e. Discuss the solution in context.

An adult ticket costs \$10.25, a youth ticket costs \$7.50, and a senior ticket costs \$8.75.
f. How much would it cost your family to attend the movie?

Answers will vary.
3. The system of equations is given:

$$
\begin{gathered}
5 w-2 x+y+3 z=2 \\
4 w-x+6 y+2 z=0 \\
w-x-y-z=3 \\
2 w+7 x-3 y+5 z=12
\end{gathered}
$$

a. Write the system using a matrix equation in the form $A x=b$.

$$
\left[\begin{array}{cccl}
5 & -2 & 1 & 3 \\
4 & -1 & 6 & 2 \\
1 & -1 & -1 & -1 \\
2 & 7 & -3 & 5
\end{array}\right]\left[\begin{array}{l}
w \\
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{c}
2 \\
0 \\
3 \\
12
\end{array}\right]
$$

b. Write the matrix equation that could be used to solve for $x$. Then use technology to solve for $x$.

$$
\left[\begin{array}{l}
w \\
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{cccc}
5 & -2 & 1 & 3 \\
4 & -1 & 6 & 2 \\
1 & -1 & -1 & -1 \\
2 & 7 & -3 & 5
\end{array}\right]^{-1}\left[\begin{array}{c}
2 \\
0 \\
3 \\
12
\end{array}\right]=\left[\begin{array}{cccc}
\frac{-8}{159} & \frac{22}{159} & \frac{85}{159} & \frac{13}{159} \\
\frac{-38}{159} & \frac{25}{159} & \frac{46}{159} & \frac{22}{159} \\
\frac{-11}{106} & \frac{17}{106} & \frac{-9}{106} & \frac{-1}{53} \\
\frac{31}{106} & \frac{-19}{106} & \frac{-71}{106} & \frac{-2}{53}
\end{array}\right]\left[\begin{array}{c}
2 \\
0 \\
3 \\
12
\end{array}\right]=\left[\begin{array}{c}
\frac{395}{159} \\
\frac{326}{159} \\
\frac{-73}{106} \\
\frac{-199}{106}
\end{array}\right]
$$

| Lesson 15: | Solving Equations Involving Linear Transformations of the Coordinate |
| :--- | :--- |
|  | Space |
| Date: | $1 / 24 / 15$ |

c. Verify your solution using back substitution.

$$
\begin{gathered}
5\left(\frac{395}{159}\right)-2\left(\frac{326}{159}\right)+\frac{-73}{106}+3\left(\frac{-199}{106}\right)=2 \\
4\left(\frac{395}{159}\right)-\frac{326}{159}+6\left(\frac{-73}{106}\right)+2\left(\frac{-199}{106}\right)=0 \\
\frac{395}{159}-\frac{326}{159}-\frac{-73}{106}-\frac{-199}{106}=3 \\
2\left(\frac{395}{159}\right)+7\left(\frac{326}{159}\right)-3\left(\frac{-73}{106}\right)+5\left(\frac{-199}{106}\right)=12
\end{gathered}
$$

d. Based on your experience solving this problem and others like it in this lesson, what conclusions can you draw about the efficiency of using technology to solve systems of equations compared to using algebraic methods?

Answers will vary. An example of an appropriate response would be that, in general, as the number of equations and variables in a system increases, the more efficient it is to use matrices and technology to solve systems when compared to using algebraic methods.
4. In three-dimensional space, a point $x$ is reflected over the $x z$ plane resulting in an image point of $\left[\begin{array}{c}-3 \\ 1 \\ -2\end{array}\right]$.
a. Write the transformation as an equation in the form $A x=b$, where $A$ represents the transformation of point $x$ resulting in image point $b$.

$$
\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{c}
-3 \\
1 \\
-2
\end{array}\right]
$$

b. Use technology to calculate $\boldsymbol{A}^{\mathbf{- 1}}$.

$$
A^{-1}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

c. Calculate $\boldsymbol{A}^{-1} \boldsymbol{b}$ to solve the equation for x .

$$
x=\left[\begin{array}{l}
-3 \\
-1 \\
-2
\end{array}\right]
$$

d. Verify that this solution makes sense geometrically.

Reflecting a point over the $x z$ plane effectively reflects it over the plane $y=0$, which will change the sign of the $y$-coordinate and will leave the $x$-and $z$-coordinates unchanged.
5. Jamie needed money and decided it was time to open her piggy bank. She had only nickels, dimes, and quarters. The total value of the coins was $\$ \mathbf{8 5}$. 50. The number of quarters was 39 less than the number of dimes. The total value of the nickels and dimes was equal to the value of the quarters. How many of each type of coin did Jamie have? Write a system of equations and solve.

Let $x=$ the number of quarters, $y=$ the number of dimes, and $z=$ the number of nickels.
$0.25 x+0.10 y+0.05 z=85.50$
$x=y-39$
$0.05 z+0.10 y=0.25 x$
$x=171, y=210, z=435$
Jamie has 171 quarters, 210 dimes, and 435 nickels.

## Student Outcomes

- Students represent complicated systems of equations, including $4 \times 4$ and $5 \times 5$ systems of equations, using matrix equations in the form $A x=b$, where $A$ represents the coefficient matrix, $x$ is the solution to the system, and $b$ represents the constant matrix.
- Students use technology to calculate the inverse of matrices and use inverse matrix operations to solve complex systems of equations.


## Lesson Notes

In this lesson, students apply their understanding from Lesson 14 and Lesson 15 to systems that are higher than 3-by-3 and systems in two- and three-dimensional space that are more complicated than those presented in previous lessons. They discover that, while it is difficult to geometrically describe linear transformations in four- or higher-dimensional space, the mathematics behind representing systems of equations as linear transformations using matrices is valid for higher degree space. They apply this reasoning to represent complicated systems of equations using matrices and use technology to solve the systems.

## Classwork

Example 1 (15 minutes)
In this example, students see that a cubic function can be used to model scientific data comparing side length to volume (MP.4). Students should complete part (a) with a partner. After a few minutes, a selected pair should display its solution, demonstrating how to use substitution of the data points into the cubic equation to find the equations for the system. After verifying the correct system, the students should complete parts (b)-(d) with a partner. Technology that enables students to calculate the inverse of a matrix will be needed to complete part (c). Each student should write and solve the matrix equation independently and verify the answer with a partner, but students can work in small groups or pairs, especially if access to technology is limited. Part (e) should be completed as a teacher-led discussion. Students should be encouraged to critically assess the usefulness of the model. If time permits, students could create a twocolumn chart for display that lists positives and negatives for the model.

- In the problem, what do the ordered pairs represent?
- For each ordered pair, the first number is the greatest linear measurement of the irregularly-shaped object, and the second number represents the volume in cubic centimeters measured by water displacement.
- So, how could we use the information from the problem to write a system of equations?
- Substitute each ordered pair into the $v(x)$ equation, and simplify the resulting equations. Specifically, substitute the first coordinate for $x$ and the second coordinate for $v(x)$.
- How would this look for the first point?
- $3=a\left(1^{3}\right)+b\left(1^{2}\right)+c(1)+d$.
- Once we have performed substitution with all the ordered pairs, what are we left with?
- Four equations with four unknowns.
- And how can we represent our system of equations as a matrix equation?
- Use the equation $A x=b$, where $A$ represents the coefficient matrix for the system when all the equations are written in standard form, $x=\left[\begin{array}{l}a \\ b \\ c \\ d\end{array}\right]$ and $b$ represents the $v(x)$ values for the equations.
- How can we use the matrix equation to write a model for the scientist's data?
- Isolate $x$ by applying $A^{-1}$ to both sides of the matrix equation.
- Describe the measurements of the first object measured.
- Its greatest linear measurement is 1 cm , and its volume is 3 cubic centimeters.
- Why did the scientist select a cubic equation to model the data?
- He generalized that there is usually a cubic relationship between the linear dimensions of an object and its volume.
- How else could we determine whether a cubic equation would serve as a good model for the data?
- Answers will vary. Examples of appropriate responses would be that the points could be plotted to see if they seem to fit the pattern of a cubic function or that the data could be entered into a software program and a cubic regression performed.
- How can you find the inverse of matrix $A$ ?
- The students should mention the software or application that has been used in previous lessons to calculate the inverse of 3-by-3 matrices.
- Does $A^{-1}$ have an inverse? If so, what is it?
- Yes, it is $\left[\begin{array}{cccc}\frac{-1}{15} & \frac{1}{8} & \frac{-1}{12} & \frac{1}{40} \\ \frac{4}{5} & \frac{-11}{8} & \frac{3}{4} & \frac{7}{40} \\ \frac{-44}{15} & \frac{17}{4} & \frac{-5}{3} & \frac{7}{20} \\ 5 & \frac{-3}{1} & \frac{-1}{5}\end{array}\right]$.
- And how do we use the inverse matrix to solve the system?

$$
x=A^{-1} b=\left[\begin{array}{c}
0.175 \\
-1.225 \\
4.45 \\
-0.4
\end{array}\right]
$$

- How do we use the solution to the system to write a cubic equation to model the data?
- Substitute the solution for $a, b, c$, and $d$ in the equation. The cubic equation that models the data is $v(x)=0.175 x^{3}-1.225 x^{2}+4.45 x-0.4$.
- How well does this model represent the scientist's data?
- Well because all the points fit the model.
- Does this mean that a cubic model is the best model for this data?
- Not necessarily. It is possible that other models might be able to fit the data also.
- What are some limitations of the model?
- Answers will vary but might include that there are only four data points and no indication that they are based on repeated measurements.
- What could we recommend to the scientist if he wanted to strengthen his argument that a cubic model should represent the relationship between the greatest linear measure and the volume of the irregular objects?
- Answers will vary but might include having him take additional measurements and assess the fit of several types of models to the data set to see which model is the best fit.


## Example 1

A scientist measured the greatest linear dimension of several irregular metal objects. He then used water displacement to calculate the volume of each of the objects. The data he collected are $(1,3),(2,5),(4,9)$, and $(6,20)$, where the first coordinate represents the linear measurement of the object in centimeters, and the second coordinate represents the volume in cubic centimeters. Knowing that volume measures generally vary directly with the cubed value of linear measurements, he wants to try to fit this data to a curve in the form of $v(x)=a x^{3}+b x^{2}+c x+d$.
a. Represent the data using a system of equations.

$$
\begin{aligned}
3 & =a+b+c+d \\
5 & =8 a+4 b+2 c+d \\
9 & =64 a+16 b+4 c+d \\
20 & =216 a+36 b+6 c+d
\end{aligned}
$$

b. Represent the system using a matrix equation in the form $A x=b$.

$$
\left[\begin{array}{cccc}
1 & 1 & 1 & 1 \\
8 & 4 & 2 & 1 \\
64 & 16 & 4 & 1 \\
216 & 36 & 6 & 1
\end{array}\right]\left[\begin{array}{l}
a \\
b \\
c \\
d
\end{array}\right]=\left[\begin{array}{c}
3 \\
5 \\
9 \\
20
\end{array}\right]
$$

c. Use technology to solve the system.

$$
\left[\begin{array}{l}
a \\
b \\
c \\
d
\end{array}\right]=\left[\begin{array}{cccc}
1 & 1 & 1 & 1 \\
8 & 4 & 2 & 1 \\
64 & 16 & 4 & 1 \\
216 & 36 & 6 & 1
\end{array}\right]^{-1}\left[\begin{array}{c}
3 \\
5 \\
9 \\
20
\end{array}\right]=\left[\begin{array}{c}
0.175 \\
-1.225 \\
4.45 \\
-0.4
\end{array}\right]
$$

d. Based on your solution to the system, what cubic equation models the data?

$$
v(x)=0.175 x^{3}-1.225 x^{2}+4.45 x-0.4
$$

e. What are some of the limitations of the model?

It is based off of only four points, so the equation goes exactly through all the points.

## Exercises 1-3 (20 minutes)

Allow students to complete the work in pairs. They should each complete the work and then compare answers. Some pairs may need more help or additional instruction. After a few minutes, discuss Exercise 1 part (a) to ensure that students are able to represent the situation using a matrix. After about 5 minutes, Exercise 1 could be reviewed and discussed to help students who are struggling with understanding how to apply the matrix to find the solutions in parts (b) and (c). After the discussion, students could complete Exercise 2 in pairs or small groups. It might be necessary to model for students how to compute matrices to powers (e.g., how to use software to find $A^{30}$ ).

## Exercises 1-3

1. An attendance officer in a small school district noticed a trend among the four elementary schools in the district. This district used an open enrollment policy, which means any student within the district could enroll at any school in the district. Each year, $\mathbf{1 0} \%$ of the students from Adams Elementary enrolled at Davis Elementary, and 10\% of the students from Davis enrolled at Adams. In addition, 10\% of the students from Brown Elementary enrolled at Carson Elementary, and $\mathbf{2 0} \%$ of the students from Brown enrolled at Davis. At Carson Elementary, about 10\% of students enrolled at Brown, and 10\% enrolled at Davis, while at Davis, 10 \% enrolled at Brown, and 20\% enrolled at Carson. The officer noted that this year, the enrollment was 490,250,300, and 370 at Adams, Brown, Carson, and Davis, respectively.
a. Represent the relationship that reflects the annual movement of students among the elementary schools using a matrix.

$$
A=\left[\begin{array}{cccc}
0.9 & 0 & 0 & 0.1 \\
0 & 0.7 & 0.1 & 0.2 \\
0 & 0.1 & 0.8 & 0.1 \\
0.1 & 0.1 & 0.2 & 0.6
\end{array}\right]
$$

b. Write an expression that could be used to calculate the attendance one year prior to the year cited by the attendance officer. Find the enrollment for that year.

Expression $=A^{-1} b$, where

$$
\begin{gathered}
b=\left[\begin{array}{l}
490 \\
250 \\
300 \\
370
\end{array}\right] \\
A^{-1} b=\left[\begin{array}{l}
500 \\
200 \\
300 \\
400
\end{array}\right]
\end{gathered}
$$

Enrollment one year prior to cited data: 500 at Adams, 250 at Brown, 300 at Carson, and 400 at Davis.
c. Assuming that the trend in attendance continues, write an expression that could be used to calculate the enrollment two years after the year cited by the attendance officer. Find the attendance for that year.

Expression $=A^{2} b$

$$
A^{2} b=\left[\begin{array}{l}
465.8 \\
296.7 \\
305.1 \\
349.7
\end{array}\right]
$$

## Scaffolding:

- Define $A$ as applying the enrollment trend from one year to the next year. Then have students determine an expression to represent applying the enrollment trend for 2 years, 3 years, and $n$ years. Work with students to define $A^{-1}$ as representing applying the enrollment trend one year backwards.
- Work step-by-step through part (a) with struggling students. Then encourage them to use similar reasoning to what was applied in Exercise 1 parts (b) and (c) to find the values in Exercise 2 parts (b) and (c).
d. Interpret the results to part (c) in context.

The approximate enrollment at the schools would be 466, 297, 305, and 350 for Adams, Brown, Carson, and Davis, respectively.
2. Mrs. Kenrick is teaching her class about different types of polynomials. They have just studied quartics, and she has offered 5 bonus points to anyone in the class who can determine the quartic that she has displayed on the board.
The quartic has 5 points identified: $(-6,25),(-3,1),\left(-2, \frac{7}{3}\right),(0,-5)$, and $(3,169)$. Logan really needs those bonus points and remembers that the general form for a quartic is $y=a x^{4}+b x^{3}+c x^{2}+d x+e$. Can you help Logan determine the equation of the quartic?
a. Write the system of equations that would represent this quartic.

$$
\begin{aligned}
25 & =1296 a-216 b+36 c-6 d+e \\
1 & =81 a-27 b+9 c-3 d+e \\
\frac{7}{3} & =16 a-8 b+4 c-2 d+e \\
-5 & =e \\
169 & =81 a+27 b+9 c+3 d+e
\end{aligned}
$$

b. Write a matrix that would represent the coefficients of this quartic.

$$
A=\left[\begin{array}{ccccc}
1296 & -216 & 36 & -6 & 1 \\
81 & -27 & 9 & -3 & 1 \\
16 & -8 & 4 & -2 & 1 \\
0 & 0 & 0 & 0 & 1 \\
81 & 27 & 9 & 3 & 1
\end{array}\right]
$$

c. Write an expression that could be used to calculate coefficients of the equation.

Expression $=A^{-1} b$, where

$$
\begin{gathered}
b=\left[\begin{array}{c}
25 \\
1 \\
7 \\
3 \\
-5 \\
169
\end{array}\right] \\
A^{-1} b=\left[\begin{array}{c}
\frac{1}{3} \\
3 \\
7 \\
1 \\
-5
\end{array}\right]
\end{gathered}
$$

d. Explain the answer in the context of this problem.

$$
a=\frac{1}{3}, b=3, c=7, d=1, c=-5
$$

These are the coefficients of the quartic. The equation of the quartic is

$$
y=\frac{1}{3} x^{4}+3 x^{3}+7 x^{2}+x-5
$$

3. The Fibonacci numbers are the numbers $1,1,2,3,5,8,13,21,34, \ldots$. Each number beyond the second is the sum of the previous two.
Let $u_{1}=\left[\begin{array}{l}1 \\ 1\end{array}\right], u_{2}=\left[\begin{array}{l}1 \\ 2\end{array}\right], u_{3}=\left[\begin{array}{l}2 \\ 3\end{array}\right], u_{4}=\left[\begin{array}{l}3 \\ 5\end{array}\right], u_{5}=\left[\begin{array}{l}5 \\ 8\end{array}\right]$, and so on.
a. Show that $u_{n+1}=\left[\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right] u_{n}$.

If we define the terms in the Fibonacci sequence as $f_{n}$, where $n=1,2,3,4,5, \ldots$, then $u_{1}=\left[\begin{array}{l}f_{1} \\ f_{2}\end{array}\right], u_{2}=\left[\begin{array}{l}f_{2} \\ f_{3}\end{array}\right]$, $u_{3}=\left[\begin{array}{l}f_{3} \\ f_{4}\end{array}\right], u_{n}=\left[\begin{array}{c}f_{n} \\ f_{n+1}\end{array}\right]$, and $u_{n+1}=\left[\begin{array}{l}f_{n+1} \\ f_{n+2}\end{array}\right]$. By definition, $f_{n+2}=f_{n+1}+f_{n}$, so $u_{n+1}=\left[\begin{array}{c}f_{n+1} \\ f_{n+1}+f_{n}\end{array}\right]$. Now,
$\left[\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right] u_{n}=\left[\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right]\left[\begin{array}{c}f_{n} \\ f_{n+1}\end{array}\right]=\left[\begin{array}{c}f_{n+1} \\ f_{n+1}+f_{n}\end{array}\right]$, which is equivalent to $u_{n+1}$.
b. How could you use matrices to find $\boldsymbol{u}_{30}$ ? Use technology to find $\boldsymbol{u}_{30}$.

$$
u_{30}=\left[\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right]^{29}\left[\begin{array}{l}
1 \\
1
\end{array}\right]=\left[\begin{array}{c}
832040 \\
1346269
\end{array}\right]
$$

c. If $u_{n}=\left[\begin{array}{l}165580141 \\ 267914296\end{array}\right]$, find $u_{n-1}$. Show your work.

$$
u_{n-1}=\left[\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right]^{-1} u_{n}=\left[\begin{array}{c}
63245986 \\
102334155
\end{array}\right]
$$

## Closing (5 minutes)

Review Exercise 3 as a teacher-led discussion. Students should be encouraged to respond in writing to the questions provided:

- In what ways can matrix operations be useful in modeling real-world situations?
- Answers will vary but might include that they can be used to create models for data or to project trends forward and backward in time.
- In what ways can matrix operations be useful in representing mathematical relationships like the numbers in the Fibonacci sequence?
- Answers will vary but might include that matrix operations can be used to determine values that would be very cumbersome to calculate by hand.


## Exit Ticket (5 minutes)

Name
Date $\qquad$

## Lesson 16: Solving General Systems of Linear Equations

## Exit Ticket

1. Anabelle, Bryan, and Carl are playing a game using sticks of gum. For each round of the game, Anabelle gives half of her sticks of gum to Bryan and one-fourth to Carl. Bryan gives one-third of his sticks to Anabelle and keeps the rest. Carl gives 40 percent of his sticks of gum to Anabelle and 10 percent to Bryan. Sticks of gum can be cut into fractions when necessary.
a. After one round of the game, the players count their sticks of gum. Anabelle has 525 sticks of gum, Bryan has 600 , and Carl has 450 . How many sticks of gum will each player have after 2 more rounds of the game? Use a matrix equation to represent the situation, and explain your answer in context.
b. How many sticks of gum did each player have at the start of the game? Use a matrix equation to represent the situation, and explain your answer in context.

## Exit Ticket Sample Solutions

1. Anabelle, Bryan, and Carl are playing a game using sticks of gum. For each round of the game, Anabelle gives half of her sticks of gum to Bryan and one-fourth to Carl. Bryan gives one-third of his sticks to Anabelle and keeps the rest. Carl gives 40 percent of his sticks of gum to Anabelle and 10 percent to Bryan. Sticks of gum can be cut into fractions when necessary.
a. After one round of the game, the players count their sticks of gum. Anabelle has 525 sticks of gum, Bryan has 600 , and Carl has 450 . How many sticks of gum will each player have after 2 more rounds of the game? Use a matrix equation to represent the situation, and explain your answer in context.

$$
\begin{gathered}
{\left[\begin{array}{ccc}
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\frac{1}{3} & \frac{2}{3} & 0 \\
\frac{2}{5} & \frac{1}{10} & \frac{1}{2}
\end{array}\right]^{2}\left[\begin{array}{l}
525 \\
600 \\
450
\end{array}\right]=x} \\
x=\left[\begin{array}{l}
547 \frac{3}{16} \\
564 \frac{7}{12} \\
522 \frac{1}{2}
\end{array}\right]
\end{gathered}
$$

Anabelle would have $547 \frac{3}{16}$ sticks of gum, Bryan would have $564 \frac{7}{12}$ sticks of gum, and Carl would have $522 \frac{1}{2}$ sticks of gum.
b. How many sticks of gum did each player have at the start of the game? Use a matrix equation to represent the situation, and explain your answer in context.

$$
\left.\begin{array}{c}
{\left[\begin{array}{ccc}
\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\
\frac{1}{3} & \frac{2}{3} & 0 \\
\frac{2}{5} & \frac{1}{10} & \frac{1}{2}
\end{array}\right]^{-1}\left[\begin{array}{l}
525 \\
600 \\
450
\end{array}\right]=x} \\
x
\end{array}\right]\left[\begin{array}{l}
600 \\
600 \\
300
\end{array}\right]=\$
$$

At the start of the game, Anabelle and Bryan each had 600 sticks of gum, and Carl had 300 sticks of gum.

Solving General Systems of Linear Equations $1 / 24 / 15$

## Problem Set Sample Solutions

1. The system of equations is given:

$$
\begin{aligned}
& 1.2 x+3 y-5 z+4.2 w+v=0 \\
& 6 x=5 y+2 w \\
& 3 y+4.5 z-6 w+2 v=10 \\
& 9 x-y+z+2 v=-3 \\
& -4 x+2 y-w+3 v=1
\end{aligned}
$$

a. Represent this system using a matrix equation.

$$
\left[\begin{array}{ccccc}
1.2 & 3 & -5 & 4.2 & 1 \\
6 & -5 & 0 & -2 & 0 \\
0 & 3 & 4.5 & -6 & 2 \\
9 & -1 & 1 & 0 & 2 \\
-4 & 2 & 0 & -1 & 3
\end{array}\right]=\left[\begin{array}{c}
x \\
y \\
z \\
w \\
v
\end{array}\right]=\left[\begin{array}{c}
0 \\
0 \\
10 \\
-3 \\
1
\end{array}\right]
$$

b. Use technology to solve the system. Show your solution process, and round your entries to the tenths place.

$$
\left[\begin{array}{l}
x \\
y \\
z \\
w \\
v
\end{array}\right]=\left[\begin{array}{ccccc}
1.2 & 3 & -5 & 4.2 & 1 \\
6 & -5 & 0 & -2 & 0 \\
0 & 3 & 4.5 & -6 & 2 \\
9 & -1 & 1 & 0 & 2 \\
-4 & 2 & 0 & -1 & 3
\end{array}\right]^{-1}\left[\begin{array}{c}
0 \\
0 \\
10 \\
-3 \\
1
\end{array}\right]=\left[\begin{array}{c}
0.2 \\
1.3 \\
-1.5 \\
-2.5 \\
-1.1
\end{array}\right]
$$

2. A caterer was preparing a fruit salad for a party. She decided to use strawberries, blackberries, grapes, bananas, and kiwi. The total weight of the fruit was 10 pounds. Based on guidelines from a recipe, the weight of the grapes was equal to the sum of the weight of the strawberries and blackberries; the total weight of the blackberries and kiwi was 2 pounds; half the total weight of fruit consisted of kiwi, strawberries, and bananas; and the weight of the grapes was twice the weight of the blackberries.
a. Write a system of equations to represent the constraints placed on the caterer when she made the fruit salad. Be sure to define your variables.
$S=$ pounds of strawberries
$B=$ pounds of blackberries
$G=$ pounds of grapes
$K=$ pounds of kiwi
$B a=$ pounds of bananas

$$
\begin{aligned}
S+B+G+K+B a & =10 \\
G & =S+B \\
B+K & =2 \\
K+S+B a & =5 \\
G & =2 B
\end{aligned}
$$

b. Represent the system using a matrix equation.

$$
\left[\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 \\
0 & -2 & 1 & 0 & 0
\end{array}\right]\left[\begin{array}{c}
S \\
B \\
G \\
K \\
B a
\end{array}\right]=\left[\begin{array}{c}
10 \\
0 \\
2 \\
5 \\
0
\end{array}\right]
$$

c. Solve the system using the matrix equation. Explain your solution in context.

$$
\left[\begin{array}{c}
S \\
B \\
G \\
K \\
B a
\end{array}\right]=\left[\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
-1 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 \\
0 & -2 & 1 & 0 & 0
\end{array}\right]^{-1}\left[\begin{array}{c}
10 \\
0 \\
2 \\
5 \\
0
\end{array}\right]=\left[\begin{array}{c}
\frac{5}{3} \\
5 \\
\frac{5}{3} \\
\frac{10}{3} \\
\frac{1}{3} \\
3
\end{array}\right]
$$

The fruit salad consisted of $\frac{5}{3}$ pounds of strawberries, $\frac{5}{3}$ pounds of blackberries, $\frac{10}{3}$ pounds of grapes, $\frac{1}{3}$ pound of kiwi, and 3 pounds of bananas.
d. How helpful would the solution to this problem likely be to the caterer as she prepares to buy the fruit?

It is useful as a general guideline, but the caterer is unlikely to buy the fruit in exactly the amount indicated by the problem. For instance, it is unlikely that she could purchase exactly $\frac{1}{3}$ pound of kiwi because it generally has to be purchased per fruit, not per ounce.
3. Consider the sequence $1,1,1,3,5,9,17,31,57, \ldots$ where each number beyond the third is the sum of the previous three. Let $w_{n}$ be the points with the $n^{\text {th }},(n+1)^{\mathrm{th}}$, and $(n+2)^{\text {th }}$ terms of the sequence.
a. Find a $3 \times 3$ matrix $A$ so that $A w_{n}=w_{n+1}$ for each $n$.

$$
A=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{array}\right]
$$

b. What is the $30^{\text {th }}$ term of the sequence?

$$
\begin{gathered}
w_{30}=A^{29} w_{1} \\
=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{array}\right]^{29}\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]=\left[\begin{array}{c}
20603361 \\
37895489 \\
697006711
\end{array}\right]
\end{gathered}
$$

c. What is $A^{-1}$ ? Explain what $A^{-1}$ represents in terms of the sequence. In other words, how can you find $w_{n-1}$ if you know $w_{n}$ ?

$$
A^{-1}=\left[\begin{array}{ccc}
-1 & -1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right]
$$

The first entry of $w_{n-1}$ is the third entry from $w_{n}$ minus the sum of the first two entries of $w_{n}$, the second entry of $w_{n-1}$ is the first entry of $w_{n}$ and the third entry of $w_{n-1}$ is the second entry of $w_{n}$.
d. Could you find the $-5^{\text {th }}$ term in the sequence? If so, how? What is its value? Yes.

$$
w_{-5}=\left[\begin{array}{ccc}
-1 & -1 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right]^{-6}\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]=\left[\begin{array}{c}
1 \\
3 \\
-3
\end{array}\right]
$$

4. Mr. Johnson completed a survey on the number of hours he spends weekly watching different types of television programs. He determined that he spends 30 hours a week watching programs of the following types: comedy, drama, movies, competition, and sports. He spends half as much time watching competition shows as he does watching dramas. His time watching sports is double his time watching dramas. He spends an equal amount of time watching comedies and movies. The total amount of time spent watching comedies and movies is the same as the total amount of time spent watching dramas and competition shows.

Write and solve a system of equations to determine how many hours Mr. Johnson watches each type of programming each week.

Let $f=$ hours watching comedy, $d=$ hours watching drama, $m=$ hours watching movies, $c=$ hours watching competition shows, and $s=$ hours watching sports.

\[

\]

Mr. Johnson spends 4.5 hours watching comedies, 6 hours watching dramas, 4.5 hours watching movies, 3 hours watching competition shows, and 12 hours watching sports each week.
5. A copper alloy is a mixture of metals having copper as their main component. Copper alloys do not corrode easily and conduct heat. They are used in all types of applications including cookware and pipes. A scientist is studying different types of copper alloys and has found one containing copper, zinc, tin, aluminum, nickel, and silicon. The alloy weighs 3.2 kilograms. The percentage of aluminum is triple the percentage of zinc. The percentage of silicon is half that of zinc. The percentage of zinc is triple that of nickel. The percentage of copper is fifteen times the sum of the percentages of aluminum and zinc combined. The percentage of copper is nine times the combined percentages of all the other metals.
a. Write and solve a system of equations to determine the percentage of each metal in the alloy.

Let $c=$ percentage of copper, $z=$ percentage of zinc, $t=$ percentage of tin, $a=$ percentage of aluminum, $n=$ percentage of nickel, $s=$ percentage of silicon:

$$
\begin{gathered}
c+z+t+a+n+s=3.2 \\
a=3 z \\
s=0.5 z \\
z=3 n \\
c=15(a+z) \\
c=9(z+t+a+n+s) \\
c=90 \%, z=1.5 \%, t=2.75 \%, a=4.5 \%, N=0.5 \%, s=0.75 \%
\end{gathered}
$$

The alloy has $\mathbf{9 0} \%$ copper, $1.5 \%$ zinc, $2.75 \%$ tin, $4.5 \%$ aluminum, $\mathbf{0 . 5} \%$ nickel, and $0.75 \%$ silicon.
$\left|\begin{array}{cccccc}1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & 0 & 1 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 & 0 \\ 1 & -15 & 0 & -15 & 0 & 0 \\ 1 & -9 & -9 & -9 & -9 & -9\end{array}\right|\left|\begin{array}{c}c \\ z \\ t \\ a \\ n \\ s\end{array}\right|=\left|\begin{array}{c}3.2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0\end{array}\right|$
b. How many kilograms of each alloy are present in the sample?

| Copper: | $0.90(3.2)=2.88 \mathrm{~kg}$ |
| :--- | :--- |
| Zinc: | $0.015(3.2)=0.048 \mathrm{~kg}$ |
| Tin: | $0.0275(3.2)=0.088 \mathrm{~kg}$ |
| Aluminum: | $0.045(3.2)=0.144 \mathrm{~kg}$ |
| Nickel: | $0.005(3.2)=0.016 \mathrm{~kg}$ |
| Silicon: | $0.0075(3.2)=0.024 \mathrm{~kg}$ |

Solving General Systems of Linear Equations $1 / 24 / 15$

## Topic D:

## Vectors in Plane and Space

N-VM.A.1, N-VM.A.2, N-VM.A.3, N-VM.B.4, N-VM.B.5, N-VM.C. 11

Focus Standards:

N-VM.A. 1
(+) Recognize vector quantities as having both magnitude and direction. Represent vector quantities by directed line segments, and use appropriate symbols for vectors and their magnitudes (e.g. $\mathbf{v},|\mathbf{v}|,| | \mathbf{v} \|, v)$.
N-VM.A. $2 \quad(+)$ Find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.

N-VM.A. $3 \quad(+)$ Solve problems involving velocity and other quantities that can be represented by vectors.
N-VM.B. $4 \quad(+)$ Add and subtract vectors.
a. Add vectors end-to-end, component-wise, and by the parallelogram rule. Understand that magnitude of a sum of two vectors is typically not the sum of the magnitudes.
b. Given two vectors in magnitude and direction form, determine the magnitude and direction of their sum.
c. Understand vector subtraction $\mathbf{v}-\mathbf{w}$ as $\mathbf{v}+(-\mathbf{w})$, where $-\mathbf{w}$ is the additive inverse of $\mathbf{w}$, with the same magnitude as $\mathbf{w}$ and pointing in the opposite direction. Represent vector subtraction graphically by connecting the tips in the appropriate order, and perform vector subtraction component-wise.
N-VM.B. $5 \quad(+)$ Multiply a vector by a scalar.
a. Represent scalar multiplication graphically by scaling vectors and possibly reversing their direction; perform scalar multiplication component-wise, e.g., as $c\left(v_{x}, v_{y}\right)=\left(c v_{x}, c v_{y}\right)$.
b. Compute the magnitude of a scalar multiple $c \mathbf{v}$ using $\|c \mathbf{v}\|=|c| v$. Compute the direction of $c \mathbf{v}$ knowing that when $|c| v \neq 0$, the direction of $c \mathbf{v}$ is either along $\mathbf{v}$ for $(c>0)$ or against $\mathbf{v}$ (for $c<0$ ).
N-VM.C. $11 \quad(+)$ Multiply a vector (regarded as a matrix with one column) by a matrix of suitable dimensions to produce another vector. Work with matrices as transformations of vectors.

Instructional Days:
8

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    Lesson 17: Vectors in the Coordinate Plane (P)}\mp@subsup{}{}{1
Lesson 18: Vectors and Translation Maps (P)
Lesson 19: Directed Line Segments and Vectors (P)
Lesson 20: Vectors and Stone Bridges (E)
Lesson 21: Vectors and the Equation of a Line (S)
Lesson 22: Linear Transformations of Lines (S)
Lessons 23-24: Why Are Vectors Useful? (P, S)
```

Topic D opens with a formal definition of a vector (the motivation and context for it is well in place at this point), and the arithmetical work for vector addition, subtraction, scalar multiplication, and vector magnitude is explored along with the geometrical frameworks for these operations (N-VM.A.1, N-VM.A.2, N-VM.B.4, N-VM.B.5).

Lesson 17 introduces vectors in terms of translations. Students use their knowledge of transformations to represent vectors as arrows with an initial point and a terminal point. They calculate the magnitude of a vector, add and subtract vectors, and multiply a vector by a scalar. Students interpret these operations geometrically and compute them componentwise (N-VM.A.1, N-VM.A.3, N-VM.B.4, N-VM.B.5). Lesson 18 builds on vectors as shifts by relating them to translation maps studied in prior lessons. The connection between matrices and vectors becomes apparent in this lesson as a notation for vectors is introduced that recalls matrix notation. This lesson's focus extends vector addition, subtraction, and scalar multiplication to $\mathbb{R}^{3}$.

Lesson 19 introduces students to directed line segments and how to subtract initial point coordinates from terminal point coordinates to find the components of a vector. Vector arithmetic operations are reviewed, and the parallelogram rule is introduced. Students study the magnitude and direction of vectors. In Lesson 20 , students apply vectors to real-world applications as they look at vector effects on ancient stone arches and try to create their own.

In Lesson 21, students describe a line in the plane using vectors and parameters, and then they apply this description to lines in $\mathbb{R}^{3}$. The shift to describing a line using vectors to indicate the direction of the line requires that students think geometrically about lines in the plane instead of algebraically. Lesson 21 poses the question, "Is the image of a line under a linear transformation a line?" Students must extend the process of finding parametric equations for a line in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$ (N-VM.C.11). Lessons 21 and 22 help students see the coherence between the work that they have done with functions and how that relates to vectors written using parametric equations. This sets the mathematical foundation that students will need to understand the definition of vectors. Vectors are generally described as a quantity that has both a magnitude and a direction. In Lessons 21 and 22, students perform linear transformations in two and three dimensions by writing parametric equations. The most basic definition of a vector is that it is a description of a shift or translation. Students will see that any physical operation that induces a shift of some kind is often thought of as a vector. Hence, vectors are prevalent in mathematics, science, and engineering. For example, force is often interpreted as a vector in physics because a force exerted on an object is a push of some magnitude that causes the object to shift in some direction.

In Lessons 23 and 24, students will solve problems involving velocity as well as other quantities, such as force, that can be represented by vectors (N-VM.A.3). Students work with non-right triangles and will continue to

[^5]work on adding and subtracting vectors (N-VM.B.4) but will interpret the resulting magnitude and direction within a context, for example with particle motion. Students discover the notion (using pictures only) that the method of transforming systems of linear equations while preserving the solution can be rephrased in terms of a series of linear transformations and translations.

In Topic D, students are making sense of vectors and using them to represent real world situations such as earthquakes, velocity, and force (MP.4). Students focus on vector arithmetic, determining the magnitude and direction of vectors and vector sums and products (MP.6), and constructing arguments about why the operations work the way that they do (MP.3).

## Lesson 17: Vectors in the Coordinate Plane

## Student Outcomes

- Students add and subtract vectors and understand those operations geometrically and component-wise.
- Students understand scalar multiplication graphically and perform it component-wise.


## Lesson Notes

This lesson introduces translation by a vector in the coordinate plane. In this lesson, we represent a vector as an arrow with an initial point and a terminal point. Students learn vector notation and the idea that a vector can represent a shift (i.e., a translation). They calculate the magnitude of a vector, add and subtract vectors, and multiply a vector by a scalar. Students interpret these operations geometrically and compute them component-wise. This lesson focuses on several N.VM standards including N-VM.A.1, N-VM.A.3, N-VM.B.4a and N-VM.B.4c, and N-VM.B.5. However, students are not yet working with directed line segments or vectors in $\mathbb{R}^{3}$. Later lessons will also represent vectors in magnitude and direction form. Students are making sense of vectors and relating them to a real-world situation (MP.2). Later, the lesson focuses on vector arithmetic and making sense of why the operations work the way that they do (MP. 6 and MP.3)

The study of vectors is a vital part of this course; notation for vectors varies across different contexts and curricula. These materials will refer to a vector as $\mathbf{v}$ (lowercase, bold, non-italicized); or as $\langle 4,5\rangle$ which, in column format is $\binom{4}{5}$ or $\left[\begin{array}{l}4 \\ 5\end{array}\right]$.
We will use "let $\mathbf{v}=\langle 4,5\rangle$ " to establish a name for the vector $\langle 4,5\rangle$.
When naming a vector, this curriculum will avoid stating $\mathbf{v}=\langle 4,5\rangle$ without the word "let" preceding the equation, unless it is absolutely clear from the context that we are naming a vector. However, as we have done in other grades, we will continue using $=$ to describe vector equations, such as $\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v}$.

We will refer to the vector from $A$ to $B$ as vector $\overrightarrow{A B}$; notice that this is a ray with a full arrow. This notation is consistent with how vectors were introduced in Grade 8 and is also widely used in post-secondary textbooks to describe both rays and vectors, depending on the context. To avoid confusion, this curriculum will provide (or strongly imply) the context, to make it clear whether the full arrow indicates a vector or a ray. For example, when referring to a ray from $A$ passing through $B$, we will say "ray $\overrightarrow{A B}$," and when referring to a vector from $A$ to $B$, we will say "vector $\overrightarrow{A B}$." Students should be encouraged to think about the context of the problem and not just rely on a hasty inference based on the symbol.

The magnitude of a vector will be signified as $\|\mathbf{v}\|$ (lowercase, bold, non-italicized).
Since Grade 6, we have been using the term vector in two slightly different ways. Up until now, the difference was subtle and didn't matter where the term was used in the discussions. However, in Precalculus and Advanced Topics, we need to distinguish between a bound vector and a free vector:

Bound Vector: A bound vector is a directed line segment (i.e., an "arrow"). For example, the directed line segment $\overrightarrow{A B}$ is a bound vector whose initial point (i.e., tail) is $A$ and terminal point (i.e., tip) is $B$.

Bound vectors are "bound" to a particular location in space. A bound vector $\overrightarrow{A B}$ has a magnitude given by the length of segment $\overline{A B}$ and direction given by the ray $\overrightarrow{A B}$. Many times only the magnitude and direction of a bound vector matters, not its position in space. In such a case, we consider any translation of that bound vector to represent the same free vector.

Free vector: A free vector is the equivalence class of all directed line segments (i.e., arrows) that are equivalent to each other by translation. For example, scientists often use free vectors to describe physical quantities that have magnitude and direction only, "freely" placing an arrow with the given magnitude and direction anywhere it is needed in a diagram. For any directed line segment in the equivalence class defining a free vector, the directed line segment is said to be a representation of the free vector (i.e., it represents the free vector).

Free vectors are usually notated by a lowercase letter with an arrow $\vec{v}$ or by a boldface lowercase letter $\mathbf{v}$. Any other representation of a free vector is also labeled by the same lowercase letter (i.e., any two arrows in a diagram/Euclidean plane/Cartesian plane with the same magnitude and direction are labeled $\mathbf{v}$ ). The notation $\overrightarrow{A B}$ can also be used to represent a free vector $\mathbf{v}$, but it still is the specific directed line segment from $A$ to $B$. (One wouldn't label another representation of the free vector $\mathbf{v}$ at a different location by $\overrightarrow{A B}$, for example).

## Unless specifically stated, the term "vector" will refer to free vector throughout this module.

Cartesian coordinates are useful for representing free vectors as points in coordinate space. Such representations provide simple formulas for adding/subtracting vectors and for finding a scalar multiple of a vector, as well as making it easy to write formulas for translations.

## Classwork

## Opening (2 minutes)

Have students read the opening paragraph, and then ask them what they know about earthquakes. Hold a brief discussion to activate prior knowledge. Emphasize the shifting ground that can occur during an earthquake, and explain that in this lesson, we will consider a way to represent such a shift mathematically. You may want to reference this article (http://www.cnn.com/2011/WORLD/asiapcf/03/12/japan.earthquake.tsunami.earth/) that discusses the 2011 Japan earthquake that shifted the coastline by eight feet.

## Opening Exercise ( 2 minutes)

Students should complete this exercise working with a partner. Have one or two students share their responses with the whole class. The idea is to help students see that the shift is independent of a starting point and that all points in the plane are shifted by the same amount. If students mention the force of the earthquake, make a connection back to that language when discussing magnitude.

## Opening Exercise

When an earthquake hits, the ground shifts abruptly due to forces created when the tectonic plates along fault lines rub together. As the tectonic plates shift and move, the intense shaking can even cause the physical movement of objects as large as buildings.

Suppose an earthquake causes all points in a town to shift 10 feet to the north and 5 feet to the east.

## Scaffolding:

- Use a graphic organizer (like a Frayer model) to help students make sense of this new concept, vector. See Precalculus and Advanced Topics, Module 1, Lesson 5 for an example of a Frayer diagram.
- Have advanced students draw vectors that represent different shifts that are not whole numbers such as S (2.25 feet) and W (7 feet)
$N$ (9 feet) and E (1.5 feet)
S (1.75 feet) and E ( 5.5 feet)
a. Explain how the diagram shown above could be said to represent the shifting caused by the earthquake.

The arrows show the amount and direction of the shift. Each point, not just the ones represented in this diagram, would be shifted 5 feet east and 10 feet north.
b. Draw another arrow that shows the same shift. Explain how you drew your arrow.

The new arrow would be shifted 5 feet east and 10 feet north from its initial point.

## Discussion (5 minutes)

Lead a brief discussion to introduce the notion and notation of a vector. Clarify the vector notation used in this Module. Vectors are denoted by a letter bold in text with the components enclosed in angled brackets. Some texts simply use parentheses. When writing a vector by hand, direct students to draw an arrow over the symbol that represents the vector.

In mathematics, a shift like the one described in the Opening can be represented by a vector. This vector has a horizontal component of 5 and a vertical component of 10 .

We use the following notation for a vector:

$$
\mathbf{v}=\langle 5,10\rangle
$$

When writing a vector that starts at point $A$ and goes through point $B$, use this notation:

$$
\overrightarrow{A B}=\langle 5,10\rangle
$$

- Are the points in the diagram above the only points that were shifted during the earthquake? Explain your thinking.
- No, every point in the area affected by the earthquake would have been shifted. In reality, how far a point shifts depends on how far it is from a quake's epicenter
- When we consider vectors, the precise location of the arrow that represents the vector does not matter because the vector represents the translation of an object (in this case points in the plane) by the given horizontal and vertical components. What other objects could be translated by a vector?
- You could shift any figure by a vector, such as a line, a circle, or another geometric figure.


## Exercises 1-3 (10 minutes)

The next exercises let students compare different vectors in the coordinate plane. These exercises show students that vectors are defined as having both a magnitude and a direction, and that two arrows with the same magnitude and direction represent the same vector. Students will explore the notion of magnitude in Exercise 3, and the example that follows will formally define the magnitude of a vector. Have students work in small groups or with a partner, but give them time to work individually first so that each student has a chance to think and make sense of these ideas.

## Exercises 1-3

Several vectors are represented in the coordinate plane below using arrows.


1. Which arrows represent the same vector? Explain how you know.

Arrows $\mathrm{w}, \mathrm{u}$, and a represent the same vector because they indicate a translation of 1 unit right and 3 units up. Arrows v and b represent the same vector because these arrows indicate a translation of 3 units right and 1 unit up. Arrows c and d represent the same vector because they represent a translation of 1 unit left and 3 units down.

## 2. Why do arrows $\mathbf{c}$ and $\mathbf{u}$ not represent the same vector? <br> These arrows have the same magnitude but opposite directions.

Discuss the following questions after Exercise 2 with your students.

- How many different vectors are in the diagram? What are the components of each one?
- There are really just three vectors: $\mathbf{u}=\langle 1,3\rangle, \mathbf{v}=\langle 3,1\rangle$, and $\mathbf{d}=\langle-1,-3\rangle$
- Why might we only draw one arrow to represent a vector even though all points are shifting by the components of the vector?
- Since a vector represents a translation, no matter where in the coordinate plane we represent it, it will still describe the same shifting.
- Why does the location of the tip of the arrow matter when representing a vector in the coordinate plane?
- The tip is used to indicate direction. Otherwise vectors like $\mathbf{u}=\langle 1,3\rangle$ and $\mathbf{d}=\langle-1,-3\rangle$ would appear to be the same.

If needed, before starting Exercise 3, redirect students' attention to the Opening. Give students time to collaborate on their approach to part (b) in groups or with a partner. Students may benefit from guided questioning like that listed below if they appear stuck.

- What quantities would you need to measure to determine which earthquake shifted the points further?
- For each earthquake, you would need to find the distance between the starting location and the stopping location for one point, since all points shifted the same amount.
- How could you use a familiar formula to determine these quantities?
- Since we know the horizontal and vertical displacement, we can use the Pythagorean theorem (or distance formula) to calculate the distance, which is represented by the length of the arrow.

3. After the first earthquake shifted points $\mathbf{5}$ feet east and $\mathbf{1 0}$ feet north, suppose a second earthquake hits the town and all points shift 6 feet east and 9 feet south.
a. Write and draw a vector $t$ that represents this shift caused by the second earthquake.

b. Which earthquake, the first one or the second one, shifted all the points in the town further? Explain your reasoning.

The length of the arrow that represents the vector $v$ is $\sqrt{5^{2}+10^{2}}=\sqrt{125}$ feet. The length of the arrow that represents the vector t is $\sqrt{6^{2}+(-9)^{2}}=\sqrt{117}$ feet. The first quake shifted the points further. You can also see from the diagram that if we rotated t from the tip of v to align with v that t would be slightly shorter.

- Explain what you have just learned to your neighbor. Use this as an informal way to check student understanding.


## Example 1: The Magnitude of a Vector (3 minutes)

This example introduces the formula to calculate the magnitude of a vector. Students already likely used the Pythagorean theorem in their work in Exercise 3. This example provides a way to formalize what we mean when we refer to the magnitude of a vector. For now, the direction of a vector will simply be indicated by the arrow tip showing the direction as well as the sign and size of the components. Later in this module students will use trigonometry to calculate an angle that represents the direction of a vector.

## Example 1: The Magnitude of a Vector

The magnitude of a vector $v=\langle a, b\rangle$ is the length of the line segment from the origin to the point $(a, b)$ in the coordinate plane, which we denote by $\|v\|$. Using the language of translation, the magnitude of $v$ is the distance between any point and its image under the translation $a$ units horizontally and $b$ units vertically. It is denoted $\|v\|$.
a. Find the magnitude of $v=\langle 5,10\rangle$ and $t=\langle 6,-9\rangle$. Explain your reasoning.

$$
\begin{gathered}
\|\mathrm{v}\|=\sqrt{5^{2}+\mathbf{1 0}^{2}}=\sqrt{\mathbf{1 2 5}} \\
\|\mathrm{t}\|=\sqrt{(6)^{2}+(-9)^{2}}=\sqrt{117}
\end{gathered}
$$



We use the Pythagorean theorem (or distance formula) to find the length of the hypotenuse of a triangle with sides of length $a$ and $b$.
b. Write the general formula for the magnitude of a vector.

$$
\|v\|=\sqrt{a^{2}+b^{2}}
$$

- How does this example confirm or refute your work in Exercise 3, part (b)?
- We got the same results and we can now see that the magnitude of a vector is a measure of how much shifting occurred.


## Discussion: Vector Addition (5 minutes)

- Explain why this drawing shows the overall shifting caused by both earthquakes.
- The points shift from the starting point or pre-image to the image translated 5 right and 10 up and then those points are translated again, 6 right and 9 down to their final location.

- What is the resulting vector that represents the new location of all the points? Explain how you got your answer.
- The components of the new vector $\mathbf{e}$ are $\langle 11,1\rangle$. You can see that the total horizontal shift is 11 units right and the total vertical shift is 1 unit up.

- Did you need to grid lines to calculate the components of the overall shift caused by the two earthquakes? Explain why or why not.
- No. We can simply add the horizontal components together followed by the vertical components to find the resulting components of the final vector.

Next, introduce the idea of vector addition. Students can add this information to their notes.

$$
\begin{gathered}
\text { Vector addition is defined by the rule: } \\
\text { If } \mathbf{v}=\langle a, b\rangle \text { and } \mathbf{w}=\langle c, d\rangle \text { then } \mathbf{v}+\mathbf{w}=\langle a+c, b+d\rangle .
\end{gathered}
$$

The rule stated above shows that to add two vectors you simply add their horizontal and vertical components. Explain the definition of vector addition geometrically using transformations and by modeling, in general, that the addition of two vectors has the same effect as two horizontal and two vertical translations of a point or other object in the coordinate plane. The diagram shown below illustrates this idea. The CCSS-M refers to this method of adding vectors as end-to-end (See N-VM.B.4a). The parallelogram rule for adding vectors will be presented in Lesson 19.


## Exercises 4-7 (5 minutes)

Students should work these exercises with their small group. As students are working, circulate around the room to check their progress. Lead a short debriefing by having one or two students share their answers with the group. Students should realize that, as with matrices and complex numbers, properties of arithmetic such as the associative property can be extended to vector operations. The next lesson makes these connections even more explicit. Collegelevel mathematics courses use matrices frequently to represent higher dimension vectors because they make calculations simpler and more efficient.

Exercises 4-10
4. Given that $v=\langle 3,7\rangle$ and $t=\langle-5,2\rangle$.
a. What is $v+t$ ?

$$
\mathrm{v}+\mathrm{t}=\langle 3+(-5), 7+2)\rangle=\langle-2,9\rangle
$$

b. Draw a diagram that represents this addition and shows the resulting sum of the two vectors.

c. What is $t+v$ ?

$$
t+v=\langle-2,9\rangle
$$

d. Draw a diagram that represents this addition and shows the resulting sum of the two vectors.

5. Explain why vector addition is commutative.

Since we are combining two horizontal and two vertical translations when we add vectors, the end result will be the same regardless of the order in which we apply the translations. Thus, when we add two vectors it doesn't matter which comes first and which comes second. Using the rule we were given, we can see that the components represent real numbers and thus the associative property should apply to each component of the resulting sum vector.
6. Given $\mathbf{v}=\langle 3,7\rangle$ and $\mathbf{t}=\langle-5,2\rangle$.
a. Show numerically that $\|\mathrm{v}\|+\|\mathbf{t}\| \neq\|v+\mathbf{t}\|$.
$\|v\|=\sqrt{3^{2}+7^{2}}=\sqrt{58}$ and $\|t\|=\sqrt{(-5)^{2}+2^{2}}=\sqrt{29}$
$\mathrm{v}+\mathrm{t}=\langle-2,9\rangle$ and $\|\mathrm{v}+\mathrm{t}\|=\sqrt{(-2)^{2}+(9)^{2}}=\sqrt{85}$
$\sqrt{58}+\sqrt{29} \neq \sqrt{85}$. This can be confirmed quickly using approximations for each square root.
b. Provide a geometric argument to explain in general, why the sum of the magnitudes of two vectors is not equal to the magnitude of the sum of the vectors.

When added end to end, two vectors and the resulting sum vector lie on the sides of a triangle. Since the sum of any two sides of a triangle must be longer than the third side and the magnitude of the vectors would correspond to the lengths of the sides of the triangle, this statement cannot be true.
c. Can you think of an example when the statement would be true? Justify your reasoning.

This statement would be true of one of the vectors had a magnitude of 0 . The sum of the vectors would be equal to the original non-zero vector so they would have the same magnitude.
7. Why is the vector $0=\langle 0,0\rangle$ called the zero vector? Describe its geometric effect when added to another vector.

The magnitude of this vector is o , and its components are 0 . It maps the pre-image vector onto itself and essentially have no translational effect on the original vector. It has the same effect as adding the real-number 0 to any other real number.

## Exercises 8-10 (8 minutes)

These problems introduce the scalar multiplication of a vector. Students have already examined the effect of scalar multiplication with complex numbers and matrices. They should be familiar with the idea of dilation from their work in Grade 8 and Geometry. We want them to see that if we multiply a vector $\mathbf{v}$ by a scalar $c$, that it produces a new vector $c \mathbf{v}$ which is a dilation of the vector $\mathbf{v}$ by a scale factor $c$. This effect maintains the direction of the vector and multiplies its magnitude by a factor of $c$. Students may need the terminology 'initial point' clarified before you begin. Explain that when the initial point is $(0,0)$ then the components of the vector $\langle\mathrm{a}, \mathrm{b}\rangle$ correspond to the point in the Cartesian plane $(a, b)$. For this reason, we often draw arrows that represent vectors from the origin.

## Exercises 8-10

8. Given the vectors shown below.

$$
\begin{gathered}
\mathrm{v}=\langle\mathbf{3 , 6 \rangle} \\
\mathrm{u}=\langle 9,18\rangle \\
\mathrm{w}=\langle-3,-6\rangle \\
\mathrm{s}=\langle 1,2\rangle \\
\mathrm{t}=\langle-1.5,-3\rangle \\
\mathrm{r}=\langle 6,12\rangle
\end{gathered}
$$

a. Draw each vector with its initial point located at $(0,0)$. The vector $v$ is already shown. How are all of these vectors related?



All of these vectors lie on the same line that passes through the origin. If you dilate with center ( 0,0 ), then each vector is a dilation of every other vector in the list.
b. Which vector is 2 v ? Explain how you know.

The vector 2 v is twice as long as $v$ and points in the same direction. The components would be doubled so $r=2 \mathrm{v}$.

Introduce scalar multiplication at this point. Have students record the rule in their notes.

> Vector multiplication by a scalar is defined by the rule:
> If $\mathbf{v}=\langle a, b\rangle$ and $c$ is a real number, then $c \mathbf{v}=\langle c a, c b\rangle$.

If time permits, early finishers can be asked to show that $\|c \mathbf{v}\|=c \sqrt{a^{2}+b^{2}}$ by showing the following:
If $c \mathbf{v}=\langle c a, c b\rangle$ then by the definition of magnitude of a vector, the properties of real numbers and the properties of radicals,

$$
\begin{aligned}
\|c \mathbf{v}\| & =\sqrt{(c a)^{2}+(c b)^{2}} \\
& =\sqrt{c^{2}\left(a^{2}+b^{2}\right)} \\
& =c \sqrt{a^{2}+b^{2}}
\end{aligned}
$$

c. Describe the remaining vectors as a scalar multiple of $v=\langle 3,6\rangle$ and explain your reasoning.

$$
\begin{aligned}
\mathbf{u} & =3 \mathbf{v} \\
\mathbf{w} & =-\mathbf{v} \\
\mathbf{s} & =\frac{1}{3} \mathbf{v} \\
\mathbf{t} & =-\frac{1}{2} \mathbf{v}
\end{aligned}
$$

d. Is the vector $p=\langle 3 \sqrt{2}, 6 \sqrt{2}\rangle$ a scalar multiple of $v$ ? Explain.

Yes; $p=\sqrt{2} \mathrm{v}$. You can see that if the initial point were located at $(0,0)$ then the vector $p$ would also lie on the line through the origin that contains the other vectors in this exercise.
9. Which vector from Exercise 8 would it make sense to call the opposite of $v=\langle 3,6\rangle$ ?

You could call w the opposite of v , because w has the same length as v and it has the opposite direction.
10. Describe a rule that defines vector subtraction. Use the vectors $v=\langle 5,7\rangle$ and $u=\langle 6,3\rangle$ to support your reasoning.

Just like addition of two real numbers is adding the opposite of the second number to the first number, it would make sense that vector subtraction would work in a similar way. To subtract two vectors, you add the opposite of the second vector or more simply, just subtract the components.

We can create the opposite of v by multiplying by the scalar -1 .

$$
\mathbf{v}-\mathbf{u}=\mathbf{v}+(-\mathbf{u})=\langle 5+(-6), 7+(-3)\rangle=\langle-1,4\rangle
$$

Or, simply subtracting the components gives

$$
v-u=\langle 5-6,7-3\rangle=\langle-1,4\rangle
$$

If time permits you can discuss the geometric effect of subtraction as adding the opposite of the second vector when placing the vectors end to end.

## Closing (2 minutes)

Give students one minute to brainstorm the top three things they learned about vectors in this lesson. Have them share briefly with a partner, and then ask for a few volunteers to share with the entire class. Use the Lesson Summary below to clarify any misunderstandings that may arise when students report out.

## Lesson Summary

A vector can be used to describe a translation of an object. It has a magnitude and a direction based on its horizontal and vertical components. A vector $v=\langle\boldsymbol{a}, \boldsymbol{b}\rangle$ can represent a translation of $a$ units horizontally and $b$ units vertically with magnitude given by $\|\mathrm{v}\|=\sqrt{a^{2}+b^{2}}$.

- To add two vectors, add their respective horizontal and vertical components.
- To subtract two vectors, subtract their respective horizontal and vertical components.
- Multiplication of a vector by a scalar multiplies the horizontal and vertical components of the vector by the value of the scalar.

Exit Ticket (3 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 17: Vectors in the Coordinate Plane

## Exit Ticket

1. Vector $\mathbf{v}=\langle 3,4\rangle$, and the vector $\mathbf{u}$ is represented by the arrow shown below. How are the vectors the same? How are they different?

2. Let $\mathbf{u}=\langle 1,5\rangle$ and $\mathbf{v}=\langle 3,-2\rangle$. Write each vector in component form and draw an arrow to represent the vector.
a. $\mathbf{u}+\mathbf{v}$
b. $\mathbf{u}-\mathbf{v}$
c. $2 \mathbf{u}+3 \mathbf{v}$
3. For $\mathbf{u}=\langle 1,5\rangle$ and $\mathbf{v}=\langle 3,-2\rangle$ as in 2(a), what is the magnitude of $\mathbf{u}+\mathbf{v}$ ?

## Exit Ticket Sample Solutions

1. Vector $v=\langle 3,4\rangle$ and the vector $u$ is represented by the arrow shown below. How are the vectors the same? How are they different?
Both u and v have the same length and direction, so they are different representations of the same vector. They both represent a translation of 3 units right and 4 units up.
2. Let $u=\langle 1,5\rangle$ and $v=\langle 3,-2\rangle$. Write each vector in component form and draw an arrow to represent the vector.
a. $\mathbf{u}+\mathbf{v}$

$$
\mathbf{u}+\mathbf{v}=\langle\mathbf{4}, \mathbf{3}\rangle
$$


b. $u-v$

$$
\mathbf{u}-\mathbf{v}=\mathbf{u}+(-\mathbf{v})=\langle-2,7\rangle
$$


c. $\quad \mathbf{2 u}+3 \mathbf{v}$

$$
2 u+3 v=\langle 11,4\rangle
$$


3. For $\mathbf{u}=\langle 1,5\rangle$ and $\mathbf{v}=\langle 3,-2\rangle$ as in Problem 2, part (a), what is the magnitude of $+\mathbf{v}$ ?

$$
\|\mathbf{u}+\mathbf{v}\|=\|\langle 4,3\rangle\|=5
$$

## Problem Set Sample Solutions

1. Sasha says that a vector has a direction component in it; therefore, we cannot add two vectors or subtract one from the other. His argument is that we cannot add "east" to "north" nor subtract "east" from "north," for instance. Therefore, he claims, we cannot add or subtract vectors.
a. Is he correct? Explain your reasons.

No, Sasha is not correct. Although a vector has a magnitude and direction, and it is numerically suited to do translation of an object, it has horizontal and vertical components indicating how many units for translation. Therefore, we can add and subtract vectors.
b. What would you do if you need to add two vectors, $u$ and $v$, or subtract vector $v$ from vector u arithmetically?

For addition, we add the same corresponding vector components. For example, $\mathbf{u}=\left\langle\mathbf{u}_{1}, \mathbf{u}_{2}\right\rangle, \mathbf{v}=\left\langle\mathbf{v}_{1}, \mathbf{v}_{\mathbf{2}}\right\rangle$
$\mathbf{u}+\mathbf{v}=\left\langle\mathbf{u}_{1}+\mathbf{v}_{1}, \mathbf{u}_{2}+\mathbf{v}_{2}\right\rangle$.
For subtraction $\mathbf{u}-\mathbf{v}=\mathbf{u}+(-\mathbf{v})=\left\langle\mathbf{u}_{1}+\left(-\mathbf{v}_{\mathbf{1}}\right), \mathbf{u}_{2}+\left(-\mathbf{v}_{2}\right)\right\rangle=\left\langle\mathbf{u}_{1}-\mathbf{v}_{1}, \mathbf{u}_{2}-\mathbf{v}_{2}\right\rangle$.
2. Given $\mathbf{u}=\langle 3,1\rangle$ and $\mathbf{v}=\langle-4,2\rangle$, write each vector in component form, graph it, and explain the geometric effect. a. $3 \mathbf{u}$

$$
3 \mathbf{u}=\langle 9,3\rangle
$$

The vector is dilated by a factor of 3 and the direction stays the same.

b. $\quad \frac{1}{2} v$

$$
\frac{1}{2} v=\langle-2,1\rangle
$$

The vector is dilated by a factor of $\frac{1}{2}$, and the direction stays the same.

c. $\quad-\mathbf{2 u}$

$$
2 \mathrm{u}=\langle-6,-2\rangle
$$

The vector is dilated by a factor of 2 and the direction is reversed.

d. $-\mathbf{v}$

$$
-v=\langle 4,-2\rangle
$$

The length of the vector is unchanged, and the direction is reversed.

e. $\mathbf{u}+\mathbf{v}$

$$
\mathbf{u}+\mathbf{v}=\langle-1,3\rangle
$$

When adding vector $\mathrm{u}=\langle 3,1\rangle$ onto vector $\mathrm{v}=\langle-4,2\rangle$, from the tip of vector v , we move 3 units to the right and 1 unit upward, and the resultant vector is $\langle-1,3\rangle$.

f. $\quad \mathbf{2 u}+3 \mathbf{v}$

$$
2 u+3 v=\langle-6,8\rangle
$$

When adding vector $2 \mathrm{u}=\langle 6,2\rangle$ onto vector $3 \mathrm{v}=\langle-12,6\rangle$, from the tip of vector 3 v , we move 6 units to the right and 2 units upward, and the resultant vector is $\langle-6,8\rangle$.

g. $\quad 4 u-3 v$

$$
4 u-3 v=4 u+(-3 v)=\langle 24,-2\rangle
$$

When adding vector $4 \mathrm{u}=\langle 12,4\rangle$ onto vector $-3 \mathrm{v}=\langle 12,-6\rangle$, from the tip of vector -3 v , we move 12 units to the right and 4 units upward. The resultant vector is $\langle 24,-2\rangle$.

h. $\frac{1}{2} u-\frac{1}{3} v$

$$
\frac{1}{2} u-\frac{1}{3} v=\frac{1}{2} u+\left(-\frac{1}{3} v\right)=\left\langle\frac{17}{6},-\frac{1}{6}\right\rangle
$$

When adding vector $\frac{1}{2} \mathrm{u}=\left\langle\frac{3}{2}, \frac{1}{2}\right\rangle$ onto vector $-\frac{1}{3} \mathrm{v}=\left\langle-\frac{4}{3}, 1\right\rangle$, from the tip of vector $-\frac{1}{3} \mathrm{v}$, we move $\frac{3}{2}$ units to the right and $\frac{1}{2}$ unit upward, and the resultant vector is $\left\langle\frac{17}{6},-\frac{1}{6}\right\rangle$

3. Given $\mathbf{u}=\langle 3,1\rangle$ and $\mathbf{v}=\langle-4,2\rangle$, find the following.
a. $\|\mathbf{u}\|$.

$$
\sqrt{\mathbf{1 0}}
$$

b. $\|\mathbf{v}\|$.

$$
2 \sqrt{5}
$$

c. $\quad\|2 \mathbf{u}\|$ and $2\|\mathbf{u}\|$.

$$
\|2 u\|=\|6,2\|=\sqrt{40}=2 \sqrt{10} .2\|u\|=2 \sqrt{10}
$$

d. $\quad\left\|\frac{1}{2} v\right\|$ and $\frac{1}{2}\|v\|$

$$
\frac{1}{2} v=\langle-2,1\rangle, \quad\left\|\frac{1}{2} v\right\|=\sqrt{5} . \quad \frac{1}{2}\|v\|=\frac{1}{2} \sqrt{5}
$$

e. Is $\|\mathbf{u}+\mathbf{u}\|$ equal to $\|\mathbf{u}\|+\|\mathbf{u}\|$ ? Explain how you know.

Yes. We have
and

$$
\begin{gathered}
\|\mathbf{u}+\mathbf{u}\|=\|2 \mathbf{u}\|=2 \sqrt{10} \\
\|\mathbf{u}\|+\|\mathbf{u}\|=2\|\mathbf{u}\|=2 \sqrt{10}
\end{gathered}
$$

f. Is $\|\mathbf{u}+\mathbf{v}\|$ equal to $\|\mathbf{u}\|+\|\mathbf{v}\|$ ? Explain how you know.

No. We have

$$
\mathrm{u}+\mathrm{v}=\langle-1,3\rangle\|u+v\|=\sqrt{10} \text { and }\|u\|+\|v\|=\sqrt{10}+2 \sqrt{5}
$$

g. Is \|u $\mathbf{-} \mathbf{v} \|$ equal to $\|\mathbf{u}\|-\|\mathbf{v}\|$ ? Explain how you know.

No. We have

$$
\mathbf{u}-\mathrm{v}=\langle 7,-1\rangle
$$

4. Given $u=\langle 1,2\rangle, v=\langle 3,-4\rangle$, and $w=\langle-4,6\rangle$, show that $(u+v)+w=u+(v+w)$.
$(u+v)+w=\langle 4,-2\rangle+\langle-4,6\rangle=\langle 0,4\rangle$
$u+(v+w)=\langle 1,2\rangle+\langle-1,2\rangle=\langle 0,4\rangle$
5. Tyiesha says that if the magnitude of a vector $u$ is zero, then $u$ has to be a zero vector. Is she correct? Explain how you know.
Yes. Suppose that $\mathbf{u}=\left\langle\mathbf{u}_{1}, \mathbf{u}_{2}\right\rangle$, and $\|\mathbf{u}\|=\sqrt{\left(\mathbf{u}_{1}\right)^{2}+\left(\mathbf{u}_{2}\right)^{2}}=\mathbf{0}$, so $\left(\mathbf{u}_{1}\right)^{2}+\left(\mathbf{u}_{2}\right)^{2}=\mathbf{0}$. Then $\left(\mathbf{u}_{1}\right)^{2}$ and $\left(\mathbf{u}_{2}\right)^{2}$ are positive numbers, and if two positive numbers sum to zero then both numbers must be zero. Then $u_{1}=0$ and $\mathbf{u}_{2}=0$, which proves $u$ is a zero vector.
6. Sergei experienced one of the biggest earthquakes when visiting Taiwan in 1999. He noticed that his refrigerator moved on the wooden floor and made marks on it. By measuring the marks he was able to trace how the refrigerator moved. The first move was northeast with a distance of 20 cm . The second move was northwest with a distance of 10 cm . The final move was northeast with a distance of 5 cm . Find the vectors that would re-create the refrigerator's movement on the floor and find the distance that the refrigerator moved from its original spot to its resting place. Draw a diagram of these vectors.

The first vector is
the second vector is
the third vector is
The resultant vector is

$$
v=v_{1}+v_{2}+v_{3}=\left\langle\frac{15 \sqrt{2}}{2}, \frac{35 \sqrt{2}}{2}\right\rangle
$$

the magnitude is

$$
\|v\|=\sqrt{\left(\frac{15 \sqrt{2}}{2}\right)^{2}+\left(\frac{35 \sqrt{2}}{2}\right)^{2}}=5 \sqrt{29} \mathrm{~cm}
$$



## Lesson 18: Vectors and Translation Maps

## Student Outcomes

- Students use vectors to define a translation map and translate geometric figures in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$.
- Students represent and perform calculations with three-dimensional vectors and understand operations in three dimensions that are analogous to operations in two dimensions.


## Lesson Notes

This lesson builds on the use of vectors to represent shifts by using vectors to define translation maps studied in prior lessons. The connection between matrices and vectors should become more apparent in this lesson, and we even use a notation for vectors that recalls matrix notation. This lesson focuses on several N.VM standards including N-VM.A.1, N-VM.A.3, N-VM.B.4a, N-VM.V.4c, and N-VM.B.5, extending vector addition, subtraction, and scalar multiplication to $\mathbb{R}^{3}$. This lesson lends itself to working in a program capable of creating three-dimensional graphs, such as GeoGebra (v5.0 or later). You can model these lessons on one computer or, if you have access to a lab, have students work on their own computers in the latter portion of this lesson.

## Classwork

## Opening Exercise (5 minutes)

Activate student knowledge about the previous day's lesson with this short Opening Exercise. Students should work independently at first, and then check their solutions with a partner. Students could draw their vectors using GeoGebra. If using GeoGebra, type vector $[(\mathbf{a}, \mathbf{b})]$ in the input box where $\mathbf{a}$ and $\mathbf{b}$ are the horizontal and vertical components of the vector. In GeoGebra, the vector $\mathbf{v}=\langle 1,2\rangle$ would be shown as $\mathbf{v}=\binom{1}{2}$.

## Opening Exercise

Write each vector described below in component form and find its magnitude. Draw an arrow originating from ( $\mathbf{0}, \mathbf{0}$ ) to represent each vector's magnitude and direction.
a. Translate 3 units right and 4 units down.

$$
u=\langle 3 .-4\rangle,\|u\|=5
$$

b. Translate 6 units left.

$$
v=\langle-6,0\rangle,\|v\|=6
$$

c. Translate 2 units left and 2 units up.

$$
w=\langle-2,2\rangle,\|w\|=2 \sqrt{2}
$$



## d. Translate 5 units right and 7 units up.

$$
\mathbf{t}=\langle 5,7\rangle,\|\mathbf{t}\|=\sqrt{25+49}=\sqrt{74}
$$

## Discussion (5 minutes)

Lead a short discussion to model how to write a vector as a translation map. Introduce the column notation for vectors and ask students to recall when they have seen something similar in previous lessons. They should recall it from Module 1 and from the beginning of this module.

- We can use a translation map to represent a vector. In fact, we can make this the definition of a vector: A vector is a translation map.
- For example, the vector $\mathbf{v}=\langle 2,1\rangle$ is identified with the translation map, $T_{\mathrm{v}}$ that maps a point $\boldsymbol{x}=\left[\begin{array}{l}x \\ y\end{array}\right]$ to the point $\boldsymbol{x}+\left[\begin{array}{l}2 \\ 1\end{array}\right]=\left[\begin{array}{l}x \\ y\end{array}\right]+\left[\begin{array}{l}2 \\ 1\end{array}\right]=\left[\begin{array}{l}x+2 \\ y+1\end{array}\right]$; thus

$$
T_{\mathbf{v}}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{l}
x+2 \\
y+1
\end{array}\right]
$$

- Why would it make sense to think of a vector as a translation map?
- Because vectors accomplish the same thing as a translation map. An object is translated by the components of the vector in the horizontal and vertical direction. The object's size and shape would remain unchanged under this type of transformation.
- Explain how writing vectors in a column makes it easier to think of them as translation maps.
- We are using the same kind of notation we used when working with translation maps and linear transformations in the previous module and earlier in this module.
Give students time to summarize this information in their notes or on the student materials.


## Exercises 1-3 (5 minutes)

Provide students with the opportunity to work in small groups on the next exercises. Check for precision and accuracy in their work with the translation map notation applied to vectors. Make sure they are thinking carefully about the transformations of the circle under the mapping in Exercises 2 and 3 and check to make sure they are creating the equations of the image figures correctly.

## Scaffolding:

- GeoGebra or other graphing software can be a powerful tool for scaffolding this lesson, particularly for students who are struggling to quickly sketch graphs of circles and lines by hand
- Post the formulas for a line and a circle on the board prior to this lesson and have students pair share with a partner what they recall about these equations and the related graphs.
- A line with slope $-\frac{a}{b}$ and $y$-intercept $\left(0, \frac{c}{b}\right)$ is given by $a x+b y=c$.
- A circle with radius $r$ and center $(h, k)$ is given by $(x-h)^{2}+(y-k)^{2}=r^{2}$
- For the following equations, have students sketch the graph and state the key features.

$$
\begin{gathered}
2 x-6 y=12 \\
x+2 y=-5 \\
(x-1)^{2}+(y-3)^{2}=9 \\
(x+3)^{2}+(y-2)^{2}=5
\end{gathered}
$$

## Exercises 1-3

1. Write a translation map defined by each vector from the opening.

Consider the vector $\mathrm{v}=\langle-2,5\rangle$, and its associated translation map:
$T_{\mathrm{v}}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x-2 \\ y+5\end{array}\right]$

| Vector in Component Form | Translation Map, $T_{v}$ |
| :---: | :---: |
| $u=\langle 3 .-4\rangle$ | $T_{u}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x+3 \\ y-4\end{array}\right]$ |
| $v=\langle-6.0\rangle$ | $T_{v}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x-6 \\ y\end{array}\right]$ |
| $w=\langle-2.2\rangle$ | $T_{w}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x-2 \\ y+2\end{array}\right]$ |
| $t=\langle 5,7\rangle$ | $T_{t}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x+5 \\ y+7\end{array}\right]$ |

2. Suppose we apply the translation map $T_{v}$ to each point on the circle $(x+4)^{2}+(y-3)^{2}=25$.
a. What is the radius and center of the original circle?

The radius is 5 and the center is $(-4,3)$.
b. Show that the image points satisfy the equation of another circle.

Suppose that $T_{v}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x^{\prime} \\ y^{\prime}\end{array}\right]$. Then $x^{\prime}=x-2$ and $y^{\prime}=y+5$, so $x=x^{\prime}+2$ and $y=y^{\prime}-5$.
Since $(x+4)^{2}+(y-3)^{2}=25$, we have

$$
\begin{aligned}
(x+4)^{2}+(y-3)^{2} & =25 \\
\left(\left(x^{\prime}+2\right)+4\right)^{2}+\left(\left(y^{\prime}-5\right)-3\right)^{2} & =25 \\
\left(x^{\prime}+6\right)^{2}+\left(y^{\prime}-8\right)^{2} & =25
\end{aligned}
$$

So the image points $\left[\begin{array}{l}x^{\prime} \\ y^{\prime}\end{array}\right]$ lie on a circle.
c. What is center and radius of this image circle?


The radius of 5 will remain unchanged. The new center will be $(-6,8)$.
3. Suppose we apply the translation map $T_{v}$ to each point on the line $2 x-3 y=10$.
a. What are the slope and $y$-intercept of the original line?

The slope is $\frac{2}{3}$ and the $y$-intercept is $\left(0,-\frac{10}{3}\right)$.
b. Show that the image points satisfy the equation of another line.

Suppose that $T_{v}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x^{\prime} \\ y^{\prime}\end{array}\right]$. Then $x^{\prime}=x-2$ and $y^{\prime}=y+5$, so $x=x^{\prime}+2$ and $y=y^{\prime}-5$.
Since $2 x-3 y=10$, we have

$$
\begin{gathered}
2\left(x^{\prime}+2\right)-3\left(y^{\prime}-5\right)=10 \\
2 x^{\prime}+4-3 y^{\prime}+15=10 \\
2 x^{\prime}-3 y=-9
\end{gathered}
$$

c. What are the slope and $y$-intercept of this image line?

The slope is $\frac{2}{3}$ and the $y$-intercept is $(0,3)$.


## Discussion (5 minutes): Vectors in Three Dimensions

A vector in two dimensions can be used to define a translation map that translates objects in the coordinate plane by the horizontal and vertical components of the vector.

- What types of objects could we translate by a vector in two dimensions?
- Any geometric figure could be translated by a vector.

This idea is easily extended to $\mathbb{R}^{3}$, the Cartesian coordinate system in three dimensions. For example, the vector $\mathbf{v}=$ $\langle 1,3,5\rangle$ is a translation in space 1 unit in the $x$ direction, 3 units in the $y$ direction and 5 units in the $z$ direction.

- What would be the associated translation map for the vector $\mathbf{v}=\langle 1,3,5\rangle$ ?

$$
\text { - The associated translation map would be } \boldsymbol{T}_{\mathbf{v}}\left(\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]\right)=\left[\begin{array}{l}
x+1 \\
y+3 \\
z+5
\end{array}\right] \text {. }
$$

- What types of objects could we represent in three dimensions?
- Points, planes, spheres, cubes, cylinders, pyramids, or other solid figures could be represented in three dimensions. Two-dimensional figures can also be represented in three dimensions.

Give students an opportunity to summarize this information in their notes or on the student pages before starting the exercises that follow.

## Example 1 (8 minutes): Vectors and Translation Maps in $\mathbb{R}^{3}$

If technology is available, you can model this example using GeoGebra by selecting 3-D Graphs from the view menu. Type each figure into the input bar and then type in vector $[(1,3,5)]$. Use the "translate by vector" feature in the translate menu. Students will be able to see the equations updating and the image graph displayed along with the preimage when you use this feature. If technology is not available, use 3-dimensional graph paper or isometric graph paper. 3-D graph paper is included in Lesson 5 or can be

## Scaffolding:

Help students graph in 3-D by using 3-D or isometric graph paper. 3-D graph paper is included in Lesson 5. downloaded for free at http://www.waterproofpaper.com/graph-paper/isometric-graphing-paper.pdf.

Take time to help students make sense of the graphs they are seeing of each object. Point out the similarities between points, planes, and spheres in three dimensions and points, lines, and circles in two dimensions.

- How do the coordinates of a point in two dimensions compare to coordinates in three dimensions?
- You just add another coordinate for the distance from the origin in the $z$ direction.
- How does the equation of a line in two dimensions compare to the equation of a plane in three dimensions?
- You just add another linear term using the variable z. The intercepts are similar with two coordinates being 0 and the third coordinate giving the point where the plane crosses the given axis.
- How does the equation of a circle in two dimensions compare to the equation of a sphere in three dimensions?
- You just add another quadratic term using the variable $z$. The radius is the cube root of the constant and the center is the same except three coordinates.

Example 1: Vectors and Translation Maps in $\mathbb{R}^{3}$
Translate by the vector $v=\langle 1,3,5\rangle$ by applying the translation map $T_{v}$ to the following objects in $\mathbb{R}^{3}$. A sketch of the original object and the vector is shown. Sketch the image.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]\right)=\left[\begin{array}{l}
x+1 \\
y+3 \\
z+5
\end{array}\right]
$$

a. The point $A(2,-2,4)$


The new point will be $(2+1,-2+3,4+5)=(3,1,9)$
b. The plane $2 x+3 y-z=0$


The new plane will be $2(x-1)+3(y-3)-(z-5)=0$, which is equivalent to $2 x+3 y-z=6$.
c. $\quad$ The sphere $(x-1)^{2}+(y-3)^{2}+z^{2}=9$.



The new sphere will be $(x-(1+1))^{2}+(y-(3+3))^{2}+(z-(0+5))^{2}=9$, which is equivalent to $(x-2)^{2}+(y-6)^{2}+(x-5)^{2}=9$.

After sharing the discussing the solutions to Example 1, conclude with a brief discussion of the results.

- When we translated a line by a vector, why was the image parallel to the original line?
- All points in the plane were moved by the same amount in the same direction due to the translation map.
- Why does it make sense that when you translate a plane by a vector the image plane is parallel to the preimage?
- If every point moves the same distance in the same direction, then the planes will have the same orientation in space but a different location.
- When translating any geometric figure by a vector will the image be congruent to the pre-image? Explain how you know.
- Since translations are rigid transformations, the image will be congruent to the pre-image.


## Exercise 4 (2 minutes)

This short exercise will allow you to determine whether or not students are able to extend their thinking about vectors to three dimensions.

## Exercise 4

4. $\quad$ Given the sphere $(x+3)^{2}+(y-1)^{2}+(z-3)^{2}=10$.
a. What are its center and radius?

The center is $(-3,1,3)$, and the radius is $\sqrt{\mathbf{1 0}}$.
b. Write a vector and its associated translation map that would take this sphere to its image centered at the origin.
We need to translate the center from the point $(-3,1,3)$ to the point $(0,0,0)$. This represents a translation of 3 units in the $x$ direction, -1 unit in the $y$ direction, and -3 units in the $z$ direction.

The vector is $\mathrm{v}=\langle 3,-1,-3\rangle$ and the translation map would be

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]\right)=\left[\begin{array}{l}
x+3 \\
y-1 \\
z-3
\end{array}\right]
$$

## Example 2 ( 5 minutes): What is the Magnitude of a Vector in $\mathbb{R}^{3}$ ?

You can provide less scaffolding by asking students to determine an expression for the magnitude of $v$ shown in the diagram below by simply showing the diagram on the board or document camera without the additional questions. Notice that $\mathbf{v}$ is a vector, but that $x, y, z$, and $w$ are real numbers that represent lengths. For students that need a more concrete approach, label the terminal point of the vector $\mathbf{v}$ with an actual set of coordinates such as $(2,8,6)$ and then model how to find the magnitude of the vector before working through the problem in general.

- Use the Pythagorean theorem to write an equation relating $x, y$, and $w$.

$$
x^{2}+y^{2}=w^{2}
$$

- Use the Pythagorean theorem to write an equation relating $w, z$, and $\|\mathbf{v}\|$.
- $w^{2}+z^{2}=\|\mathbf{v}\|^{2}$
- Now write an equation relating $\|\mathbf{v}\|, x, y$, and $z$.
- $x^{2}+y^{2}=w^{2}$

ㅁ $w^{2}=\|\mathbf{v}\|^{2}-z^{2}$
ㅁ $x^{2}+y^{2}=\|\mathbf{v}\|^{2}-z^{2}$

- $x^{2}+y^{2}+z^{2}=\|\mathbf{v}\|^{2}$

Example 2: What is the Magnitude of a Vector in $\mathbb{R}^{3}$ ?

a. Find a general formula for $\|\mathbf{v}\|^{2}$.

$$
\|\mathbf{v}\|^{2}=x^{2}+y^{2}+z^{2}
$$

b. Solve this equation for $\|v\|$ to find the magnitude of the vector.

$$
\|v\|=\sqrt{x^{2}+y^{2}+z^{2}}
$$

## Exercises 5-8 (5 minutes)

In Exercise 5, students practice finding the magnitude of a vector in three dimensions. Direct students to work independently on Exercise 5 first, and then have them work with a partner to check their results.

## Exercises 5-8

5. Which vector has greater magnitude, $v=\langle 0,5,-4\rangle$ or $u=\langle 3,-4,4\rangle$ ? Show work to support your answer.

$$
\begin{aligned}
& \|v\|=\sqrt{0^{2}+5^{2}+(-4)^{2}}=\sqrt{41} \\
& \|u\|=\sqrt{3^{2}+(-4)^{2}+4^{2}}=\sqrt{41}
\end{aligned}
$$

These vectors have equal magnitude.
6. Explain why vectors can have equal magnitude but not be the same vector.

A vector has both a magnitude and a direction. If you graphed the vectors from Exercise 5, you can see they point in different directions so they cannot be the same even though they had equal magnitude.

Give students time to consider how the rules and representations learned in the previous lesson extend to three dimensions. If needed, scaffold this exercise by providing specific example of vectors for students to work with such as those shown in the table below before asking them to write general rules.

|  | Vectors in $\mathbb{R}^{2}$ | Vectors in $\mathbb{R}^{3}$ |
| :---: | :---: | :---: |
| Component Form | $\langle 2,3\rangle$ | $\langle 2,3,4\rangle$ |
| Column Form | $\left[\begin{array}{l}2 \\ 3\end{array}\right]$ | $\left[\begin{array}{l}2 \\ 3 \\ 4\end{array}\right]$ |
| Magnitude | $\\|\mathbf{v}\\|=\sqrt{2^{2}+3^{2}}=\sqrt{13}$ | $\\|\mathbf{v}\\|=\sqrt{2^{2}+3^{2}+4^{2}}=\sqrt{29}$ |
| Addition | $\begin{gathered} \text { If } \mathbf{v}=\langle 2,3\rangle \text { and } \mathbf{u}=\langle 2,-4\rangle, \\ \text { Then } \\ \mathbf{v}+\mathbf{u}=\langle 2+2,3+(-4)\rangle=\langle 4,-1\rangle \end{gathered}$ | If $\mathbf{v}=\langle 2,3,4\rangle$ and $\mathbf{u}=\langle 2,-4,1\rangle$, <br> Then $\mathbf{v}+\mathbf{u}=\langle 2+2,3+(-4), 4+1\rangle=\langle 4,-1,5\rangle$ |
| Subtraction | $\begin{gathered} \text { If } \mathbf{v}=\langle 2,3\rangle \text { and } \mathbf{u}=\langle 2,-4\rangle \\ \text { Then } \mathbf{v}+\mathbf{u}=\langle 2-2,3-(-4)\rangle=\langle 0,7\rangle \end{gathered}$ | If $\mathbf{v}=\langle 2,3,4\rangle$ and $\mathbf{u}=\langle 2,-4,1\rangle$, <br> Then $\mathbf{v}+\mathbf{u}=\langle 2-2,3-(-4), 4-1\rangle=\langle 0,7,3\rangle$ |
| Scalar <br> Multiplication | $\begin{gathered} \text { If } \mathbf{v}=\langle 2,3\rangle \text { then } \\ 2 \mathbf{v}=\langle 2 \cdot 2,2 \cdot 3\rangle=\langle 4,6\rangle \end{gathered}$ | $\begin{gathered} \text { If } \mathbf{v}=\langle 2,3,4\rangle \text { then } \\ 2 \mathbf{v}=\langle 2 \cdot 2,2 \cdot 3,2 \cdot 4\rangle=\langle 4,6,8\rangle \end{gathered}$ |

7. Vector arithmetic in $\mathbb{R}^{3}$ is analogous to vector arithmetic in $\mathbb{R}^{2}$. Complete the graphic organizer to illustrate these ideas.

|  | Vectors in $\mathbb{R}^{2}$ | Vectors in $\mathbb{R}^{3}$ |
| :---: | :---: | :---: |
| Component Form | $\langle\boldsymbol{a}, \boldsymbol{b}\rangle$ | $\langle a, b, c\rangle$ |
| Column Form | $\left[\begin{array}{l}a \\ b\end{array}\right]$ | $\left[\begin{array}{l}a \\ b \\ c\end{array}\right]$ |
| Magnitude | $\\|\mathrm{v}\\|=\sqrt{a^{2}+b^{2}}$ | $\\|\mathrm{v}\\|=\sqrt{a^{2}+b^{2}+c^{2}}$ |
| Addition | If $\mathbf{v}=\langle\boldsymbol{a}, \boldsymbol{b}\rangle$ and $\mathbf{u}=\langle\boldsymbol{c}, \boldsymbol{d}\rangle$, <br> Then $\mathbf{v}+\mathbf{u}=\langle\boldsymbol{a}+\boldsymbol{c}, \boldsymbol{b}+\boldsymbol{d}\rangle$ | If $\mathrm{v}=\langle a, b, c\rangle$ and $\mathrm{u}=\langle\boldsymbol{d}, \boldsymbol{e}, \boldsymbol{f}\rangle$, <br> Then $\mathrm{v}+\mathrm{u}=\langle a+d, b+e, c+f\rangle$ |
| Subtraction | If $\mathbf{v}=\langle\boldsymbol{a}, \boldsymbol{b}\rangle$ and $\mathbf{u}=\langle\boldsymbol{c}, \boldsymbol{d}\rangle$, <br> Then $\mathbf{v}-\mathbf{u}=\langle\boldsymbol{a}-\boldsymbol{c}, \boldsymbol{b}-\boldsymbol{d}\rangle$ | If $\mathrm{v}=\langle a, b, c\rangle$ and $\mathrm{u}=\langle d, e, f\rangle$, <br> Then $\mathrm{v}-\mathrm{u}=\langle a-d, b-e, c-f\rangle$ |
| Scalar <br> Multiplication | If $v=\langle a, b\rangle$ and $k$ is a real number $k v=\langle k a, k b\rangle$ | $\begin{aligned} & \text { If } \mathrm{v}=\langle a, b, c\rangle \text { and } k \text { is a real number } \\ & k v=\langle k a, k b, k c\rangle \end{aligned}$ |

Before starting Exercise 8, make sure students have correct information in the graphic organizer. You can show a completed one to the class and if time permits have different groups explain how they got their answers for each row. Allow students time to correct any errors they may have made.
8. Given $v=\langle 2,0,-4\rangle$ and $u=\langle-1,5,3\rangle$.
a. Calculate the following.
i. $\quad \mathbf{v}+\mathbf{u}$

$$
\mathbf{v}+\mathbf{u}=\langle 2+(-1), 0+5,-4+3\rangle=\langle 1,5,-1\rangle
$$

ii. $\quad \mathbf{2 v}-u$

$$
2 v-u=\langle 2 \cdot 2-(-1), 2 \cdot 0-5,2 \cdot(-4)-3\rangle=\langle 5,-5,-11\rangle
$$

iii. $\|v\|$

$$
\|v\|=\sqrt{(-1)^{2}+5^{2}+3^{2}}=\sqrt{35}
$$

b. Suppose the point $(1,3,5)$ is translated by $v$ and then by $u$. Determine a vector $w$ that would return the point back to its original location $(1,3,5)$.

From part (a), we have $\mathbf{v}+\mathbf{u}=\langle 1,5,-1\rangle$. The point will be translated from $(1,3,5)$ to $(2,8,4)$ since $1+1=2,5+3=8$, and $5-1=4$. The vector that will return this point to its original location will be the opposite of $\mathrm{v}+\mathrm{u}$.

$$
-(v+u)=\langle-1,-5,1\rangle
$$

And this vector will translate $(2,8,4)$ back to $(1,3,5)$ because $2-1=1,8-5=3$, and $4+1=5$.

## Closing (2 minutes)

Give students time to respond to the questions below individually in writing or with a partner.

- Why can we represent vectors as translation maps?
- They mean the same thing geometrically.
- How are two-dimensional vectors the same as three-dimensional vectors, and how are they different?
- Either dimension has a magnitude and direction and represents a translation or shift. They are different because you need one additional component to describe a vector in three dimensions.

The graphic organizer in Exercise 7 can serve as a lesson summary of vector arithmetic in three dimensions along with the information about translation maps shown below.

## Lesson Summary

A vector $v$ can define a translation map $T_{v}$ that takes a point to its image under the translation. Applying the map to the set of all points that make up a geometric figure serves to translate the figure by the vector.

Exit Ticket (3 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 18: Vectors and Translation Maps

## Exit Ticket

1. Given the vector $\mathbf{v}=\langle 2,-1\rangle$, find the image of the line $3 x-2 y=2$ under the translation map $T_{\mathbf{v}}$. Graph the original line and its image, and explain the geometric effect of the map $T_{\mathbf{v}}$.
2. Given the vector $\mathbf{v}=\langle-1,2\rangle$, find the image of the circle $(x-2)^{2}+(y+1)^{2}=4$ under the translation $\operatorname{map} T_{\mathbf{v}}$. Graph the original circle and its image, and then explain the geometric effect of the map $T_{\mathbf{v}}$.

## Exit Ticket Sample Solutions

1. Given the vector $v=\langle 2,-1\rangle$, find the image of the line $3 x-2 y=2$ under the translation map $T_{v}$. Graph the original line and its image, and explain the geometric effect of the map $T_{\mathrm{v}}$.
$T_{v}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\binom{x-2}{y+1} ;$ image of the original line: $3(x+2)-2(y-1)=2$.
$3 x+6-2 y+2=2,3 x-2 y=-6$.
Every point on the line is shifted 2 units to the left and 1 unit upward.

2. Given the vector $v=\langle-1,2\rangle$, find the image of the circle $(x-2)^{2}+(y+1)^{2}=4$ under the translation map $T_{\mathrm{v}}$. Graph the original circle and its image, and explain the geometric effect of the map $T_{\mathrm{v}}$.
$T_{v}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\binom{x-1}{y+2} ;$ image of original circle: $(x-1)^{3}+(y-1)^{2}=4$.
Every point on the circle is shifted one unit to the left and two units upward. The new center is $(1,1)$ and the radius $r=2$ stays the same.


## Problem Set Sample Solutions

1. Myishia says that when applying the translation map $T_{\mathrm{v}}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x+1 \\ y-2\end{array}\right]$ to a set of points given by an equation relating $x$ and $y$, we should replace every $x$ that is in the equation by $x+1$, and $y$ by $y-2$. For example, the equation of the parabola $y=x^{2}$ would become $y-2=(x+1)^{2}$. Is she correct? Explain your answer.
No, she is not correct. What she did will translate the points on the parabola $y=x^{2}$ in the opposite direction-one unit to the left and 2 units upward, which is not the geometric effect of $T_{v}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x+1 \\ y-2\end{array}\right]$. In order to have the correct translation based on $T_{v}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x+1 \\ y-2\end{array}\right]$, we need to set $x^{\prime}=x+1$ and $y^{\prime}=y-2$, which is equivalent to $x=x^{\prime}-1$ and $y=y^{\prime}+2$. This process gives the transformed equation $y^{\prime}+2=\left(x^{\prime}-1\right)^{2}$, which we write as

$$
y+2=(x-1)^{2}
$$

2. Given the vector $\mathrm{v}=\langle-1,3\rangle$, find the image of the line $x+y=1$ under the translation map $T_{\mathrm{v}}$. Graph the original line and its image, and explain the geometric effect of the map $T_{\mathrm{v}}$ on the line.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{l}
x-1 \\
y+3
\end{array}\right],(x-(-1))+(y-(3))=1, x+1+y-3=1, x+y=3
$$

Every point on the line is shifted one unit left and three units upward. The slopes of the lines remain -1.

3. Given the vector $\mathrm{v}=\langle 2,1\rangle$, find the image of the parabola $y-1=x^{2}$ under the translation map $T_{\mathrm{v}}$. Draw a graph of the original parabola and its image, and explain the geometric effect of the map $T_{v}$ on the parabola. Find the vertex and $x$-intercepts of the graph of the image.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{l}
x+2 \\
y+1
\end{array}\right], y-2=(x-2)^{2}
$$

Every point on the parabola is shifted two units to the right and one unit upward.
The vertex is $(2,2)$, and there are no $x$-intercepts.

4. Given the vector $\mathrm{v}=\langle 3,2\rangle$, find the image of the graph of $y+1=(x+1)^{3}$ under the translation map $T_{\mathrm{v}}$. Draw the original graph and its image, and explain the geometric effect of the map $T_{v}$ on the graph. Find the $x$-intercepts of the graph of the image.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{l}
x+3 \\
y+2
\end{array}\right], y-1=(x-2)^{3}
$$

Every point on the curve is shifted three units to the right and two units upward.

The $x$-intercept is 1 .

5. Given the vector $v=\langle 3,-3\rangle$, find the image of the graph of $y+2=\sqrt{x+1}$ under the translation map $T_{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_{\mathrm{v}}$ on the graph. Find the $x$-intercepts of the graph of the image.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{l}
x+3 \\
y-3
\end{array}\right], y+5=\sqrt{x-2}
$$

Every point on the curve is shifted three units to the right and three units downward.
The $x$-intercept is 27 .

6. Given the vector $\mathrm{v}=\langle-1,-2\rangle$, find the image of the graph of $y=\sqrt{9-x^{2}}$ under the translation map $T_{\mathrm{v}}$. Draw the original graph and its image, and explain the geometric effect of the map $T_{v}$ on the graph. Find the $x$-intercepts of the graph of the image.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{l}
x-1 \\
y-2
\end{array}\right], y+2=\sqrt{9-(x+1)^{2}}
$$

Every point on the semicircle is shifted one unit to the left and two units downward.
$x$-intercepts: $1 \pm \sqrt{5}$.

7. Given the vector $v=\langle 1,3\rangle$, find the image of the graph of $y=\frac{1}{x+2}+1$ under the translation map $T_{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_{v}$ on the graph. Find the equations of the asymptotes of the graph of the image.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{l}
x+1 \\
y+3
\end{array}\right], y=\frac{1}{x+1}+4
$$

Every point on the curve is shifted one unit to the right and three units upward.
The new vertical asymptote is $x=-1$, the new horizontal asymptote is $y=4$.

8. Given the vector $v=\langle-1,2\rangle$, find the image of the graph of $y=|x+2|+1$ under the translation map $T_{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_{v}$ on the graph. Find the $x$-intercepts of the graph of the image.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{l}
x-1 \\
y+2
\end{array}\right], y=|x+3|+3
$$

Every point on the graph is shifted one unit to the left and two units upward.
There are no $x$-intercepts.

9. Given the vector $v=\langle 1,-2\rangle$, find the image of the graph of $y=2^{x}$ under the translation map $T_{v}$. Draw the original graph and its image, and explain the geometric effect of the map $T_{v}$ on the graph. Find the $x$-intercepts of the graph of the image.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{l}
x+1 \\
y-2
\end{array}\right], y=2^{x-1}-2
$$

Every point on the curve is shifted one unit to the right and two units downward.
$x$-intercept is 2.

10. Given the vector $\mathrm{v}=\langle-1,3\rangle$, find the image of the graph of $y=\log _{2} x$, under the translation map $T_{\mathrm{v}}$. Draw the original graph and its image, and explain the geometric effect of the map $T_{v}$ on the graph. Find the $x$-intercepts of the graph of the image.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{l}
x-1 \\
y+3
\end{array}\right], y=\log _{2}(x+1)+3
$$

Every point on the curve is shifted one unit to the left and three units upward.

$$
x \text {-intercept: }-\frac{7}{8}
$$


11. Given the vector $v=\langle 2,-3\rangle$, find the image of the graph of $\frac{x^{2}}{4}+\frac{y^{2}}{16}=1$ under the translation map $T_{\mathrm{v}}$. Draw the original graph and its image, and explain the geometric effect of the map $T_{v}$ on the graph. Find the new center, major and minor axis of the graph of the image.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y
\end{array}\right]\right)=\left[\begin{array}{l}
x+2 \\
y-3
\end{array}\right], \frac{(x-2)^{2}}{4}+\frac{(y+3)^{2}}{16}=1
$$

Every point on the ellipse is shifted two units to the right and three units downward.
The new center is $(2,-3)$, the major axis is 4, minor axis is 2 .

12. Given the vector v , find the image of the given point $P$ under the translation map $T_{\mathrm{v}}$. Graph $P$ and its image.
a. $\quad \mathbf{v}=\langle 3,2,1\rangle, P=\left[\begin{array}{l}1 \\ 2 \\ 3\end{array}\right]$

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]\right)=\left[\begin{array}{l}
x+3 \\
y+2 \\
z+1
\end{array}\right] \text {, the new image: }\left[\begin{array}{l}
4 \\
4 \\
4
\end{array}\right]
$$


b. $\quad v=\langle-2,1,-1\rangle, P=\left[\begin{array}{c}2 \\ -1 \\ -4\end{array}\right]$

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]\right)=\left[\begin{array}{l}
x-2 \\
y+1 \\
z-1
\end{array}\right] \text {, the new image: }\left[\begin{array}{c}
0 \\
0 \\
-5
\end{array}\right]
$$


13. Given the vector v , find the image of the given plane under the translation map $T_{\mathrm{v}}$. Sketch the original vector and its image.
a. $\quad v=\langle 2,-1,3\rangle, 3 x-2 y-z=0$,

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]\right)=\left[\begin{array}{l}
x+2 \\
y-1 \\
z+3
\end{array}\right], 3 x-2 y-z=5
$$


b. $\quad v=\langle-1,2,-1\rangle, 2 x-y+z=1$.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]\right)=\left[\begin{array}{l}
x+2 \\
y-1 \\
z+3
\end{array}\right], 3 x-y+z=-4
$$


14. Given the vector v , find the image of the given sphere under the translation map $T_{\mathrm{v}}$. Sketch the original sphere and its image.
a. $\quad v=\langle-1,2,3\rangle, x^{2}+y^{2}+z^{2}=9$.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]\right)=\left[\begin{array}{l}
x-1 \\
y+2 \\
z+3
\end{array}\right],(x+1)^{2}+(y-2)^{2}+(z-3)^{2}=9
$$


b. $\quad v=\langle-3,-2,1\rangle,(x+2)^{2}+(y-3)^{2}+(z+1)^{2}=1$.

$$
T_{v}\left(\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]\right)=\left[\begin{array}{l}
x-3 \\
y-2 \\
z+1
\end{array}\right],(x+5)^{2}+(y-1)^{2}+(z)^{2}=1
$$


15. Find a vector v and translation map $T_{\mathrm{v}}$ that will translate the line $x-y=1$ to the line $x-y=-3$. Sketch the original vector and its image.
Answers vary. For example, $\mathrm{v}=\left[\begin{array}{l}0 \\ 4\end{array}\right]$ and $T_{v}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x+0 \\ y+4\end{array}\right]$.

16. Find a vector v and translation map $T_{\mathrm{v}}$ that will translate the parabola $y=x^{2}+4 x+1$ to the parabola $y=x^{2}$ Because $y=x^{2}+4 x+1$ can be written as $y+3=(x+2)^{2}, v=\left[\begin{array}{c}-2 \\ 3\end{array}\right]$ and $T_{v}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x+2 \\ y-3\end{array}\right]$.
17. Find a vector v and translation map $T_{\mathrm{v}}$ that will translate the circle with equation $x^{2}+y^{2}-4 x+2 y-4=0$ to the circle with equation $(x+3)^{2}+(y-4)^{2}=9$

Because $x^{2}-4 x+y^{2}+2 y=4$ can be written as $(x-2)^{2}+(y+1)^{2}=9, v=\left[\begin{array}{c}5 \\ -5\end{array}\right]$ and $T_{v}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x-5 \\ y+5\end{array}\right]$.
18. Find a vector v and translation map $T_{\mathrm{v}}$ that will translate the graph of $y=\sqrt{x-3}+2$ to the graph of $y=\sqrt{x+2}-3$.
$\mathbf{v}=\left[\begin{array}{c}5 \\ -5\end{array}\right]$ and $T_{v}\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{l}x-5 \\ y+5\end{array}\right]$
19. Find a vector $v$ and translation map $T_{v}$ that will translate the sphere $(x+2)^{2}+(y-3)^{2}+(z+1)^{2}=1$ to the sphere $(x-3)^{2}+(y+1)^{2}+(z+2)^{2}=1$
$\mathrm{v}=\left[\begin{array}{c}-5 \\ 4 \\ 1\end{array}\right]$ and $T_{v}\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=\left[\begin{array}{c}x+5 \\ y-4 \\ z-1\end{array}\right]$
20. Given vectors $u=\langle 2,-1,3\rangle, v=\langle 2,0,-2\rangle$, and $w=\langle-3,6,0\rangle$, find the following.
a. $\quad \mathbf{3 u}+\mathbf{v}+\mathbf{w}$

$$
\langle 5,4,7\rangle
$$

b. $\quad w-2 v-u$

$$
\langle-9,7,1\rangle
$$

c. $\quad 3\left(2 u-\frac{1}{2} v\right)-\frac{1}{3} w$

$$
\langle 10,-8,21\rangle
$$

d. $\quad-2 u-3(5 v-3 w)$.
$\langle-61,56,24\rangle$
e. $\quad\|\mathbf{u}\|,\|v\|$, and $\|w\|$.

$$
\|u\|=\sqrt{14}, \quad\|v\|=2 \sqrt{2},\|w\|=\sqrt{45}
$$

f. Show that $2\|\mathbf{v}\|=\|2 \mathbf{v}\|$.

$$
2\|v\|=2(2 \sqrt{2})=4 \sqrt{2}, \quad\|2 v\|=\|\langle 4,0,-4\rangle\|=\sqrt{32}=4 \sqrt{2}
$$

g. Show that $\|\mathbf{u}+\mathbf{v}\| \neq\|\mathbf{u}\|+\|\mathbf{v}\|$.

$$
\|u+v\|=\|\langle 4,-1,1\rangle\|=\sqrt{18}, \quad\|u\|+\|v\|=\sqrt{14}+2 \sqrt{2}
$$

h. Show that $\|\mathbf{v}-\mathbf{w}\| \neq\|\mathbf{v}\|-\|\mathbf{w}\|$.

$$
\|v-w\|=\|\langle 5,-6,-2\rangle\|=\sqrt{65}, \quad\|v\|-\|w\|=2 \sqrt{2}-\sqrt{45}
$$

i. $\quad \frac{1}{\|\mathbf{u}\|} \mathbf{u}$ and $\left\|\frac{1}{\|\mathbf{u}\|} \mathbf{u}\right\|$.

$$
\begin{aligned}
\frac{1}{\|\mathbf{u}\|} \mathbf{u}=\frac{\langle 2,-1,3\rangle}{\sqrt{14}}= & \left\langle\frac{2}{\sqrt{14}},-\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right\rangle,\left\|\frac{1}{\|\mathbf{u}\|} \mathbf{u}\right\|=\sqrt{\left(\frac{2}{\sqrt{14}}\right)^{2}+\left(-\frac{1}{\sqrt{14}}\right)^{2}+\left(\frac{3}{\sqrt{14}}\right)^{2}}=\sqrt{\frac{4}{14}+\frac{1}{14}+\frac{9}{14}} \\
& =1
\end{aligned}
$$

## Student Outcomes

- Students find the components of a vector by subtracting the coordinates of an initial point from the coordinates of a terminal point.
- Students add vectors end-to-end using the parallelogram rule or component-wise. They understand vector arithmetic from a geometric perspective.


## Lesson Notes

This lesson introduces students to directed line segments and how to subtract initial point coordinates from terminal point coordinates to find the components of a vector. Vector arithmetic operations are reviewed and the parallelogram rule is introduced. Students should be able to do much of this lesson without the aid of technology. Continue to emphasize the connections between vector arithmetic and transformations.

## Classwork

## Discussion (3 minutes)

Start this lesson by introducing the new terminology relating to directed line segments. Have students read the text in the student materials, and then use the discussion questions below to clarify their understanding of vectors defined by an initial point and a terminal point.

A vector can be used to represent a translation that takes one point to an image point. The starting point is called the initial point, and the image point under the translation is called the terminal point.

initial point

If we know the coordinates of both points, we can easily determine the horizontal and vertical components of the vector.

## Scaffolding:

- Revise the Frayer model from Lesson 17 to include information about directed line segments, initial point, terminal point, and how to find the components by subtracting coordinates.
- For advanced learners, give them a vector in three dimensions and have them determine an initial point, terminal point, and magnitude.

$$
A(3,-2,1), B(-1,4,-3)
$$

Show the diagram below, and ask students the following questions:

- Which point is the initial point? Which point is the terminal point? How do you know?
- The initial point is $A$ and the terminal point is $B$. The end without the arrow tip is the initial point. The arrow indicates the direction of the translation of point $A$ to its new location at $B$.

- How many units is the initial point translated horizontally? How many units vertically? How do you know?
- 4 units horizontally and 2 units vertically. You can subtract corresponding coordinates or count the units on the grid.
- What are the components of the vector?
- $\quad \mathbf{v}=\langle 4,2\rangle$
- What is the magnitude of the vector?
- The magnitude is $\sqrt{20}=2 \sqrt{5}$.


## Exercises 1-3 (9 minutes)

Start students on the next exercise. Have them work independently for a few minutes, and then have them team up with a partner to share their work. Call on individual students to explain how they got each answer. Emphasize that we must calculate the components by starting from the initial point and ending at the terminal point. The signs of the components matter because they indicate direction.

## Exercises 1-3

1. Several vectors, represented by arrows, are shown below. For each vector, state the initial point, terminal point, component form of the vector and magnitude.


For $\mathbf{v}$,
Initial point $(2,4)$ and terminal point $(-2,5) ; \mathbf{v}=\langle-4,1\rangle$ and $\|v\|=\sqrt{17}$
For $\mathbf{u}$,
Initial point $(0,0)$ and terminal point $(4,2) ; u=\langle 4,2\rangle$ and $\|u\|=\sqrt{20}=2 \sqrt{5}$
For w,
Initial point $(-3,2)$ and terminal point $(-5,-2) ; w=\langle-2,-4\rangle$ and $\|w\|=\sqrt{20}=2 \sqrt{5}$
For a ,
Initial point $(-4,-5)$ and terminal point $(1,-2) ; \mathbf{a}=\langle 5,3\rangle$ and $\|\mathbf{a}\|=\sqrt{34}$
For b,
Initial point $(3,0)$ and terminal point $(3,-3) ; \mathbf{b}=\langle 0,-3\rangle$ and $\|\mathrm{b}\|=3$

If students did not struggle with Exercise 1, move them on to the next exercise which takes away their ability to count segments on the grid to determine the components of the vectors. Exercises 1-3 are designed to lead students to knowing that they can find the components by subtracting the coordinates of the initial point from the coordinates of the terminal point.
2. Several vectors, represented by arrows, are shown below. For each vector, state the initial point, terminal point, component form of the vector, and magnitude.


For v ,
Initial point $(-11,19)$ and terminal point $(15,24) ; v=\langle 26,5\rangle$ and $\|v\|=\sqrt{701}$
For $\mathbf{u}$,
Initial point $(5,15)$ and terminal point $(28,-7) ; \mathbf{u}=\langle 23,-22\rangle$ and $\|\mathbf{u}\|=\sqrt{1013}$
For $\mathbf{w}$,
Initial point $(0,0)$ and terminal point $(-10,-25) ; \mathbf{w}=\langle-10,-25\rangle$ and $\|w\|=\sqrt{725}$
3. Write a rule for the component form of the vector $v$ shown in the diagram. Explain how you got your answer.


The component form is $\mathrm{v}=\left\langle x_{2}-x_{1}, y_{2}-y_{1}\right\rangle$ The components of the vector are the distance $A$ is translated vertically and horizontally to get to $B$. To find the horizontal distance from $A$ to $B$, subtract the $x$-coordinates. To find the vertical distance from $A$ to $B$, subtract the $y$-coordinates.

Give students time to struggle with Exercise 3 before intervening. Emphasize that they must generalize the process used in Exercises 1 and 2 to write a rule in Exercise 3.

## Discussion (3 minutes)

Introduce the term directed line segment before students begin Exercises 4-6. Students use the results of Exercise 3 to find the components of vectors. The next exercises use the directed line segment notation.

When we use the initial and terminal points to describe a vector, we often refer to the vector as a directed line segment. $A$ vector or directed line segment with initial point $A$ and terminal point $B$ is denoted $\overrightarrow{A B}$.

## Exercises 4-7 (8 minutes)

Exercises 4-7
4. Write each vector in component form.
a. $\overrightarrow{A E}$

$$
\overrightarrow{A E}=\langle-2,6\rangle
$$

b. $\overrightarrow{B H}$

$$
\overrightarrow{B H}=\langle-8,-12\rangle
$$

c. $\overrightarrow{\boldsymbol{D C}}$

$$
\overrightarrow{D C}=\langle 4,-2\rangle
$$

d. $\overrightarrow{\boldsymbol{G F}}$

$\overrightarrow{G F}=\langle-4,2\rangle$
e. $\overrightarrow{I J}$
$\overrightarrow{I J}=\langle 6,8\rangle$
5. Consider points $P(2,1), Q(-3,3)$ and $R(1,4)$.
a. Compute $\overrightarrow{P Q}$ and $\overrightarrow{Q P}$ and show that $\overrightarrow{P Q}+\overrightarrow{Q P}$ is the zero vector. Draw a diagram to show why this makes sense geometrically.
$\overrightarrow{P Q}=\langle-5,2\rangle$ and $\overrightarrow{Q P}=\langle 5,-2\rangle$
$\overrightarrow{P Q}+\overrightarrow{Q P}=\langle-5+5,2-2\rangle=\langle 0,0\rangle$

This makes sense because one vector translates $P$ to $Q$ and then the other vector translates the point back from $Q$ to $P$. The sum would be the zero vector because the original point is returned to its starting location.

b. Plot the points $P, Q$, and $R$. Use the diagram to explain why $\overrightarrow{P Q}+\overrightarrow{Q R}+\overrightarrow{R P}$ is the zero vector. Show that this is true by computing the sum $\overrightarrow{P Q}+\overrightarrow{Q R}+\overrightarrow{R P}$.

The diagram shows that the sum should be the zero vector because point $P$ is translated back to its original coordinates when you add the three vectors.

$$
\overrightarrow{P Q}+\overrightarrow{Q R}+\overrightarrow{R P}=\langle-5+4+1,2+1-3\rangle=\langle 0,0\rangle
$$


6. Show for any two points $A$ and $B$ that $-\overrightarrow{A B}=\overrightarrow{B A}$.

Consider $A\left(x_{1}, y_{1}\right)$ and $B\left(x_{2}, y_{2}\right)$. Then $\overrightarrow{A B}=\left\langle x_{2}-x_{1}, y_{2}-y_{2}\right\rangle$
and $-\overrightarrow{A B}=\left\langle-x_{2}+x_{1},-y_{2}+y_{2}\right\rangle=\left\langle x_{1}-x_{2}, y_{1}-y_{2}\right\rangle$
This is the component form of $\overrightarrow{B A}=\left\langle x_{1}-x_{2}, y_{1}-y_{2}\right\rangle$.
7. Given the vectors $v=\langle 2,-3\rangle, w=\langle-5,1\rangle, u=\langle 4,-2\rangle$ and $t=\langle-1,4\rangle$.
a. Verify that the sum of these four vectors is the zero vector.
$\mathrm{v}+\mathrm{w}=\langle-3,-2\rangle$ and $\mathrm{v}+\mathrm{w}+\mathrm{u}=\langle 1,-4\rangle$ and $\mathrm{v}+\mathrm{w}+\mathrm{u}+\mathrm{t}=\langle\mathbf{0}, 0\rangle$
b. Draw a diagram representing the vectors as arrows placed end-to-end to support why their sum would be the zero vector.
If we position v to have an initial point at $(0,0)$ then v translates the point $(0,0)$ to $(2,-3)$. Each additional vector translates the point by its components. The result of all four translations returns the point back to the origin.


## Example 1 (5 minutes): The Parallelogram Rule for Vector Addition

This exercise introduces the parallelogram rule for vector addition. Depending on the level of your students, you can provide more or less support with this example. Point out the use of the word position vector to denote a vector whose initial point is the origin, and then point out that the terminal point coordinates are the same as the horizontal and vertical components of the vector.

## Example 1: The Parallelogram Rule for Vector Addition

When the initial point of a vector is the origin, then the coordinates of the terminal point will correspond to the horizontal and vertical components of the vector. This type of vector, with initial point at the origin, is often called a position vector.
a. Draw arrows to represent the vectors $v=\langle 5,3\rangle$ and $u=\langle 1,7\rangle$ with the initial point of each vector at $(0,0)$.
b. Add $v+u$ end-to-end. What is $v+u$ ? Draw the arrow that represents $v+u$ with initial point at the origin.
c. Add $\mathbf{u}+\mathbf{v}$ end-to-end. What is $\mathbf{u}+\mathbf{v}$ ? Draw the arrow that represents $\mathbf{u}+\mathbf{v}$ with an initial point at the origin.


Lead a discussion to introduce the parallelogram rule for adding two vectors. Make sure students' diagrams clearly show the parallelogram.

- Notice that the vectors lie on the sides of a parallelogram and the sum of the two vectors is the diagonal of this parallelogram with initial point at the same location as the original two vectors.
- How do you know the vectors would lie on the sides of a parallelogram?
- We can calculate the slope of the arrow that represents the vector. You can see that opposite sides would have the same slope and thus lie on parallel lines so the opposite sides of the quadrilateral are parallel, which gives us a parallelogram.
- How does this example support the fact that the operation (i.e., vector addition) is commutative?
- We got the same result regardless of the order in which we added the two vectors.
- Suppose we translated all these vectors by another vector $\mathbf{t}$. Would the components of $\mathbf{v}+\mathbf{u}$ change? Explain your reasoning.
- No. Translation is a rigid transformation so the size and shape would be preserved, which means the components would remain the same.
- Explain how to use the parallelogram rule to add two vectors.
- To use the parallelogram rule, draw both vectors with the same initial point, and then construct a parallelogram with adjacent sides corresponding to the arrows that represent each vector. The vector sum will be the diagonal from the initial point.


## Exercise 8 (5 minutes)

Exercises 8-10
8. Let $u=\langle-2,5\rangle$ and $v=\langle 4,3\rangle$.
a. Draw a diagram to illustrate $\mathbf{v}$ and $\mathbf{u}$ and then find $\mathbf{v}+\mathbf{u} u \operatorname{sing}$ the parallelogram rule.


$$
\mathbf{v}+\mathbf{u}=\langle\mathbf{2}, \mathbf{8}\rangle
$$

b. Draw a diagram to illustrate $2 v$ and then find $2 v+u$ using the parallelogram rule.


$$
2 v+u=\langle 6,11\rangle
$$

c. Draw a diagram to illustrate $-\mathbf{v}$ and then find $\mathbf{u}-\mathbf{v}$ using the parallelogram rule.


$$
u-v=\langle-6,2\rangle
$$

d. Draw a diagram to illustrate $3 v$ and $-3 v$.

i. How do the magnitudes of these vectors compare to one another and to that of $v$ ?

The magnitude of 3 v and -3 v are the same. The magnitude of these vectors is 3 times the magnitude of $v$. You can see that because scalar multiplication dilates the vector by the scalar multiple.
ii. How do the directions of $3 v$ and $-3 v$ compare to the direction of $v$ ?

When the scalar is positive the direction of the scalar multiple is the same going along v . When the scalar is negative, the direction is opposite the direction of $v$ going against v .

## Discussion (3 minutes)

Have students read the paragraph in their student materials and summarize it with a partner in their own words. Then lead a brief discussion before starting students on the exercises that follow.

- How are directed line segments in $\mathbb{R}^{2}$ the same as directed line segments in $\mathbb{R}^{3}$ ?
- They still have an initial point and a terminal point. The components are still found by subtracting the initial point coordinates from the terminal point coordinates.
- How are directed line segments in $\mathbb{R}^{2}$ different from directed line segments in $\mathbb{R}^{3}$ ?
- Instead of a point being described by an ordered pair we must use an ordered triple.

Directed line segments can also be represented in $\mathbb{R}^{3}$. Instead of two coordinates like we use in $\mathbb{R}^{2}$, we simply use three to locate a point in space relative to the origin, denoted $(0,0,0)$. Thus the vector $\overrightarrow{\overrightarrow{A B}}$ with initial point $A\left(x_{1}, y_{1}, z_{1}\right)$ and terminal point $B\left(x_{2}, y_{2}, z_{2}\right)$ would have component form

$$
\overrightarrow{A B}=\left\langle x_{2}-x_{1}, y_{2}-y_{1}, z_{2}-z_{1}\right\rangle .
$$

## Exercises 9-10 (3 minutes)

## Exercises 9-10

9. Consider points $A(1,0,-5)$ and $B(2,-3,6)$.
a. What is the component form of $\overrightarrow{A B}$ ?

$$
\stackrel{\rightharpoonup}{A B}=\langle 2-1,-3-0,6-(-5)\rangle=\langle 1,-3,11\rangle
$$

b. What is the magnitude of $\overrightarrow{A B}$ ?

$$
\|\stackrel{\rightharpoonup}{A B}\|=\sqrt{1^{2}+(-3)^{2}+11^{2}}=\sqrt{131}
$$

10. Consider points $A(1,0,-5), B(2,-3,6)$, and $C(3,1,-2)$.
a. Show that $\overrightarrow{A B}+\overrightarrow{B A}=0$. Explain your answer using geometric reasoning.
$\overrightarrow{A B}=\langle 1,-3,11\rangle$ and $\overrightarrow{B A}=\langle-1,3,-11\rangle$

$$
\overrightarrow{A B}+\overrightarrow{B A}=\langle 1-1,-3+3,11-11\rangle=\langle 0,0,0\rangle
$$

$\overrightarrow{A B}$ translates $A$ to $B$ and the $\overrightarrow{B A}$ translates $B$ back to $A$. The net translation effect is zero.
b. Show that $\overrightarrow{A B}+\overrightarrow{B C}+\overrightarrow{C A}=\mathbf{0}$. Explain your answer using geometric reasoning.

We have $\overrightarrow{A B}=\langle 1,-3,11\rangle, \overrightarrow{B C}=\langle 1,4,-8\rangle$, and $\overrightarrow{C A}=\langle-2,-1,-3\rangle$
Adding all the respective components together gives

$$
\overrightarrow{A B}+\overrightarrow{B C}+\overrightarrow{C A}=\langle 1+1-2,-3+4-1,11-8-3\rangle=\langle 0,0,0\rangle
$$

The sum of these three vectors translates $A$ to $B$, the $B$ to $C$ and then $C$ to $A$. The net translation effect is zero.

## Closing (3 minutes)

Have students respond to these questions in writing or with a partner to summarize their learning for this lesson.

- How do you use the parallelogram rule to add vectors?
- Construct a parallelogram with the added vectors along adjacent sides. The sum will be the diagonal from the initial points of opposite vertex on the parallelogram.
- How do you find the components of a vector when you know the coordinates of the initial and terminal points?
- You subtract the initial $x$-coordinate from the terminal $x$-coordinate to find the horizontal components. You subtract the initial $y$-coordinate from the terminal $y$-coordinate to find the vertical components.
- Why when you add the vectors that represent the directed line segments between the three vertices of a triangle do you get the zero vector?
- The net effect of the transformations that result from adding these three vectors takes the initial point of the first vector back to its original location in the coordinate plane.

Lesson Summary
A vector v can be used to represent a directed line segment $\overrightarrow{A B}$. If the initial point is $A\left(x_{1}, y_{1}\right)$ and the terminal point is $B\left(x_{2}, y_{2}\right)$, then the component form of the vector is $\mathrm{v}=\overrightarrow{A B}=\left\langle x_{2}-x_{1}, y_{2}-y_{1}\right\rangle$

Vectors can be added end-to-end or using the parallelogram rule.

## Exit Ticket (3 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 19: Directed Line Segments and Vectors

## Exit Ticket

1. Consider vectors with their initial and terminal points as shown below. Find the components of the specified vectors and their magnitudes.
a. $\quad \mathbf{u}=\overrightarrow{E F}$ and $\|\mathbf{u}\|$
b. $\quad \mathbf{v}=\overrightarrow{A B}$ and $\|\mathbf{v}\|$
c. $\quad \mathbf{w}=\overrightarrow{C D}$ and $\|\mathbf{w}\|$

d. $\quad \mathbf{t}=\overrightarrow{G F}$ and $\|\mathbf{t}\|$
2. For vectors $\mathbf{u}$ and $\mathbf{v}$ as in Question 1, explain how to find $\mathbf{u}+\mathbf{v}$ using the parallelogram rule. Support your answer graphically below.


## Exit Ticket Sample Solutions

1. Consider vectors with their initial and terminal points as shown below. Find the components of the specified vectors and their magnitudes.
a. $\mathbf{u}=\overrightarrow{\boldsymbol{E F}}$ and $\|\mathbf{u}\|$
$u=\langle 4,2\rangle,\|u\|=2 \sqrt{5}$
b. $\quad v=\overrightarrow{A B}$ and $\|v\|$
$\mathrm{v}=\langle-1,-3\rangle,\|\mathrm{v}\|=\sqrt{\mathbf{1 0}}$

c. $\quad \mathbf{w}=\overrightarrow{C D}$ and $\|\mathbf{w}\|$
$\mathrm{w}=\langle 0,-3\rangle,\|\mathrm{w}\|=3$
d. $\quad \mathbf{t}=\overrightarrow{\boldsymbol{G F}}$ and $\|\mathbf{t}\|$
$t=\langle 2,1\rangle,\|t\|=\sqrt{5}$
2. For vectors $\mathbf{u}$ and $\mathbf{v}$ as in Question 1, explain how to find $\mathbf{u}+\mathbf{v}$ using the parallelogram rule. Support your answer graphically below.

To find $\mathrm{u}+\mathrm{v}$ using the parallelogram rule, form a parallelogram with vectors u and v as sides. The diagonal of the parallelogram is the sum of the two vectors.


## Problem Set Sample Solutions

1. Vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{a}$, and $\mathbf{b}$ are shown at right.
a. Find the component form of $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{a}$, and $\mathbf{b}$.
$\mathrm{u}=\langle-4,1\rangle, \mathrm{v}=\langle 1,2\rangle, \mathrm{w}=\langle 3,1\rangle, \mathrm{a}=\langle 2,0\rangle, \mathrm{b}=\langle-3,-2\rangle$
b. Find the magnitudes $\|\mathbf{u}\|,\|\mathbf{v}\|,\|\mathbf{w}\|,\|\mathbf{a}\|$, and $\|\mathbf{b}\|$.
$\|u\|=\sqrt{17},\|v\|=\sqrt{5},\|w\|=\sqrt{10},\|a\|=2,\|b\|=\sqrt{13}$
c. Find the component form of $\mathbf{u}+\mathbf{v}$ and calculate $\|\mathbf{u}+\mathbf{v}\|$.
$\langle-3,3\rangle,\|u+v\|=3 \sqrt{2}$

d. Find the component form of $\mathbf{w}-\mathbf{b}$ and calculate $\|\mathbf{w}-\mathbf{b}\|$.
$\langle 6,3\rangle,\|w-b\|=3 \sqrt{5}$
e. Find the component form of $3 \mathbf{u}-2 \mathbf{v}$.
$\langle-14,-1\rangle$
f. Find the component form of $v-2(u+b)$.
$\langle 15,4\rangle$
g. Find the component form of $2(u-3 v)-a$.
$\langle-16,-10\rangle$
h. Find the component form of $\mathbf{u}+\mathbf{v}+\mathbf{w}+\mathbf{a}+\mathbf{b}$.
$\langle-1,2\rangle$
i. Find the component form of $\mathbf{u}-\mathbf{v}-\mathbf{w}-\mathbf{a}+\mathbf{b}$.
$\langle-13,-4\rangle$
j. Find the component form of $2(u+4 v)-3(w-3 a+2 b)$.
$\langle 27,27\rangle$
2. Given points $A(1,2,3), B(-3,2,-4), C(-2,1,5)$, find component forms of the following vectors.
a. $\quad \overrightarrow{A B}$ and $\overrightarrow{B A}$.
$\overrightarrow{A B}=\langle-4,0,-7\rangle, \overrightarrow{B A}=\langle 4,0,7\rangle$
b. $\overrightarrow{B C}$ and $\overrightarrow{C B}$
$\overrightarrow{B C}=\langle 1-1,9\rangle, \overrightarrow{C B}=\langle 1,1,-9\rangle$
c. $\overrightarrow{C A}$ and $\overrightarrow{A C}$
$\overrightarrow{C A}=\langle 3,1,-2\rangle, \overrightarrow{A C}=\langle-3,-1,2\rangle$
d. $\overrightarrow{A B}+\overrightarrow{B C}-\overrightarrow{A C}$
$\langle 0,0,0\rangle$
e. $-\overrightarrow{\boldsymbol{B C}}+\overrightarrow{\boldsymbol{B A}}+\overrightarrow{\boldsymbol{A C}}$
$\langle 0,0,0\rangle$
f. $\overrightarrow{A B}-\overrightarrow{C B}+\overrightarrow{\overrightarrow{C A}}$
$\langle\mathbf{0}, \mathbf{0}, \mathbf{0}\rangle$
3. Given points $A(1,2,3), B(-3,2,-4), C(-2,1,5)$, find the following magnitudes.
a. $\|\overrightarrow{A B}\|$
$\|\overrightarrow{A B}\|=\sqrt{65}$
b. $\quad\|\overrightarrow{A B}+\overrightarrow{B C}\|$.
$\|\overrightarrow{A B}+\overrightarrow{B C}\|=\sqrt{\mathbf{1 4}}$
c. $\|\overrightarrow{A B}+\overrightarrow{B C}+\overrightarrow{C A}\|$

0
4. Given vectors $u=\langle-3,2\rangle, v=\langle 2,4\rangle, w=\langle 5,-3\rangle$, use the parallelogram rule to graph the following vectors.
a. $\mathbf{u}+\mathbf{v}$

b. $\quad v+w$

c. $\mathbf{u}-\mathbf{v}$

d. $\quad \mathbf{v}-\mathbf{w}$

e. $2 w+u$

f. $3 \mathbf{u}-\mathbf{2 v}$

g. $\mathbf{u}+\mathbf{v}+\mathbf{w}$

5. Points $A, B, C, D, E, F, G$ and $H$ and vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are shown below. Find the components of the following vectors.

a. $\overrightarrow{A B}+\overrightarrow{B C}+\overrightarrow{C A}$
$\overrightarrow{A B}=\langle 3,6\rangle, \overrightarrow{B C}=\langle 9,-3\rangle, \overrightarrow{C A}=\langle-12,-3\rangle, \overrightarrow{A B}+\overrightarrow{B C}+\overrightarrow{C A}=0$
b. $\quad \mathbf{u}+\mathbf{v}+\mathbf{w}$
$\mathbf{u}=\langle 1,2\rangle, \mathbf{v}=\langle 3,-1\rangle, \mathbf{w}=\langle-4,-1\rangle, u+v+\mathbf{w}=0$
c. $\overrightarrow{A D}+\overrightarrow{B E}+\overrightarrow{C G}$
$\overrightarrow{A D}=\langle 2,4\rangle, \overrightarrow{B E}=\langle 3,-1\rangle, \overrightarrow{C G}=\langle-4,-1\rangle, \overrightarrow{A B}+\overrightarrow{B C}+\overrightarrow{C A}=\langle 1,2\rangle$
6. Consider Example 5, part (b) in the lesson and Problem 5, part (a) above. What can you conclude about three vectors that form a triangle when placed tip-to-tail? Explain by graphing.

If three vectors form a triangle when placed tip-to-tail, then the sum of those three vectors is zero.

7. Consider the vectors shown below.
a. Find the components of vectors $\mathbf{u}=\overrightarrow{\mathbf{A C}}, \mathbf{w}=\overrightarrow{\mathbf{A D}}, \mathbf{v}=\overrightarrow{\mathbf{A B}}$, and $\mathbf{c}=\overrightarrow{\mathbf{E F}}$.
$\mathrm{u}=\overrightarrow{A C}=\langle 2,1\rangle, \mathrm{w}=\overrightarrow{A D}=\langle 4,2\rangle, \mathrm{v}=\overrightarrow{A B}=\langle 8,4\rangle, \mathrm{c}=\overrightarrow{E F}=\langle 2,1\rangle$.
b. Is vector u equal to vector $\boldsymbol{c}$ ?

Yes, $\mathbf{u}=\langle 2,1\rangle=c$, they have the same components and direction.
c. Jens says that if two vectors $u$ and $v$ have the same initial point $A$ and lie on the same line, then one vector is a scalar multiple of the other. Do you agree with him? Explain how you know. Given an example to support your answer.

Yes. Suppose that the terminal point of $u$ is $C$ and the terminal point of $v$ is $B$. Then $u$ has length $A C \neq 0$ and v has magnitude $A B$, where $A B \neq 0$. If u and v point in the same direction, then $\mathrm{v}=\left(\frac{A B}{A C}\right) \mathrm{u}$. If u and v point in opposite directions, then $\mathrm{v}=-\left(\frac{A B}{A C}\right) \mathbf{u}$ For example in the diagram above, vector $\mathbf{u}=\overrightarrow{A C}=\langle 2,1\rangle$ and vector $\mathrm{v}=\overrightarrow{A B}=\langle 8,4\rangle$, and we have $\mathrm{v}=\frac{\sqrt{80}}{\sqrt{5}} \mathrm{u}=4 \mathrm{u}$.

## Q Lesson 20: Vectors and Stone Bridges

## Student Outcomes

- Students understand the forces involved in constructing a stone arch.
- Students add and subtract vectors given in magnitude and direction form.
- Students solve problems that can be represented by vectors.


## Lesson Notes

Lesson 19 specified vectors by using either the initial and terminal point, such as $\overrightarrow{A B}$, or by its components, such as $\mathbf{v}=\langle 1,2\rangle$. In this lesson, students apply the vector and magnitude form of a vector as they explore how tall a stone arch can be built without the thrust forces causing the structure to collapse.

Magnitude and direction are both properties of vectors and can be used together to describe a vector. There is no universal way to represent a vector using its magnitude and direction. In this lesson and the ensuing problem set, we will use two different but equivalent methods. The magnitude of a vector is clearly defined and described as a positive number that measures the length of the directed line segment that defines the vector. The direction can be specified either by using a geographical description, or by measuring the amount of rotation $\theta$ that the positive $x$-axis undergoes to align with the vector when its tail is placed at the origin. For example,
 consider the vector $\mathbf{v}$ shown above. We can use magnitude and direction to describe $\mathbf{v}$ in the following ways:

1. Magnitude 2 and direction $32^{\circ}$ north of west.
2. Magnitude 2 and direction $148^{\circ}$ from the positive $x$-axis.

There is some ambiguity in using the geographical description of direction, because $32^{\circ}$ north of west is equivalent to $58^{\circ}$ west of north. Thus, there is more than one valid way to use a geographic description to specify a vector. However, in application problems in physics, this type of geographic description of direction is the norm, so students should learn both approaches, and see that they are equivalent.

The construction of a stone arch is illustrated in the interactive app "Physics of Stone Arches", available from PBS/Nova here: http://www.pbslearningmedia.org/resource/nv37.sci.engin.design.arches/physics-of-arches/.

If students have access to computers for classroom use, then adapt the opening to let them experiment with the software themselves. Otherwise, demonstrate this interactive app at the front of the classroom. The Physics of Stone Arches app will emulate the construction of a stone arch on top of a stack of blocks we will call the base column. The goal of the app is to construct an arch on as tall a base column as possible. Although the app allows users to experiment with different fortification strategies, this lesson only addresses buttressing.

There is also a GeoGebra App, NineStoneArch (http://eureka-math.org/G12M2L20/geogebra-NineStoneArch), available that you may choose to either demonstrate at the front of the room, or allow students to experiment with during the lesson. This app shows a simplified version of the force vectors in a stone arch with nine stones, allowing users to increase the weight of the individual stones as well as the height of the base columns.

This lesson addresses standards N-VM.A.3, N-VM.B.4a, and N-VM.B.4b. To extend this lesson, consider asking students to construct physical arch bridges from a substance such as floral foam or Styrofoam. You may consider extending this lesson to two days, allowing a full day for the Exploratory Challenge.

Materials needed: Protractor marked in degrees, a ruler marked in centimeters, and a computer with access to GeoGebra and the Internet.

## Classwork

## Opening (3 minutes)

- The ancient Romans were the first to recognize the potential of arches for bridge construction. In 1994, Vittorio Galliazzo counted 931 surviving ancient Roman bridges scattered throughout 26 different countries. Most of these bridges were made of stone, and many have survived for more than 2,000 years. Roman arch bridges were usually constructed using semicircular arcs, although some used arcs less than a semicircle. The Pont Julien (French for "Julien Bridge") in southeast France, built in the year 3 BC, is based on semicircular arcs. It was used for all traffic, including car traffic, until 2005, when a replacement bridge was constructed to preserve the old bridge. This bridge was used for more than 2,000 years!


Photograph by Veronique Pagnier, courtesy of Wikimedia Commons/Public domain

- Stone arch bridge construction centers on the idea that the arch distributes the force of gravity acting on the stone (i.e., the thrust) through the curve of the arch and into the column of stones beneath it, which we will call the base columns. If the resultant forces were not contained in the stones or the ground, the structure would collapse.

Demonstrate the Physics of Stone Arches app. Let the students guide you in constructing an arch, and watching it inevitably collapse as you add more and more blocks to the base columns. Experiment with the different types of fortification: buttresses, pinnacles, and flying buttresses. Although there is an option to select a pointed or rounded arch, this lesson only addresses the rounded arch and simple buttresses, so use your own discretion whether to explore the other possibilities.

- The Physics of Stone Arches app demonstrates an arch with nine stones. The Pont Julien contains many more stones in its three arches. To keep things simple, we will experiment with arches made of nine or fewer stones.
- Additionally, our work will not completely align with the Physics of Stone Arches app because we need to simplify the model to make it accessible to students.
- Our biggest simplification is in assuming that each stone pushes on another with a force of equal magnitude. In reality, the angle of the stone affects the magnitude of these force vectors.


## Discussion (12 minutes)

This discussion relies on the GeoGebra app, G12-M2-L20-NineStoneArch.ggb. Ideally, students will be able to access this app directly, but if there are not enough available computers, then demonstrate at the front of the classroom.

- What forces are acting on the stones in the bridge?
- Gravity, friction, compression.

Note to teacher: It is likely that students will be able to name gravity as a force acting on the stone, but possibly not friction or compression. Friction is the attractive force between stones caused by the roughness of the surface that keeps the stones from sliding away from each other. Compression is the force of one stone pushing on the one next to it. We will disregard the force of friction in our model, which is common with mathematical models.

- Have you heard the saying "For every action, there is an equal and opposite reaction?"
- (Students will answer yes or no. Continue to explain the meaning of this statement either way.)
- "For every action there is an equal and opposite reaction" is a restatement of Newton's third law of motion, which says "When one body exerts a force on a second body, the second body simultaneously exerts a force equal in magnitude and opposite in direction on the first body." What does this mean for the stones in our bridge?
- If a stone pushes down on the stone under it, the lower stone also pushes up on the stone above it in the opposite direction but with the same magnitude.
- Let's consider the keystone, that is, the stone at the top of the bridge. This stone is often shaped differently than the others for aesthetic reasons, but for our purposes all the stones in the arch are the same shape and size. If the bridge is standing, is the keystone moving?
- No. If the keystone is moving, then the bridge is falling down.
- What forces are acting on the keystone, and in what direction?
- Gravity is pulling the keystone straight downward. The stone on the left is pushing upward on the left edge of the keystone in the direction perpendicular to the edge between the stones. The stone on the right is pushing upward on the right edge of the keystone in the direction perpendicular to the edge between the stones.

Draw or display a figure of the trapezoidal stone shown at right, marking the three force vectors acting on the stone. We do not have a way to accurately represent the magnitude of these vectors, but the two vectors on the sides should mirror each other. We will make some assumptions later in the lesson about these vectors, which will enable us to do some calculations with their magnitude and direction.


- What do we know about the sum of these three vectors acting on the keystone? Why?
- Because the keystone is not moving, the net force acting on it is zero. That is, the three force vectors sum to zero. This means that if placed tip-to-tail, the three vectors would form
 a triangle.
- We are going to greatly simplify our model by assuming that all of the stones except the keystone push on each other with compression forces of equal magnitude. The weight of the keystone is carried by both the left and right sides of the arch, so it pushes on the stones to the left and right with compression forces of half of the magnitude of the other stones.
- Thus, the force vectors for each stone push downward in a direction perpendicular to the joint between the stones and, except for the forces from the keystone, the force vectors are assumed to have the same magnitude.

Display the GeoGebra app, G12-M2-L20-NineStoneArch.ggb, and show the force vectors. Except for the one from the keystone, the blue vectors all have the same magnitude, and each is perpendicular to the edge that it crosses. In the diagram, the blue vectors have an initial point at the center of mass of each stone. The green vector shows the result of adding up all of the blue force vectors. The bridge will stand or collapse based on whether or not the tip of this resultant vector is contained either in the arch itself or in the ground. The arch will stand if the vector is green, and will fall if it is red.


Figure 1: The resultant force is contained within the arch. The arch will stand.


Figure 2: The resultant force is not contained within the arch. The arch will collapse.

Use the GeoGebra app to explore what happens as the height $h$ of the base columns changes using the slider on the left.

- If there is time, ask the students what should happen if they use the slider on the right to change the weight $w$ of the stones.
- The resultant force is stronger but points in the same direction, so the weight of the stones in this model does not affect the outcome.

Check the box to add buttresses to the model and explore the result.

- When we add the buttresses, the width of each base column doubles. Does that double the maximum height of the base columns before the structure collapses?
- No. Without the buttresses, the structure stands with base columns of height up to 4 units. With the buttresses, the structure stands with base columns of height up to 6.8 units. Adding buttresses makes the arch stronger, but does not double the maximum height of the base columns.


## Discussion (3 minutes)

This discussion describes the magnitude and direction form of a vector, which is key to N-VM.A.3, N-VM.B.4a and N-VM.B.4b. Highlight this description before having the students start the challenge.

- In Lesson 19, we described vectors by either specifying the coordinates of their endpoints or by specifying their components. We can also describe vectors using magnitude and direction: if we know the length of a vector and the direction in which it points, then we have uniquely identified that vector in


## Scaffolding:

Use a visual approach to magnitude and direction form. Show a vector and ask students to describe it in magnitude and direction form. magnitude and direction form.

- However—there are two acceptable ways for us to identify the direction of a vector.
a. We can describe the direction of the vector relative to the compass points north, east, south, and west. Then we can describe the direction of the vector $\mathbf{v}=\langle-1,1\rangle$ as $45^{\circ}$ north of west.
b. We can describe the direction of the vector as the amount of rotation $\theta$, measured in degrees, that the positive horizontal axis must undergo to align with the vector when its tail is placed at the origin, for $-180<\theta \leq 180$. Then the direction of the vector $\mathbf{v}=\langle-1,1\rangle$ is $135^{\circ}$ from the positive horizontal axis.
- Then the magnitude and direction form of the vector $\mathbf{v}=\langle-1,1\rangle$ can be described in any of these ways:
a. Magnitude $\sqrt{2}$, direction $45^{\circ}$ north of west.
b. Magnitude $\sqrt{2}$, direction $45^{\circ}$ west of north.
c. Magnitude $\sqrt{2}$, direction $135^{\circ}$ from the positive horizontal axis.


## Exploratory Challenge ( 20 minutes)

In these exercises, students calculate the magnitude and direction of the force vectors acting on the stones in the arch for a bridge with five arch stones. They can then test the stability of the arch by finding the sum of the three vectors acting on one side of it. To test their understanding of magnitude and direction form of vectors, this challenge does not ask students to find the component form of the vectors; the arch stability is tested geometrically.

## Exploratory Challenge

1. For this Exploratory Challenge, we will consider an arch made with five trapezoidal stones on top of the base columns as shown. We will focus only on the stones labeled 1, 2 and 3.

a. We will study the force vectors acting on the keystone (stone 1 ) and stones $\mathbf{2}$ and 3 on the left side of the arch. Why is it acceptable for us to disregard the forces on the right side of the arch?

Due to symmetry, the forces on the right side of the arch will be the same magnitude as the forces on the left, but with directions reflected across the vertical line through the center of the keystone.
b. We will first focus on the forces acting on the keystone. Stone 2 pushes on the left side of the keystone with force vector $\mathbf{p}_{1 \mathrm{~L}}$. The stone to the right of the keystone pushes on the right of the keystone with force vector $\mathbf{p}_{1 \mathrm{R}}$. We know that these vectors push perpendicular to the sides of the stone, but we do not know their magnitude. All we know is that vectors $p_{1 L}$ and $p_{1 R}$ have the same magnitude.
i. Find the measure of the acute angle formed by $p_{1 L}$ and the horizontal.


We need to consider the angles in the triangles formed by the trapezoidal stones. Since there are five stones that form the arch, each trapezoid creates a triangle with angles that measure
$36^{\circ}, 72^{\circ}$, and $72^{\circ}$. The vector $p_{1 L}$ is shown in green. Looking more closely at just the keystone, we see that the $36^{\circ}$ angle is bisected by the vertical line through the center of the keystone. Thus, the angle $p_{1 \mathrm{~L}}$ makes with the vertical direction is $90^{\circ}+18^{\circ}$. Therefore, vector $\mathrm{p}_{1 \mathrm{~L}}$ makes an $18^{\circ}$ angle with the horizontal axis.
ii. Find the measure of the acute angle formed by $\mathbf{p}_{1 \mathrm{R}}$ and the horizontal.

Due to symmetry, the acute angle formed by $\mathrm{p}_{1 \mathrm{R}}$ and the horizontal is congruent to the acute angle formed by $p_{1 \mathrm{~L}}$ and the horizontal. Thus, this angle measures $18^{\circ}$.
c. Move vectors $p_{1 L}, p_{1 R}$ and g tip-to-tail. Why must these vectors form a triangle?

Because the keystone does not move, we know that the forces acting on the stone sum to zero. Thus, the vectors that represent these forces will form a triangle when placed tip-to-tail, as we saw in the previous lesson.
d. Suppose that vector $g$ has magnitude 1. Use triangle trigonometry together with the measure of the angles you found in part (b) to find the magnitudes of vectors $p_{1 L}$ and $p_{1 R}$ to the nearest tenth of a unit.
i. Find the magnitude and direction form of $g$.

We know that g has magnitude 1 and direction - $90^{\circ}$ from the positive horizontal axis.
ii. Find the magnitude and direction form of $\mathbf{p}_{1 \mathrm{~L}}$.

Using the figure at right, we see that there are two right triangles with angles $18^{\circ}$ and $72^{\circ}$. Since $g$ has magnitude
 1 , these triangles have a short leg of length $1 / 2$. We will use $x$ to represent the magnitude of $\mathbf{p}_{1 \mathrm{~L}}$. Using triangle trigonometry, we see that
$\sin \left(18^{\circ}\right)=\frac{1 / 2}{x}$ and then $x=\frac{1}{2 \sin \left(18^{\circ}\right)} \approx 1.6$.
Thus, vector $\mathrm{p}_{1 \mathrm{~L}}$ has magnitude 1.6 and direction $18^{\circ}$ from the positive horizontal axis.
iii. Find the magnitude and direction form of $p_{1 R}$.

Due to symmetry, we know that $\left\|\mathrm{p}_{1 \mathrm{R}}\right\|=\left\|\mathrm{p}_{1 \mathrm{~L}}\right\|$ so the magnitude of $p_{1 R}$ is also 1.6. Then the vector $p_{1 R}$ has magnitude 1.6 and direction $162^{\circ}$ from the positive horizontal axis.
e. Vector $\mathbf{p}_{1 \mathrm{~L}}$ represents the force of stone 1 pushing on the keystone, and by Newton's third law of motion, there is an equal and opposite reaction. Thus, there is a force of the keystone acting on stone 2 that has the same magnitude as $p_{1 L}$ and the opposite direction. Call this vector $\mathbf{v}_{1 \mathrm{~L}}$.
i. Find the magnitude and direction form of $\mathbf{v}_{1 \mathrm{~L}}$.

The vector $\mathrm{v}_{1 \mathrm{~L}}$ has magnitude 1.6 and direction $-162^{\circ}$ from the positive horizontal axis.
ii. Carefully draw vector $\mathbf{v}_{1 \mathrm{~L}}$ on the arch below, with initial point at the point marked $O$, which is the center of mass of the keystone. Use a protractor measured in degrees and a ruler measured in centimeters.

See the final drawing in part i.
f. We will assume that the forces $v_{2 L}$ of stone 2 acting on stone 3 and $v_{3 L}$ of stone $\mathbf{3}$ acting on the base column have the same magnitude as each other, and twice the magnitude as the force $v_{1 L}$. Why does it make sense that the force vector $\mathbf{v}_{1 \mathrm{~L}}$ is significantly shorter than the other two force vectors?
The keystone compresses the stones on both the left and right sides of the arch equally, so the gravitational pull on that stone is split in half down the right and left sides. Thus, it is reasonable to assume that the stones 2 and 3 act with twice the compressive force as stone 1.
g. Find the magnitude and direction form of vector $v_{2 L}$, the force of stone 2 pressing on stone 3. Carefully draw vector $\mathbf{v}_{2 L}$ on the arch on page 152 , placing its initial point at the terminal point of $v_{1 L}$.

Vector $\mathrm{v}_{2 \mathrm{~L}}$ has twice the magnitude of vector $\mathrm{v}_{1 \mathrm{~L}}$, so its magnitude is 3.2. Because each stone is rotated $36^{\circ}$ from the neighboring stones, vector $\mathrm{v}_{2 \mathrm{~L}}$ is rotated $-162^{\circ}+$ $36^{\circ}=-126^{\circ}$ from the horizontal. Thus, vector $v_{2 L}$ has magnitude 3.2 and direction $-126^{\circ}$ from the positive horizontal axis.
h. Find the magnitude and direction form of vector $v_{3 L}$, the force of stone 3 pressing on the base column. Carefully draw vector $\mathbf{v}_{3 \mathrm{~L}}$ on the arch on page 152, placing its initial point at the terminal point of $\mathbf{v}_{2 \mathrm{~L}}$.

Vector $\mathrm{v}_{3 \mathrm{~L}}$ has the same magnitude as vector $\mathrm{v}_{2 \mathrm{~L}}$ and it is rotated an additional $36^{\circ}$ counterclockwise. Thus, vector $\mathrm{v}_{3 \mathrm{~L}}$ has magnitude 3.2 and direction $-126^{\circ}+36^{\circ}=$ $-90^{\circ}$ from the positive horizontal axis. Thus, vector $\mathrm{v}_{3 \mathrm{~L}}$ points straight downward.
i. Use the parallelogram method to find the sum of the force vectors $v_{1 L}, v_{2 L}$, and $v_{3 L}$ on the left side of the arch.
(The diagram of the force vectors is shown below.)

j. Will the arch stand or fall? Explain how you know.


[^6]
## Scaffolding:

- Challenge advanced students or rapid finishers to calculate the maximum height of the base columns before the arch will collapse.
- Also, challenge these students to consider the effect of adding buttresses to the arch; with buttresses, what is the maximum height of the base columns before the arch will collapse?

Plot the force vectors acting on the arch on this diagram to determine whether or not this arch will be able to stand or if it will collapse.


## Closing (3 minutes)

Ask students to turn to a partner and explain how they found the following forces in the arch. Partner 1 can explain (a), and partner 2 can explain (b).
a. The magnitude of the force of stone 2 pushing on stone 1 , which we called $\mathbf{p}_{1 \mathrm{~L}}$.
b. The total force acting on the left side of the arch.

Lesson Summary
A vector can be described using its magnitude and direction.
The direction of a vector $v$ can be described either using geographical description, such as $32^{\circ}$ north of west, or by the amount of rotation the positive $x$-axis must undergo to align with the vector v , such as rotation by $148^{\circ}$ from the positive $x$-axis.

## Exit Ticket (4 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 20: Vectors and Stone Bridges

## Exit Ticket

We saw in the lesson that the forces acting on a stone in a stable arch must sum to zero since the stones do not move.
Now, we will consider the upper-left stone in a stable arch made of six stones. We will denote this stone by $S$. In the image below, $\mathbf{p}_{\mathbf{L}}$ represents the force acting on stone $S$ from the stone on the left. Vector $\mathbf{p}_{\mathbf{R}}$ represents the force acting on stone $S$ from the stone on the right. Vector $\mathbf{g}$ represents the downward force of gravity.

a. Describe the directions of vectors $\mathbf{g}, \mathbf{p}_{\mathbf{L}}$, and $\mathbf{p}_{\mathbf{R}}$ in terms of rotation from the positive $x$-axis by $\theta$ degrees, for $-180<\theta<180$.
b. Suppose that vector $\mathbf{g}$ has a magnitude of 1 . Find the magnitude of vectors $\mathbf{p}_{\mathbf{L}}$ and $\mathbf{p}_{\mathrm{R}}$.
c. Write vectors $\mathbf{g}, \mathbf{p}_{\mathrm{L}}$, and $\mathbf{p}_{\mathrm{R}}$ in magnitude and direction form.

## Exit Ticket Sample Solutions

We saw in the lesson that the forces acting on a stone in a stable arch must sum to zero since the stones do not move. We will consider the upper left stone in a stable arch made of six stones. We will denote this stone by $S$. In the image below, $p_{\mathrm{L}}$ represents the force acting on stone $S$ from the stone on the left. Vector $\mathrm{p}_{\mathrm{R}}$ represents the force acting on stone $S$ from the stone on the right. Vector $g$ represents the downward force of gravity.
a. Describe the directions of vectors $g, p_{L}$, and $p_{R}$ in terms of rotation from the positive $\boldsymbol{x}$-axis by $\boldsymbol{\theta}$ degrees, for $-180<\theta \leq 180$.


Vector $g$ is vertical, pointing downward, so $g$ aligns with the terminal ray of rotation by $-90^{\circ}$.

Vector $\mathrm{p}_{\mathrm{R}}$ is perpendicular to the vertical edge of the stone at point $N$, so it is horizontal and pointing to the left. Thus, $\mathrm{p}_{\mathrm{R}}$ aligns with the terminal ray of rotation by $180^{\circ}$.

Vector $\mathbf{p}_{\mathrm{L}}$ is perpendicular to line $\overleftrightarrow{M O}$ shown at left. Thus, vector $\mathrm{p}_{\mathrm{L}}$ is rotated $120^{\circ}$ from vertical. Hence, $p_{\mathrm{L}}$ aligns with the terminal

b. $\quad$ Suppose that vector $g$ has a magnitude of 1 . Find the magnitude of vectors $p_{L}$ and $p_{R}$.

We need to move the vectors tip-to-tail, and since the forces sum to zero, the vectors should make a triangle. Since g is vertical and $\mathrm{p}_{\mathrm{R}}$ is horizontal, the vectors will make a right triangle.

By part (a), the angle made by vectors $\mathrm{p}_{\mathrm{R}}$ and $\mathrm{p}_{\mathrm{L}}$ measures $30^{\circ}$. We are given that $\|\mathrm{g}\|=1$. Then we know that $\sin \left(30^{\circ}\right)=\frac{1}{\left\|\mathrm{p}_{\mathrm{L}}\right\|}$,
so $\frac{1}{2}=\frac{1}{\left\|p_{L}\right\|}$ and then $\left\|\mathrm{p}_{\mathrm{L}}\right\|=2$.


Then $\cos \left(30^{\circ}\right)=\frac{\left\|\mathrm{p}_{\mathrm{R}}\right\|}{\left\|\mathrm{p}_{\mathrm{L}}\right\|^{\prime}}$ so $\frac{\sqrt{3}}{2}=\frac{\left\|\mathrm{p}_{\mathrm{R}}\right\|}{2}$ and $\left\|\mathrm{p}_{\mathrm{R}}\right\|=\sqrt{3}$.
c. Write vectors $g, p_{L}$, and $p_{R}$ in magnitude and direction form.

Vector g has magnitude 1 and direction $-90^{\circ}$ from the positive horizontal axis
Vector $\mathrm{p}_{\mathrm{L}}$ has magnitude 2 and direction $30^{\circ}$ from the positive horizontal axis.
Vector $\mathrm{p}_{\mathrm{R}}$ has magnitude $\sqrt{3}$ and direction $180^{\circ}$ from the positive horizontal axis.

## Problem Set Sample Solutions

1. Vectors $\mathbf{v}$ and $\mathbf{w}$ are given in magnitude and direction form. Find the coordinate representation of the sum $\mathbf{v}+\mathbf{w}$ and the difference $\mathbf{v}-\mathbf{w}$. Give coordinates to the nearest tenth of a unit.
a. $\quad v$ : magnitude 12 , direction $50^{\circ}$ east of north
$w$ : magnitude 8, direction $30^{\circ}$ north of east

$$
\begin{gathered}
v=\left\langle 12 \cos \left(40^{\circ}\right), 12 \sin \left(40^{\circ}\right)\right\rangle ; w=\left\langle 8 \cos \left(30^{\circ}\right), 8 \sin \left(30^{\circ}\right)\right\rangle \\
v+w \approx\langle 16.1,11.7\rangle \\
v-w \approx\langle 2.3,3.7\rangle
\end{gathered}
$$

b. $\quad v$ : magnitude 20 , direction $54^{\circ}$ south of east w: magnitude 30 , direction $18^{\circ}$ west of south

$$
\begin{aligned}
\mathrm{v}=\left\langle 20 \cos \left(-54^{\circ}\right), 20 \sin \left(-54^{\circ}\right)\right\rangle ; \mathrm{w} & =\left\langle 30 \cos \left(-108^{\circ}\right), 30 \sin \left(-108^{\circ}\right)\right\rangle \\
\mathrm{v}+\mathrm{w} & \approx\langle 2.5,-44.7\rangle \\
\mathrm{v}-\mathrm{w} & \approx\langle 21.0,14.6\rangle
\end{aligned}
$$

2. Vectors $v$ and $w$ are given by specifying the length $r$ and the amount of rotation from the positive $x$-axis. Find the coordinate representation of the sum $v+w$ and the difference $v-w$. Give coordinates to the nearest tenth of a unit.
a. $\quad v$ : length $r=3$, rotated $12^{\circ}$ from the positive $x$-axis
w : length $r=4$, rotated $18^{\circ}$ from the positive $x$-axis

$$
\begin{gathered}
v=\left\langle 3 \cos \left(12^{\circ}\right), 3 \sin \left(12^{\circ}\right)\right\rangle ; w=\left\langle 4 \cos \left(18^{\circ}\right), 4 \sin \left(18^{\circ}\right)\right\rangle \\
v+w \approx\langle 6.7,1.9\rangle \\
v-w \approx\langle-0.9,-0.6\rangle
\end{gathered}
$$

b. $\quad v$ : length $r=16$, rotated $162^{\circ}$ from the positive $x$-axis $w$ : length $r=44$, rotated $-18^{\circ}$ from the positive $x$-axis

$$
\begin{gathered}
v=\left\langle 16 \cos \left(162^{\circ}\right), 16 \sin \left(116^{\circ}\right)\right\rangle ; w=\left\langle 44 \cos \left(-18^{\circ}\right), 44 \sin \left(-18^{\circ}\right)\right\rangle \\
v+w \approx\langle 26.6,0.78\rangle \\
v-w \approx\langle-57.1,27.9\rangle
\end{gathered}
$$

3. Vectors $\mathbf{v}$ and $\mathbf{w}$ are given in magnitude and direction form. Find the magnitude and direction of the sum $\mathbf{v}+\mathbf{w}$ and the difference $v-w$. Give the magnitude to the nearest tenth of a unit and the direction to the nearest tenth of a degree.
a. $\quad \mathrm{v}$ : magnitude 20 , direction $45^{\circ}$ north of east
w: magnitude 8, direction $45^{\circ}$ west of north
The tip of the vector v has coordinates
$(10 \sqrt{2}, 10 \sqrt{2})$ and tip of vector w has
coordinates $(-4 \sqrt{2}, 4 \sqrt{2})$. Then the sum has
tip
$v+w=(6 \sqrt{2}, 14 \sqrt{2})$. The rotation of $v+w$
is $\theta=\arctan \left(\frac{14 \sqrt{2}}{6 \sqrt{2}}\right)=\arctan \left(\frac{7}{3}\right)$, so $\theta \approx$
66.8 ${ }^{\circ}$. The length of $v+w$ is $|v+w|=$
$\sqrt{(6 \sqrt{2})^{2}+(14 \sqrt{2})^{2}}=\sqrt{464} \approx 21.5$. Thus,
$\mathrm{v}+\mathrm{w}$ has magnitude approximately 21.5
and direction $66.8^{\circ}$ north of east.
The difference has tip $\mathrm{v}-\mathrm{w}=(14 \sqrt{2}, 6 \sqrt{2})$.
The rotation of $\mathrm{v}-\mathrm{w}$ is $\theta=\arctan \left(\frac{6 \sqrt{2}}{14 \sqrt{2}}\right)=$
$\arctan \left(\frac{3}{7}\right)$, so $\theta \approx 23.2^{\circ}$. The length of
$v-w$ is $|v-w|=\sqrt{(14 \sqrt{2})^{2}+(6 \sqrt{2})^{2}}=\sqrt{464} \approx 21.5$. Thus,
$\mathrm{v}-\mathrm{w}$ has magnitude approximately 21.5 and direction $23.2^{\circ}$ north of east.
b. v: magnitude 12.4 , direction $54^{\circ}$ south of west
w : magnitude 16.0 , direction $36^{\circ}$ west of south

Since $54^{\circ}$ south of west and $36^{\circ}$ west of south are the same direction, vectors v and w are collinear. Thus, the vector $\mathrm{v}+\mathrm{w}$ has length $12.4+16=28.4$ and direction $54^{\circ}$ south of west.

The vector $\mathrm{v}-\mathrm{w}$ has length $|12.4-16|=$ 3.6 and direction $54^{\circ}$ north of east.

Lesson 20: Date:
4. Vectors v and w are given by specifying the length $r$ and the amount of rotation from the positive $x$-axis. Find the length and direction of the sum $v+w$ and the difference $v-w$. Give the magnitude to the nearest tenth of a unit and the direction to the nearest tenth of a degree.
a. $\quad \mathrm{v}$ : magnitude $r=1$, rotated $102^{\circ}$ from the positive $x$-axis
w : magnitude $r=\frac{1}{2}$, rotated $18^{\circ}$ from the positive $x$-axis

$$
\begin{gathered}
v=\left\langle\cos \left(102^{\circ}\right), \sin \left(102^{\circ}\right)\right\rangle ; w=\left\langle\frac{1}{2} \cos \left(18^{\circ}\right), \frac{1}{2} \sin \left(18^{\circ}\right)\right\rangle \\
v+w \approx\langle 0.3,1.1\rangle \\
\|v+w\| \approx \sqrt{0.3^{2}+1.1^{2}} \approx 1.2 \\
\arctan \left(\frac{1.1}{0.3}\right) \approx 74.7^{\circ}
\end{gathered}
$$

The sum $v+w$ has magnitude approximately 1.2 and direction $74.7^{\circ}$ from the positive $x$-axis.

$$
\begin{gathered}
v-w \approx\langle-0.7,0.8\rangle \\
\|v-w\| \approx \sqrt{0.7^{2}+0.8^{2}} \approx 1.1 \\
\arctan \left(-\frac{0.8}{0.7}\right) \approx-48.8^{\circ}
\end{gathered}
$$

Since $\mathrm{v}-\mathrm{w}$ lies in the second quadrant, it aligns with the terminal ray of rotation by $180^{\circ}-48.8^{\circ}=131.2^{\circ}$. The vector $v-w$ has magnitude approximately 1.1 and direction $131.2^{\circ}$ from the positive $x$-axis.
b. $\quad \mathrm{v}$ : magnitude $r=1000$, rotated $-126^{\circ}$ from the positive $x$-axis
w : magnitude $r=500$, rotated $-18^{\circ}$ from the positive $x$-axis

$$
\begin{gathered}
v=\left\langle 1000 \cos \left(-126^{\circ}\right), 1000 \sin \left(-126^{\circ}\right)\right\rangle ; w=\left\langle 500 \cos (-18)^{\circ}, 500 \sin \left(-18^{\circ}\right)\right\rangle \\
v+w \approx\langle-112.3,-963.5\rangle \\
\|v+w\| \approx \sqrt{112.3^{2}+963.5^{2}} \approx 970.0 \\
\arctan \left(\frac{-963.5}{-112.3}\right) \approx 83.4^{\circ}
\end{gathered}
$$

Since $\mathrm{v}+\mathrm{w}$ lies in the third quadrant, it aligns with the terminal ray of rotation by $-\left(180^{\circ}-83.4^{\circ}\right)=-96.6^{\circ}$. The sum $v+w$ has magnitude approximately 970 and direction $-96.6^{\circ}$ from the positive $x$-axis.

$$
\begin{gathered}
v-w \approx\langle-1063.3,-654.5\rangle \\
\|v-w\| \approx \sqrt{1063.3^{2}+654.5^{2}} \approx 1248.6 \\
\arctan \left(\frac{-654.5}{-1063.3}\right) \approx 31.6^{\circ}
\end{gathered}
$$

Since $\mathrm{v}-\mathrm{w}$ lies in the third quadrant, it aligns with the terminal ray of rotation by $-\left(180^{\circ}-31.6^{\circ}\right)=-148.4^{\circ}$. The difference $v-w$ has magnitude approximately 1248.6 and direction-148.4 ${ }^{\circ}$ from the positive $x$-axis.
5. You hear a rattlesnake while out on a hike. You abruptly stop hiking at point $S$ and take eight steps. Then you take another six steps. For each distance below, draw a sketch to show how the sum of your two displacements might add so that you find yourself that distance from point $S$. Assume that your steps are a uniform size.
a. 14 steps

b. $\quad 10$ steps

c. 2 steps


8
6. A delivery driver travels 2.6 km due north, then 5.0 km due west, and then $4.2 \mathrm{~km} 45^{\circ}$ north of west. How far is he from his starting location? Include a sketch with your answer.

He is about 9.72 km from his starting location.

7. Morgan wants to swim directly across a river, from the east to the west side. She swims at a rate of $\mathbf{1} \mathbf{~ m} / \mathrm{s}$. The current in the river is flowing due north at a rate of $3 \mathrm{~m} / \mathrm{s}$. Which direction should she swim so that she travels due west across the river?

There is no way for Morgan to swim due west if she can only swim at a rate of $1 \mathrm{~m} / \mathrm{s}$. She would need to cancel out the north vector by swimming south at an equal rate ( $3 \mathrm{~m} / \mathrm{s}$ ). If she swims due south, she will still be swept downstream at a rate of $2 \mathrm{~m} / \mathrm{s}$.
8. A motorboat traveling at a speed of $4.0 \mathrm{~m} / \mathrm{s}$ pointed east encounters a current flowing at a speed $3.0 \mathrm{~m} / \mathrm{s}$ north.
a. What is the speed and direction that the motorboat travels?

The motorboat is traveling $5 \mathrm{~m} / \mathrm{s}$ at a direction $36.87^{\circ} \mathrm{N}$ of E .

b. What distance downstream does the boat reach the opposite shore?

Since the vectors are perpendicular, the north flowing current does not affect the easterly direction and vice versa. It takes $\frac{20}{4}=5$ seconds to reach the other side, and in that time the boat will have moved $3 \cdot 5=15 \mathrm{~m}$ downstream.
9. A ball with mass 0.5 kg experiences a force F due to gravity of 4.9 Newtons directed vertically downward. If this ball is rolling down a ramp that is $30^{\circ}$ inclined from the horizontal, what is the magnitude of the force that is directed parallel to the ramp? Assume that the ball is small enough so that all forces are acting at the point of contact of the ball and the ramp.

The diagram at right shows the forces that arise in this situation. We know that the magnitude of F is 4.9 N , and we need to find the magnitude of vector v . The three force vectors form a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle, so we know that

$$
\sin \left(30^{\circ}\right)=\frac{\|F\|}{\|v\|}
$$

Thus,


$$
\begin{aligned}
\mathbf{v} & =\frac{\|\mathbf{F}\|}{\sin \left(30^{\circ}\right)} \\
& =2(4.9 \mathrm{~N}) \\
& =9.8 \mathrm{~N} .
\end{aligned}
$$

Therefore, the force along the ramp has magnitude 9. 8 Newtons.

10. The stars in the Big Dipper may all appear to be the same distance from Earth, but they are, in fact, very far from each other. Distances between stars are measured in light years, the distance that light travels in one year. The star Alkaid at one end of the Big Dipper is 138 light years from Earth, and the star Dubhe at the other end of the Big Dipper is 105 light years from earth. From the Earth, it appears that Alkaid and Dubhe are $25.7^{\circ}$ apart. Find the distance in light years between stars Alkaid and Dubhe.


The angle of elevation does not matter and the only thing that does is the angle between them. The distance between the two stars will be the same no matter what angle we observe them, so we can treat the Alkaid star as being at a direction of $0^{\circ}$ and the Dubhe star as being at a direction of $25.7^{\circ}$. Then we find the difference between the two vectors, and we find that the distance between them is about $\mathbf{6 2}$. 9 light years.
11. A radio station has selected three listeners to compete for a prize buried in a large, flat field. Starting in the center, the contestants were given a meter stick, a compass, a calculator, and a shovel. Each contestant was given the following three vectors, in a different order for each contestant.
$64.2 \mathrm{~m}, 36^{\circ}$ east of north
$42.5 \mathrm{~m}, 20^{\circ}$ south of west
18. 2 m due south.

The three displacements led to the point where the prize was buried. The contestant that found the prize first won. Instead of measuring immediately, the winner began by doing calculations on paper. What did she calculate?

The winner calculated the sum of the three vectors. The prize is -2.2 m to the west and 19.2 m to the north. This is only about 19.3 m away from the starting position, while someone following the directions blindly would travel a total of 124.9 m .


## 8 <br> Lesson 21: Vectors and the Equation of a Line

## Student Outcomes

- Students write the equation for a line in $\mathbb{R}^{2}$ or $\mathbb{R}^{3}$ using vectors.
- Students write the parametric equations for a line in $\mathbb{R}^{2}$ or $\mathbb{R}^{3}$.
- Students convert between parametric equations and the slope-intercept form of a line in $\mathbb{R}^{2}$.


## Lesson Notes

In Algebra I, students wrote equations in the following forms to represent lines in the plane:

- Slope-intercept form: $y=m x+b$
- Point-slope form: $y-y_{1}=m\left(x-x_{1}\right)$
- Standard form: $a x+b y=c$.

In Algebra II students studied functions, this lesson introduces parametric equations to students showing the connection of functions to vectors (N-VM.C.11). We are looking for ways to describe a line in $\mathbb{R}^{3}$ so we are not restricted to twodimensions and can model real-life scenarios. This requires using vectors and parameters and writing parametric equations which give us a way to move and model three dimensional models with our two-dimensional system. Thus, we start by reconsidering how to describe a line in the plane using vectors and parameters, and then we apply this description to lines in $\mathbb{R}^{3}$. The shift to describing a line using vectors to indicate the direction of the line requires that students think geometrically about lines in the plane instead of algebraically.

Lessons 21 and 22 are important as they set the mathematical foundation for students to understand the definition of vectors.

## Classwork

## Opening Exercise (3 minutes)

The purpose of the second exercise below is to remind students how to graph a line by using the slope to generate points on the line. Encourage students to think geometrically, not algebraically, for this exercise.

## Opening Exercise

a. Find three different ways to write the equation that represents the line in the plane that passes through points $(1,2)$ and ( $2,-1$ ).

The following four equations show different forms of the equation that represents the line through $(1,2)$ and $(2,-1)$.

$$
\begin{aligned}
(y-2) & =-3(x-1) \\
y+1 & =-3(x-2) \\
y & =-3 x+5 \\
3 x+y & =5
\end{aligned}
$$

b. Graph the line through point $(1,1)$ with slope 2.


## Discussion (12 minutes)

- In the Opening Exercise, you found three different equations to represent a specific line in the plane.
- Consider the point-slope form of a line: $y-y_{1}=m\left(x-x_{1}\right)$. With the equation in this form, we know that the line passes through point $\left(x_{1}, y_{1}\right)$ and has slope $m$. Using this information, we can draw the line as we did in Opening Exercise 2.
- Suppose that the equation of a line $\ell$ is $y-3=\frac{1}{2}(x-4)$. Then we know that $\ell$ passes through the point $(4,3)$ and has slope $\frac{1}{2}$. This means that if we start at point $(4,3)$ and move $t$ units horizontally, we need to move $\frac{1}{2} t$ units vertically to arrive at a new point on the line.


That is, all points on line $\ell$ can be found by moving $t$ units right and $\frac{1}{2} t$ units up from $(4,3)$ or by moving $t$ units left and $\frac{1}{2} t$ units down from $(4,3)$.

- How could the process of finding a new point on line $\ell$ be found using vectors?
- A new point on line $\ell$ can be found by adding a multiple of the vector $\mathbf{v}=\left[\begin{array}{l}1 \\ \frac{1}{2}\end{array}\right]$ to the vector $\left[\begin{array}{l}4 \\ 3\end{array}\right]$ that represents point $(4,3)$.
- What point do we find when we let $t=-1$ ? When we let $t=6$ ? When we let $t=0$ ? When we let $t=-4$ ?
- If $t=-1$, then the new point is represented by $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}4 \\ 3\end{array}\right]-\left[\begin{array}{l}1 \\ \frac{1}{2}\end{array}\right]=\left[\begin{array}{c}3 \\ 2 \frac{1}{2}\end{array}\right]$.
- If $t=6$, then the new point is represented by $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}4 \\ 3\end{array}\right]+\left[\begin{array}{l}1 \\ \frac{1}{2}\end{array}\right] \cdot 6=\left[\begin{array}{l}4 \\ 3\end{array}\right]+\left[\begin{array}{l}6 \\ 3\end{array}\right]=\left[\begin{array}{c}10 \\ 6\end{array}\right]$.
- If $t=0$, then the new point is represented by $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}4 \\ 3\end{array}\right]+\left[\begin{array}{l}1 \\ \frac{1}{2}\end{array}\right] \cdot 0=\left[\begin{array}{l}4 \\ 3\end{array}\right]$.
- If $t=-4$, then the new point is represented by $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}4 \\ 3\end{array}\right]+\left[\begin{array}{l}1 \\ \frac{1}{2}\end{array}\right] \cdot(-4)=\left[\begin{array}{l}0 \\ 1\end{array}\right]$.

- Based on the calculations of different values of $t$, what is an equation that uses vectors to represent the points on line $\ell$ ? Explain your reasoning.
- The points on line $\ell$ can be represented by $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}4 \\ 3\end{array}\right]+\left[\begin{array}{l}1 \\ \frac{1}{2}\end{array}\right]$ t for real numbers $t$. This is a vector equation of the line $\ell$. The vector $\overrightarrow{\mathbf{v}}=\left[\begin{array}{l}4 \\ 3\end{array}\right]$ is a direction vector for the line. There are many different ways to choose the starting point and the direction vector, so the vector form of a line is not unique.
- Both $x$ and $y$ are both functions of the real number $t$, which is called a parameter. We can rewrite
$\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}4 \\ 3\end{array}\right]+\left[\begin{array}{l}1 \\ \frac{1}{2}\end{array}\right] t$, as $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}4+t \\ 3+\frac{1}{2} t\end{array}\right]$, which means that as functions of $t$, we have

$$
\begin{aligned}
& x(t)=4+t \\
& y(t)=3+\frac{1}{2} t
\end{aligned}
$$

## Scaffolding:

Have students create a Frayer diagram for parametric equations. (See Module 1, Lesson 5 for an example.)

These equations for $x$ and $y$ as functions of $t$ are parametric equations of line $\ell$.

- Do these parametric equations agree with our original equation in point-slope form that represents line $\ell$ ?

That equation is $y-3=\frac{1}{2}(x-4)$. Let's see, using our parameterized equations:

$$
\begin{aligned}
y-3 & =\left(3+\frac{1}{2} t\right)-3 \\
& =\frac{1}{2} t \\
\frac{1}{2}(x-4) & =\frac{1}{2}((4+t)-4) \\
& =\frac{1}{2} t
\end{aligned}
$$

So, we see that for any point $(x(t), y(t))$ that satisfies the parametric equations $x(t)=4+t$ and $y(t)=3+\frac{1}{2} t$, we have $y-3=\frac{1}{2}(x-4)$, so the point $(x(t), y(t))$ is on line $l$.

- Take a few minutes and explain to your neighbor what you have learned about parametric equations.

Use the summary box below to debrief parametric equations as a class and use this as an informal assessment of student knowledge.

Let $\ell$ be a line in the plane that contains point $\left(x_{1}, y_{1}\right)$ and has direction vector $\overrightarrow{\mathbf{v}}=\left[\begin{array}{l}a \\ b\end{array}\right]$. If the slope of line $\ell$ is defined, then $m=\frac{b}{a}$.

- A vector form of the equation that represents line $\ell$ is

$$
\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
x_{1} \\
y_{1}
\end{array}\right]+\left[\begin{array}{l}
a \\
b
\end{array}\right] t
$$

- Parametric equations that represent line $\ell$ are

$$
\begin{aligned}
& x(t)=x_{1}+a t \\
& y(t)=y_{1}+b t
\end{aligned}
$$

## Exercises 1-3 (10 minutes)

Have students work on this exercise in pairs or small groups. Take the time to debrief Exercise 2 and emphasize that choosing a different starting point $\left(x_{1}, y_{1}\right)$ on the line $\ell$ or different values $a$ and $b$ so that $\frac{b}{a}$ is the slope of line $\ell$ will produce equivalent equations of the line that look significantly different. That is, there are multiple correct forms of the vector and parametric equations of a line.

## Exercises

1. Consider the line $\ell$ in the plane given by the equation $3 x-2 y=6$.
a. Sketch a graph of line $\ell$ on the axes provided.

b. Find a point on line $\ell$ and the slope of line $\ell$.

Student responses for the point will vary; common choices include $(0,-3)$ or $(2,0)$. The slope of the line is $\frac{3}{2}$.
c. Write a vector equation for line $\ell$ using the information you found in part (b).

Student responses will vary. Sample responses are

$$
\begin{aligned}
& {\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{c}
0 \\
-3
\end{array}\right]+\left[\begin{array}{c}
1 \\
1.5
\end{array}\right] t} \\
& {\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
2 \\
0
\end{array}\right]+\left[\begin{array}{c}
1 \\
1.5
\end{array}\right] t .}
\end{aligned}
$$

d. Write parametric equations for line $\boldsymbol{\ell}$.

Student responses will vary. Sample responses are

$$
\begin{aligned}
& x(t)=t \\
& y(t)=-3+1.5 t
\end{aligned}
$$

or

$$
\begin{aligned}
& x(t)=2+t \\
& y(t)=1.5 t
\end{aligned}
$$

e. Verify algebraically that your parametric equations produce points on line $\boldsymbol{\ell}$.

$$
\begin{aligned}
3 x-2 y & =3(t)-2(-3+1.5 t) \\
& =3 t+6-3 t \\
& =6
\end{aligned}
$$

Thus, the parametric equations $x(t)=t$ and $y(t)=-3+1.5 t$ produce points on line $\ell$.
2. Olivia wrote parametric equations $x(t)=4+2 t$ and $y(t)=3+3 t$. Are her equations correct? What did she do differently from you?

Her equations are also correct:

$$
\begin{aligned}
3 x-2 y & =3(4+2 t)-2(3+3 t) \\
& =12+6 t-6-6 t \\
& =6
\end{aligned}
$$

She chose the point $(4,3)$ on the line and used the vector $\left[\begin{array}{l}2 \\ 3\end{array}\right]$.
3. Convert the parametric equations $x(t)=2-3 t$ and $y(t)=4+t$ into slope-intercept form.

One vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}2 \\ 4\end{array}\right]+\left[\begin{array}{c}-3 \\ 1\end{array}\right] t$, so the line passes through $(2,4)$ with slope $m=-\frac{1}{3}$. Then the line has equation

$$
\begin{aligned}
& y-4=-\frac{1}{3}(x-2) \\
& y=-\frac{1}{3} x+\frac{2}{3}+4 \\
& y=-\frac{1}{3} x+\frac{14}{3}
\end{aligned}
$$

## Discussion (4 minutes)

- A line is uniquely determined in the plane if we know a point that it passes through and its slope. Do a point and a slope provide enough to uniquely identify a line in $\mathbb{R}^{3}$ ?
- No, we have no sense of a slope in space, so a point and a slope won't clearly identify a line in $\mathbb{R}^{3}$.
- How can we uniquely specify a line $\ell$ in space?
- If we know a point that $\ell$ passes through and the direction in which the line points, then we can uniquely specify that line.
- That is, we need to know a point $\left(x_{1}, y_{1}, z_{1}\right)$ on line $\ell$ and a vector that is pointed in the same direction as $\ell$. Then we can start at that point $\left(x_{1}, y_{1}, z_{1}\right)$, and move in the direction of the vector to find new points on $\ell$.


## Example (5 minutes)

In this Example, we extend the process from lines in the plane to lines in space.

- Consider the line $\ell$ in space that passes through point $(1,1,2)$ and has direction vector $\overrightarrow{\mathbf{v}}=\left[\begin{array}{c}1 \\ -3 \\ 2\end{array}\right]$, as shown at right.
- Can you find another point on $\ell$ by moving $t$ steps in the direction of $\overrightarrow{\mathbf{v}}$ from point $(1,1,2)$ ?

$$
\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{l}
1 \\
1 \\
2
\end{array}\right]+\left[\begin{array}{c}
1 \\
-3 \\
2
\end{array}\right] t .
$$

- What are the parametric equations of $\ell$ ?

$$
\begin{aligned}
& x=1+t \\
& y=1-3 t \\
& z=2+2 t .
\end{aligned}
$$



- Debrief this activity in class using the summary box below.

Let $\ell$ be a line in space that contains point $\left(x_{1}, y_{1}, z_{1}\right)$ and has direction vector $\overrightarrow{\mathbf{v}}=\left[\begin{array}{l}a \\ b \\ c\end{array}\right]$.

- A vector form of the equation that represents line $\ell$ is

$$
\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right]+\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right] t
$$

- Parametric equations that represent line $\ell$ are

$$
\begin{aligned}
& x(t)=x_{1}+a t \\
& y(t)=y_{1}+b t \\
& z(t)=z_{1}+c t .
\end{aligned}
$$

## Exercises 4-5 (4 minutes)

Keep students working in the same pairs or small groups as in the previous exercise.
4. Find parametric equations to represent the line that passes through point $(4,2,9)$ and has direction vector

$$
\begin{gathered}
\vec{v}=\left[\begin{array}{c}
2 \\
-1 \\
-3
\end{array}\right] . \\
{\left[\begin{array}{c}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{c}
4 \\
2 \\
9
\end{array}\right]+\left[\begin{array}{c}
2 \\
-1 \\
-3
\end{array}\right] t} \\
x(t)=4+2 t \\
y(t)=2-t \\
z(t)=9-3 t
\end{gathered}
$$

5. Find a vector form of the equation of the line given by the parametric equations

$$
\begin{gathered}
x(t)=3 t \\
y(t)=-4-2 t \\
z(t)=3-t . \\
{\left[\begin{array}{c}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{c}
0 \\
-4 \\
3
\end{array}\right]+\left[\begin{array}{c}
3 \\
-2 \\
-1
\end{array}\right] t}
\end{gathered}
$$

## Closing (3 minutes)

Ask students to summarize the key points of the lesson in writing or to a partner. Some important summary elements are listed below.

## Lesson Summary

Lines in the plane and lines in space can be described by either a vector equation or a set of parametric equations.

- Let $\ell$ be a line in the plane that contains point $\left(x_{1}, y_{1}\right)$ and has direction vector $\overrightarrow{\mathrm{v}}=\left[\begin{array}{l}a \\ b\end{array}\right]$. If the slope of line $\ell$ is defined, then $m=\frac{b}{a}$.
A vector form of the equation that represents line $\ell$ is

$$
\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
x_{1} \\
y_{1}
\end{array}\right]+\left[\begin{array}{l}
a \\
b
\end{array}\right] t .
$$

Parametric equations that represent line $\ell$ are

$$
\begin{aligned}
& x(t)=x_{1}+a t \\
& y(t)=y_{1}+b t .
\end{aligned}
$$

- Let $\ell$ be a line in space that contains point $\left(x_{1}, y_{1}, z_{1}\right)$ and has direction vector $\overrightarrow{\mathrm{v}}=\left[\begin{array}{l}a \\ b \\ c\end{array}\right]$.

A vector form of the equation that represents line $\ell$ is

$$
\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right]+\left[\begin{array}{c}
a \\
b \\
c
\end{array}\right] t .
$$

Parametric equations that represent line $\ell$ are

$$
\begin{aligned}
& x(t)=x_{1}+a t \\
& y(t)=y_{1}+b t \\
& z(t)=z_{1}+c t .
\end{aligned}
$$

## Exit Ticket (4 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 21: Vectors and the Equation of a Line

## Exit Ticket

1. Find parametric equations for the line in the plane given by $y=2 x+3$.
2. Do $y=7-x$ and $x(t)=-1$ and $y(t)=1+7 t$ represent the same line? Explain why or why not.
3. Find parametric equations for the line in space that passes through point $(1,0,4)$ with direction vector $\overrightarrow{\mathbf{v}}=\left[\begin{array}{l}3 \\ 2 \\ 1\end{array}\right]$.

## Exit Ticket Sample Solutions

1. Find parametric equations for the line in the plane given by $y=2 x+3$.

This line passes through point $(0,3)$ with slope $\boldsymbol{m}=2$. Then the vector form of the line is

$$
\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
0 \\
3
\end{array}\right]+\left[\begin{array}{l}
1 \\
2
\end{array}\right] t
$$

so the parametric equations are

$$
\begin{aligned}
& x(t)=t \\
& y(t)=3+2 t
\end{aligned}
$$

2. Do $y=7-x$ and $x(t)=-1$ and $y(t)=1+7 t$ represent the same line? Explain why or why not.

We can see that if $x(t)=-1$ and $y(t)=7-t$, then $7-x=7-(-1)=8$ so $y \neq 7-x$, and thus the equations are not the same line.
3. Find parametric equations for the line in space that passes through point $(\mathbf{1}, \mathbf{0}, 4)$ with direction vector $\overrightarrow{\mathbf{v}}=\left[\begin{array}{l}3 \\ 2 \\ 1\end{array}\right]$.

$$
\begin{aligned}
& x(t)=1+3 t \\
& y(t)=0+2 t \\
& z(t)=4+t
\end{aligned}
$$

## Problem Set Sample Solutions

The vector and parametric forms of equations in the plane and in space are not unique. There are many different forms of correct answers to these questions. One correct sample response is included, but there are many other correct responses students could provide. Problems 1-6 address lines in the plane, and Problems 7-11 address lines in space.

1. Find three points on the line in the plane with parametric equations $x(t)=4-3 t$ and $y(t)=1+\frac{1}{3} t$.

Student responses will vary. Using $t=0, t=3$, and $t=-3$ gives the three points $(4,1),(-5,4)$ and $(13,0)$.
2. Find vector and parametric equations to represent the line in the plane with the given equation.
a. $y=3 x-4$

Since the slope is $3=\frac{3}{1}$ and a point on the line is $(0,-4)$, a vector form of the equation is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}0 \\ -4\end{array}\right]+\left[\begin{array}{l}1 \\ 3\end{array}\right] t$. Then the parametric equations are $x(t)=t$ and $y=-4+3 t$.
b. $\quad 2 x-5 y=10$

First, we rewrite the equation of the line in slope-intercept form: $y=\frac{2}{5} x-2$. Since the slope is $\frac{2}{5}$ and a point on the line is $(0,-2)$, a vector form of the equation is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}0 \\ -2\end{array}\right]+\left[\begin{array}{l}5 \\ 2\end{array}\right]$ t. Then the parametric equations are $x(t)=5 t$ and $y=-2+2 t$.
c. $y=-x$

Since the slope is -1 and a point on the line is $(0,0)$, a vector form of the equation is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}0 \\ 0\end{array}\right]+\left[\begin{array}{c}1 \\ -1\end{array}\right] t$. Then the parametric equations are $x(t)=t$ and $y=-t$.
d. $\quad y-2=3(x+1)$

First, we rewrite the equation of the line in slope-intercept form: $y=3 x+5$. Since the slope is 3 and a point on the line is $(0,5)$, a vector form of the equation is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}0 \\ 3\end{array}\right]+\left[\begin{array}{l}1 \\ 3\end{array}\right] t$. Then the parametric equations are $x(t)=t$ and $y=3+3 t$.
3. Find vector and parametric equations to represent the following lines in the plane.
a. the $x$-axis

A vector in the direction of the $x$-axis is $\left[\begin{array}{l}1 \\ 0\end{array}\right]$ and the line passes through the origin $(0,0)$, so a vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}1 \\ 0\end{array}\right] t$. Thus, the parametric equations are $x(t)=t$ and $y(t)=0$.
b. the $y$-axis

A vector in the direction of the $y$-axis is $\left[\begin{array}{l}0 \\ 1\end{array}\right]$ and the line passes through the origin $(0,0)$, so a vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}0 \\ 1\end{array}\right] t$. Thus, the parametric equations are $x(t)=0$ and $y(t)=t$.
c. the horizontal line with equation $y=4$

A vector in the direction of the line is $\left[\begin{array}{l}1 \\ 0\end{array}\right]$ and the line passes through $(\mathbf{0}, 4)$, so a vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}0 \\ 4\end{array}\right]+\left[\begin{array}{l}1 \\ 0\end{array}\right] t$. Thus, the parametric equations are $x(t)=t$ and $y(t)=4$.
d. the vertical line with equation $x=-2$

A vector in the direction of the line is $\left[\begin{array}{l}0 \\ 1\end{array}\right]$ and the line passes through $(-2,0)$, so a vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}-2 \\ 0\end{array}\right]+\left[\begin{array}{l}0 \\ 1\end{array}\right] t$. Thus, the parametric equations are $x(t)=-2$ and $y(t)=t$.
e. the horizontal line with equation $y=k$, for a real number $k$

A vector in the direction of the line is $\left[\begin{array}{l}1 \\ 0\end{array}\right]$ and the line passes through $(0, k)$, so a vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}0 \\ k\end{array}\right]+\left[\begin{array}{l}1 \\ 0\end{array}\right] t$. Thus, the parametric equations are $x(t)=t$ and $y(t)=k$.
f. the vertical line with equation $x=h$, for a real number $h$

A vector in the direction of the line is $\left[\begin{array}{l}0 \\ 1\end{array}\right]$ and the line passes through $(h, 0)$, so a vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}h \\ 0\end{array}\right]+\left[\begin{array}{l}0 \\ 1\end{array}\right] t$. Thus, the parametric equations are $x(t)=h$ and $y(t)=t$.
4. Find the point-slope form of the line in the plane with the given parametric equations.
a. $\quad x(t)=2-4 t, y(t)=3-7 t$

The vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}2 \\ 3\end{array}\right]+\left[\begin{array}{l}-4 \\ -7\end{array}\right]$, so the line passes through the point $(2,3)$ with slope $m=\frac{-7}{-4}=\frac{7}{4}$. Thus, the point-slope form of the line is $y-3=\frac{7}{4}(x-2)$.
b. $\quad x(t)=2-\frac{2}{3} t, y(t)=6+t$

The vector form is

$$
\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
2 \\
6
\end{array}\right]+\left[\begin{array}{c}
-\frac{2}{3} \\
6
\end{array}\right] t
$$

so the line passes through the point $(2,6)$ with slope

$$
m=\frac{6}{-\frac{2}{3}}=-9
$$

Thus, the point-slope form of the line is $y-6=-9(x-2)$.
c. $\quad x(t)=3-t, y(t)=3$

The vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}3 \\ 3\end{array}\right]+\left[\begin{array}{c}-1 \\ 0\end{array}\right] t$, so the line passes through the point $(3,3)$ with slope $m=0$. Thus the point-slope form of the line is $y-3=0(x-3)$, which is equivalent to $y=3$.
d. $\quad x(t)=t, y(t)=t$

The vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}0 \\ 0\end{array}\right]+\left[\begin{array}{l}1 \\ 1\end{array}\right] t$, so the line passes through the point $(0,0)$ with slope $m=1$. Thus the point-slope form of the line is $y=x$.
5. Find vector and parametric equations for the line in the plane through point $P$ in the direction of vector $v$.
a. $\quad P=(1,5), \overrightarrow{\mathbf{v}}=\left[\begin{array}{c}2 \\ -1\end{array}\right]$

The vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}1 \\ 5\end{array}\right]+\left[\begin{array}{c}2 \\ -1\end{array}\right] t$, so the parametric equations are $x(t)=1+2 t$ and $y(t)=5-t$.
b. $\quad P=(0,0), \vec{v}=\left[\begin{array}{l}4 \\ 4\end{array}\right]$

The vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}0 \\ 0\end{array}\right]+\left[\begin{array}{l}4 \\ 4\end{array}\right] t$, so the parametric equations are $x(t)=4 t$ and $y(t)=4 t$.
c. $\quad P=(-3,-1), \overrightarrow{\mathrm{v}}=\left[\begin{array}{l}1 \\ 2\end{array}\right]$

The vector form is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}-3 \\ -1\end{array}\right]+\left[\begin{array}{l}1 \\ 2\end{array}\right] t$, so the parametric equations are $x(t)=-3+t$ and $y(t)=-1+2 t$.
6. Determine if the point $A$ is on the line $\ell$ represented by the given parametric equations.
a. $\quad A=(3,1), x(t)=1+2 t$ and $y(t)=3-2 t$.

Point $A$ is on the line if there is a single value of $t$ so that $3=1+2 t$ and $1=3-2 t$. If $1+2 t=3$, then $t=1$. If $1=3-2 t$, then $t=1$. Thus, $A$ is on the line given by these parametric equations.
b. $\quad A=(0,0), x(t)=3+6 t$ and $y(t)=2+4 t$

Point $A$ is on the line if there is a single value of $t$ so that $0=3+6 t$ and $0=2+4 t$. If $3+6 t=0$, then $t=-\frac{1}{2}$. If $0=2+4 t$, then $t=-\frac{1}{2}$. Thus, $A$ is on the line given by these parametric equations.
c. $\quad A=(2,3), x(t)=4-2 t$ and $y(t)=4+t$

Point $A$ is on the line if there is a single value of $t$ so that $2=4-2 t$ and $3=4+t$. If $4-2 t=2$, then $t=1$. If $3=4+t$, then $t=-1$. Since there is no value of $t$ that gives $4-2 t=2$ and $3=4+t$, point $A$ is not on the line.
d. $\quad A=(2,5), x(t)=12+2 t$ and $y(t)=15+2 t$

Point $A$ is on the line if there is a single value of $t$ so that $12+2 t=2$ and $15+2 t=5$. If $12+2 t=2$, then $t=-5$. If $15+2 t=5$, then $t=-5$. Thus, $A$ is on the line given by these parametric equations.
7. Find three points on the line in space with parametric equations $x(t)=4+2 t, y(t)=6-t$, and $z(t)=t$.

Student responses will vary. Using $t=0, t=1$, and $t=-1$ gives the three points $(4,6,0),(6,5,1)$, and ( $2,7,-1$ ).
8. Find vector and parametric equations to represent the following lines in space.
a. the $x$-axis

A vector in the direction of the $x$-axis is $\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right]$ and the line passes through the origin $(0,0,0)$, so a vector form is $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right] t$. Thus, the parametric equations are $x(t)=t, y(t)=0$, and $z(t)=0$.
b. the $y$-axis

A vector in the direction of the $y$-axis is $\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]$ and the line passes through the origin $(0,0,0)$, so a vector form is $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right] t$. Thus, the parametric equations are $x(t)=0, y(t)=1$, and $z(t)=0$.
c. the $z$-axis

A vector in the direction of the $z$-axis is $\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]$ and the line passes through the origin $(0,0,0)$, so a vector form is
$\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right] t$. Thus, the parametric equations are $x(t)=0, y(t)=0$, and $z(t)=1$.
9. Convert the equation given in vector form to a set of parametric equations for the line $\ell$.
a. $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}1 \\ 1 \\ 1\end{array}\right]+\left[\begin{array}{l}2 \\ 3 \\ 4\end{array}\right] t$
$x(t)=1+2 t, y(t)=1+3 t$, and $z(t)=1+4 t$
b. $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}3 \\ 0 \\ 0\end{array}\right]+\left[\begin{array}{c}0 \\ 1 \\ -2\end{array}\right] t$
$x(t)=3, y(t)=t$, and $z(t)=-2 t$
c.
$\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}5 \\ 0 \\ 2\end{array}\right]+\left[\begin{array}{c}4 \\ -3 \\ -8\end{array}\right] t$
$x(t)=5+4 t, y(t)=-3 t$, and $z(t)=2-8 t$
10. Find vector and parametric equations for the line in space through point $P$ in the direction of vector $\overrightarrow{\mathbf{v}}$.
a. $\quad P=(1,4,3), \overrightarrow{\mathbf{v}}=\left[\begin{array}{c}3 \\ 6 \\ -2\end{array}\right]$

$$
\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{l}
1 \\
4 \\
3
\end{array}\right]+\left[\begin{array}{c}
3 \\
6 \\
-2
\end{array}\right] t ; x(t)=1+3 t, y(t)=4+6 t, \text { and } z(t)=3-2 t
$$

b. $\quad P=(2,2,2), \overrightarrow{\mathrm{v}}=\left[\begin{array}{l}1 \\ 1 \\ 1\end{array}\right]$
$\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}2 \\ 2 \\ 2\end{array}\right]+\left[\begin{array}{l}1 \\ 1 \\ 1\end{array}\right] t ; x(t)=2+t, y(t)=2+t$, and $z(t)=2+t$
c. $\quad P=(0,0,0), \vec{v}=\left[\begin{array}{c}4 \\ 4 \\ -2\end{array}\right]$
$\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]+\left[\begin{array}{c}4 \\ 4 \\ -2\end{array}\right] t ; x(t)=4 t, y(t)=4 t$, and $z(t)=-2 t$
11. Determine if the point $A$ is on the line $\ell$ represented by the given parametric equations.
a. $\quad A=(3,1,1), x(t)=5-t, y(t)=-5+3 t$, and $z(t)=9-4 t$

If $A$ is on line $\ell$, then there is a single value of $t$ so that $5-t=3,-5+3 t=1$, and $9-4 t=1$. If $5-t=$ 3 , then $t=2$. If $-5+3 t=1$, then $t=2$. If $9-4 t=1$, then $t=2$. Thus, $A$ lies on line $\ell$.
b. $\quad A=(1,0,2), x(t)=7-2 t, y(t)=3-t$, and $z(t)=4-t$

If $A$ is on line $\ell$, then there is a single value of $t$ so that $7-2 t=1,3-t=0$, and $4-t=2$. If $7-2 t=1$, then $t=3$. If $3-t=0$, then $t=3$. If $4-t=2$, then $t=2$. Thus, there is no value of $t$ that satisfies all three equations, so point $A$ is not on line $\ell$.
c. $\quad A=(5,3,2), x(t)=8+t, y(t)=-t$, and $z(t)=-4-2 t$

If $A$ is on line $\ell$, then there is a single value of $t$ so that $8+t=5,-t=3$, and $-4-2 t=2$. If $8+t=5$, then $t=-3$. If $-t=3$, then $t=-3$. If $-4-2 t=2$, then $t=-3$. Thus, $A$ lies on line $\ell$.

## Lesson 22: Linear Transformations of Lines

## Student Outcomes

- Students write parametric equations for a line through two points in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$ and for a line segment between two points in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$.
- Students write parametric equations for the image of a line under a given linear transformation in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$ and for the image of a line segment between two points under a given linear transformation in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$.


## Lesson Notes

In this lesson, students continue their work with parametric equations to see the relationship between their work with functions and vectors (N-VM.C.11). This lesson continues the work of understanding the definition of a vector.

The main question of this lesson is whether the image of a line under a linear transformation is again a line. Before we answer this, we need to extend the process of finding parametric equations for a line in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$ introduced in Lesson 21. In the previous lesson, students found vector and parametric equations for a line given a point and a vector; in this lesson, we extend the process to finding parametric equations for the line given two points on the line. We also consider the question of how to parameterize a line segment. In Topic E, students will use linear transformations to emulate 3-dimensional motion on a 2-dimensional screen, and learn that one of the fundamental qualities of linear transformations is that they preserve lines.

## Classwork

## Opening Exercise (3 minutes)

The Opening Exercise reviews the process from Lesson 21 of finding parametric equations of lines in $\mathbb{R}^{2}$ and $\mathbb{R}^{3}$ given a point and a vector. This lesson will extend this process to find parametric equations of lines through two given points and to find parametric equations of line segments.

## Opening Exercise

a. Find parametric equations of the line through point $P(1,1)$ in the direction of vector $\left[\begin{array}{c}-2 \\ 3\end{array}\right]$.

A vector form of the equation is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}1 \\ 1\end{array}\right]+\left[\begin{array}{c}-2 \\ 3\end{array}\right] t$, which gives parametric equations $x(t)=1-2 t$ and $y(t)=1+3 t$ for any real number $t$.
b. Find parametric equations of the line through point $P(2,3,1)$ in the direction of vector $\left[\begin{array}{c}4 \\ 1 \\ -1\end{array}\right]$. A vector form of the equation is $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}2 \\ 3 \\ 1\end{array}\right]+\left[\begin{array}{c}4 \\ 1 \\ -1\end{array}\right] t$, which gives parametric equations $x(t)=2+4 t$ and $y(t)=3+t$ and $z(t)=1-t$ for any real number $t$.

## Discussion (5 minutes)

- In the Opening Exercise we found parametric equations for the line $\ell$ through $P(1,1)$ with direction vector $\left[\begin{array}{c}-2 \\ 3\end{array}\right]$. How could we find parametric equations for this line if all we knew was that points $P(1,1)$ and $Q(-1,4)$ were on line $\ell$ ?
- First we find the vector that points from $P$ to $Q$, and then we apply the process from the last lesson.
- What is this direction vector?
- The direction vector $\overrightarrow{\mathbf{v}}$ is the difference between the vectors representing points $Q$ and $P$ :

$$
\overrightarrow{\mathbf{v}}=\left[\begin{array}{c}
-1 \\
4
\end{array}\right]-\left[\begin{array}{l}
1 \\
1
\end{array}\right] \text { so } \overrightarrow{\mathbf{v}}=\left[\begin{array}{c}
-2 \\
3
\end{array}\right] .
$$

- What are parametric equations for the line $\ell$ ?
- This is the same direction vector as we had in Problem 1 of the Opening Exercise, so parametric equations are $x(t)=1-2 t, y(t)=1+3 t$ for any real number $t$.
- What would happen if we swapped $P$ and $Q$ ? Do we get parametric equations for a different line?
- No. If we interchange $P$ and $Q$, then we get a direction vector $\left[\begin{array}{c}2 \\ -3\end{array}\right]$, and the parametric equations are $x(t)=1+2 t, y(t)=1-3 t$ for any real number $t$. This describes the same line, but it is being traversed backwards. Instead of moving from $P$ to $Q$ as $t$ increases, this new line locates points from $Q$ to $P$ as $t$ increases.


## Example 1 (8 minutes)

This example is analogous to the one in the previous discussion but in $\mathbb{R}^{3}$ instead of $\mathbb{R}^{2}$. It then proceeds to describe how to use parametric equations to describe a line segment $\overline{P Q}$.

- What if we had a line in $\mathbb{R}^{3}$ ? Suppose we want to find parametric equations of the line through points $P(1,2,3)$ and $Q(-4,1,0)$. How do we find these equations?


## Scaffolding:

- For struggling learners, display an image of the point $P(1,1)$ and the line $y=-3 / 2 x+5 / 2$. Place a marker on $P$ to indicate when $t=0$, and slide the marker upward to the left to illustrate the point on the line corresponding to increasing values of $t$.
- Ask advanced learners to find the parametric equations described in Example 1 in pairs without the guiding questions and then present their work to the class.
- First, we need the vector that points from $P$ to $Q: \overrightarrow{\mathbf{v}}=\left[\begin{array}{c}-4 \\ 1 \\ 0\end{array}\right]-\left[\begin{array}{l}1 \\ 2 \\ 3\end{array}\right]=\left[\begin{array}{c}-5 \\ -1 \\ -3\end{array}\right]$. Then we have the vector form of the equation $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{c}-4 \\ 1 \\ 0\end{array}\right]+\left[\begin{array}{l}-5 \\ -1 \\ -3\end{array}\right] t$, which gives three parametric equations $x(t)=1-5 t, y(t)=2-t$ and $z(t)=3-3 t$ for real numbers $t$.
- Is there a way to use parametric equations to describe just the line segment $\overline{P Q}$ instead of the entire line $\overleftrightarrow{P Q}$ ?
- Give students time to figure this out on their own or with a partner and then discuss later with the class as shown below.
- What is the value of $t$ in $x(t)=1-5 t, y(t)=2-t$ and $z(t)=3-3 t$ that produces point $P$ ?
- If $t=0$, then $\left[\begin{array}{l}x(0) \\ y(0) \\ z(0)\end{array}\right]=\left[\begin{array}{l}1 \\ 2 \\ 3\end{array}\right]=P$.
- What value of $t$ in $x(t)=1-5 t, y(t)=2-t$ and $z(t)=3-3 t$ produces point $Q$ ?
- If $t=1$, then $\left[\begin{array}{l}x(1) \\ y(1) \\ z(1)\end{array}\right]=\left[\begin{array}{c}-4 \\ 1 \\ 0\end{array}\right]=Q$.
- So, how can we use parametric equations to describe just segment $\overline{P Q}$ ?
- Use the same parametric equations as for line $\overleftrightarrow{P Q}$, but restrict $0 \leq t \leq 1$.
- In general, describe the process for finding parametric equations of the line through $P$ and $Q$.
- First, find the direction vector $\overrightarrow{\mathbf{v}}$ by subtracting the vector associated with $P$ from the vector associated with $Q$. Then find the vector form of the equation of the line and the parametric form. Let take on any real number value.
- In general, describe the process for finding parametric equations of the segment $\overline{P Q}$.
- First, find the direction vector $\overrightarrow{\mathbf{v}}$ by subtracting the vector associated with $P$ from the vector associated with $Q$. Then find the vector form of the equation of the line and the parametric form. The segment $\overline{P Q}$ corresponds to the part of the line with $0 \leq t \leq 1$.


## Discussion (12 minutes)

This discussion starts with an example that shows that the image of a particular line in $\mathbb{R}^{3}$ under a given linear transformation is again a line in $\mathbb{R}^{3}$. Once this example has been established, the discussion proceeds to establish this fact for any line in $\mathbb{R}^{2}$ or $\mathbb{R}^{3}$ and any linear transformation $L$.

- Now, we want to explore what happens when we transform a line using a linear transformation. What do you expect the image of a line to be under a linear transformation? Why?
- I don't know. Linear transformations include things like rotation, dilation, and reflection. All of these operations will transform a line into another line. But, there might be a linear transformation that does something else that might distort or bend a line.
- Suppose that the line passes through points $P(1,0,1)$ and $Q(3,-3,2)$ and we have a linear transformation $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=\left[\begin{array}{lll}1 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 2\end{array}\right]\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$. Then what are the transformed points $L(P)$ and $L(Q)$ ?

$$
\quad L(P)=L\left(\left[\begin{array}{l}
1 \\
0 \\
1
\end{array}\right]\right)=\left[\begin{array}{lll}
1 & 2 & 1 \\
2 & 1 & 3 \\
1 & 1 & 2
\end{array}\right]\left[\begin{array}{l}
1 \\
0 \\
1
\end{array}\right]=\left[\begin{array}{l}
2 \\
5 \\
3
\end{array}\right] \text { and } L(Q)=L\left(\left[\begin{array}{c}
3 \\
-3 \\
2
\end{array}\right]\right)=\left[\begin{array}{ccc}
1 & 2 & 1 \\
2 & 1 & 3 \\
1 & 1 & 2
\end{array}\right]\left[\begin{array}{c}
3 \\
-3 \\
2
\end{array}\right]=\left[\begin{array}{c}
-1 \\
9 \\
4
\end{array}\right]
$$

- How can we describe a point on the line $\overleftrightarrow{P Q}$ ?
- We can use the parametric equations for $\overleftrightarrow{P Q}$ : First, a direction vector is $\left[\begin{array}{c}3 \\ -3 \\ 2\end{array}\right]-\left[\begin{array}{l}1 \\ 0 \\ 1\end{array}\right]=\left[\begin{array}{c}2 \\ -3 \\ 1\end{array}\right]$. Then, $a$ vector form of the equation of $\overleftrightarrow{P Q}$ is $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}1 \\ 0 \\ 1\end{array}\right]+\left[\begin{array}{c}2 \\ -3 \\ 1\end{array}\right]$ t. Finally, parametric equations for $\overleftrightarrow{P Q}$ are $x(t)=1+2 t, y(t)=-3 t$ and $z(t)=1+t$ for all real numbers $t$.
- Since we know that $\left[\begin{array}{c}1+2 t \\ -3 t \\ 1+t\end{array}\right]$ is a generic point on the line $\overleftrightarrow{P Q}$, we can transform this point under $L$ :

$$
\begin{aligned}
L\left(\left[\begin{array}{c}
1+2 t \\
-3 t \\
1+t
\end{array}\right]\right) & =\left[\begin{array}{lll}
1 & 2 & 1 \\
2 & 1 & 3 \\
1 & 1 & 2
\end{array}\right]\left[\begin{array}{c}
1+2 t \\
-3 t \\
1+t
\end{array}\right] \\
& =\left[\begin{array}{c}
1(1+2 t)+2(-3 t)+1(1+t) \\
2(1+2 t)+1(-3 t)+3(1+t) \\
1(1+2 t)+1(-3 t)+2(1+t)
\end{array}\right] \\
& =\left[\begin{array}{c}
2-3 t \\
5+4 t \\
3+t
\end{array}\right]
\end{aligned}
$$

But, this is how we express a line in vector form. So, any point on the line $P Q$ is transformed into a point on the line $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}2 \\ 5 \\ 3\end{array}\right]+\left[\begin{array}{c}-3 \\ 4 \\ 1\end{array}\right]$. We saw earlier that $L(P)=\left[\begin{array}{l}2 \\ 5 \\ 3\end{array}\right]$. Is this a coincidence?

- No, it's probably not a coincidence, because the starting point is when $t=0$ and when $t=0$ in our parametric equation, we get the initial point.
- Now, let's generalize this result to any transformation $L$ and any line $\ell$ through points $P$ and $Q$ in $\mathbb{R}^{3}$. Let $\ell$ be a line in either $\mathbb{R}^{2}$ or $\mathbb{R}^{3}$, and let $L$ be a linear transformation on that space that can be represented by multiplication by matrix $A$. Let point $P$ be represented by vector $\left[\begin{array}{l}p_{1} \\ p_{2} \\ p_{3}\end{array}\right]$ and let point $Q$ be represented by vector $\left[\begin{array}{l}q_{1} \\ q_{2} \\ q_{3}\end{array}\right]$. Then, we can find the direction vector $\overrightarrow{\mathbf{v}}$ by $\overrightarrow{\mathbf{v}}=\left[\begin{array}{l}q_{1}-p_{1} \\ q_{2}-p_{2} \\ q_{3}-p_{3}\end{array}\right]$. Any point $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ on line $\overleftrightarrow{P Q}$ is given by $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}p_{1} \\ p_{2} \\ p_{3}\end{array}\right]+\left[\begin{array}{l}q_{1}-p_{1} \\ q_{2}-p_{2} \\ q_{3}-p_{3}\end{array}\right] t$ for some real number $t$. Then the transformed point is given by

$$
\begin{aligned}
L\left(\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]\right) & =A\left(\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]\right) \\
& =A\left(\left[\begin{array}{l}
p_{1} \\
p_{2} \\
p_{3}
\end{array}\right]+\left[\begin{array}{l}
q_{1}-p_{1} \\
q_{2}-p_{2} \\
q_{3}-p_{3}
\end{array}\right] t\right) \\
& =A\left[\begin{array}{l}
p_{1} \\
p_{2} \\
p_{3}
\end{array}\right]+\left(A\left[\begin{array}{l}
q_{1}-p_{1} \\
q_{2}-p_{2} \\
q_{3}-p_{3}
\end{array}\right]\right) t \\
& =A\left[\begin{array}{l}
p_{1} \\
p_{2} \\
p_{3}
\end{array}\right]+\left(A\left[\begin{array}{l}
q_{1} \\
q_{2} \\
q_{3}
\end{array}\right]-A\left[\begin{array}{l}
p_{1} \\
p_{2} \\
p_{3}
\end{array}\right]\right) t \\
& =L(P)+(L(Q)-L(P)) t
\end{aligned}
$$

Since $L(P)$ and $(L(Q)-L(P))$ are vectors that represent points in space, this is the vector form of a line that passes through $L(P)$ and has direction vector $(L(Q)-L(P))$. Therefore, the image of any line in $\mathbb{R}^{3}$ under a linear transformation $L$ is again a line.

## Exercises 1-3 (8 minutes)

Have students work in pairs or small groups on these exercises.

## Exercises 1-3

1. Consider points $P(2,1,4)$ and $Q(3,-1,2)$, and define a linear transformation by $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=\left[\begin{array}{ccc}1 & 2 & -1 \\ 0 & 1 & 2 \\ 3 & -1 & 1\end{array}\right]\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$. Find parametric equations to describe the image of line $\overleftrightarrow{P Q}$ under the transformation $L$.
Direction vector: $\left[\begin{array}{c}3 \\ -1 \\ 2\end{array}\right]-\left[\begin{array}{l}2 \\ 1 \\ 4\end{array}\right]=\left[\begin{array}{c}1 \\ -2 \\ -2\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}2 \\ 1 \\ 4\end{array}\right]+\left[\begin{array}{c}1 \\ -2 \\ -2\end{array}\right] t$ for all real numbers $t$.
Parametric Equations: $x(t)=2+t, y(t)=1-2 t$, and $z(t)=4-2 t$ for all real numbers $t$.
2. The process that we developed for images of lines in $\mathbb{R}^{3}$ also applies to lines in $\mathbb{R}^{2}$. Consider points $P(2,3)$ and $\boldsymbol{Q}(-1,4)$. Define a linear transformation by $L\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{cc}1 & 2 \\ 3 & -1\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]$. Find parametric equations to describe the image of line $\overleftrightarrow{P Q}$ under the transformation $L$.
Direction vector: $\left[\begin{array}{c}-1 \\ 4\end{array}\right]-\left[\begin{array}{l}2 \\ 3\end{array}\right]=\left[\begin{array}{c}-3 \\ 1\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}2 \\ 3\end{array}\right]+\left[\begin{array}{c}-3 \\ 1\end{array}\right]$ t for all real numbers $t$
Parametric Equations: $x(t)=2-3 t$ and $y(t)=3+t$ for all real numbers $t$.
3. Not only is the image of a line under a linear transformation another line, but the image of a line segment under a linear transformation is another line segment. Let $P, Q$, and $L$ be as specified in Exercise 2. Find parametric equations to describe the image of segment $\overline{P Q}$ under the transformation $L$.
Direction vector: $\left[\begin{array}{c}-1 \\ 4\end{array}\right]-\left[\begin{array}{l}2 \\ 3\end{array}\right]=\left[\begin{array}{c}-3 \\ 1\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}2 \\ 3\end{array}\right]+\left[\begin{array}{c}-3 \\ 1\end{array}\right]$ t for $0 \leq t \leq 1$
Parametric Equations: $x(t)=2-3 t$ and $y(t)=3+t$ for $0 \leq t \leq 1$.

## Closing (4 minutes)

Ask students to summarize the key points of the lesson in writing or to a partner. Some important summary elements are listed below.

## Lesson Summary

We can find vector and parametric equations of a line in the plane or in space if we know two points that the line passes through, and we can find parametric equations of a line segment in the plane or in space by restricting the values of $t$ in the parametric equations for the line.

- Let $\boldsymbol{\ell}$ be a line in the plane that contains points $P\left(x_{1}, y_{1}\right)$ and $Q\left(x_{2}, y_{2}\right)$. Then a direction vector is given by $\left[\begin{array}{l}x_{2}-x_{1} \\ y_{2}-y_{1}\end{array}\right]$, and an equation in vector form that represents line $\ell$ is

$$
\left[\begin{array}{l}
x \\
y
\end{array}\right]=\left[\begin{array}{l}
x_{1} \\
y_{1}
\end{array}\right]+\left[\begin{array}{l}
x_{2}-x_{1} \\
y_{2}-y_{1}
\end{array}\right] t, \text { for all real numbers } t
$$

Parametric equations that represent line $\ell$ are

$$
\begin{aligned}
& x(t)=x_{1}+\left(x_{2}-x_{1}\right) t \\
& y(t)=y_{1}+\left(y_{2}-y_{1}\right) t \text { for all real numbers } t .
\end{aligned}
$$

Parametric equations that represent segment $\overline{P Q}$ are

$$
\begin{aligned}
& x(t)=x_{1}+\left(x_{2}-x_{1}\right) t \\
& y(t)=y_{1}+\left(y_{2}-y_{1}\right) t \text { for } t \leq t \leq 1
\end{aligned}
$$

- Let $\ell$ be a line in space that contains points $P\left(x_{1}, y_{1}, z_{1}\right)$ and $Q\left(x_{2}, y_{2}, z_{2}\right)$. Then a direction vector is given by $\left[\begin{array}{l}x_{2}-x_{1} \\ y_{2}-y_{1} \\ z_{2}-z_{1}\end{array}\right]$, and an equation in vector form that represents line $\ell$ is

$$
\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{l}
x_{1} \\
y_{1} \\
z_{1}
\end{array}\right]+\left[\begin{array}{l}
x_{2}-x_{1} \\
y_{2}-y_{1} \\
z_{2}-z_{1}
\end{array}\right] t \text {, for all real numbers } t
$$

Parametric equations that represent line $\boldsymbol{\ell}$ are

$$
\begin{aligned}
& x(t)=x_{1}+\left(x_{2}-x_{1}\right) t \\
& y(t)=y_{1}+\left(y_{2}-y_{1}\right) t \\
& z(t)=z_{1}+\left(z_{2}-z_{1}\right) t \text { for all real numbers } t .
\end{aligned}
$$

Parametric equations that represent segment $\overline{P Q}$ are

$$
\begin{aligned}
& x(t)=x_{1}+\left(x_{2}-x_{1}\right) t \\
& y(t)=y_{1}+\left(y_{2}-y_{1}\right) t \\
& z(t)=z_{1}+\left(z_{2}-z_{1}\right) t \text { for } 0 \leq t \leq 1
\end{aligned}
$$

- The image of a line $\overleftrightarrow{P Q}$ in the plane under a linear transformation $L$ is given by

$$
\left[\begin{array}{l}
x \\
y
\end{array}\right]=L(P)+(L(Q)-L(P)) t, \text { for all real numbers } t
$$

- The image of a line $\overleftrightarrow{P Q}$ in space under a linear transformation $L$ is given by

$$
\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=L(P)+(L(Q)-L(P)) t, \text { for all real numbers } t \text {. }
$$

## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 22: Linear Transformations of Lines

## Exit Ticket

1. Consider points $P(2,1)$ and $Q(2,5)$. Find parametric equations that describe points on the line segment $\overline{P Q}$.
2. Suppose that points $P(2,1)$ and $Q(2,5)$ are transformed under the linear transformation $L\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{cc}1 & -3 \\ 0 & 1\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]$. Find parametric equations that describe the image of line $\overleftrightarrow{P Q}$ under this transformation.

## Exit Ticket Sample Solutions

1. Consider points $P(2,1)$ and $Q(2,5)$. Find parametric equations that describe points on the line segment $\overline{P Q}$.

A direction vector $\overrightarrow{\mathrm{v}}$ is given by $\overrightarrow{\mathrm{v}}=\left[\begin{array}{l}2 \\ 5\end{array}\right]-\left[\begin{array}{l}2 \\ 1\end{array}\right]=\left[\begin{array}{l}0 \\ 4\end{array}\right]$, so a vector form of the segment $\overline{P Q}$ is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}2 \\ 1\end{array}\right]+\left[\begin{array}{l}0 \\ 4\end{array}\right] t$ for $0 \leq t \leq 1$. This gives the parametric equations $x(t)=2$ and $y(t)=1+4 t$ for $0 \leq t \leq 1$.
2. Suppose that points $P(2,1)$ and $Q(2,5)$ are transformed under the linear transformation $L\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=\left[\begin{array}{cc}1 & -3 \\ 0 & 1\end{array}\right]\left[\begin{array}{l}x \\ y\end{array}\right]$. Find parametric equations that describe the image of line $\overleftrightarrow{P Q}$ under this transformation.
The images of $P$ and $Q$ are

$$
\begin{aligned}
& L(P)=\left[\begin{array}{cc}
1 & -3 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
2 \\
1
\end{array}\right]=\left[\begin{array}{c}
-1 \\
1
\end{array}\right] \\
& L(Q)=\left[\begin{array}{cc}
1 & -3 \\
0 & 1
\end{array}\right]\left[\begin{array}{l}
2 \\
5
\end{array}\right]=\left[\begin{array}{c}
-13 \\
5
\end{array}\right] .
\end{aligned}
$$

The direction vector $\overrightarrow{\mathrm{v}}$ is then $\overrightarrow{\mathrm{v}}=\left[\begin{array}{c}-13 \\ 5\end{array}\right]-\left[\begin{array}{c}-1 \\ 1\end{array}\right]=\left[\begin{array}{c}-12 \\ 4\end{array}\right]$, so the vector form of the image of $\overleftrightarrow{P Q}$ is $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}-1 \\ 1\end{array}\right]+\left[\begin{array}{c}-12 \\ 4\end{array}\right] t$ for all real numbers $t$.

Parametric equations that represent the limit of line $\overleftrightarrow{P Q}$ are $x(t)=-1-12 t$ and $y(t)=1+4 t$ for all real numbers $t$.

## Problem Set Sample Solutions

1. Find parametric equations of the line $\overleftrightarrow{P Q}$ through points $P$ and $Q$ in the plane.
a. $P(1,3), Q(2,-5)$

Direction vector: $\vec{v}=\left[\begin{array}{c}2 \\ -5\end{array}\right]-\left[\begin{array}{l}1 \\ 3\end{array}\right]=\left[\begin{array}{c}1 \\ -8\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}1 \\ 3\end{array}\right]+\left[\begin{array}{c}1 \\ -8\end{array}\right] t$ for all real numbers $t$.
Parametric Equations: $x(t)=1+t$ and $y(t)=3-8 t$ for all real numbers $t$.
b. $\quad P(3,1), Q(0,2)$

Direction vector: $\vec{v}=\left[\begin{array}{l}0 \\ 2\end{array}\right]-\left[\begin{array}{l}3 \\ 1\end{array}\right]=\left[\begin{array}{c}-3 \\ 1\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}3 \\ 1\end{array}\right]+\left[\begin{array}{c}-3 \\ 1\end{array}\right] t$ for all real numbers $t$.
Parametric Equations: $x(t)=3-3 t$ and $y(t)=1+t$ for all real numbers $t$.
c. $\quad P(-2,2), Q(-3,-4)$

Direction vector: $\vec{v}=\left[\begin{array}{c}-3 \\ -4\end{array}\right]-\left[\begin{array}{c}-2 \\ 2\end{array}\right]=\left[\begin{array}{l}-1 \\ -6\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}-2 \\ 2\end{array}\right]+\left[\begin{array}{l}-1 \\ -6\end{array}\right] t$ for all real numbers $t$.
Parametric Equations: $x(t)=-2-t$ and $y(t)=2-6 t$ for all real numbers $t$.
2. Find parametric equations of the line $\overleftrightarrow{P Q}$ through points $P$ and $Q$ in space.
a. $\quad P(1,0,2), Q(4,3,1)$

Direction vector: $\vec{v}=\left[\begin{array}{l}4 \\ 3 \\ 1\end{array}\right]-\left[\begin{array}{l}1 \\ 0 \\ 2\end{array}\right]=\left[\begin{array}{c}3 \\ 3 \\ -1\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}1 \\ 0 \\ 2\end{array}\right]+\left[\begin{array}{c}3 \\ 3 \\ -1\end{array}\right] t$ for all real numbers $t$.
Parametric Equations: $x(t)=1+3 t, y(t)=3 t$, and $z(t)=2-t$ for all real numbers $t$.
b. $\quad P(3,1,2), Q(2,8,3)$

Direction vector: $\vec{v}=\left[\begin{array}{l}2 \\ 8 \\ 3\end{array}\right]-\left[\begin{array}{l}3 \\ 1 \\ 2\end{array}\right]=\left[\begin{array}{c}-1 \\ 7 \\ 1\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}3 \\ 1 \\ 2\end{array}\right]+\left[\begin{array}{c}-1 \\ 7 \\ 1\end{array}\right] t$ for all real numbers $t$.
Parametric Equations: $x(t)=3-t, y(t)=1+7 t$, and $z(t)=2+t$ for all real numbers $t$.
c. $\quad P(1,4,0), Q(-2,1,-1)$

Direction vector: $\vec{v}=\left[\begin{array}{c}-2 \\ 1 \\ -1\end{array}\right]-\left[\begin{array}{l}1 \\ 4 \\ 0\end{array}\right]=\left[\begin{array}{l}-3 \\ -3 \\ -1\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}1 \\ 4 \\ 0\end{array}\right]+\left[\begin{array}{l}-3 \\ -3 \\ -1\end{array}\right] t$ for all real numbers $t$.
Parametric Equations: $x(t)=1-3 t, y(t)=4-3 t$, and $z(t)=-t$ for all real numbers $t$.
3. Find parametric equations of segment $\overline{P Q}$ through points $P$ and $Q$ in the plane.
a. $\quad P(2,0), Q(2,10)$

Direction vector: $\vec{v}=\left[\begin{array}{c}2 \\ 10\end{array}\right]-\left[\begin{array}{c}2 \\ 0\end{array}\right]=\left[\begin{array}{c}0 \\ 10\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}2 \\ 0\end{array}\right]+\left[\begin{array}{c}0 \\ 10\end{array}\right] t$ for $0 \leq t \leq 1$
Parametric Equations: $x(t)=2$ and $y(t)=10 t$ for $0 \leq t \leq 1$
b. $\quad P(1,6), Q(-3,5)$

Direction vector: $\vec{v}=\left[\begin{array}{c}-3 \\ 5\end{array}\right]-\left[\begin{array}{l}1 \\ 6\end{array}\right]=\left[\begin{array}{l}-4 \\ -1\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}1 \\ 6\end{array}\right]+\left[\begin{array}{l}-4 \\ -1\end{array}\right]$ t for $0 \leq t \leq 1$
Parametric Equations: $x(t)=1-4 t$ and $y(t)=6-t$ for $0 \leq t \leq 1$
c. $\quad P(-2,4), Q(6,9)$

Direction vector: $\vec{v}=\left[\begin{array}{l}6 \\ 9\end{array}\right]-\left[\begin{array}{c}-2 \\ 4\end{array}\right]=\left[\begin{array}{l}8 \\ 5\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}-2 \\ 4\end{array}\right]+\left[\begin{array}{l}8 \\ 5\end{array}\right] t$ for $0 \leq t \leq 1$
Parametric Equations: $x(t)=-2+8 t$ and $y(t)=4+5 t$ for $0 \leq t \leq 1$
4. Find parametric equations of segment $\overline{P Q}$ through points $P$ and $Q$ in space.
a. $P(\mathbf{1}, \mathbf{1}, \mathbf{1}), Q(\mathbf{0}, \mathbf{0}, \mathbf{0})$

$$
\text { Direction vector: } \vec{v}=\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]-\left[\begin{array}{l}
1 \\
1 \\
1
\end{array}\right]=\left[\begin{array}{l}
-1 \\
-1 \\
-1
\end{array}\right]
$$

Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}1 \\ 1 \\ 1\end{array}\right]+\left[\begin{array}{l}-1 \\ -1 \\ -1\end{array}\right] t$ for $0 \leq t \leq 1$
Parametric Equations: $x(t)=1-t, y(t)=1-t$ and $z(t)=1-t$ for $0 \leq t \leq 1$
b. $\quad P(2,1,-3), Q(1,1,4)$

Direction vector: $\vec{v}=\left[\begin{array}{l}1 \\ 1 \\ 4\end{array}\right]-\left[\begin{array}{c}2 \\ 1 \\ -3\end{array}\right]=\left[\begin{array}{c}-1 \\ 0 \\ 7\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{c}2 \\ 1 \\ -3\end{array}\right]+\left[\begin{array}{c}-1 \\ 0 \\ 7\end{array}\right] t$ for $0 \leq t \leq 1$
Parametric Equations: $x(t)=1+3 t, y(t)=3 t$ and $z(t)=2-t$ for $0 \leq t \leq 1$
c. $\quad P(3,2,1), Q(1,2,3)$

Direction vector: $\vec{v}=\left[\begin{array}{l}1 \\ 2 \\ 3\end{array}\right]-\left[\begin{array}{l}3 \\ 2 \\ 1\end{array}\right]=\left[\begin{array}{c}-2 \\ 0 \\ 2\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}3 \\ 2 \\ 1\end{array}\right]+\left[\begin{array}{c}-2 \\ 0 \\ 2\end{array}\right] t$ for $0 \leq t \leq 1$
Parametric Equations: $x(t)=3-2 t, y(t)=2$, and $z(t)=1+2 t$ for $0 \leq t \leq 1$
5. Jeanine claims that the parametric equations $x(t)=3-t$ and $y(t)=4-3 t$ describe the line through points $P(2,1)$ and $\boldsymbol{Q}(3,4)$. Is she correct? Explain how you know.

Yes, she is correct. If $t=1$, then $x(t)=2$ and $y(t)=1$, so the line passes through point $P$. If $t=0$, then $x(t)=3$ and $y(t)=4$, so the line passes through point $Q$.
6. Kelvin claims that the parametric equations $x(t)=3+t$ and $y(t)=4+3 t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is he correct? Explain how you know.
Yes, he is correct. If $t=-1$, then $x(t)=2$ and $y(t)=1$, so the line passes through point $P$. If $t=0$, then $x(t)=3$ and $y(t)=4$, so the line passes through point $Q$.
7. LeRoy claims that the parametric equations $x(t)=1+3 t$ and $y(t)=-2+9 t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is he correct? Explain how you know.

Yes, he is correct. If $=\frac{1}{3}$, then $x(t)=2$ and $y(t)=1$, so the line passes through point $P$. If $t=\frac{2}{3}$, then $x(t)=3$ and $y(t)=4$, so the line passes through point $Q$.
8. Miranda claims that the parametric equations $x(t)=-2+2 t$ and $y(t)=3-t$ describe the line through points $P(2,1)$ and $Q(3,4)$. Is she correct? Explain how you know.

No, she is not correct. If $t=2$, then $x(t)=2$ and $y(t)=1$, so the line passes through point $P$. However, when we solve $-2+2 t=3$ we find $t=\frac{5}{2}$ and when we solve $3-t=4$, we find that $t=-1$. Thus, there is no value of $t$ so that $(x(t), y(t))=(3,4)$ so this line does not pass through point $Q$.
9. Find parametric equations of the image of the line $\overleftrightarrow{P Q}$ under the transformation $L\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=A\left[\begin{array}{l}x \\ y\end{array}\right]$ for the given points $P, Q$, and matrix $A$.
a. $P(2,4), Q(5,-1), A=\left[\begin{array}{ll}1 & 3 \\ 1 & 2\end{array}\right]$
$L(P)=\left[\begin{array}{ll}1 & 3 \\ 1 & 2\end{array}\right]\left[\begin{array}{l}2 \\ 4\end{array}\right]=\left[\begin{array}{l}14 \\ 10\end{array}\right]$ and $L(Q)=\left[\begin{array}{ll}1 & 3 \\ 1 & 2\end{array}\right]\left[\begin{array}{c}5 \\ -1\end{array}\right]=\left[\begin{array}{l}2 \\ 3\end{array}\right]$ so $v=\left[\begin{array}{l}2 \\ 3\end{array}\right]-\left[\begin{array}{l}14 \\ 10\end{array}\right]=\left[\begin{array}{c}-12 \\ -7\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}14 \\ 10\end{array}\right]+\left[\begin{array}{c}-12 \\ -7\end{array}\right] t$ for all real numbers $t$.
Parametric equations: $x(t)=14-12 t$ and $y(t)=10-7 t$ for all real numbers $t$.
b. $\quad P(1,-2), Q(0,0), A=\left[\begin{array}{cc}1 & 2 \\ -2 & 1\end{array}\right]$
$L(P)=\left[\begin{array}{cc}1 & 2 \\ -2 & 1\end{array}\right]\left[\begin{array}{c}1 \\ -2\end{array}\right]=\left[\begin{array}{c}-3 \\ -4\end{array}\right]$ and $L(Q)=\left[\begin{array}{cc}1 & 2 \\ -2 & 1\end{array}\right]\left[\begin{array}{l}0 \\ 0\end{array}\right]=\left[\begin{array}{l}0 \\ 0\end{array}\right]$ so $\overrightarrow{\mathbf{v}}=\left[\begin{array}{l}-3 \\ -4\end{array}\right]-\left[\begin{array}{l}0 \\ 0\end{array}\right]=\left[\begin{array}{l}-3 \\ -4\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}1 \\ -2\end{array}\right]+\left[\begin{array}{l}-3 \\ -4\end{array}\right] t$ for all real numbers $t$.
Parametric equations: $x(t)=1-3 t$ and $y(t)=-2-4 t$ for all real numbers $t$.
c. $\quad P(2,3), Q(1,10), A=\left[\begin{array}{ll}1 & 4 \\ 0 & 1\end{array}\right]$
$L(P)=\left[\begin{array}{ll}1 & 4 \\ 0 & 1\end{array}\right]\left[\begin{array}{l}2 \\ 3\end{array}\right]=\left[\begin{array}{c}14 \\ 3\end{array}\right]$ and $L(Q)=\left[\begin{array}{ll}1 & 4 \\ 0 & 1\end{array}\right]\left[\begin{array}{c}1 \\ 10\end{array}\right]=\left[\begin{array}{l}14 \\ 10\end{array}\right]$ so $\overrightarrow{\mathrm{v}}=\left[\begin{array}{c}14 \\ 10\end{array}\right]-\left[\begin{array}{c}14 \\ 3\end{array}\right]=\left[\begin{array}{l}0 \\ 7\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{c}14 \\ 3\end{array}\right]+\left[\begin{array}{l}0 \\ 7\end{array}\right]$ t for all real numbers $t$.
Parametric equations: $x(t)=14$ and $y(t)=3+7 t$ for all real numbers $t$.
10. Find parametric equations of the image of the line $\overleftrightarrow{P Q}$ under the transformation $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=A\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ for the given points $P, Q$, and matrix $A$.
a. $\quad P(1,-2,1), Q(-1,1,3), A=\left[\begin{array}{lll}2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3\end{array}\right]$
$L(P)=\left[\begin{array}{lll}2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3\end{array}\right]\left[\begin{array}{c}1 \\ -2 \\ 1\end{array}\right]=\left[\begin{array}{c}0 \\ -1 \\ 0\end{array}\right]$ and $L(Q)=\left[\begin{array}{lll}2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3\end{array}\right]\left[\begin{array}{c}-1 \\ 1 \\ 3\end{array}\right]=\left[\begin{array}{c}-1 \\ 4 \\ 10\end{array}\right]$ so $\vec{v}=\left[\begin{array}{c}-1 \\ 4 \\ 10\end{array}\right]-\left[\begin{array}{c}0 \\ -1 \\ 0\end{array}\right]=\left[\begin{array}{c}-1 \\ 5 \\ 10\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{c}0 \\ -1 \\ 0\end{array}\right]+\left[\begin{array}{c}-1 \\ 5 \\ 10\end{array}\right] t$ for all real numbers $t$.
Parametric equations: $x(t)=-t$ and $y(t)=-1+5 t$ and $z(t)=10 t$ for all real numbers $t$.
b. $\quad P(2,1,4), Q(1,-1,-3), A=\left[\begin{array}{lll}1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1\end{array}\right]$
$L(P)=\left[\begin{array}{lll}1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1\end{array}\right]\left[\begin{array}{l}2 \\ 1 \\ 4\end{array}\right]=\left[\begin{array}{c}7 \\ 11 \\ 6\end{array}\right]$ and $L(Q)=\left[\begin{array}{lll}1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1\end{array}\right]\left[\begin{array}{c}1 \\ -1 \\ -3\end{array}\right]=\left[\begin{array}{c}-3 \\ -6 \\ -2\end{array}\right]$ so $\vec{v}=\left[\begin{array}{c}-3 \\ -6 \\ -2\end{array}\right]-\left[\begin{array}{c}7 \\ 11 \\ 6\end{array}\right]=\left[\begin{array}{c}-10 \\ -17 \\ -8\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{c}7 \\ 11 \\ 6\end{array}\right]+\left[\begin{array}{c}-10 \\ -17 \\ -8\end{array}\right] t$ for all real numbers $t$.
Parametric equations: $x(t)=7-10 t$ and $y(t)=11-17 t$ and $z(t)=6-8 t$ for all real numbers $t$.
c. $\quad P(0,0,1), Q(4,2,3), A=\left[\begin{array}{lll}1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1\end{array}\right]$
$L(P)=\left[\begin{array}{lll}1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1\end{array}\right]\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]=\left[\begin{array}{l}0 \\ 1 \\ 1\end{array}\right]$ and $L(Q)=\left[\begin{array}{lll}1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1\end{array}\right]\left[\begin{array}{l}4 \\ 2 \\ 3\end{array}\right]=\left[\begin{array}{c}10 \\ 9 \\ 7\end{array}\right]$ so $\vec{v}=\left[\begin{array}{c}10 \\ 9 \\ 7\end{array}\right]-\left[\begin{array}{l}0 \\ 1 \\ 1\end{array}\right]=\left[\begin{array}{l}9 \\ 8 \\ 6\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}0 \\ 1 \\ 1\end{array}\right]+\left[\begin{array}{l}9 \\ 8 \\ 6\end{array}\right] t$ for all real numbers $t$.
Parametric equations: $x(t)=9 y(t)=1+8 t$ and $z(t)=1+6 t$ for all real numbers $t$.
11. Find parametric equations of the image of the segment $\overline{P Q}$ under the transformation $L\left(\left[\begin{array}{l}x \\ y\end{array}\right]\right)=A\left[\begin{array}{l}x \\ y\end{array}\right]$ for the given points $P, Q$, and matrix $A$.
a. $\quad P(2,1), Q(-1,-1), A=\left[\begin{array}{ll}1 & 3 \\ 1 & 2\end{array}\right]$
$L(P)=\left[\begin{array}{ll}1 & 3 \\ 1 & 2\end{array}\right]\left[\begin{array}{l}2 \\ 1\end{array}\right]=\left[\begin{array}{l}5 \\ 4\end{array}\right]$ and $L(Q)=\left[\begin{array}{ll}1 & 3 \\ 1 & 2\end{array}\right]\left[\begin{array}{l}-1 \\ -1\end{array}\right]=\left[\begin{array}{l}-4 \\ -3\end{array}\right]$ so $\vec{v}=\left[\begin{array}{l}-4 \\ -3\end{array}\right]-\left[\begin{array}{l}5 \\ 4\end{array}\right]=\left[\begin{array}{l}-9 \\ -7\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}5 \\ 4\end{array}\right]+\left[\begin{array}{l}-9 \\ -7\end{array}\right] t$ for $0 \leq t \leq 1$
Parametric equations: $x(t)=5-9 t$ and $y(t)=4-7 t$ for $0 \leq t \leq 1$
b. $\quad P(0,0), Q(4,2), A=\left[\begin{array}{cc}1 & 2 \\ -2 & 1\end{array}\right]$
$L(P)=\left[\begin{array}{cc}1 & 2 \\ -2 & 1\end{array}\right]\left[\begin{array}{l}0 \\ 0\end{array}\right]=\left[\begin{array}{l}0 \\ 0\end{array}\right]$ and $L(Q)=\left[\begin{array}{cc}1 & 2 \\ -2 & 1\end{array}\right]\left[\begin{array}{l}4 \\ 2\end{array}\right]=\left[\begin{array}{c}8 \\ -6\end{array}\right]$ so $\overrightarrow{\mathrm{v}}=\left[\begin{array}{c}8 \\ -6\end{array}\right]-\left[\begin{array}{l}0 \\ 0\end{array}\right]=\left[\begin{array}{c}8 \\ -6\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}0 \\ 0\end{array}\right]+\left[\begin{array}{c}8 \\ -6\end{array}\right] t$ for $0 \leq t \leq 1$
Parametric equations: $x(t)=8 t$ and $y(t)=-6 t$ for $0 \leq t \leq 1$
c. $\quad P(3,1), Q(1,-2), A=\left[\begin{array}{ll}1 & 4 \\ 0 & 1\end{array}\right]$
$L(P)=\left[\begin{array}{ll}1 & 4 \\ 0 & 1\end{array}\right]\left[\begin{array}{l}3 \\ 1\end{array}\right]=\left[\begin{array}{l}7 \\ 1\end{array}\right]$ and $L(Q)=\left[\begin{array}{ll}1 & 4 \\ 0 & 1\end{array}\right]\left[\begin{array}{c}1 \\ -2\end{array}\right]=\left[\begin{array}{c}-7 \\ -2\end{array}\right]$ so $\overrightarrow{\mathrm{v}}=\left[\begin{array}{c}-7 \\ -2\end{array}\right]-\left[\begin{array}{l}7 \\ 1\end{array}\right]=\left[\begin{array}{c}-14 \\ -3\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y\end{array}\right]=\left[\begin{array}{l}7 \\ 1\end{array}\right]+\left[\begin{array}{c}-14 \\ -3\end{array}\right] t$ for $0 \leq t \leq 1$
Parametric equations: $x(t)=7-14 t$ and $y(t)=1-3 t$ for $0 \leq t \leq 1$
12. Find parametric equations of the image of the segment $\overline{P Q}$ under the transformation $L\left(\left[\begin{array}{l}x \\ y \\ z\end{array}\right]\right)=A\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ for the given points $P, Q$ and matrix $A$.
a. $\quad P(0,1,1), Q(-1,1,2), A=\left[\begin{array}{lll}2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3\end{array}\right]$
$L(P)=\left[\begin{array}{lll}2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3\end{array}\right]\left[\begin{array}{l}0 \\ 1 \\ 1\end{array}\right]=\left[\begin{array}{l}1 \\ 2 \\ 5\end{array}\right]$ and $L(Q)=\left[\begin{array}{lll}2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 3\end{array}\right]\left[\begin{array}{c}-1 \\ 1 \\ 2\end{array}\right]=\left[\begin{array}{c}-1 \\ 3 \\ 7\end{array}\right]$ so $\vec{v}=\left[\begin{array}{c}-1 \\ 3 \\ 7\end{array}\right]-\left[\begin{array}{l}1 \\ 2 \\ 5\end{array}\right]=\left[\begin{array}{c}-2 \\ 1 \\ 2\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}1 \\ 2 \\ 5\end{array}\right]+\left[\begin{array}{c}-2 \\ 1 \\ 2\end{array}\right] t$ for $0 \leq t \leq 1$
Parametric equations: $x(t)=1-2 t, y(t)=2+t$, and $z(t)=5+2 t$ for $0 \leq t \leq 1$
b. $\quad P(2,1,1), Q(1,1,2), A=\left[\begin{array}{lll}1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1\end{array}\right]$
$L(P)=\left[\begin{array}{lll}1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1\end{array}\right]\left[\begin{array}{l}2 \\ 1 \\ 1\end{array}\right]=\left[\begin{array}{l}4 \\ 5 \\ 3\end{array}\right]$ and $L(Q)=\left[\begin{array}{lll}1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1\end{array}\right]\left[\begin{array}{l}1 \\ 1 \\ 2\end{array}\right]=\left[\begin{array}{l}4 \\ 6 \\ 3\end{array}\right]$ so $\vec{v}=\left[\begin{array}{l}4 \\ 6 \\ 3\end{array}\right]-\left[\begin{array}{l}4 \\ 5 \\ 3\end{array}\right]=\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}4 \\ 5 \\ 3\end{array}\right]+\left[\begin{array}{l}0 \\ 1 \\ 0\end{array}\right] t$ for $0 \leq t \leq 1$
Parametric equations: $x(t)=4, y(t)=5+t$, and $z(t)=3$ for $0 \leq t \leq 1$
c. $\quad P(0,0,1), Q(1,0,0), A=\left[\begin{array}{lll}1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1\end{array}\right]$
$L(P)=\left[\begin{array}{lll}1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1\end{array}\right]\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]=\left[\begin{array}{l}0 \\ 1 \\ 1\end{array}\right]$ and $L(Q)=\left[\begin{array}{lll}1 & 3 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1\end{array}\right]\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right]=\left[\begin{array}{l}1 \\ 1 \\ 0\end{array}\right]$ so $\overrightarrow{\mathrm{v}}=\left[\begin{array}{l}1 \\ 1 \\ 0\end{array}\right]-\left[\begin{array}{l}0 \\ 1 \\ 1\end{array}\right]=\left[\begin{array}{c}1 \\ 0 \\ -1\end{array}\right]$
Vector equation: $\left[\begin{array}{l}x \\ y \\ z\end{array}\right]=\left[\begin{array}{l}0 \\ 1 \\ 1\end{array}\right]+\left[\begin{array}{c}1 \\ 0 \\ -1\end{array}\right] t$ for $0 \leq t \leq 1$
Parametric equations: $x(t)=t, y(t)=1$, and $z(t)=1-t$ for $0 \leq t \leq 1$

## Student Outcomes

- Students solve problems involving physical phenomena that can be represented by vectors.


## Lesson Notes

Vectors are generally described as a quantity that has both a magnitude and a direction. In the next two lessons, students will work on examples that give a context to that description. The most basic definition of a vector is that it is a description of a shift or translation. Students will see that any physical operation that induces a shift of some kind is often thought of as a vector. Hence, vectors are prevalent in mathematics, science, and engineering. For example, force is often interpreted as a vector in physics because a force exerted on an object is a push of some magnitude that causes the object to shift in some direction. In this lesson, students will solve problems involving velocity as well as other quantities, such as force, that can be represented by a vector (N-VM.A.3). Students will make use of the law of cosines and the law of sines when working with non-right triangles (G-SRT.D.11). Students will continue to work on adding and subtracting vectors (N-VM.B.4) but will interpret the resulting magnitude and direction within a context. The focus of this lesson is using vectors to model real-world phenomena, in particular focusing on relating abstract representations to real-world aspects (MP.2).

## Classwork

## Opening (5 minutes)

Ask students to brainstorm real-world situations where vectors might be useful. Engage students by showing the vector video from NBC Learn on the science of NFL football (http://www.nbclearn.com/nfl/cuecard/51220). Remind students that vectors are used to describe anything that has both a direction and a magnitude.

- List these phenomena on the board, an interactive board, or index cards given to students. Have students sort the phenomena into two categories-those that can be described by vectors and those that cannot-and then explain the rationale behind their sorting. Discuss student choices and rationales as a class.
- Position of a moving object
- Wind
- Position of a ball that has been thrown
- Temperature
- Mass
- Velocity of a ball that has been thrown
- Volume
- Water current
- Phenomena that could be described using a vector:
- Wind, position of a ball that has been thrown, velocity of a ball that has been thrown, water current because all have a direction and a magnitude.
- Phenomena that could not be described using a vector:
- Temperature, mass, volume because these are just measurements with no magnitude and direction.
- Why would the position of a moving object need to be described as a vector rather than a scalar?
- Just knowing how far it moved would not be enough information to locate the object. You would also need to know the direction.


## Opening Exercise (5 minutes)

Give students time to work on the opening exercise independently, and then discuss as a group.

Opening Exercise
Suppose a person walking through South Boston, MA travels from point $A$ due east along $E$ 3rd Street for 0.3 miles and then due south along $K$ Street for 0.4 miles to end on point $B$ (as shown on the map).
a. Find the magnitude and direction of vector $\overrightarrow{A B}$.
$|A B|=\sqrt{0.3^{2}+0.4^{2}}=0.5$ miles
$\theta=\tan ^{-1}\left(\frac{0.4}{0.3}\right)=53.1^{\circ}$
b. What information does vector $\overrightarrow{\boldsymbol{A B}}$ provide?


The magnitude tells us that the person's displacement was 0.5 miles. In other words, the person ended up 0.5 miles from the point where he or she started. The angle tells us that the person traveled $53.1^{\circ}$ south of east.

- Why is a vector a useful way to map a person's location?
- It tells us the actual distance between the starting and ending point and also the direction in which the person traveled.
- If I knew just the direction, could I map the person's position?
- No. We could draw an arrow in the correct direction but wouldn't know where to put the endpoint.
- If I knew just the magnitude, could I map the person's position?
- No. The person could have traveled in any direction.


## Example 1 (10 minutes)

Discuss the example as a class before giving students time to work on solving the problem. Provide students with grid paper as needed. Debrief as a class, and guide students as necessary.

## Example 1

An airplane flying from Dallas-Fort Worth to Atlanta veers off course to avoid a storm. The plane leaves Dallas-Fort Worth traveling $50^{\circ}$ east of north and flies for $\mathbf{4 5 0}$ miles before turning to travel $70^{\circ}$ east of south for 350 miles. What is the resultant displacement of the airplane? Include both the magnitude and direction of the displacement.
$x_{1}=450 \cos \left(40^{\circ}\right)=344.720$
$y_{1}=450 \sin \left(40^{\circ}\right)=289.254$
$x_{2}=344.720+328.892=673.612$
$y_{2}=289.254-119.707=169.547$
$d=\sqrt{673.612^{2}+169.547^{2}}=694.622$ miles
$\theta=\tan ^{-1}\left(\frac{169.547}{673.612}\right)=14.1^{\circ}$



The resultant displacement is 694.622 miles at $75.9^{\circ}$ east of north.

## Exercises 1-4 (17 minutes)

Allow students time to work in groups stopping to debrief as students complete the exercises.

## Exercises 1-4

1. A motorized robot moves across the coordinate plane. Its position $\binom{x(t)}{y(t)}$ at time $t$ seconds is given by $\binom{x(t)}{y(t)}=\mathbf{a}+t v$ where $a=\binom{4}{-10}$ and $=\binom{-4}{3}$. The units of distance are measured in meters.
a. Where is the robot at time $\boldsymbol{t}=\mathbf{0}$ ?

$$
\binom{4}{-10}
$$

## Scaffolding:

For students who are struggling, use a graphing utility to reinforce the concept. Have them graph $x(t)=4-4 t$ and $y(t)=-10+3 t$ in parametric mode on the graphing
b. Plot the path of the robot.

c. Describe the path of the robot.

The robot is moving along the line $y=-\frac{3}{4} x-7$.
d. Where is the robot $\mathbf{1 0}$ seconds after it starts moving?
$\binom{-36}{20}$
e. Where is the robot when it is $\mathbf{1 0}$ meters from where it started?
$\binom{-4}{-4}$
f. Is the robot traveling at a constant speed? Explain, and if the speed is constant, state the robot's speed.

Yes, the robot is traveling at a constant speed of $5 \mathrm{~m} / \mathrm{s}$. For every one second that elapses, the robot moves 3 units up and 4 units to the left which means the robot moves 5 meters every second.

- What was the speed of the robot?
- The speed was $5 \mathrm{~m} / \mathrm{s}$.
- How did you determine the speed?
- I knew the robot moved 3 units up and 4 units to the left each second. I used the Pythagorean theorem to find the actual distance the robot traveled.
- What is the relationship between the speed and vector $\mathbf{v}$ ?
- The magnitude of a vector is its length. The length can be determined by the product of the constant speed for a given number of time units.

Make the point that vector $\mathbf{v}$ is the velocity of the robot, and the magnitude of the velocity vector is the speed of the robot. Velocity is a vector because it has both a magnitude and a direction. Speed and time results in the magnitude of a vector.

- How could we interpret vector $\mathbf{v}$ in terms of the velocity of the robot?
- Vector $\mathbf{v}$ is the velocity of the robot. It tells us both the speed and the direction of the robot.
- How could we interpret vector $\mathbf{v}$ in terms of the speed of the robot?
- The magnitude of vector $\mathbf{v}$ is the distance traveled or the length of the vector. It tells us how fast the robot is moving but not the direction in which the robot moves.

2. A row boat is crossing a river that is $\mathbf{5 0 0} \mathbf{~ m}$ wide traveling due east at a speed of $2.2 \mathrm{~m} / \mathrm{s}$. The river's current is $0.8 \mathrm{~m} / \mathrm{s}$ due south.
a. What is the resultant velocity of the boat?
$|\mathrm{v}|=\sqrt{2.2^{2}+0.8^{2}}=2.34 \mathrm{~m} / \mathrm{s}$
$\theta=\tan ^{-1}\left(\frac{0.8}{2.2}\right)=20.0^{\circ}$
b. How long does it take for the boat to cross the river?
$t=\frac{500}{2.2}=227.273$ seconds
c. How far downstream is the boat when it reaches the other side?
$d=0.8(227.273)=181.818 \mathrm{~m}$

3. Consider the airplane from Example 1 that leaves Dallas-Fort Worth with a bearing of $50^{\circ}$. (Note that the bearing is the number of degrees east of north.) The plane is traveling at a speed of 550 mph . There is a crosswind of $\mathbf{4 0} \mathbf{~ m p h}$ due east. What is the resultant velocity of the airplane?
$|\mathrm{v}|=\sqrt{550^{2}+40^{2}-2(550)(40) \cos 130^{\circ}}=576.526 \mathrm{mph}$


Another physical operation that is often interpreted in physics as a vector is force. A force exerted on an object is a push of some magnitude in some direction which causes the object to have a tendency to shift.

Force is often defined in units of newtons ( N ). One newton is defined as the force required to accelerate an object with a mass of one kilogram ( 1 kg ) one meter per second ( $1 \mathrm{~m} / \mathrm{sec}^{2}$ ).
4. A raft floating in the water experiences an eastward force of $\mathbf{1 0 0} \mathbf{N}$ due to the current of the water and a southeast force of 400 N due to wind.
a. In what direction will the boat move?
$36.5^{\circ}$ south of east
b. What is the magnitude of the resultant force on the boat?
475.992 N
c. If the force due to the wind doubles, does the resultant force on the boat double? Explain or show work that supports your answer.

No. If the force of the wind doubles, the resultant force on the boat is 873.577 N

## Closing ( 3 minutes)

Ask students to write a brief answer to the question, "Why are vectors useful?" and then share responses with a partner. Then, share responses as a class.

- Why are vectors useful?
- Vectors can be used to describe any sort of physical phenomena that have both a magnitude and a direction. They are useful for describing a moving object's displacement or velocity where just a single number would not provide an adequate description.


## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 23: Why Are Vectors Useful?

## Exit Ticket

A hailstone is traveling through the sky. Its position $\left(\begin{array}{c}x(t) \\ y(t) \\ z(t)\end{array}\right)$ in meters is given by $\left(\begin{array}{l}x(t) \\ y(t) \\ z(t)\end{array}\right)=\left(\begin{array}{c}0 \\ 0 \\ 2160\end{array}\right)+\left(\begin{array}{c}3 \\ -2 \\ -9\end{array}\right) t$ where $t$ is the time in seconds since the hailstone began being tracked.
a. If $x(t)$ represents an east-west location, how quickly is the hailstone moving to the east?
b. If $y(t)$ represents a north-south location, how quickly is the hailstone moving to the south?
c. What could be causing the east-west and north-south velocities for the hailstone?
d. If $z(t)$ represents the height of the hailstone, how quickly is the hailstone falling?
e. At what location will the hailstone hit the ground (assume $z=0$ is the ground)? How long will this take?
f. What is the overall speed of the hailstone? To the nearest meter, how far did the hailstone travel from $t=0$ to the time it took to hit the ground?

## Exit Ticket Sample Solutions

A hailstone is traveling through the sky. Its position $\left(\begin{array}{l}x(t) \\ y(t) \\ z(t)\end{array}\right)$ in meters is given by $\left(\begin{array}{l}x(t) \\ y(t) \\ z(t)\end{array}\right)=\left(\begin{array}{c}0 \\ 0 \\ 2160\end{array}\right)+\left(\begin{array}{c}3 \\ -2 \\ -9\end{array}\right) t$ where $t$
is the time in seconds since the hailstone began being tracked.
a. If $x(t)$ represents its east-west location, how quickly is the hailstone moving to the east?
$3 \mathrm{~m} / \mathrm{s}$
b. If $y(t)$ represents its north-south location, how quickly is the hailstone moving to the south?
$2 \mathrm{~m} / \mathrm{s}$
c. What could be causing the east-west and north-south velocities for the hailstone?

Wind is probably the number one factor affecting the speed of the hailstone.
d. If $z(t)$ represents the height of the hailstone, how quickly is the hailstone falling?
$9 \mathrm{~m} / \mathrm{s}$
e. At what location will the hailstone hit the ground (assume $z=0$ is the ground)? How long will this take?

The hailstone will hit the ground at $t=2160 \div 9=240$ seconds, which is 4 minutes. In 4 minutes it will move 720 meters to the east and 480 meters to the south to impact at $\left(\begin{array}{c}720 \\ 480 \\ 0\end{array}\right)$.
f. What is the overall speed of the hailstone? To the nearest meter, how far did the hailstone travel from $t=0$ to when it hit the ground?

The hailstone is traveling at $\sqrt{3^{2}+2^{2}+9^{2}}=\sqrt{94}$ meters per second. It traveled $240 \sqrt{94} \approx 2327$ meters.

## Problem Set Sample Solutions

1. Suppose Madison is traveling due west for 0.5 miles and then due south for 1.2 miles.
a. Draw a picture of this scenario with her starting point labeled $A$, ending point $B$, and include the vector $\overrightarrow{A B}$.

b. State the value of $\overrightarrow{A B}$.
$\overrightarrow{A B}=\binom{-0.5}{-1.2}$
c. What is the magnitude and direction of $\overrightarrow{A B}$ ?
$\left\|\binom{-0.5}{-1.2}\right\|=\sqrt{0.5^{2}+1.2^{2}}=\sqrt{1.69}=1.3$
$\tan ^{-1}\left(\frac{12}{5}\right) \approx 67.380 \Rightarrow 247.380^{\circ}$
The vector has a magnitude of 1.3 and a direction of $247.380^{\circ}$ from east, or $67.380^{\circ}$ south of west.
2. An object's azimuth is the angle of rotation of its path measured clockwise from due north. For instance, an object traveling due north would have an azimuth of $\mathbf{0}^{\circ}$, and due east would have an azimuth of $90^{\circ}$.
a. What are the azimuths for due south and due west?
$180^{\circ}$ and $270^{\circ}$
b. Consider a craft on an azimuth of $215^{\circ}$ traveling $\mathbf{3 0}$ knots.
i. Draw a picture representing the situation.

ii. Find the vector representing this craft's speed and direction.

This vector is $215^{\circ}$ from north, which is the same as saying $55^{\circ}$ south of west. Thus, the $x$-coordinate of the vector will be $30 \cos (55) \approx 17.207$, and the $y$-coordinate will be $30 \sin (55) \approx 24.575$. The vector is represented by $\binom{-17.207}{-24.575}$.
3. Bearings can be given from any direction, not just due north. For bearings, like azimuths, clockwise angles are represented by positive degrees and counterclockwise angles are represented by negative degrees. A ship is traveling $30^{\circ}$ east of north at 18 kn , then turns $\mathbf{2 0}^{\circ}$, maintaining its speed.
a. Draw a picture representing the situation.

b. Find vectors v and w representing the first and second bearing.
$v=\binom{9}{9 \sqrt{3}} \approx\binom{9}{15.588}, w=\binom{18 \sin (50)}{18 \cos (50)} \approx\binom{13.789}{11.570}$
c. Find the sum of $v$ and $w$. What does $v+w$ represent?
$\mathrm{v}+\mathrm{w} \approx\binom{22.789}{27.159}$
The sum represents the ship's position relative to its starting point.
d. If the ship travels for one hour along each bearing, then how far north of its starting position has it traveled? How far east has it traveled?

The ship has traveled 22.789 nautical miles north and 27.159 nautical miles east.
4. A turtle starts out on a grid with coordinates $\binom{4}{-6.5}$ where each unit is one furlong. Its horizontal location is given by the function $x(t)=4+-2 t$, and its vertical location is given by $y(t)=-6.5+3 t$ for $t$ in hours.
a. Write the turtle's location using vectors.
$\binom{x(t)}{y(t)}=\binom{4}{-6.5}+\binom{-2}{3} t$
b. What is the speed of the turtle?

The turtle is traveling at $\sqrt{2^{2}+3^{2}}=\sqrt{13}$ furlongs per hour.
c. If a hare's location is given as $\binom{x_{h}(t)}{y_{h}(t)}=a+t v$ where $a=\binom{23}{-35}$ and $v=\binom{-8}{12}$, then what is the speed of the hare? How much faster is the hare traveling than the turtle?

The hare is traveling at a speed of $\sqrt{8^{2}+12^{2}}=\sqrt{208}$ furlongs per hour.
$\sqrt{208}=4 \sqrt{13}$ which is 4 times faster than the turtle.
d. Which creature will reach $\binom{-1}{1}$ first?

The turtle will reach $\binom{-1}{1}$ at $t=2.5$, and the hare will reach it at $t=3$, so the turtle will beat the hare by half an hour.
5. A rocket is launched at an angle of $33^{\circ}$ from the ground at a rate of $50 \mathrm{~m} / \mathrm{s}$.
a. How fast is the rocket traveling up to the nearest $\mathrm{m} / \mathrm{s}$ ?

The rocket is traveling up at a rate of $50 \cdot \sin (33) \approx 27 \mathrm{~m} / \mathrm{s}$.
b. How fast is the rocket traveling to the right to the nearest $\mathrm{m} / \mathrm{s}$ ?

The rocket is traveling to the right at a rate of $50 \cdot \cos (33) \approx 42 \mathrm{~m} / \mathrm{s}$.
c. What is the rocket's velocity vector?

The velocity vector is $\binom{42}{27}$.
d. Does the magnitude of the velocity vector agree with the set-up of the problem? Why or why not?
$\left\|\binom{42}{27}\right\|=\sqrt{42^{2}+27^{2}}=\sqrt{2493} \approx 49.92995$
This is effectively the original speed and is only different because of rounding errors.
e. If a laser is in the path of the rocket and would like to strike the rocket, in what direction does the laser need to be aimed? Express your answer as a vector.

If the laser is in the path of the rocket, then that means it needs to be aimed in the opposite direction the rocket is traveling, which is $\binom{-42}{-27}$.
6. A boat is drifting downriver at a rate of 5 nautical miles per hour. If the occupants of the boat want to travel to the shore, do they need to overcome the current downriver? Use vectors to explain why or why not.

The occupants do not need to overcome the stream since the stream is moving downriver, and they only want to move perpendicular to that. Since they will be moving perpendicular, their motions will have no impact on the downriver rate, nor should they. Any effort fighting the current will only waste resources. Orthogonal vectors do not affect the position.
7. A group of friends moored their boats together and fell asleep on the lake. Unfortunately, their lashings came undone in the night, and they have drifted apart. Gerald's boat traveled due west along with the current of the lake which moves at a rate of $\frac{1}{2} \mathrm{mi} / \mathrm{hr}$ and Helena's boat was pulled southeast by some pranksters and set drifting at a rate of $2 \mathbf{m i} / \mathrm{hr}$.
a. If the boats came untied three hours ago, how far apart are the boats?

Gerald traveled 1.5 miles to the west, and Helena traveled 6 miles to the southeast. Using the law of cosines, we see that they are $\sqrt{(1.5)^{2}+6^{2}-2 \cdot 1.5 \cdot 6 \cdot \cos (135)}=\sqrt{2.25+36+9 \sqrt{2}} \approx 7.13$ miles away from each other.
b. If Gerald drops anchor, then in what direction does Helena need to travel in order to reunite with Gerald?

$$
\binom{1.5+3 \sqrt{2}}{3 \sqrt{2}}
$$

8. Consider any two vectors in space, $\mathbf{u}$ and $\mathbf{v}$ with $\boldsymbol{\theta}$ the angle between them.
a. Use the law of cosines to find the value of $\|u-v\|$.
$\|\mathbf{u}-\mathbf{v}\|^{2}=\|\mathbf{u}\|^{2}+\|\mathbf{v}\|^{2}-2\|\mathbf{u}\|\|\mathbf{v}\| \cos (\theta)$
$\|\mathbf{u}-\mathbf{v}\|=\sqrt{\|\mathbf{u}\|^{2}+\|\mathbf{v}\|^{2}-2\|\mathbf{u}\|\|v\| \cos (\theta)}$
b. Use the law of sines to find the value of $\psi$, the angle between $u-v$ and $u$. State any restrictions on the variables.

$$
\begin{aligned}
\frac{\sin (\theta)}{\|u-v\|} & =\frac{\sin (\psi)}{\|u\|} \\
\psi & =\sin ^{-1}\left(\frac{\|\mathbf{u}\| \sin (\theta)}{\|u-v\|}\right)
\end{aligned}
$$

The inverse sine only returns values between $-90^{\circ}$ and $90^{\circ}$, so if the angle is greater than this, then trig identities need to be used to help find the value.

## Lesson 24: Why Are Vectors Useful?

## Student Outcomes

- Students apply linear transformations and vectors to understand the conditions required for a sequence of transformations to preserve the solution set to the system of equations.


## Lesson Notes

Vectors are useful for representing systems of equations. Vectors (and matrices) are essential to the representation of systems of equations in higher dimensions that students will encounter in more advanced I mathematics classes. In this lesson we work with systems of linear equations in $\mathbb{R}^{2}$, but the ideas can be extended to higher-order linear systems. Vectors are extremely useful in a wide variety of settings as a means of representing and manipulating geometric objects as well as real-world quantities. In this lesson, students will see how vectors can be applied to solving a system of equations in two variables by exploring the geometry of solving systems of equations. Students will be guided to the understanding(using pictures only) that the method of transforming systems of linear equations while preserving the solution (A-REI.C.5) can be rephrased in terms of a series of linear transformations and translations. Again, students write vectors as parametric equations to understand the coherence between their work with functions, linear transformations, and vectors. The work in this lesson will help set the stage for further application of vectors and matrices in the last three lessons of this module.

## Classwork

## Opening Exercise (7 minutes)

Give students time to work on the problem individually, and then compare work with a partner. Encourage students to analyze the problem in a variety of ways: algebraically, graphically, and numerically. A graphing utility could also be used.

## Opening Exercise

Two particles are moving in a coordinate plane. Particle 1 is at the point $\binom{2}{1}$ and moving along the velocity vector $\binom{-2}{1}$. Particle $\mathbf{2}$ is at the point $\binom{-1}{1}$ and moving along the velocity vector $\binom{1}{2}$. Are the two particles going to collide? If so, at what point, and at what time? Assume that time is measured in seconds.

Particle 1: $\binom{x(t)}{y(t)}=\binom{2}{1}+\binom{-2}{1} t \quad x(t)=2-2 t$, and $y(t)=1+t \quad y=\frac{3}{2}-\frac{1}{2} x$
Particle 2: $\binom{x(t)}{y(t)}=\binom{-1}{1}+\binom{1}{2} t \quad x(t)=-1+t$, and $y(t)=1+2 t \quad y=3+2 x$
The particles will not collide. While they do cross the same point at $\binom{-0.6}{1.8}$, they do not cross this point at the same time. Particle 1 crosses this point at $t=0.4$, and Particle 2 does not cross this point until $t=0.8$.

- The two lines intersect at $(-0.6,1.8)$. Why do the particles not collide at that point? - They do not collide because the two particles reach that point at two different times.

Particle 1 travels along the line $y=\frac{3}{2}-\frac{1}{2} x$, and Particle 2 travels along the line $y=3+2 x$.
The graphs of these lines intersect at the point $(-0.6,1.8)$ as shown below. The blue portions of the graph show the parametric equations that represent the path of each particle on the interval $-3 \leq t \leq 3$.


To show this result dynamically, enter the parametric equations for each particle on a graphing calculator or other graphing software, and set the $t$-interval to be $-3 \leq t \leq 3$. Be sure the calculator is set to graph the two equations simultaneously. Then, it can be seen that the particles do not arrive at the intersection point at the same $t$-value.

The table below demonstrates numerically that two particles do not pass through the intersection point at the same time. When $t=0$, both particles are at $y=1$. The intersection point of the graphs of the lines occurs when $y=1.8$, thus the two particles will never be at that $y$-value at the same time.

| $t$ | $x_{1}(t)=2-2 t$ | $y_{1}(t)=1+t$ | $x_{2}(t)=-1+t$ | $y_{2}(t)=1+2 t$ |
| :---: | :---: | :---: | :---: | :---: |
| -3 | 7 | -2 | -4 | -5 |
| -2 | 5 | -1 | -3 | -3 |
| -1 | 3 | 0 | -2 | -1 |
| 0 | 1 | 1 | -1 | 1 |
| 1 | -1 | 2 | 0 | 3 |
| 2 | -3 | 3 | 1 | 5 |
| 3 | -5 | 4 | 2 | 7 |

We can solve this system algebraically by solving the equation $\frac{3}{2}-\frac{1}{2} x=2 x+3$ for $x$. The solution is -0.6 .
Substituting -0.6 for $x$ into the parametric equations and solving for $t$ gives the following solutions:

$$
\begin{aligned}
2-2 t & =-0.6 \\
t & =1.3
\end{aligned}
$$

and

$$
\begin{aligned}
-1+t & =-0.6 \\
t & =0.4
\end{aligned}
$$

You can see that these particles do not reach the $x$-coordinate of the intersection point at the same time and therefore will not collide.

Have students present their solution approaches to the class, highlighting different approaches.

## Exercise 1 (5 minutes)

Have students continue to work with a partner through Exercise 1. Then, debrief as a class, and hold the discussion that follows. In this exercise, we represent lines using vectors as shown in previous lessons. If students are struggling to make sense of part (a), encourage them to write the parametric equations for each line, and then eliminate the $t$ parameter to write the equation of the line in terms of the variables $x$ and $y$.

## Exercise 1

Consider lines $\boldsymbol{\ell}=\{(x, y) \mid\langle x, y\rangle=t\langle 1,-2\rangle\}$, and $m=\{(x, y) \mid\langle x, y\rangle=t\langle-1,3\rangle\}$.
a. To what graph does each line correspond?

Line $\ell$ corresponds to the graph of $y=-2 x$. Line $m$ corresponds to the graph of $y=3 x$.
b. Describe what happens to the vectors defining these lines under the transformation $A=\left(\begin{array}{ll}3 & 1 \\ 2 & 1\end{array}\right)$.

Each vector is mapped to a new vector. $\langle 1,-2\rangle \rightarrow\langle 1,0\rangle$, and $\langle-1,3\rangle \rightarrow\langle 0,1\rangle$.

Students may need to be reminded that to apply the transformation $A$, they must compute $A \cdot \mathbf{v}$, where $\mathbf{v}$ is the vector that defines each line.

$$
\left(\begin{array}{ll}
3 & 1 \\
2 & 1
\end{array}\right) \cdot\binom{1}{-2}=\binom{1}{0}
$$

and

$$
\left(\begin{array}{ll}
3 & 1 \\
2 & 1
\end{array}\right) \cdot\binom{-1}{3}=\binom{0}{1}
$$

## c. Show this transformation graphically.

The vector that defined $\ell$ shown in the diagram is $\mathbf{u}=\langle 1,-2\rangle$, and the vector that defined $m$ shown in the diagram is $v=\langle-1,3\rangle$. In the diagram the image of $\ell$ when transformed by $A$ results in the graph of the line $y=0$,' and the image of $m$ when transformed by $A$ results in the graph of the line $x=0$.


## Scaffolding:

- Challenge advanced students to come up with their own example that illustrates the point that a linear transformation will preserve the solution set for lines that pass through the origin.
- For struggling students, provide another example showing that if the solution set is the origin, it will still be the origin after a transformation.
- Let $\ell=\langle 2,1\rangle$ and $m=\langle 1,4\rangle$. Use a transformation of
$A=\left(\begin{array}{cc}1 & 4 \\ 0 & -1\end{array}\right)$.
$\langle 2,1\rangle \rightarrow\langle 6,-1\rangle$
and $\langle 1,4\rangle \rightarrow\langle 17,-4\rangle$
The solution is still $(0,0)$.


## Discussion (5 minutes)

- What is the solution to the original system of equations given by $\ell$ and $m$ ?
- The origin $(0,0)$.
- What is the solution to the system after the transformation given by $A \ell$ and $A m$ ?
- The solution is still the origin.
- Can we say that linear transformations will preserve the solution set to a system of linear equations?
- It seems this is true for lines that pass through the origin.
- What will happen if we apply a linear transformation to two lines that do not intersect at the origin? Do you think the solution set of the system of equations will still be preserved?
- Answers will vary. The solution set may be preserved because the lines are simply rotated about the intersection point. Another conjecture would be that the solution set may not be preserved because the transformation rotates the lines about the origin which will move the intersection point.


## Exercise 2 (5 minutes)

This exercise has students test their conjectures by having them consider a system of equations where the graphs of the two lines do not intersect at the origin. Have students continue to work with a partner through Exercise 2. Then, debrief as a class, and continue the discussion. Use the questions below to help students begin to think about the transformations that have been applied.

- How does line $\ell$ in this exercise compare to line $\ell$ from Exercise 1? How does line $m$ in this exercise compare to line $m$ in Exercise 1?
- The lines have the same slope but a different initial point. The new lines $\ell$ and $m$ have the same point $(1,1)$ when $t=0$.
- Describe these lines as transformations of the lines from Exercise 1.
- Both lines in Exercise 2 are a translation of the original lines by the vector $\binom{1}{1}$.
- Knowing the solution to Exercise 1, how can you quickly find the solution to the system of equations represented by the lines in this exercise?
- Since the solution to the original system in Exercise 1 is $(0,0)$, when the two lines are both translated by the vector $\binom{1}{1}$, all points on the graph of the lines will shift 1 unit right and 1 unit up; therefore, the intersection point of the graphs of the lines which is the solution to the system will become $(1,1)$.

```
Exercise 2
Consider lines }\ell={(x,y)|\langlex,y\rangle=\langle1,1\rangle+t\langle1,-2\rangle}, and m={(x,y)|\langlex,y\rangle=\langle1,1\rangle+t\langle-1,3\rangle}
```

a. What is the solution to the system of equations given by lines $\ell$ and $m$ ?

The solution is $(1,1)$.
b. Describe what happens to the lines under the transformation $A=\left(\begin{array}{ll}3 & 1 \\ 2 & 1\end{array}\right)$.

Line $\ell$ becomes $A \boldsymbol{\ell}=\{(x, y) \mid\langle x, y\rangle=\langle 4,3\rangle+t\langle 1,0\rangle\}$.
Line $m$ becomes $A m=\{(x, y) \mid\langle x, y\rangle=\langle 4,3\rangle+t\langle 0,1\rangle\}$.
c. What is the solution to the system of equations after the transformation?

The solution is the point $(4,3)$.

## Discussion (10 minutes)

- What happened to the solution to the system when we applied a linear transformation to two lines that did not intersect at the origin?
- The solution to the system changed.
- Why did the solution change when we applied the transformation $A$ to this system?
- The linear transformation A not only transforms the velocity vectors for each line, but it also transforms the position vector $\binom{1}{1}$
- Can we say that linear transformations will preserve the solution set to a system of linear equations?
- If the lines intersect at a point other than the origin, then the solution set is NOT preserved. In Example 2 , the solution set changed after the linear transformation.
- In Exercise 2, we translated each linear equation by the vector $\binom{1}{1}$ to put the solution at ( 1,1 ). Could we translate any system of linear equations so that the intersection point of the graphs of the lines is $(0,0)$ ? Explain your reasoning.
- The coordinates of the intersection point is the terminal point of a position vector. If we translate the linear equations by the opposite of this vector, the intersection point will be $(0,0)$.
- What if I translated a system of equations to the origin before applying a linear transformation to the system? Explain your reasoning.
- Then I would know that the transformation would not change the solution to the system. (The solution would be the origin.)

The diagrams below can show the sequence of transformations that will preserve the solution set to a system of equations. As you continue the discussion, draw the pictures below on the board. We are not concerned with using this as an algebraic means for solving a system of equations. We are interested in the ideas behind performing these transformations which have some important ramifications in future math courses. Some of these ideas will be explored further in the Problem Set.


- What will happen to the solution of this new system when we apply a linear transformation?
- If the intersection point of the graphs of the lines is $(0,0)$, then the solution to the system will not change when we apply a linear transformation. Further, there will exist a matrix that will take the vectors that define each line to the $x$ - and $y$-axes when we apply the linear transformation.
- But this is not the solution to the original system of equations. What would we need to do next to find the solution?
- We would need to reverse the translation of the transformed system.
- How can we transform this system into a system with the same solution set as the original one?
- We can translate the system by the original vector that represented the solution to the system.

The diagram below shows the result of applying the linear transformation $A$ to the new system of equations whose graphs intersect at the origin and then the subsequent translation $T_{P}$ to return the intersection point back to the original intersection point.

Apply $A$ to $T_{-p}$ to take the lines to the $x$ - and $y$-axes.


Then, translate by $T_{P}$ to return the system to the original intersection point.

## Exercise 3 (5 minutes)

The next exercise lets students practice this approach with a fairly simple system of linear equations. The lines are at right angles to one another, so the linear transformation that will map the translated system to the $x$ - and $y$-axes will not be that difficult to determine. Have students continue working with their partner.

## Exercise 3

The system of equations is given below. A graph of the equations and their intersection point is also shown.

$$
\begin{aligned}
& x+y=6 \\
& x-y=2
\end{aligned}
$$


a. Write each line in the form $L(t)=p+v t$ where $p$ is the position vector whose terminal point is the solution of the system, and $v$ is the velocity vector that defines the path of a particle traveling along the line such that when $t=0$, the solution to the system is $(x(0), y(0))$.
$L_{1}(t)=\binom{4}{2}+\binom{-1}{1} t$
$L_{2}(t)=\binom{4}{2}+\binom{1}{1} t$
b. Describe a translation that will take the point $(x(0), y(0))$ to the origin. What is the new system?

Translate the system by the vector $\langle-4,-2\rangle$.

$$
\begin{gathered}
y=-x \\
y=x
\end{gathered}
$$

c. Describe a transformation matrix $A$ that will rotate the lines to the $x$-and $y$-axes. What is the new system?

Let $A=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)$. We want $A \cdot\binom{-1}{1}=\binom{1}{0}$, and $A \cdot\binom{1}{1}=\binom{0}{1}$. Thus

$$
\begin{aligned}
& -a+b=1 \\
& -c+d=0
\end{aligned}
$$

and

$$
\begin{aligned}
& a+b=\mathbf{0} \\
& c+d=1
\end{aligned}
$$

Solving for $a, b, c$, and $d$ gives

$$
A=\left(\begin{array}{cc}
-\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{array}\right)
$$

The new system is

$$
\begin{aligned}
& x=0 \\
& y=0
\end{aligned}
$$

d. Describe a translation that will result in a system that has the same solution set as the original system. What is the new system of equations?

Translate by the vector $\langle 4,2\rangle$. The new system is

$$
\begin{aligned}
& x=4 \\
& y=2
\end{aligned}
$$

which does intersect at the point $(4,2)$, so the solution is $(4,2)$.

The diagram below details the transformations graphically.


## Closing (3 minutes)

Ask students to summarize the key points of the lesson first in writing and then with a partner. Share key points as a class.

- A linear transformation preserves the solution set of a system of linear equations if they intersect at the origin.
- If a system of linear equations does not intersect at the origin, the solution set may not be preserved by a linear transformation.
- A sequence of transformation can be applied to a system of equations to preserve the solution set.


## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 24: Why Are Vectors Useful?

## Exit Ticket

1. Consider the system of equations $\left\{\begin{array}{c}y=5 x+2 \\ y=3 x\end{array}\right.$, and perform the following operations on an arbitrary point $\binom{x}{y}$ :
a. Rotate around the origin by $\theta$.
b. Translate by the opposite of the solution to the system.
c. Apply a dilation of $2 / 3$.
2. What effect does each of the transformations in Problem 1 have on the solution of the system and on the origin?

## Exit Ticket Sample Solutions

1. Consider the system of equations $\left\{\begin{array}{c}y=5 x+2 \\ y=3 x\end{array}\right.$, and perform the following operations on an arbitrary point $\binom{x}{y}$ :
a. Rotate around the origin by $\boldsymbol{\theta}$.

$$
\left(\begin{array}{cc}
\cos (\theta) & -\sin (\theta) \\
\sin (\theta) & \cos (\theta)
\end{array}\right)\binom{x}{y}=\binom{x \cos (\theta)-y \sin (\theta)}{x \sin (\theta)+y \cos (\theta)}
$$

b. Translate by the opposite of the solution to the system.

$$
\binom{x}{y}-\binom{-1}{-3}=\binom{x+1}{y+3}
$$

c. Apply a dilation of 2/3.

$$
\left(\begin{array}{ll}
\frac{2}{3} & 0 \\
0 & \frac{2}{3}
\end{array}\right)\binom{x}{y}=\binom{\frac{2}{3} x}{\frac{2}{3} x}
$$

2. What effect does each of the transformations in Problem 1 have on the solution of the system and on the origin?

The solution of the system $\binom{-1}{-3}$ maps to $\binom{-\cos (\theta)+3 \sin (\theta)}{-\sin (\theta)-3 \cos (\theta)},\binom{0}{0}$, and $\binom{-\frac{2}{3}}{-2}$ respectively.
The origin maps to itself for the first and third transformations, which were linear transformations, and maps to the opposite of the system's solution for the second transformation: $\binom{1}{3}$.

## Problem Set Sample Solutions

1. Consider the system of equations $\left\{\begin{array}{c}y=3 x+2 \\ y=-x+14\end{array}\right.$.
a. Solve the system of equations.

The point $(3,11)$ is a simultaneous solution to the two equations.
b. Ilene wants to rotate the lines representing this system of equations about their solution and wishes to apply the matrix $\left(\begin{array}{cc}\cos (\theta) & -\sin (\theta) \\ \sin (\theta) & \cos (\theta)\end{array}\right)$ to any point $A$ on either of the lines. If Ilene is correct, then applying a rotation to the solution will map the solution to itself. Let $\boldsymbol{\theta}=\mathbf{9 0}^{\circ}$, and find where llene's strategy maps the solution you found in part (a). What is wrong with llene's strategy?

$$
\begin{aligned}
\left(\begin{array}{cc}
\cos (90) & -\sin (90) \\
\sin (90) & \cos (90)
\end{array}\right)\binom{3}{11} & =\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right)\binom{3}{11} \\
& =\binom{-11}{3}
\end{aligned}
$$

Her transformation does not map the pivot point to itself, so the system is not rotating around the pivot point.
c. Jasmine thinks that in order to apply a rotation to some point on either of these two lines, the entire system needs to be shifted so that the pivot point is translated to the origin. For an arbitrary point $A$ on either of the two lines, what transformation needs to be applied so that the pivot point is mapped to the origin?

To map the pivot point to the origin, we can take a translation equal to the opposite of the pivot point. In this case, that means for $A=\binom{x}{y}$ on either of the two lines, we do the following: $\binom{x}{y}+\binom{-3}{-11}$.
We see that $\binom{3}{11}+\binom{-3}{-11}=\binom{0}{0}$.
d. After applying your transformation in part (c), apply llene's rotation matrix for $\boldsymbol{\theta}=90^{\circ}$. Show that the pivot point remains on the origin. What happens to the point $(0,2)$ after both of these transformations?

$$
\begin{aligned}
\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right)\binom{0}{0} & =\binom{0}{0} \\
\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right)\left(\binom{0}{2}+\binom{-3}{-11}\right) & =\left(\begin{array}{cc}
0 & -1 \\
1 & 0
\end{array}\right)\binom{-3}{-9} \\
& =\binom{9}{-3}
\end{aligned}
$$

The point is translated and then rotated $90^{\circ}$.
e. Although Jasmine and Ilene were able to work together to rotate the points around the pivot point, now their lines are nowhere near the original lines. What transformation will bring the system of equations back so that the pivot point returns to where it started and all other points have been rotated? Find the final image of the point $(0,2)$.

Applying the inverse translation that we did in part (c) will bring the points back to where they should be. That is, for $A^{\prime}=\binom{x}{y}$, a point on the transformed system, add $\binom{x}{y}+\binom{3}{11}$. For $\binom{0}{2}$, we get

$$
\binom{9}{-3}+\binom{3}{11}=\binom{12}{8}
$$

f. Summarize your results in parts (a)-(e).

To rotate a system of equations around its solution $\binom{a}{b}$, we apply the following transformations to any arbitrary point $\binom{x}{y}$ :

$$
\left(\begin{array}{cc}
\cos (\theta) & -\sin (\theta) \\
\sin (\theta) & \cos (\theta)
\end{array}\right)\left(\binom{x}{y}-\binom{a}{b}\right)+\binom{x}{y}
$$

We translate the system so that the solution is mapped to the origin, then rotate the system, and then apply the inverse translation.

Extension:

1. Let $b_{1}=\binom{1}{0}$ and $b_{2}=\binom{0}{1}$. Then answer the following questions.
a. Find $1 \cdot b_{1}+0 \cdot b_{2}$.
$1 \cdot\binom{1}{0}+0 \cdot\binom{0}{1}=\binom{1}{0}+\binom{0}{0}=\binom{1}{0}=b_{1}$
b. Find $0 \cdot b_{1}+1 \cdot b_{2}$.
$0 \cdot\binom{1}{0}+1 \cdot\binom{0}{1}=\binom{0}{0}+\binom{0}{1}=\binom{0}{1}=b_{2}$
c. Find $1 \cdot b_{1}+1 \cdot b_{2}$.
$1 \cdot\binom{1}{0}+1 \cdot\binom{0}{1}=\binom{1}{0}+\binom{0}{1}=\binom{1}{1}$
d. Find $3 \cdot b_{1}+2 \cdot b_{2}$.
$3 \cdot\binom{1}{0}+2 \cdot\binom{0}{1}=\binom{3}{0}+\binom{0}{2}=\binom{3}{2}$
e. Find $0 \cdot b_{1}+0 \cdot b_{2}$.
$0 \cdot\binom{1}{0}+0 \cdot\binom{0}{1}=\binom{0}{0}+\binom{0}{0}=\binom{0}{0}$
f. Find $x \cdot b_{1}+y \cdot b_{2}$ for $x, y$ real numbers.
$x \cdot\binom{1}{0}+y \cdot\binom{0}{1}=\binom{x}{0}+\binom{0}{y}=\binom{x}{y}$
g. Summarize your results from parts (a)-(f). Can you use $b_{1}$ and $b_{2}$ to define any point in $\mathbb{R}^{2}$ ?

In each case, the resultant vector was always equal to the scalar multiplied by the first vector for the $x$-coordinate and the scalar multiplied by the second vector for the $y$-coordinate.
2. Let $b_{1}=\binom{3}{2}$ and $b_{2}=\binom{-2}{3}$. Then answer the following questions.
a. Find $1 \cdot b_{1}+1 \cdot b_{2}$.
$1 \cdot\binom{3}{2}+1 \cdot\binom{-2}{3}=\binom{3}{2}+\binom{-2}{3}=\binom{1}{5}$
b. Find $0 \cdot b_{1}+1 \cdot b_{2}$.
$0 \cdot\binom{3}{2}+1 \cdot\binom{-2}{3}=\binom{0}{0}+\binom{-2}{3}=\binom{-2}{3}=b_{2}$
c. Find $1 \cdot b_{1}+0 \cdot b_{2}$.
$1 \cdot\binom{3}{2}+0 \cdot\binom{-2}{3}=\binom{3}{2}+\binom{0}{0}=\binom{3}{2}=b_{1}$
d. Find $-4 \cdot b_{1}+2 \cdot b_{2}$.

$$
-4 \cdot\binom{3}{2}+2 \cdot\binom{-2}{3}=\binom{-12}{-8}+\binom{-4}{6}=\binom{-16}{-2}
$$

e. Solve $r \cdot b_{1}+s \cdot b_{2}=\mathbf{0}$.

$$
\begin{aligned}
r \cdot\binom{3}{2}+s \cdot\binom{-2}{3} & =\binom{0}{0} \\
\binom{3 r}{2 r}+\binom{-2 s}{3 s} & =\binom{0}{0} \\
\binom{3 r-2 s}{2 r+3 s} & =\binom{0}{0} \\
3 r=2 s \Rightarrow r & =\frac{2}{3} s \\
2\left(\frac{2}{3} s\right)+3 s & =0 \\
\frac{4}{3} s+3 s & =0 \\
4 s+9 s & =0 \\
13 s & =0 \\
s & =0 \\
r=\frac{2}{3} \cdot 0 & =0
\end{aligned}
$$

So both $r$ and $s$ are zero.
f. Solve $r \cdot b_{1}+s \cdot b_{2}=\binom{22}{-7}$.

$$
\begin{aligned}
r \cdot\binom{3}{2}+s \cdot\binom{-2}{3} & =\binom{22}{-7} \\
\binom{3 r}{2 r}+\binom{-2 s}{3 s} & =\binom{22}{-7} \\
\binom{3 r-2 s}{2 r+3 s} & =\binom{22}{-7}
\end{aligned}
$$

Thus, $r=\frac{22}{3}+\frac{2}{3} s$, and we get $2\left(\frac{22}{3}+\frac{2}{3} s\right)+3 s=-7$.

$$
\begin{aligned}
\frac{44}{3}+\frac{4}{3} s+3 s & =-7 \\
44+4 s+9 s & =-21 \\
13 s & =-65 \\
s & =-5 \\
r & =\frac{22}{3}+\frac{2}{3}(-5)=\frac{22}{3}-\frac{10}{3}=\frac{12}{3}=4
\end{aligned}
$$

So $r=4$ and $s=-5$.
g. Is there any point $\binom{x}{y}$ that cannot be expressed as a linear combination of $b_{1}$ and $b_{2}$ (i.e., where $r \cdot b_{1}+s \cdot b_{2}=\binom{x}{y}$ has real solutions, for $x, y$ real numbers)?

No. There was nothing special about the point $\binom{22}{-7}$. We could have substituted in any other two values, performed the same steps, and arrived at a valid solution.
h. Explain your response to part (g) geometrically.

Answers may vary. $b_{1}$ and $b_{2}$ are two vectors that intersect at the origin. They are not parallel, so through linear transformations, we can arrive at any point $\binom{x}{y}$. Alternatively, you can think of the equations $3 r-2 s=x$ and $2 r-3 s=y$ as lines in terms of $r$ and $s$; then these lines are not parallel and thus are always consistent when used in a system of linear equations.

## New York State Common Core

## Mathematics Curriculum

## Topic E:

## First-Person Video Games-Projection <br> Matrices

N-VM.C.8, N-VM.C.9, N-VM.C.10, N-VM.C. 11



The module ends with Topic E , in which students apply their knowledge developed in this module to understand how first-person video games use matrix operations to project three-dimensional objects onto two-dimensional screens and animate those images to give the illusion of motion (N-VM.C.8, N-VM.C.9, N-VM.C.10, N-VM.C.11).

Throughout this topic, students explore the projection of three-dimensional objects onto two-dimensional space. In Lesson 25, students create and manipulate objects in three dimensions and attempt a projection of a cube onto a flat screen. Through a discussion of the historical development of perspective drawing, students will learn how vanishing points can be used to create realistic two-dimensional representations of three-dimensional objects. In Lesson 26, students engage the mathematics that underlies animations like the kind seen in video games. In particular, students learn to take points in three-dimensional space and map

[^7]them onto a two-dimensional plane in the same way that a programmer would seek to model the 3-D world on a 2-D screen. Students analyze the 2-D projection of a path followed by an object being rotated through three-dimensional space.

Topic E and Module 2 end with Lesson 27, where students create a brief scene in a video game including onestep turn and roll procedures. Students describe the motions of the characters with respect to the scene and as rotations about specific axes. Students apply the mathematics from the previous lessons to use matrices to represent the motions of the characters in their scenes. Students make characters move in a straight line, turn, and flip in all directions by exploring rotations about each axis, consistently relating these motions back to their matrix representations.

The materials support the use of geometry and game-creating software, such as GeoGebra and the freely available ALICE 3.1.

In Topic E, students are making sense of vectors, understanding them as abstract representations of real world situations such as programming video games (MP.2), and modeling video games with vectors and matrices (MP.4). Through the use of computer games (ALICE 3.1) and software (GeoGebra), students are also using appropriate tools strategically to apply and understand mathematical concepts; further, they come to see matrices as tools to create transformations (MP.5).

## (8. Lesson 25: First-Person Computer Games

## Student Outcomes

- Through discovery, students will understand how challenging it is to project three-dimensional objects onto a two-dimensional space.
- Students will understand that vanishing points can be used to project three-dimensional objects onto twodimensional space. They will demonstrate this process by showing how a cube is projected onto the twodimensional surface of a screen.
- Students will recognize that projecting three-dimensional images onto a two-dimensional surface can be accomplished by applying linear transformation matrices to the pre-image points.


## Lesson Notes

The ALICE program should be downloaded before starting the lesson. In homework assigned prior to this lesson, students should become familiar with the ALICE 3.1 program. After sharing their experiences creating and manipulating objects in ALICE, students will discuss the projection of three-dimensional objects onto two-dimensional space and will attempt a projection of a cube onto a flat screen, which should help them appreciate the difficulty of sketching a twodimensional projection of a three-dimensional object by hand. They will also use right-triangle trigonometry to explore how projections onto different-sized screens affect the viewer's field of vision. Through a discussion of the historical development of perspective drawing, they will learn how vanishing points can be used to create realistic twodimensional representations of three-dimensional objects. They will apply this understanding by revisiting the task of projecting a cube onto a two-dimensional screen and form conjectures about how vectors, representing points on the three-dimensional objects, can be multiplied by matrices to generate two-dimensional projections (N-VM.C.11). This will prepare them to apply matrix multiplication to generate two-dimensional projections in the next lesson.

## Classwork

## Opening (3 minutes)

Open the ALICE 3.1 program, and encourage students to share briefly the types of items they created or manipulated while exploring the program. Allow a few volunteers to demonstrate their new skills.

## Discussion (2 minutes)

- Are the objects created in ALICE actually three-dimensional, and if not, how can we describe them?
- They are representations or projections of three-dimensional objects onto a two-dimensional screen.
- How do people create realistic looking three-dimensional objects on a flat surface?
- Answers may vary and could include the concept of depicting the object from a certain vantage point.


## Exploratory Challenge (10 minutes)

The ALICE program Projection_No_Wires should be displayed. Note: If the program is not available, the example could be completed using a clear plastic sheet (such as a sheet protector or transparency sheet), a small cube, and a larger ball. Student volunteers could change the position of the objects to represent the different vantage points shown in the scene from the Alice program.

When the program opens, the instructor should click "run." Typing 5 should display a blue cube to the right of a gray rectangle.


Explain to the students that the cube is a three-dimensional object that needs to be projected onto a screen represented by the gray rectangle. By typing 2 when the program is run, students will be able to see the viewpoint of an observer's eye looking forward through the screen to the cube and a red sphere behind it. Note: If the real objects are being used, the sheet would be directly in front of the viewer's eye, with the cube in the foreground and the larger ball directly behind the cube.


Typing 4 will provide a wide view of the scene.


After discussing the different viewpoints, provide each student with a copy of the image shown in the example. Have the students attempt the construction. After a few minutes, they should share their ideas with a partner. Then the teacher should facilitate a discussion that emphasizes the difficulty of projecting the cube onto the screen by hand. The program projection should be shown to help students visualize what the projection of the cube onto the screen represents, that is, the points where the line segments from the viewer's eye to the cube intersect the surface of the screen. This is viewpoint 5, where the "strings" represent the line of sight from the viewer to the cube.


- The first screen we saw is viewpoint 5. The blue cube represents the three-dimensional object to be projected onto a two-dimensional screen, which is represented by the gray rectangle. Now when we look at viewpoint 2, how did the appearance of the scene change?
- The screen is in the foreground, with the blue cube and a red sphere behind it.
- So viewpoint 2 could represent the vantage point of a viewer looking directly at a TV screen. You could think of the screen as a window through which the viewer looks at the real-world objects on the other side. Now let's look at viewpoint 4. How does this viewpoint change the appearance of the scene?
- Answers may vary but will probably indicate that this seems to be a wide view from the side, and the sphere looks much larger than the cube.
- How is it possible that viewpoints 2 and 4 could represent the same scene?
- They show the objects from different positions. Viewpoint 2 is from the perspective of a viewer looking at the two objects directly in front of him or her, and viewpoint 4 is shown from a perspective that is to the side of the objects.
- Why does the relative size of the sphere look so different between the viewpoints?
- Answers may vary, but students might recognize that when two objects are aligned in a viewer's line of sight, a much smaller object in the foreground can appear to be similar in size to a much larger object in the background (e.g., one's index finger placed a few inches directly in front of one's eyes might look similar in height to a building in front of the viewer a mile away).
- Now look at the example. It is drawn from a similar vantage point to viewpoint 5 in the ALICE program. By drawing line segments from the point in the lower left corner of the diagram (which represents the viewer's eye) to the vertices of the cube, we can represent the viewer's line of sight. Once we draw the line segments in, how can we draw an image of the cube on the screen?
- Answers will vary but might include that the vertices of the projection are located where the line segments intersect the rectangle.
- Now that you have tried to draw the image of the cube projected on the screen, how would you characterize the activity, and what challenges did you face?
- Most students will struggle with the construction and may not complete it. They might suggest that it is hard to see the points where the line segments intersect the screen. (Note: They will complete this activity later in the lesson through step-by-step instructions-the diagram later in the lesson shows the correctly drawn projection.)
- Let's look at another program in ALICE. In this program, the cube is "attached" to the viewer's eye by strings. In viewpoint 5, you can see where the strings intersect the screen. The intersection of the viewer's line of sight and the screen represents the projection of the cube onto the screen. This is difficult to represent on paper.

[^8]\(\left.\begin{array}{l}Scaffolding: <br>
Advanced students could be asked <br>
to project the cube onto the screen <br>

without further prompting.\end{array}\right\}\)| Model how the first vertex is drawn |
| :--- |
| to the "eye," and have students |
| sketch the remaining line segments |
| independently. |

## Example (5 minutes)

This example allows students to use right-triangle trigonometry to explore an additional challenge programmers have in projecting images onto a screen: limitations on the field of view. The students will apply trigonometry to determine the effect of screen size on a horizontal field of view. This example should be completed as part of a teacher-led discussion. Alternatively, the students could complete the problem in pairs and share their responses after a few minutes.

- How does a person's peripheral vision compare with the field of view of a screen? Explain your answer.
- The field of view of the screen is smaller. When you look at a screen, you can generally see objects in your peripheral vision that are to the right and left of the screen.
- We have experienced the limiting effect of a screen on our field of view. If we model the field of view of a television screen with a diagram, what components do we need to include?
- The screen; the field of view, which is the angle formed by the rays representing the line of sight of the viewer to the right and left ends of the screen; the width of the screen; the distance from the middle of the screen to the viewer's eyes
- What does our model look like when all these components are included?
- Two right triangles are formed, each with leg lengths of $\frac{w}{2}$ and $d$, acute angle $\frac{\theta}{2}$ formed by $d$, and the line of sight to the right or left side of the screen.
- How can we use trigonometry to find the line of sight of the person sitting in front of the television screen?
- We can apply the inverse tangent function to the ratio of $\frac{w}{2}$ and $d$ to find $\frac{\theta}{2}$.
- How does your own experience with screen viewing compare with our findings regarding the relationship between screen size, distance from a screen, and field of view?
- Answers will vary but should address that larger screens, such as movie theater screens, require a greater viewing distance for a comfortable field of view than smaller screens (similarly a comfortable viewing distance from a television screen is usually greater than that for a computer screen).


## Example

When three-dimensional objects are projected onto screens with finite dimensions, it often limits the field of view (FOV), or the angle the scene represents. This limiting effect can vary based on the size of the screen and position of the observer.
a. Sketch a diagram that could be used to calculate a viewer's field of view $\boldsymbol{\theta}$ in relation to the horizontal width of the screen $w$ and the distance the viewer is from the screen $d$.

b. Assume that a person is sitting directly in front of a television screen whose width is 48 inches at a distance of 8 feet from the screen. Use your diagram and right-triangle trigonometry to find the viewer's horizontal field of view $\boldsymbol{\theta}$.

$$
\begin{aligned}
\tan \left(\frac{\theta}{2}\right) & =\frac{\frac{w}{2}}{d} \\
\tan \left(\frac{\theta}{2}\right) & =\frac{\overline{4}}{8}=0.25 \\
\tan ^{-1}(0.25) & =\frac{\theta}{2} \\
14^{\circ} & \approx \frac{\theta}{2} \\
\theta & \approx 28^{\circ}
\end{aligned}
$$

c. How far would a viewer need to be from the middle of a computer screen with a width of $\mathbf{1 5}$ inches to produce the same field of view as the person in front of the television?
$\tan \left(\frac{\theta}{2}\right)=\frac{\frac{w}{2}}{d}$
$\tan \left(\frac{28}{2}\right)=\frac{\frac{15}{2}}{d}$
$d=\frac{\frac{15}{2}}{\tan 14^{\circ}} \approx 30$ inches
d. Write a general statement about the relationship between screen size and field of view.

The smaller the screen, the closer a viewer must be to the screen to result in a given horizontal FOV.

## Discussion (8 minutes): Development of Perspective in Renaissance Art

This discussion will expose students to the difficulties artists have faced historically in representing three-dimensional objects on a two-dimensional surface.

- We have now experienced the challenges of trying to represent a three-dimensional object on a twodimensional surface and explored some of the visual challenges in representing scenes on a surface with finite dimensions. Let us see how artists addressed the challenge of projecting images during the medieval and Renaissance periods.

Display an image that is similar to the one found at this site as well as the two images shown:
http://c300221.r21.cf1.rackcdn.com/medieval-painting-1384196679 org.jpg


Have the students, in small groups, compare, and contrast the paintings, focusing on the depiction of three-dimensional images. Facilitate a discussion based on the students' findings, and provide a brief explanation of the history of the development of linear perspective in art:

- How do the images differ in presenting realistic images of three-dimensional objects?
- Answers will vary, but should address that the image of the meal does not realistically represent the table, e.g., it looks like the food is falling off the table; the second image provides a sense of a top and bottom of the hill but some figures look unrealistic, e.g., the black sheep looks like it is floating on the hill; the painting with the eagle realistically portrays the objects.
- This first painting you looked at is an example of a painting from the Middle Ages. In the early 1300s, artists generally created works that appeared flat, without distinguishing between the foreground and background or accounting for differences in the size of objects leading to scenes that looked unrealistic, e.g., floors and tables that tilted up.
- In the early 1300s, an artist named Giotto di Bondone was the first Italian painter to attempt to paint using perspective, trying to represent objects with a realistic impression of their size and depth. He is credited with applying algebra to help place distant lines in his paintings. An example of his work is the second painting you looked at. In what ways does this painting show an improvement in representing realistic three-dimensional objects in two-dimensional space when you compare it to the first painting?
- Answers will vary but might address the illusion of depth between the left (upper) and right (lower) parts of the painting
- In the early 1400s, the artist Filipo Brunelleschi was attributed with advancing the use of linear perspective in art. In particular, he made use of a vanishing point, a point to which all lines in a painting would converge. The use of these lines helped artists to create the illusion of depth. Here is a simple example of using a vanishing point to create perspective:


A rectangle with a vanishing point at its center.


A room with floor, back wall, and side door constructed using the vanishing point.


A room with extraneous parts of the line segments erased.

- How was the vanishing point used to create the elements in the room? Share your ideas with a partner.
- Answers will vary but should include that each of the line segments for the floor, ceiling, and door were created using line segments that intersect the vanishing point when extended. For the back wall, the vertices of the rectangle lie along line segments that intersect with the vanishing point.
- By the late 1400s, artists like Leonardo da Vinci were using multiple vanishing points in their artwork to create images that were very realistic, such as the following. How can you see the use of a vanishing point in this picture?
- The lines in the drawing all converge at the vanishing point, which is at the eagle. This draws the eye to the eagle as the focal point of the drawing, in addition to creating a realistic sense of depth and size for the figures in the drawing.
- We can see the use of vanishing points in many realistic images.

One Vanishing Point

"Inside Greenwich Foot Tunnel" by C.G.P Grey, is licensed under CC BY 2.0
http://creativecommons.org/licenses/by/2.0

Two Vanishing Points

"Perspective 1" created by Ejahng,
is licensed under CC BY-SA 3.0
http://creativecommons.org/licenses/by-sa/3.0/deed.en

Three Vanishing Points


Berenice Abbott, 1934

Multiple Vanishing Points

"UT band during Rice game" by Mary Estrada, is licensed under CC BY 2.0
http://creativecommons.org/licenses/by/2.0

- How might the use of vanishing points to create linear perspective relate to transformations of objects in three-dimensional space?
- Answers will vary but might address that the rescaling of objects is similar to performing linear transformations.
- And how have we transformed objects in three-dimensional space?
- We have multiplied points by transformation matrices.
- One way to use multiple vanishing points to project a three-dimensional object into two-dimensional space by hand is to create a horizon line through the two-dimensional surface and plot one or more vanishing points along that line. We are going to use this idea to try again to project the cube onto the screen.


## Exercise 1 (6 minutes)

Students should try to create the two-dimensional projection of the cube onto the screen, this time by using the horizon line and vanishing points provided. Point out to the students that the first projected point is selected arbitrarily, and the rest of the points can be found using the vanishing points. Note: It is important that the instructor works through the example prior to modeling it with students in class. This is likely to take more than one attempt. Once the students have completed the projection, have them paraphrase the procedure in writing, including an interpretation of the object they created. The students should share their projections and written responses with a partner.

## Exercise 1

In this drawing task, the "eye" or the "camera" is the point, and the shaded figure is the "TV screen." The cube is in the 3-D universe of the computer game.

By using lines drawn from each vertex of the cube to the point, draw the image of the 3-D cube on the screen. A horizon line and two additional vanishing points have been included to help you. The image point of the first vertex is shown.


## Scaffolding:

- Advanced students could be prompted to create a horizon line with two endpoints as vanishing points outside the screen. They could be allowed, then, to choose the location of the line.
- Work with struggling students step-by-step through the construction as shown.
- Draw line segments connecting the "eye" $\left(V_{1}\right)$ with the vertices of the cube.
- Draw a vertical line segment on the screen from the given image point $\left(P_{1}{ }^{\prime}\right)$ to the point where it intersects the line segment from $V_{1}$ to the vertex beneath it $\left(P_{2}{ }^{\prime}\right)$.

- Draw dashed line segments connecting the two image points to vanishing points 2 and 3.

- Plot a point where each of the line segments from $V_{1}$ to the vertices intersect the line segments connecting the dashed line segments. These are the remaining image points for the vertices of the cube.

- Connect the image points to form the projected image of the cube.



## Exercise 2 ( 3 minutes)

This exercise should be completed as part of a teacher-led discussion. It will provide the opportunity for students to explore how matrix operations can be used to find the coordinates for projected points of threedimensional objects.

## Exercise 2

Let's assume that the point $V_{1}$ in our projection diagram is at the origin and the upper right vertex of the cube is located
at $\left(\begin{array}{l}5 \\ 8 \\ 4\end{array}\right)$. If our screen represents the plane $y=2$, use matrix multiplication to determine the vector that represents the line of sight from the observer to the projected point on the screen. Explain your thinking.
The vector from the viewer to the vertex on the actual object is $\left[\begin{array}{l}5 \\ 8 \\ 4\end{array}\right]$. Since the projected point and the actual vertex are
along the same line of sight, the vector from the line of sight of the viewer to the projected point can be found by resizing the vector to the vertex so that the $y$-coordinate of the projected point is 2 . This can be accomplished by rescaling with a factor of 0.25 using the matrix multiplication $\left[\begin{array}{ccc}0.25 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.25\end{array}\right]\left[\begin{array}{l}5 \\ 8 \\ 4\end{array}\right]=\left[\begin{array}{c}1.25 \\ 2 \\ 1\end{array}\right]$.

## Closing (3 minutes)

Students should respond to the question in writing. After a minute, they could share their predictions with a partner.

- What do you predict that computer programmers might do to represent and manipulate three-dimensional objects in two-dimensional space?
- Answers will vary but might include determining image points by multiplying vectors by transformation matrices and applying matrix operations to represent linear transformations of the points.


## Exit Ticket (5 minutes)

 COREName $\qquad$ Date $\qquad$

## Lesson 25: First-Person Computer Games

## Exit Ticket

In a computer game, the camera eye is at the origin, and the tip of a dog's nose has coordinates $\left(\begin{array}{c}2 \\ 10 \\ 3\end{array}\right)$. If the computer screen represents the plane $y=1$, determine the coordinates of the projected point that represents the tip of the dog's nose.

## Exit Ticket Sample Solutions

In a computer game, the camera eye is at the origin, and the tip of a dog's nose has coordinates $\left(\begin{array}{c}2 \\ 10 \\ 3\end{array}\right)$. If the computer screen represents the plane $y=1$, determine the coordinates of the projected point that represents the tip of the dog's nose.
The vector representing the line of sight from the viewer to the tip of the dog's nose is $\left[\begin{array}{c}2 \\ 10 \\ 3\end{array}\right]$. To resize the vector so that the $y$-coordinate of the projected point is 1 , we need to rescale by a factor of 0.1 .
$\left[\begin{array}{ccc}0.1 & 0 & 0 \\ 0 & 0.1 & 0 \\ 0 & 0 & 0.1\end{array}\right]\left[\begin{array}{c}2 \\ 10 \\ 3\end{array}\right]=\left[\begin{array}{c}0.2 \\ 1 \\ 0.3\end{array}\right]$, so the projected point is $\left(\begin{array}{c}0.2 \\ 1 \\ 0.3\end{array}\right)$.

## Problem Set Sample Solutions

1. Projecting the image of a three-dimensional scene onto a computer screen has the added constraint of the screen size limiting our field of view, or FOV. When we speak of FOV, we wish to know what angle of view the scene represents. Humans have remarkably good peripheral vision. In New York State, the requirement for a driver's license is a horizontal FOV of no less than $140^{\circ}$. There is no restriction placed on the vertical field of vision, but humans normally have a vertical FOV of greater than $120^{\circ}$.
a. Consider the (simulated) distance the camera is from the screen as $d$, the horizontal distance of the screen as $w$, and the horizontal FOV as $\theta$, then use the diagram below and right-triangle trigonometry to help you find $\boldsymbol{\theta}$ in terms of $\boldsymbol{w}$ and $\boldsymbol{d}$.

$$
\begin{aligned}
\tan \left(\frac{\theta}{2}\right) & =\frac{\frac{w}{2}}{d} \\
\frac{\theta}{2} & =\tan ^{-1}\left(\frac{\frac{w}{2}}{d}\right) \\
\theta & =2 \tan ^{-1}\left(\frac{\frac{w}{2}}{d}\right)
\end{aligned}
$$


b. Repeat procedures from part (a), but this time let $h$ represent the height of the screen and $\psi$ represent the vertical FOV.

$$
\begin{aligned}
\tan \left(\frac{\psi}{2}\right) & =\frac{\frac{h}{2}}{d} \\
\frac{\psi}{2} & =\tan ^{-1}\left(\frac{\frac{h}{2}}{d}\right) \\
\psi & =2 \tan ^{-1}\left(\frac{\frac{h}{2}}{d}\right)
\end{aligned}
$$


c. If a particular game uses an aspect ratio of $16: 9$ as its standard view and treats the camera as though it were 8 units away, find the horizontal and vertical FOVs for this game. Round your answers to the nearest degree.

$$
\begin{aligned}
\theta & =2 \cdot \tan ^{-1}\left(\frac{\frac{16}{2}}{8}\right) \\
& =2 \cdot \tan ^{-1}(1) \\
& =2 \cdot 45 \\
& =90
\end{aligned}
$$

The horizontal field of view is $90^{\circ}$.

$$
\begin{aligned}
\psi & =2 \cdot \tan ^{-1}\left(\frac{\frac{9}{2}}{8}\right) \\
& =2 \cdot \tan ^{-1}\left(\frac{4.5}{8}\right) \\
& \approx 2 \cdot 29.36 \\
& =58.72 \\
& \approx 59
\end{aligned}
$$

So, the vertical field of view is about $59^{\circ}$.
d. When humans sit too close to monitors with FOVs less than what they are used to in real life or in other games, they may grow dizzy and feel sick. Does the game in part (c) run the risk of that? Would you recommend this game be played on a computer or on a television with these FOVs?

The game in part (c) has a smaller field of vision than most humans are probably used to. Since computer games are played much closer and that increases the chance of dizziness and illness, this game would probably be better on a television than a computer.
2. Computers regularly use polygon meshes to model three-dimensional objects. Most polygon meshes are a collection of triangles that approximate the shape of a three-dimensional object. If we define a face of a polygon mesh to be a triangle connecting three vertices of the shape, how many faces at minimum do the following shapes require?
a. A cube.

A cube normally has 6 square faces, but it requires 2 triangles to make a square, so a polygon mesh of a cube would have 12 faces.
b. A pyramid with a square base.

The pyramid has 4 triangular faces already, and would require 2 more to make up the base, so 6 faces.
c. A tetrahedron.

A tetrahedron is composed of 4 triangles, so 4 faces for its polygon mesh.
d. A rectangular prism.

A rectangular prism is the same as a cube: it has 12 faces.
e. A triangular prism.

A triangular prism has 2 faces make up the ends, and each side requires 2 more faces, so $2+2 \cdot 3=8$ faces.
f. An octahedron.

An octahedron is $\mathbf{8}$ triangular pieces, so $\mathbf{8}$ faces.
g. A dodecahedron.

A dodecahedron normally has 12 pentagonal faces. Each pentagon requires a minimum of 3 triangles to construct it, so a dodecahedron would have 36 faces in its polygon mesh.
h. An icosahedron.

An icosahedron has 20 triangular faces, so its polygon mesh does as well.
i. How many faces should a sphere have?

Answers will vary. An icosahedron is a rough approximation of a sphere, so 20 is a good estimate, especially for graphics in the early years of computers and video games. Some students might suggest 360, which is a good estimate of the number of lines to approximate a circle. As many faces as the computer can handle without slowing down would be the best polygon mesh for a sphere.

| Lesson 25: | First-Person Computer Games |
| :--- | :--- |
| Date: | $1 / 24 / 15$ | $1 / 24 / 15$

3. In the beginning of 3-D graphics, objects were created only using the wireframes from a polygon mesh without shading or textures. As processing capabilities increased, 3-D models became more advanced, and shading and textures were incorporated into 3-D models. One technique that helps viewers visualize how shading works on a 3-D figure is to include both an "eye" and a "light source." Vectors are drawn from the eye to the figure, and then reflected to the light (this technique is called ray tracing). See the diagram below.

a. Using this technique, the hue of the object depends on the sum of the magnitudes of the vectors. Assume the eye in the picture above is located at the origin, $v_{s}$ is the vector from the eye to the location $\left(\begin{array}{l}4 \\ 5 \\ 3\end{array}\right)$, and the light source is located at $\left(\begin{array}{l}5 \\ 6 \\ 8\end{array}\right)$. Then find $v_{1}$, the vector from $v_{s}$ to the light source, and the sum of the magnitudes of the vectors.
$\mathrm{v}_{\mathrm{l}}$ is the difference between the light source and the object, so $\mathrm{v}_{\mathrm{l}}=\left(\begin{array}{l}5 \\ 6 \\ 8\end{array}\right)-\left(\begin{array}{l}4 \\ 5 \\ 3\end{array}\right)=\left(\begin{array}{l}1 \\ 1 \\ 5\end{array}\right)$.
$\left\|v_{1}\right\|=\sqrt{1^{2}+1^{2}+5^{2}}=\sqrt{27}=3 \sqrt{3}$
$\left\|v_{s}\right\|=\sqrt{4^{2}+5^{2}+3^{2}}=\sqrt{16+25+9}=\sqrt{50}=5 \sqrt{2}$
$\left\|v_{1}\right\|+\left\|v_{s}\right\|=3 \sqrt{3}+5 \sqrt{2}$
b. What direction does light travel in real life, and how does this compare to the computer model portrayed above? Can you think of any reason why the computer only traces the path of vectors that start at the "eye"?

In reality, light travels from the light source, bouncing off objects until it reaches our eyes. If the computer starts drawing vectors at the light source, then it will draw many vectors that will never reach our "eye," wasting resources.

# T <br> <br> Lesson 26: Projecting a 3-D Object onto a 2-D Plane 

 <br> <br> Lesson 26: Projecting a 3-D Object onto a 2-D Plane}

## Student Outcomes

- Students learn to project points in 3-dimensional space onto a 2-dimensional plane.
- Students learn how to simulate 3-dimensional turning on a 2-dimensional screen.


## Lesson Notes

Students engage the mathematics that provides a foundation for animations like the kind seen in video games. In particular, students learn to take points in 3-dimensional space and map them onto a 2-dimensional plane in the same way that a programmer would seek to model the 3-D world on a 2-D screen. Students also analyze the 2-D projection of a path followed by an object being rotated through 3-dimensional space.

## Classwork

In the context of this lesson, the statement $y=1$ will be used to describe a plane in 3-dimensional space that is parallel to the $x z$-plane. However, the same statement could be used to describe a line in the coordinate plane that is parallel to the $x$-axis, or to describe a point on the number line. This series of exercises gives students an opportunity to look for and make use of structure as they learn to differentiate between these situations based on context. Students may represent their responses to the questions using words, pictures, or graphs.

## Discussion (4 minutes): Visualizing $y=1$

- On the real number line, there is exactly one point $y$ whose coordinate is 1 :

- Describe the set of points $(x, y)$ in $\mathbb{R}^{2}$ whose $y$-coordinate is 1 . Give several specific examples of such points.
- The set of points whose $y$-coordinate is 1 consists of points that are 1 unit above the $x$-axis. This set of points forms a line that is parallel to the $x$-axis. For example, this set includes the points $(-1,1),(2,1)$, and $(5,1)$.

- Describe the set of points $(x, y, z)$ in $\mathbb{R}^{3}$ whose $y$-coordinate is 1 . Give several specific examples of such points.
- The set of points whose $y$-coordinate is 1 consists of points that are 1 unit to the right of the $x z$-plane. This set of points forms a plane that is parallel to the $x z$-plane. For example, this set includes the points $(-1,1,6),(0,1,0)$, and $(5,1,3)$.



## Discussion (11 minutes): Projections

- Now imagine that the plane $y=1$ represents a TV screen, with the origin located where your eye is. Suppose we have a point $A$ in 3-dimensional space that we wish to project on to the screen. We will call the image $A^{\prime}$. How should we go about locating $A^{\prime}$ ?
- $\quad A^{\prime}$ must be located at a point on the screen that lies on the line connects $O$ to $A$.

- Good! We would like to be able to describe the line through $O$ and $A$, then find out where it hits our screen. Before we attempt to do this, let's examine the 2-dimensional case to see if we can get any ideas.
- Suppose you have a screen located along the line $y=1$, and the origin is located where your eye is. Let's take the point $A(7,5)$ and project it onto this screen. Where is the image of $A$ ? Take a few moments to think about this, then discuss your thoughts with the students around you. In a few minutes, please be ready to present your argument to the class.

- Any point $P$ on the line joining $O$ and $A$ makes a triangle that is similar to $\triangle A O B$ in the figure below. Thus the coordinates of $P$ must satisfy the proportion $\frac{y}{x}=\frac{5}{7}$. In the case of $A^{\prime}$, which lies on our screen, the $x$-coordinate must be 1 . Thus its $y$-coordinate can be found by solving $\frac{y}{1}=\frac{5}{7}$, so $y=\frac{5}{7}$. In summary, the image of point $A$ is the point $A^{\prime}=\left(1, \frac{5}{7}\right)$.



## Scaffolding:

Show students the diagram at the left, and assist them with cues such as the following:

- What can we say about the triangles in the figure?
- Can you write an equation that shows the relationship between the lengths in the figure?
- Next we will attempt to describe the set of points on the line through $O$ and $A$ using a parameter $t$. In other words, describe the points on this line as $(x(t), y(t))$. Justify your answer.
- We have $(x(t), y(t))=(7 t, 5 t)$. We know this is correct because the coordinates satisfy the proportion $\frac{5 t}{7 t}=\frac{5}{7}$.
- To round out our discussion, we will look at this from one more point of view. Give a geometric description of the transformation $(7,5) \mapsto(7 t, 5 t)$.
- This is a dilation of the point $(7,5)$, where the origin is the center of the dilation, and the parameter $t$ represents the scale factor.
- We know that a point and its image lie on the same line through the origin when dilations are involved, so our work with the projection makes sense from that perspective as well. Do you see how we can find the image of $(7,5)$ using the parametric description $(7 t, 5 t)$ of points on the line through $O$ and $A$ ?
- Yes, we can find the value of that makes the $x$-coordinate 1 . With $7 t=1$, we have $t=\frac{1}{7}$. Thus, the $y$-coordinate must be $5\left(\frac{1}{7}\right)=\frac{5}{7}$.


## Exercises 1-3 (2 minutes)

Give students time to work on the problems below, then ask them to share their work with a partner. When they are ready, select students to present their solutions to the class.

## Exercises

1. Describe the set of points $(8 t, 3 t)$, where $t$ represents a real number.

The point $(8 t, 3 t)$ is a dilation of the point $(8,3)$ with scale factor $t$. Thus each such point lies on the line that joins $(8,3)$ to the origin.
2. Project the point $(8,3)$ onto the line $x=1$.

We need the value of $t$ that makes $8 t=1$, so $t=\frac{1}{8}$. Thus, $3 t=3\left(\frac{1}{8}\right)=\frac{3}{8}$, and so the image is $\left(1, \frac{3}{8}\right)$.
3. Project the point $(8,3)$ onto the line $x=5$.

We need the value of $t$ that makes $8 t=5$, so $t=\frac{5}{8}$. Thus, $3 t=3\left(\frac{3}{8}\right)=\frac{15}{8}$, and so the image is $\left(5, \frac{15}{8}\right)$.

## Discussion (8 minutes): Projecting a Point onto a Plane

- We have just examined the problem of projecting a point in 2-dimensional space onto a 1-dimensional screen. Now let's return to the 3-dimensional problem from the start of this discussion.
- Suppose that the point $A$ is located at $(4,10,5)$. We want to project this point onto the plane $y=1$. In order to do this, let's describe the line through $O$ and $A$ parametrically. We practiced these descriptions in a 2-dimensional context. Can you see how to extend this to the 3-dimensional case?
- We should have $(x(t), y(t), z(t))=(4 t, 10 t, 5 t)$, which is a dilation of the point $(4,10,5)$ with scale factor $t$.
- We expect points of this form to lie on the line through $O$ and $A$. Let's use similar triangles to confirm this just as we did in the 2 -dimensional case. Can you make an argument that $(4,10,5)$ and $(4 t, 10 t, 5 t)$ are in fact on the same line through the origin? The diagram below may help with this.

- The large triangle contains the points $(0,0,0),(4,10,0)$, and $(4,10,5)$.
- The small triangle contains the points $(0,0,0),(4 t, 10 t, 0)$, and $(4 t, 10 t, 5 t)$.
- The vertical segments are in the ratio $\frac{5 t}{5}=t$. The ratio of the segments in the $x y$-plane is also $t$, proving that the triangles are similar.
- The long segment in the xy-plane has length $\sqrt{4^{2}+10^{2}}$.
- $\quad$ The short segment in the $x y$-plane has length $\sqrt{(4 t)^{2}+(10 t)^{2}}=\sqrt{t^{2} \cdot 4^{2}+t^{2} \cdot 10^{2}}$.

Thus, we have $\sqrt{t^{2}\left(4^{2}+10^{2}\right)}=t \sqrt{4^{2}+10^{2}}$. This confirms that the ratio of the corresponding sides is $t$, proving that the triangles are similar. From this we conclude that the original point and its dilated image both lie on the same line through the origin.

- Now return to the problem at hand: What is the image of $(4,10,5)$ when it is projected onto the plane? $y=1$ ?
- We want the value of that takes the $y$-coordinate to 1 , so we need $10 t=1$, which gives $t=\frac{1}{10}$. So the image must be $(0.4,1,0.5)$.

- Now we will do some more practice with 3-D projections onto a 2-D screen.


## Exercises 4-5 (3 minutes)

Give students time to work on the problems below. Ask them to share their work with a partner when they are ready. Select students to present their solutions to the class at an appropriate time.
4. Project the point $(-1,4,5)$ onto the plane $y=1$.

We need the value of $t$ that makes $4 t=1$, so $t=\frac{1}{4}$. Thus, the image is $\left(-\frac{1}{4}, 1, \frac{5}{4}\right)$.
5. Project the point $(9,5,-8)$ onto the plane $z=3$.

We need the value of $t$ that makes $-8 t=3$, so $t=-\frac{3}{8}$. Thus, the image is $\left(-\frac{27}{8},-\frac{15}{8}, 3\right)$.

## Example (10 minutes): Rotations in 3D

In this example, students explore rotations around one of the coordinate axes.

- Recall that the matrix $\left(\begin{array}{ccc}\cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1\end{array}\right)$ represents a rotation in 3-dimensional space about the $z$-axis. As $\theta$ varies, what path does $(10,10,10)$ trace out in 3 -dimensional space?
- The path is a circle whose center is $(0,0,10)$.
- Now turn your attention to the screen, that is, the plane where $y=1$. What is the path traced out by the projected image of $(10,10,10)$ as $\theta$ varies?
- The rotated image is the point $\left(\begin{array}{ccc}\cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1\end{array}\right)\left(\begin{array}{l}10 \\ 10 \\ 10\end{array}\right)=\left(\begin{array}{c}10 \cos \theta-10 \sin \theta \\ 10 \sin \theta+10 \cos \theta \\ 10\end{array}\right)$.
- To calculate the image of the point when it is projected onto the screen, we need the value of that causes $t(10 \sin \theta+10 \cos \theta)=1$, which is $t=\frac{1}{10(\sin \theta+\cos \theta)}$.
- Thus, the projected image is $\binom{\frac{\cos \theta-\sin \theta}{\sin \theta+\cos \theta}}{\frac{1}{\sin \theta+\cos \theta}}$. Using graphing software we get the following path, which appears to be a hyperbola:

- If instead we wanted to model a rotation of $(10,10,10)$ around the $y$-axis, what matrix should be used?
- We should use $\left(\begin{array}{ccc}\cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta\end{array}\right)$. This leaves the $y$-coordinate fixed and has the structure of a rotation matrix.
- As $\theta$ varies, let's project the points onto the plane $y=1$. Describe the path traced out.
- The rotated image is the point $\left(\begin{array}{ccc}\cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta\end{array}\right)\left(\begin{array}{l}10 \\ 10 \\ 10\end{array}\right)=\left(\begin{array}{c}10 \cos \theta-10 \sin \theta \\ 10 \\ 10 \sin \theta+10 \cos \theta\end{array}\right)$.
- To calculate the image of the point when it is projected onto the screen, we need the value of that causes $10 t=1$, which is $t=\frac{1}{10}$.
- Thus, the projected image is $\left(\begin{array}{c}\cos \theta-\sin \theta \\ 1 \\ \sin \theta+\cos \theta\end{array}\right)$. Using graphing software we get the following path, which is a circle. This circle is simply a smaller version of the circle traced out during the rotation:

- These examples demonstrate how programmers give viewers the impression that objects are turning on the screen.


## Closing (2 minutes)

- Use your journal to write a brief summary of what you learned in today's lesson.
- We learned how to project points in 3-D space onto a plane which represents a 2-D screen.
- We learned how to model a 3-D rotation using a 2-D screen.


## Exit Ticket (5 minutes)

Name
Date $\qquad$

## Lesson 26: Projecting a 3-D Object onto a 2-D Plane

## Exit Ticket

1. Consider the plane defined by $z=2$ and the points, $x=\left(\begin{array}{l}3 \\ 6 \\ 8\end{array}\right)$ and $y=\left(\begin{array}{c}2 \\ -3 \\ 5\end{array}\right)$.
a. Find the projections of $x$ and $y$ onto the plane $z=2$ if the eye is placed at the origin.
b. Consider $w=\left(\begin{array}{c}1 \\ 2 \\ -1\end{array}\right)$; does it make sense to find the projection of $w$ onto $z=2$ ? Explain.
2. Consider an object located at $\left(\begin{array}{l}3 \\ 4 \\ 0\end{array}\right)$ and rotating around the $z$-axis. At what $\theta$ value will the object be out of sight of the plane $y=1$ ?

## Exit Ticket Sample Solutions

1. Consider the plane defined by $z=2$ and the points, $x=\left(\begin{array}{l}3 \\ 6 \\ 8\end{array}\right)$ and $y=\left(\begin{array}{c}2 \\ -3 \\ 5\end{array}\right)$.
a. Find the projections of $x$ and $y$ onto the plane $z=2$ if the eye is placed at the origin.
$x=\left(\begin{array}{c}\frac{3}{4} \\ \frac{3}{2} \\ 2\end{array}\right), y=\left(\begin{array}{c}\frac{4}{5} \\ -\frac{6}{5} \\ 2\end{array}\right)$
b. Consider $w=\left(\begin{array}{c}1 \\ 2 \\ -1\end{array}\right)$; does it make sense to find the projection of $w$ onto $z=2$ ? Explain.

No, $t=-\frac{1}{2}$.
2. Consider an object located at $\left(\begin{array}{l}3 \\ 4 \\ 0\end{array}\right)$ and rotating around the $z$-axis. At what $\theta$ value will the object be out of sight of the plane $y=1$ ?

To rotate around the $z$-axis and project onto $y=1,\left(\begin{array}{ccc}\cos (\theta) & -\sin (\theta) & 0 \\ \sin (\theta) & \cos (\theta) & 0 \\ 0 & 0 & 1\end{array}\right)\left(\begin{array}{l}3 \\ 4 \\ 0\end{array}\right)=\left(\begin{array}{c}3 \cos (\theta)-4 \cos (\theta) \\ 3 \sin (\theta)+4 \cos (\theta) \\ 0\end{array}\right)$. To project onto $y=1, t=\frac{1}{3 \sin (\theta)+4 \cos (\theta)}$. The projection would be $\left(\begin{array}{c}\frac{3 \cos (\theta)-4 \cos (\theta)}{3 \sin (\theta)+4 \cos (\theta)} \\ 1 \\ 0\end{array}\right)$. The value of $\theta$ that would make the object out of site is when $\frac{3 \cos (\theta)-4 \cos (\theta)}{3 \sin (\theta)+4 \cos (\theta)}=0$. This occurs when $3 \cos (\theta)-4 \cos (\theta)=0$ or when $\theta=\tan ^{-1}\left(\frac{3}{4}\right)$.

## Problem Set Sample Solutions

1. A cube in 3-D space has vertices $\left(\begin{array}{l}10 \\ 10 \\ 10\end{array}\right),\left(\begin{array}{l}13 \\ 10 \\ 10\end{array}\right),\left(\begin{array}{l}10 \\ 13 \\ 10\end{array}\right),\left(\begin{array}{l}10 \\ 10 \\ 13\end{array}\right),\left(\begin{array}{l}13 \\ 13 \\ 10\end{array}\right),\left(\begin{array}{l}13 \\ 10 \\ 13\end{array}\right),\left(\begin{array}{l}10 \\ 13 \\ 13\end{array}\right),\left(\begin{array}{l}13 \\ 13 \\ 13\end{array}\right)$.
a. How do we know that these vertices trace a cube?

The edges between the vertices form right angles, and all have length of 3 units.
b. What is the volume of the cube?

$$
3^{3}=27 \text { cubic units }
$$

c. Let $z=1$. Find the eight points on the screen that represent the vertices of this cube (some may be obscured).
$\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right),\left(\begin{array}{c}1.3 \\ 1 \\ 1\end{array}\right),\left(\begin{array}{c}1 \\ 1.3 \\ 1\end{array}\right),\left(\begin{array}{c}\frac{1}{1.3} \\ \frac{1}{1.3} \\ 1\end{array}\right),\left(\begin{array}{c}1.3 \\ 1.3 \\ 1\end{array}\right),\left(\begin{array}{c}1 \\ \frac{1}{1.3} \\ 1\end{array}\right),\left(\begin{array}{c}1 \\ 1.3 \\ 1 \\ 1\end{array}\right),\left(\begin{array}{l}1 \\ 1 \\ 1\end{array}\right)$
d. What do you notice about your result in part (c)?

Some of the points went to the same position. These were points that lay along the same path from the camera. Some points were dilated by a factor of $\frac{1}{10}$ and some by a factor of $\frac{1}{13}$ depending on how far they were from the plane in the z -direction.
2. An object in 3-D space has vertices $\left(\begin{array}{l}1 \\ 5 \\ 0\end{array}\right),\left(\begin{array}{l}0 \\ 6 \\ 0\end{array}\right),\left(\begin{array}{l}0 \\ 5 \\ 1\end{array}\right),\left(\begin{array}{c}-1 \\ 5 \\ 0\end{array}\right),\left(\begin{array}{l}0 \\ 4 \\ 0\end{array}\right)$.
a. What kind of shape is formed by these vertices?

It appears this is a pyramid with a square base.
b. Let $\boldsymbol{y}=1$. Find the five points on the screen that represent the vertices of this shape.

$$
\left(\begin{array}{c}
0.2 \\
1 \\
0
\end{array}\right),\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right),\left(\begin{array}{c}
0 \\
1 \\
0.2
\end{array}\right),\left(\begin{array}{c}
-0.2 \\
1 \\
0
\end{array}\right),\left(\begin{array}{l}
0 \\
1 \\
0
\end{array}\right)
$$

3. Consider the shape formed by the vertices given in Problem 2.
a. Write a transformation matrix that will rotate each point around the $y$-axis $\theta$ degrees.

$$
\left(\begin{array}{ccc}
\cos (\theta) & 0 & -\sin (\theta) \\
0 & 1 & 0 \\
\sin (\theta) & 0 & \cos (\theta)
\end{array}\right)
$$

b. Project each rotated point onto the plane $y=1$ if $\theta=45^{\circ}$.

First we rotate each point: $\left(\begin{array}{c}\frac{\sqrt{2}}{2} \\ 5 \\ \frac{\sqrt{2}}{2}\end{array}\right),\left(\begin{array}{l}0 \\ 6 \\ 0\end{array}\right),\left(\begin{array}{c}-\frac{\sqrt{2}}{2} \\ 5 \\ \frac{\sqrt{2}}{2}\end{array}\right),\left(\begin{array}{c}-\frac{\sqrt{2}}{2} \\ 5 \\ -\frac{\sqrt{2}}{2}\end{array}\right),\left(\begin{array}{l}0 \\ 4 \\ 0\end{array}\right)$.
c. Is this the same as rotating the values you obtained in Problem 3 by $45^{\circ}$ ?

Yes. Since the projection is interpreted as a dilation, order does not matter.
4. In technical drawings, it is frequently important to preserve the scale of the objects being represented. In order to accomplish this, instead of a perspective projection, an orthographic projection is used. The idea behind the orthographic projection is that the points are translated at right angles to the screen (the word stem ortho-means straight or right). To project onto the $x y$-plane for instance, we can use the matrix $\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0\end{array}\right)$.
a. Project the cube in Problem 1 onto the $x y$-plane by finding the 8 points that correspond to the vertices.

$$
\binom{10}{10},\binom{13}{10},\binom{10}{13},\binom{10}{10},\binom{13}{13},\binom{13}{10},\binom{10}{13},\binom{13}{13}
$$

b. What do you notice about the vertices of the cube after projecting?

They form a square with all 8 points being mapped to 4 points.
c. What shape is visible on the screen?

It is a square with sides of length 3.
d. Is the area of the shape that is visible on the screen what you expected from the original cube? Explain.

The area of the visible shape is 9, which is the same as the area of any of the faces of the original cube.
e. Summarize your findings from parts (a)-(d).

Orthographic projections preserve the scale of objects, but much of the original information is lost.
f. State the orthographic projection matrices for the $x z$-plane and the $y z$-plane.

For $x \mathrm{Z}$-plane: $\left(\begin{array}{lll}1 & 0 & 0 \\ 0 & 0 & 1\end{array}\right)$
For $y z$-plane: $\left(\begin{array}{lll}0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right)$
g. In regard to the dimensions of the orthographic projection matrices, what causes the outputs to be two-dimensional?

The transformation matrices are $2 \times 3$, meaning they can operate on 3-dimensional points but generate 2-dimensional outputs.
5. Consider the point $A=\left(\begin{array}{l}a_{x} \\ a_{y} \\ a_{z}\end{array}\right)$ in the field of view from the origin through the plane $z=1$.
a. Find the projection of $A$ onto the plane $z=1$.

Since the point is in the field of view, we know that $a_{z} \neq 0$, and we get: $\left(\begin{array}{c}\frac{a_{x}}{a_{z}} \\ \frac{a_{y}}{a_{z}} \\ 1\end{array}\right)$.
b. Find a $3 \times 3$ matrix $P$ such that $P A$ finds the projection of $A$ onto the plane $z=1$.
$\left(\begin{array}{ccc}\frac{1}{a_{z}} & 0 & 0 \\ 0 & \frac{1}{a_{z}} & 0 \\ 0 & 0 & \frac{1}{a_{z}}\end{array}\right)$
c. How does the matrix change if instead of projecting onto $z=1$, we project onto $z=c$, for some real number $c \neq 0$ ?

The numerator of each fraction will be $c$ instead of 1 .
d. Find the scalars that will generate the image of $A$ onto the planes $x=c$ and $y=c$, assuming the image exists. Describe the scalars in words.
$\frac{c}{a_{x}}$ and $\frac{c}{a_{y}}$. Whenever we are projecting onto a plane parallel to the $x y$-, $x z$-, or $y z$-planes, the scalar by which we multiply the point $A$ is always going to be the reciprocal of $z$-, $y$-, or $x$-coordinate times the $z$-, $y$-, or $x$-location of the plane.

## Extension:

6. Instead of considering the rotation of a point about an axis, consider the rotation of the camera. Rotations of the camera will cause the screen to rotate along with it, so that to the viewer, the screen appears immobile.
a. If the camera rotates $\boldsymbol{\theta}_{x}$ around the $\boldsymbol{x}$-axis, how does the computer world appear to move?

A rotation of $-\theta_{x}$ about the $x$-axis.
b. State the rotation matrix we could use on a point $A$ to simulate rotating the camera and computer screen by $\theta_{x}$ about the $x$-axis but in fact keeping the camera and screen fixed.
$\left(\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos \left(-\theta_{x}\right) & -\sin \left(-\theta_{x}\right) \\ 0 & \sin \left(-\theta_{x}\right) & \cos \left(-\theta_{x}\right)\end{array}\right)$
c. If the camera rotates $\boldsymbol{\theta}_{\boldsymbol{y}}$ around the $\boldsymbol{y}$-axis, how does the computer world appear to move?

A rotation of $-\theta_{y}$ about the $y$-axis.
d. State the rotation matrix we could use on a point $A$ to simulate rotating the camera and computer screen by $\theta_{y}$ about the $\boldsymbol{y}$-axis but in fact keeping the camera and screen fixed.

$$
\left(\begin{array}{ccc}
\cos \left(-\theta_{y}\right) & 0 & -\sin \left(-\theta_{y}\right) \\
0 & 1 & 0 \\
\sin \left(-\theta_{y}\right) & 0 & \cos \left(-\theta_{y}\right)
\end{array}\right)
$$

e. If the camera rotates $\boldsymbol{\theta}_{z}$ around the $\boldsymbol{z}$-axis, how does the computer world appear to move?

A rotation of $-\theta_{z}$ about the $z$-axis.
f. State the rotation matrix we could use on a point $A$ to simulate rotating the camera and computer screen by $\boldsymbol{\theta}_{z}$ about the $z$-axis but in fact keeping the camera and screen fixed.
$\left(\begin{array}{ccc}\cos \left(-\theta_{z}\right) & -\sin \left(-\theta_{z}\right) & 0 \\ \sin \left(-\theta_{z}\right) & \cos \left(-\theta_{z}\right) & 0 \\ 0 & 0 & 1\end{array}\right)$
g. What matrix multiplication could represent the camera starting at a relative angle $\left(\boldsymbol{\theta}_{x}, \boldsymbol{\theta}_{y}, \boldsymbol{\theta}_{z}\right)$ ? Apply the transformations in the order $z-y-x$. Do not find the product.

$$
\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos \left(-\theta_{x}\right) & -\sin \left(-\theta_{x}\right) \\
0 & \sin \left(-\theta_{x}\right) & \cos \left(-\theta_{x}\right)
\end{array}\right)\left(\begin{array}{ccc}
\cos \left(-\theta_{y}\right) & 0 & -\sin \left(-\theta_{y}\right) \\
0 & 1 & 0 \\
\sin \left(-\theta_{y}\right) & 0 & \cos \left(-\theta_{y}\right)
\end{array}\right)\left(\begin{array}{ccc}
\cos \left(-\theta_{z}\right) & -\sin \left(-\theta_{z}\right) & 0 \\
\sin \left(-\theta_{z}\right) & \cos \left(-\theta_{z}\right) & 0 \\
0 & 0 & 1
\end{array}\right)
$$

## Lesson 27: Designing Your Own Game

## Student Outcomes

- Students will create a short animation in ALICE 3.1 using one-step procedures.
- Students will apply their understanding of the mathematics of projecting three-dimensional images onto twodimensional surfaces by writing matrices to represent the motions in the animations they create in ALICE.


## Lesson Notes

This is the last lesson in this topic and this module. Lesson 27 is a culminating lesson where students are able to see how their work with functions, matrices, vectors, and linear transformations all come together.

Prior to this lesson, the students should have become familiar with the ALICE program, having opportunities to explore it in previous homework assignments. The ALICE 3.1 program should be available to students during the lesson. Students will use it to create a brief scene including one-step turn and roll procedures. They will describe the motions of the ALICE characters with respect to the scene and also describe them as rotations about specific axes. Then students will apply the mathematics from the previous lesson to use matrices to represent the motions of the characters in their scenes.

If there is only one copy of ALICE available, this lesson could be completed by having the class create one collective animation. Selected students could choose the setting and characters. Then, volunteers could each apply a one-step roll or turn procedure, while the rest of the students determine a matrix operation that would appropriately represent the movement. The students could be prompted to create distinct procedures, so the movements represent rotations of various degrees and around various axes. The activities and procedures addressed in this lesson will provide opportunities for students to reason abstractly about the movements of the animated objects (MP.2) and to model the movements of objects using matrix operations (MP.4).

## Classwork

## Discussion (10 minutes): Reviewing Projections of 3D objects into 2D space

Through a teacher-led series of questions, the students will review what they have learned about projecting threedimensional objects into two-dimensional space, including using matrices to represent projected images and rotations of points. If technology is not available, the question addressing the ALICE program could be skipped.

- In a sentence, explain the function of the ALICE program to a person new to the program.
- It creates 3D animations using procedures to manipulate objects within the ALICE world.
- What is actually happening to create the illusion of three-dimensional objects in a two-dimensional space?
- The points composing the three-dimensional objects are projected onto a two-dimensional screen.
- And how is this projection formed?
- It is the intersection of the viewer's line of sight to the real object with the flat surface onto which it is projected.
- How can this type of projection be drawn by hand?
- Answers may vary but should include using a horizon line and vanishing points.
- And will all projections of a point look exactly the same?
- No, they will vary based on the placement of the horizon point and the vanishing points.
- We attempted this process in Lesson 25 by projecting a cube onto a flat screen. What challenges did we encounter?
- Answers may vary but could include difficulties in seeing the points of intersection between the line of sight and the screen prior to the addition of the horizon line and vanishing points.
- What might be some additional challenges we would face in trying to project more complex three-dimensional images onto a screen by hand?
- Answers may vary and could include the need for projecting numerous points to get an accurate projection, which could be time consuming and cumbersome when done by hand.
- How can mathematics be applied to expedite the process of projecting three-dimensional objects into twodimensional space?
- We can set the viewer's eye at the origin and write an equation from the eye to each point to be projected. The image point on the screen will be located on that line at the intersection with the screen.
- So if we have the point $A=\left[\begin{array}{l}a_{1} \\ a_{2} \\ a_{3}\end{array}\right]$, how can we represent the line through the viewer's eye and point $A$ parametrically?
- The equation of the line can be represented as $\left[\begin{array}{l}t a_{1} \\ t a_{2} \\ t a_{3}\end{array}\right]$, where $t$ is a real number.
- And what does this line represent?
- It is a dilation of the point $\left[\begin{array}{l}a_{1} \\ a_{2} \\ a_{3}\end{array}\right]$ about the origin.
- Let's say we are projecting our point $A$ onto a screen defined by the plane $y=1$. How could we find the coordinates of the point?
- Since $y=1$, we need to find a value $t$ such that $t a_{2}=1$. Therefore, $t=\frac{1}{a_{2}}$, and the image point is

$$
\left[\begin{array}{l}
t a_{1} \\
t a_{2} \\
t a_{3}
\end{array}\right]=\left[\begin{array}{c}
\frac{a_{1}}{a_{2}} \\
1 \\
\frac{a_{3}}{a_{2}}
\end{array}\right]
$$

- How would our projected point change if we chose the screen to be at $y=2$ ?
- We would need to find the value $t$ so that $t a_{2}=2$, but otherwise, the procedure would be the same.
- How would our projected point change if we chose the screen to be at $z=1$ ?
- We would need to find the value $t$ so that $t a_{3}=1$, but otherwise, the procedure would be the same.
- Let's say we have created a seafloor scene in ALICE. In the scene we placed a shark whose pivot point is represented by the coordinates $(3,5,0.2)$. What would be the coordinates of this point projected onto a screen at $y=1$ ? At $z=2$ ?
- For $y=1,5 t=1$, so $t=0.2$, and the projected image point is $(0.6,1.0,0.04)$.
- For $Z=2,0.2 t=2$, so $t=10$, and the projected image point is $(30,50,2)$.
- Now let's discuss rotating points in three-dimensional space. How can you use matrices to represent the rotation of a point about the $z$-axis through an angle $\theta$ ?

$$
\left[\begin{array}{ccc}
\cos (\theta) & -\sin (\theta) & 0 \\
\sin (\theta) & \cos (\theta) & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
a_{1} \\
a_{2} \\
a_{3}
\end{array}\right]
$$

- How can you use matrices to represent the rotation of a point about the $y$-axis through an angle $\theta$ ?

$$
\left[\begin{array}{ccc}
\cos (\theta) & 0 & -\sin (\theta) \\
0 & 1 & 0 \\
\sin (\theta) & 0 & \cos (\theta)
\end{array}\right]\left[\begin{array}{l}
a_{1} \\
a_{2} \\
a_{3}
\end{array}\right]
$$

- And a rotation about the $x$-axis through an angle $\theta$ ?

$$
\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \cos (\theta) & -\sin (\theta) \\
0 & \sin (\theta) & \cos (\theta)
\end{array}\right]\left[\begin{array}{l}
a_{1} \\
a_{2} \\
a_{3}
\end{array}\right]
$$

- And what are the coordinates of point $A$ rotated about the $x$-axis?

$$
\left[\begin{array}{c}
a_{1} \\
a_{2} \cos (\theta)-a_{3} \sin (\theta) \\
a_{2} \sin (\theta)+a_{3} \cos (\theta)
\end{array}\right]
$$

- How would we project these points onto the plane $y=1$ ?
- Multiply the rotation matrix by $t$ so that $t\left(a_{2} \cos (\theta)-a_{3} \sin (\theta)\right)=1$.
- And what would that matrix look like?
- If $t\left(a_{2} \cos (\theta)-a_{3} \sin (\theta)\right)=1$, then $t=\frac{1}{a_{2} \cos (\theta)-a_{3} \sin (\theta)}$, and the projected points are represented by

$$
\left[\begin{array}{c}
\frac{a_{1}}{a_{2} \cos (\theta)-a_{3} \sin (\theta)} \\
1 \\
\frac{a_{2} \sin (\theta)+a_{3} \cos (\theta)}{a_{2} \cos (\theta)-a_{3} \sin (\theta)}
\end{array}\right]
$$

## Example 1 (10 minutes)

Open the ALICE program, and demonstrate how it works by completing Example 1. As students observe the setup of a simple scene with one character and the manipulation of the character within the ALICE world, they will be reminded of how to create and manipulate objects in ALICE, which should help them as they create their own scene with a partner in the exercises that follow. While presenting the example, use matrix operations to represent the motions of characters. If technology is not available, the students could work in pairs or small groups to create an idea for a 3D animation. They could create a scene with multiple characters and determine a few motions that each character would make in their scene. These motions could be described using captioned sketches, or the students could create paper characters, which could be manipulated to demonstrate their motions. Parameters for the motions of the characters could be set (e.g., between 2-4 characters per scene; 1-2 movements per character; at least two rotations around different axes must be represented). The scenes could be presented, and students who are not presenting their scene could use matrix operations to represent the movements of the characters (presentations discussed in the next lesson section).

## Scaffolding:

Advanced students could represent multiple movements using matrix operations, such as multiple rotations or a rotation followed by a translation.

- Once we have selected the dolphin, I am going to move it left and right. Look at the coordinates of the dolphin on the right side of the screen. What coordinate is changing, and how is it changing?
- The $x$-coordinate changes when the dolphin moves left and right. Moving to the left increases the value of $x$, and moving to the right decreases it.
- So our positive $x$-axis is to the left from the viewer's vantage point when the dolphin is facing with its nose towards the viewer. Now, if I click on the arrow pointing up, I can move the dolphin up and down. What coordinate is changing now, and how is it changing?
- The y-coordinate changes when the dolphin moves up and down, with the up direction increasing the value of $y$.
- So our $y$-axis is vertical when the dolphin is facing forward with its back to the sky. How about when we move the dolphin forward and backward?
- The z-coordinate changes. It increases as the dolphin moves back.
- Now we have moved the dolphin so its pivot point is at the origin. How can we represent translations of the dolphin using matrix operations?
- Answers should address adding the matrix $\left[\begin{array}{l}a \\ b \\ c\end{array}\right]$ to $\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]$, where a represents the number of units the dolphin is translated to the left, $b$ represents the number of units the dolphin is translated up, and $c$ represents the number of units the dolphin is translated backward.
- Now we have selected the rotation button and selected the dolphin. What does the dolphin appear to do when we rotate it about the $x$-axis?
- A forward or backward somersault.
- And the $y$-axis?
- The dolphin turns in the water.
- And the $z$-axis?
- The dolphin rolls on its side.
- And how can we generalize the rotations of the dolphin about the axes using matrices?
- Rotation about $x$-axis:
$\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos (c) & -\sin (c) \\ 0 & \sin (c) & \cos (c)\end{array}\right]$ where $c$ is the degrees of rotation counterclockwise about the axis
- Rotation about y-axis:
$\left[\begin{array}{ccc}\cos (c) & 0 & -\sin (c) \\ 0 & 1 & 0 \\ \sin (c) & 0 & \cos (c)\end{array}\right]$ where $c$ is the degrees of rotation counterclockwise about the axis
- Rotation about z-axis:
$\left[\begin{array}{ccc}\cos (c) & -\sin (c) & 0 \\ \sin (c) & \cos (c) & 0 \\ 0 & 0 & 1\end{array}\right]$ where $c$ is the degrees of rotation counterclockwise about the axis
- Now we have selected the procedure: this.dolphin roll RIGHT 0.25 as seen by this duration 2.0

BEGIN_AND_END_ABRUPTLY. What happened when we ran the program?

- The dolphin does a quarter backwards somersault out of the water in a counterclockwise rotation about the negative $z$-axis.
- How does this make sense based on the procedure we created?
- Answers will vary but should address that in the default orientation, a roll to its right would appear to the viewer as a roll onto its left side, which is a counterclockwise motion.
- Now you will get to use what we have reviewed together to create your own animation and to describe the motions using matrices.


## Example 1

a. Select the Sea Surface as your background scene, and select the setup scene mode. From the Gallery By Class Hierarchy, select Swimmer Classes, then Marine Mammals, and a Dolphin. Name your dolphin. Describe what you see.

We have a scene with an ocean surface and a dolphin facing forward in the center.
b. Click on the dolphin, and from the handle style buttons in the top right corner of the screen, select translation. By selecting the arrows on the dolphin, we can move it to different locations in the screen. Move the dolphin left and right, up and down, forward and backward. Then, move it so that its coordinates are ( $\mathbf{0}, \mathbf{0}, \mathbf{0}$ ).
c. Use a matrix to describe each of the movements of the dolphin from its location at the origin.
i. Move 2 units right.

$$
\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]+\left[\begin{array}{c}
-2 \\
0 \\
0
\end{array}\right]
$$

ii. Move 4 units down.

$$
\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]+\left[\begin{array}{c}
0 \\
-4 \\
0
\end{array}\right]
$$

iii. Move 3 units forward.

$$
\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]+\left[\begin{array}{c}
0 \\
0 \\
-3
\end{array}\right]
$$

d. Click on the rotation button from the handle style buttons, and practice rotating the dolphin about the three axes through its center. Use a matrix to represent the motion of the dolphin described. Assume the center of rotation of the dolphin is at the origin.
i. Rotation counterclockwise one full turn about the $z$-axis

$$
\left[\begin{array}{ccc}
\cos (360) & -\sin (360) & 0 \\
\sin (360) & \cos (360) & 0 \\
0 & 0 & 1
\end{array}\right]=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]
$$

ii. Rotation counterclockwise one half turn about the $x$-axis
$\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos (180) & -\sin (180) \\ 0 & \sin (180) & \cos (180)\end{array}\right]=\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1\end{array}\right]$

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| :--- | :--- |
| Date: | $1 / 24 / 15$ |

e. Select Edit Code from the screen. Drag and drop this.dolphinturn from the procedures menu and drop it into the declare procedures region on the right. Select from the drop-down menus: this.dolphin turn LEFT 0.25 as seen by this duration 2.0 BEGIN_AND_END_ABRUPTLY. Then run the program. Describe what you see. Represent the motion using a matrix.

The dolphin undergoes a quarter counterclockwise rotation in a horizontal circle about the $y$-axis.
$\left[\begin{array}{ccc}\cos (90) & 0 & -\sin (90) \\ 0 & 1 & 0 \\ \sin (90) & 0 & \cos (90)\end{array}\right]=\left[\begin{array}{ccc}0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0\end{array}\right]$
f. Drag and drop this.dolphinroll from the procedures menu, and drop it into the declare procedures region on the right beneath the turn procedure. Select from the drop-down menus: this.dolphin roll RIGHT 3.0 as seen by this duration 2.0 BEGIN_AND_END_ABRUPTLY. Then run the program. Describe what you see. Represent the motion using a matrix.

The dolphin performs three backwards counterclockwise somersaults in the vertical plane, about the negative z-axis.
$\left[\begin{array}{ccc}\cos (1080) & -\sin (1080) & 0 \\ \sin (1080) & \cos (1080) & 0 \\ 0 & 0 & 1\end{array}\right]=\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$

## Exercises (15 minutes)

Give students time to work on the problems below in ALICE. If it is possible, the students could complete the tasks in pairs. Once they have completed the exercises, they can consult with another pair to verify their solutions. Select students to present their scenes and solutions to the class as time permits; for example, perhaps each pair could demonstrate one procedure while the pair with whom they partnered displays the matrix that represents the motion in the 3D ALICE world, as well as on a projected screen. Some pairs of students may need some one-on-one assistance either setting up the scene or representing the projecting points using matrices. When students do pivots, have them assume the pivot point is at the origin. If technology is not available, continue the activity begun earlier in the lesson, where students present their ideas for animations, including descriptions or demonstrations of character movements. Have the students who are not presenting represent the movements of each character using matrix operations. These students could share their matrices and justify their responses, while other students could provide supporting or refuting evidence.

## Exercises

1. Open ALICE 3.1. Select a background and 3 characters to create a scene. Describe the scene, including the coordinates of the pivot point for each character and the direction each character is facing.

Scene description: Answers will vary. I created a sea surface scene that contains an adult walrus facing forward, a baby walrus to the left of the adult and facing it (looking to the right), and a dolphin breaching on its back, with its nose pointed up and to the right. The dolphin is behind and to the left of the walruses.

Coordinates of the pivot point for each character:
Answers will vary. For example, dolphin (4.16, 0.5,4.1); baby walrus (0.6, 0. 11, 0.55); adult walrus $(-0.46,-0.03,1.11)$.
2.
a. Describe the location of a plane $x=5$ in your scene from the perspective of the viewer.

The plane $x=5$ is a vertical plane 5 units to the left of the center of the screen.
b. Determine the coordinates of the pivot points for each character if they were projected onto the plane $x=5$.

Answers will vary. An example of an acceptable response is shown.
For the dolphin, $4.16 t=5$, so $t \approx 1.20$ and image $=\left[\begin{array}{c}5 \\ 6 \\ 4.92\end{array}\right]$.
For the baby walrus, $0.6 t=5$, so $t=\frac{25}{3}$ and image $=\left[\begin{array}{c}5 \\ 0.92 \\ 4.58\end{array}\right]$.
For the adult walrus, $-0.46 t=5$, so $t \approx-10.87$ and image $=\left[\begin{array}{c}5 \\ 0.33 \\ -12.07\end{array}\right]$.
3. Create a short scene that includes a one-step turn or roll procedure for each of the characters. The procedures should be unique. Write down the procedures in the space provided. After each procedure, describe what the character did in the context of the scene. Then describe the character's motion using transformational language. Finally, represent each procedure using matrix operations.

Answers will vary. An example of an acceptable response is shown.
Procedure 1-adult walrus: this.adult walrus TURN forward 4.0 as seen by this duration 2.0.
Character motion (in scene): The adult walrus performed four forward somersaults toward the foreground.
Transformational motion: The adult walrus performed a counterclockwise rotation of four full turns about the $x$-axis.
Matrix representation of motion: $\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos (1440) & -\sin (1440) \\ 0 & \sin (1440) & \cos (1140)\end{array}\right]$ for any point $A$ on the walrus translated to be centered around the origin.

Procedure 2-baby walrus: this.babywalrus TURN left 2.0 as seen by this duration 2.0.
Character motion (in scene): The baby walrus spun twice in a counterclockwise circle next to the adult.
Transformational motion: The baby walrus rotated about the $y$-axis counterclockwise.
Matrix representation of motion: $\left[\begin{array}{ccc}\cos (720) & 0 & -\sin (720) \\ 0 & 1 & 0 \\ \sin (720) & 0 & \cos (720)\end{array}\right] B$ for any point $B$ on the walrus translated to be centered around the origin.

Procedure 3-Dolphin: this.dolphin ROLL left 4.0 as seen by this duration 2.0.
Character motion (in scene): The dolphin performed four back somersaults out of the water.
Transformational motion: The dolphin rotated counterclockwise about the z-axis.
Matrix representation of motion:
$\left[\begin{array}{ccc}\cos (720) & -\sin (720) & 0 \\ \sin (720) & \cos (720) & 0 \\ 0 & 0 & 1\end{array}\right] C$ for any point $C$ on the dolphin translated to be centered around the origin.

## Closing ( 5 minutes)

Have students respond in writing to the prompt. After a few minutes, several students should be selected to share their thoughts. If technology is not available, the students can respond to the prompt with respect to rotations of 3D objects in any 3D animation situation.

- Make a generalization about how one-step turns or rolls in ALICE can be represented using matrix operations. Use an example from your animation to support your response.
- Answers will vary. An example of an acceptable response is included:

One-step turns and rolls can be represented as rotations of a given point on the character once it has been translated to the origin. This rotation can be represented using a rotation matrix applied to the translated point $A$ about the axis of rotation for a rotation of $\theta$ degrees. For example, the adult walrus performed a rotation about the $x$-axis, which can be represented as
$\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos (1440) & -\sin (1440) \\ 0 & \sin (1440) & \cos (1140)\end{array}\right] A$

## Exit Ticket (5 minutes)

Name $\qquad$ Date $\qquad$

## Lesson 27: Designing Your Own Game

## Exit Ticket

Consider the following set of code for the ALICE program featuring a bluebird whose center is located at $\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]$.
this.bluebird resize 2.0
this.bluebird resizeHeight 0.5
this.bluebird turn RIGHT 0.25
this.bluebird move FORWARD 1.0
For an arbitrary point $x$ on this bluebird, write the four matrices that represent the code above, and state where the point ends after the program runs.

## Exit Ticket Sample Solutions

Consider the following set of code for the ALICE program featuring a bluebird whose center is located at $\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]$.
this.bluebird resize 2.0
this.bluebird resizeHeight 0.5
this.bluebird turn RIGHT 0.25
this.bluebird move FORWARD 1.0
For an arbitrary point $x$ on this bluebird, write the four matrices that represent the code above, and state where the point ends after the program runs.

Resize 2.0: $\left[\begin{array}{lll}2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2\end{array}\right]$
ResizeHeight 0.5: $\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1\end{array}\right]$
Turn RIGHT 0.25: turning right is a rotation about the $y$-axis clockwise, so we get

$$
\left[\begin{array}{ccc}
\cos (-90) & 0 & -\sin (-90) \\
0 & 1 & 0 \\
\sin (-90) & 0 & \cos (90)
\end{array}\right]=\left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{array}\right]
$$

Move forward is a translation by 1 unit in the positive $x$ direction after the turn right. There is no linear transformation we can use, but we can write this as $x^{\prime}+\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right]$.
$x$ would end at

$$
\begin{aligned}
& {\left[\begin{array}{ccc}
0 & 0 & 1 \\
0 & 1 & 0 \\
-1 & 0 & 0
\end{array}\right]\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0.5 & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{lll}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{array}\right] x+\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]} \\
& {\left[\begin{array}{ccc}
0 & 0 & 2 \\
0 & 1 & 0 \\
-2 & 0 & 0
\end{array}\right] x+\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right]}
\end{aligned}
$$

If $x=\left[\begin{array}{l}a \\ b \\ c\end{array}\right]$, then the image of $x$ at the end of the program would be $\left[\begin{array}{c}2 c+1 \\ b \\ -2 a\end{array}\right]$.

## Problem Set Sample Solutions

1. For the following commands, describe a matrix you can use to get the desired result. Assume the character is centered at the origin, facing in the negative $z$ direction with positive $x$ on its right for each command. Let $c$ be a nonzero real number.
a. move LEFT $\boldsymbol{c}$
$\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]+\left[\begin{array}{c}-c \\ 0 \\ 0\end{array}\right]$
b. move RIGHT c

$$
\left[\begin{array}{l}
0 \\
0 \\
0
\end{array}\right]+\left[\begin{array}{l}
c \\
0 \\
0
\end{array}\right]
$$

c. move UP c
$\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]+\left[\begin{array}{l}0 \\ c \\ 0\end{array}\right]$
d. move DOWN $c$
$\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]+\left[\begin{array}{c}0 \\ -c \\ 0\end{array}\right]$
e. move FORWARD $c$
$\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]+\left[\begin{array}{c}0 \\ 0 \\ -c\end{array}\right]$
f. move BACKWARD $c$
$\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]+\left[\begin{array}{l}0 \\ 0 \\ c\end{array}\right]$
g. turn LEFT $\boldsymbol{c}$
$\left[\begin{array}{ccc}\cos (360 c) & 0 & -\sin (360 c) \\ 0 & 1 & 0 \\ \sin (360 c) & 0 & \cos (360 c)\end{array}\right]$
h. turn RIGHT c
$\left[\begin{array}{ccc}\cos (-360 c) & 0 & -\sin (-360 c) \\ 0 & 1 & 0 \\ \sin (-360 c) & 0 & \cos (-360 c)\end{array}\right]$
i. turn FORWARD c
$\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos (-360 c) & -\sin (-360 c) \\ 0 & \sin (-360 c) & \cos (-360 c)\end{array}\right]$
j. turn BACKWARD $c$
$\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos (360 c) & -\sin (360 c) \\ 0 & \sin (360 c) & \cos (360 c)\end{array}\right]$
k. roll LEFT $c$
$\left[\begin{array}{ccc}\cos (-360 c) & -\sin (-360 c) & 0 \\ \sin (-360 c) & \cos (-360 c) & 0 \\ 0 & 0 & 1\end{array}\right]$
I. roll RIGHT c
$\left[\begin{array}{ccc}\cos (360 c) & -\sin (360 c) & 0 \\ \sin (360 c) & \cos (360 c) & 0 \\ 0 & 0 & 1\end{array}\right]$
m. resize $c$
$\left[\begin{array}{lll}c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c\end{array}\right]$
n. resizeWidth $c$
$\left[\begin{array}{lll}c & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{array}\right]$
o. resizeHEIGHT c
$\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1\end{array}\right]$
p. resizeDEPTH $c$
$\left[\begin{array}{lll}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c\end{array}\right]$
2. In each of the transformations above, we have assumed that each animation will take one second of time. If $T$ is the number of seconds an animation takes and $t$ is the current running time of the animation, then rewrite the following commands as a function of $t$.
a. move RIGHT $\boldsymbol{c}$ duration $\boldsymbol{T}$
$\left[\begin{array}{l}0 \\ 0 \\ 0\end{array}\right]+\left[\begin{array}{l}c \\ 0 \\ 0\end{array}\right] \frac{t}{T}$
b. turn FORWARD $\boldsymbol{c}$ duration $\boldsymbol{T}$
$\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos (-360 c) & -\sin (-360 c) \\ 0 & \sin (-360 c) & \cos (-360 c)\end{array}\right] \frac{t}{T}$
c. resize $\boldsymbol{c}$ duration $\boldsymbol{T}$
$\left[\begin{array}{lll}c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c\end{array}\right] \frac{t}{T}$
3. For computational simplicity, we have been assuming that the pivot points of our characters occur at the origin.
a. If we apply a rotation matrix like we have been when the pivot point is at the origin, what will happen to characters that are not located around the origin?

The characters will trace out a circular path much like we saw in previous lessons when we were rotating points instead of objects.
b. Let $x=\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ be the pivot point of any three-dimensional object and $A=\left[\begin{array}{l}a \\ b \\ c\end{array}\right]$ a point on the surface of the object. Moving the pivot point to the origin has what effect on $A$ ? Find $A^{\prime}$, the image of $A$ after moving the object so that its pivot point is the origin.

$$
\begin{aligned}
A^{\prime} & =A-x \\
& =\left[\begin{array}{l}
a \\
b \\
c
\end{array}\right]-\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right] \\
& =\left[\begin{array}{l}
a-x \\
b-y \\
c-z
\end{array}\right]
\end{aligned}
$$

c. Apply a rotation of $\boldsymbol{\theta}$ about the $\boldsymbol{x}$-axis to $\boldsymbol{A}^{\prime}$. Does this transformation cause a pivot or what you described in part (a)?
$\left[\begin{array}{ccc}1 & 0 & 0 \\ 0 & \cos (\theta) & -\sin (\theta) \\ 0 & \sin (\theta) & \cos (\theta)\end{array}\right]\left[\begin{array}{l}a-x \\ b-y \\ c-z\end{array}\right]=\left[\begin{array}{c}a-x \\ \cos (\theta)(b-y)-\sin (\theta)(c-z) \\ \sin (\theta)(b-y)+\cos (\theta)(c-z)\end{array}\right]$
Since the pivot point has been shifted to the origin, the rotation rotates the object about itself instead of through space.
d. After applying the rotation, translate the object so that its pivot point returns to $x=\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$. Was the $x$-coordinate affected by the rotation? Was the pivot point?
$\left[\begin{array}{c}a-x \\ \cos (\theta)(b-y)-\sin (\theta)(c-z) \\ \sin (\theta)(b-y)+\cos (\theta)(c-z)\end{array}\right]+\left[\begin{array}{c}x \\ y \\ z\end{array}\right]=\left[\begin{array}{c}a \\ \cos (\theta)(b-y)-\sin (\theta)(c-z)+y \\ \sin (\theta)(b-y)+\cos (\theta)(c-z)+z\end{array}\right]$
No, the $x$-coordinate stayed the same once we shifted it back to the starting point. The pivot point would also have stayed the same since we shifted it to the origin, the origin does not change during rotations, and then shifted it back in the opposite direction.
e. Summarize what you found in (a)-(d).

To pivot points that are not centered on the origin, we have to translate the object so that its pivot point is at the origin, then perform the rotation, and then shift the object back. If $R$ is the rotation matrix, $A$ is the point on the surface of the object, and $x$ is the pivot point, then we do the following:

$$
R(A-x)+x
$$

for each point $A$ on the surface of the object.

## Extension:

4. In first-person computer games, we think of the camera moving left-right, forward-backward, and up-down. For computational simplicity, the camera and screen stays fixed and the objects in the game world move in the opposite direction instead. Let the camera be located at $(0,0,0)$, the screen be located at $z=1$, and the point $v=\left[\begin{array}{c}10 \\ 6 \\ 5\end{array}\right]$ represent the center of an object in the game world. If a character in ALICE is the camera, then answer the following questions.
a. What are the coordinates of the projection of $v$ on the screen?

$$
v^{\prime}=\left[\begin{array}{c}
2 \\
1.2 \\
1
\end{array}\right]
$$

b. What is the value of the image of $v$ as the character moves 4 units closer to $v$ in the $x$ direction?
$v^{\prime}=\left[\begin{array}{l}6 \\ 6 \\ 5\end{array}\right]$
c. What are the coordinates of the projection of $v^{\prime}$ ?
$v^{\prime}=\left[\begin{array}{c}1.2 \\ 1.2 \\ 1\end{array}\right]$
d. If the character jumps up 6 units, then where does the image of $v$ move on the screen?
$v$ moves to the apparent position $\left[\begin{array}{c}10 \\ 0 \\ 5\end{array}\right]$, which appears to be located at $\left[\begin{array}{l}2 \\ 0 \\ 1\end{array}\right]$ on the screen.
e. In Lesson 26, you learned that if the camera is not in the standard orientation, that rotation matrices need to be applied to the camera first. In Lesson 27, you learned that rotation matrices only pivot an object if that object is located at the origin. If a camera is in a non-standard orientation and not located at the origin, then should the rotation matrices be applied first or a translation to the origin?

The translation to the origin needs to take place before the rotation or else the camera will be tracing out a circular path around the axes instead of rotating to the correct position.

| Lesson 27: | Designing Your Own Game |
| :--- | :--- |
| Date: | $1 / 24 / 15$ |

Name $\qquad$ Date $\qquad$
1.
a. Find values for $a, b, c, d$, and $e$ so that the following matrix product equals the $3 \times 3$ identity matrix. Explain how you obtained these values.

$$
\left[\begin{array}{ccc}
a & -3 & 5 \\
c & c & 1 \\
5 & b & -4
\end{array}\right]\left[\begin{array}{lll}
1 & b & d \\
1 & c & e \\
2 & b & b
\end{array}\right]
$$

b. Represent the following system of linear equations as a single matrix equation of the form $A x=b$, where $A$ is a $3 \times 3$ matrix, and $x$ and $b$ are $3 \times 1$ column matrices.

$$
\begin{aligned}
& x+3 y+2 z=8 \\
& x-y+z=-2 \\
& 2 x+3 y+3 z=7
\end{aligned}
$$

c. Solve the system of three linear equations given in part (b).
2. The following diagram shows two two-dimensional vectors $\mathbf{v}$ and $\mathbf{w}$ in the place positioned to both have endpoint at point $P$.

a. On the diagram, make reasonably accurate sketches of the following vectors, again each with endpoint at $P$. Be sure to label your vectors on the diagram.
i. $2 \mathbf{v}$
ii. $-\mathbf{w}$
iii. $v+3 w$
iv. $w-2 v$
v. $\frac{1}{2} \mathbf{v}$

Vector $\mathbf{v}$ has magnitude 5 units, $\mathbf{w}$ has magnitude 3, and the acute angle between them is $45^{\circ}$.
b. What is the magnitude of the scalar multiple -5 v ?
c. What is the measure of the smallest angle between $-5 v$ and $3 w$ if these two vectors are placed to have a common endpoint?
3. Consider the two-dimensional vectors $\mathbf{v}=\langle 2,3\rangle$ and $\mathbf{w}=\langle-2,-1\rangle$.
a. What are the components of each of the vectors $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ ?
b. On the following diagram, draw representatives of each of the vectors $\mathbf{v}, \mathbf{w}$, and $\mathbf{v}+\mathbf{w}$, each with endpoint at the origin.

c. The representatives for the vectors $\mathbf{v}$ and $\mathbf{w}$ you drew form two sides of a parallelogram, with the vector $\mathbf{v}+\mathbf{w}$ corresponding to one diagonal of the parallelogram. What vector, directed from the third quadrant to the first quadrant, is represented by the other diagonal of the parallelogram? Express your answer solely in terms of $\mathbf{v}$ and $\mathbf{w}$, and also give the coordinates of this vector.
d. Show that the magnitude of the vector $\mathbf{v}+\mathbf{w}$ does not equal the sum of the magnitudes of $\mathbf{v}$ and of w.
e. Give an example of a non-zero vector $\mathbf{u}$ such that $\|\mathbf{v}+\mathbf{u}\|$ does equal $\|\mathbf{v}\|+\|\mathbf{u}\|$.
f. Which of the following three vectors has the greatest magnitude: $\mathbf{v}+(-\mathbf{w}), \mathbf{w}-\mathbf{v}$, or $(-\mathbf{v})-(-\mathbf{w})$ ?
g. Give the components of a vector one-quarter the magnitude of vector $\mathbf{v}$ and with direction opposite the direction of $\mathbf{v}$.
4. Vector a points true north and has magnitude 7 units. Vector boints $30^{\circ}$ east of true north. What should the magnitude of $\mathbf{b}$ be so that $\mathbf{b}$ - $\mathbf{a}$ points directly east?

State the magnitude and direction of $\mathbf{b}-\mathbf{a}$.
5. Consider the three points $A=(10,-3,5), B=(0,2,4)$, and $C=(2,1,0)$ in three-dimensional space. Let $M$ be the midpoint of $\overline{A B}$ and $N$ be the midpoint of $\overline{A C}$.
a. Write down the components of the three vectors $\overrightarrow{A B}, \overrightarrow{B C}$, and $\overrightarrow{C A}$, and verify through arithmetic that their sum is zero. Also, explain why geometrically we expect this to be the case.
b. Write down the components of the vector $\overrightarrow{M N}$. Show that it is parallel to the vector $\overrightarrow{B C}$ and half its magnitude.

Let $G=(4,0,3)$.
c. What is the value of the ratio $\frac{\|\overrightarrow{M G}\|}{\|\overrightarrow{M C}\|}$ ?
d. Show that the point $G$ lies on the line connecting $M$ and $C$. Show that $G$ also lies on the line connecting $N$ and $B$.
6. A section of a river, with parallel banks 95 ft . apart, runs true north with a current of $2 \mathrm{ft} / \mathrm{sec}$. Lashana, an expert swimmer, wishes to swim from point $A$ on the west bank to the point $B$ directly opposite it. In still water she swims at an average speed of $3 \mathrm{ft} / \mathrm{sec}$.

The diagram to the right illustrates the situation.
To counteract the current, Lashana realizes that she is to swim at some angle $\theta$ to the east/west direction as shown.

With the simplifying assumptions that Lashana's swimming speed will be a constant $3 \mathrm{ft} / \mathrm{sec}$ and that the current of the water is a uniform $2 \mathrm{ft} / \mathrm{sec}$ flow northward throughout all regions of the river (we will ignore the effects of drag at the river banks, for example), at what angle $\theta$ to east/west direction should Lashana swim in order to reach the opposite bank precisely at point $B$ ? How long will her swim take?

a. What is the shape of Lashana's swimming path according to an observer standing on the bank watching her swim? Explain your answer in terms of vectors.
b. If the current near the banks of the river is significantly less than $2 \mathrm{ft} / \mathrm{sec}$, and Lashana swims at a constant speed of $3 \mathrm{ft} / \mathrm{sec}$ at the constant angle $\theta$ to the east/west direction as calculated in part (a), will Lashana reach a point different from $B$ on the opposite bank? If so, will she land just north or just south of $B$ ? Explain your answer.
7. A 5 kg ball experiences a force due to gravity $\vec{F}$ of magnitude 49 Newtons directed vertically downwards. If this ball is placed on a ramped tilted at an angle of $45^{\circ}$, what is the magnitude of the component of this force, in Newtons, on the ball directed $45^{\circ}$ towards the bottom of the ramp? (Assume the ball is of sufficiently small radius that is reasonable to assume that all forces are acting at the point of contact of the ball with the ramp.)

8. Let $A$ be the point $(1,1,-3)$ and $B$ be the point $(-2,1,-1)$ in three-dimensional space.

A particle moves along the straight line through $A$ and $B$ at uniform speed in such a way that at time $t=0$ seconds the particle is at $A$, and at $t=1$ second the particle is at $B$. Let $P(t)$ be the location of the particle at time $t$ (so, $P(0)=A$ and $P(1)=B$ ).
a. Find the coordinates of the point $P(t)$ each in terms of $t$.
b. Give a geometric interpretation of the point $P(0.5)$.

Let $L$ be the linear transformation represented by the $3 \times 3$ matrix $\left[\begin{array}{lll}2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 1\end{array}\right]$, and let $A^{\prime}=L A$ and $B^{\prime}=L B$ be the images of the points $A$ and $B$, respectively, under $L$.
c. Find the coordinates of $A^{\prime}$ and $B^{\prime}$.

A second particle moves through three-dimensional space. Its position at time $t$ is given by $L(P(t))$, the image of the location of the first particle under the transformation $L$.
d. Where is the second particle at times $t=0$ and $t=1$ ? Briefly explain your reasoning.
e. Prove that second the particle is also moving along a straight line path at uniform speed.

A Progression Toward Mastery

| Assessment Task Item |  | STEP 1 <br> Missing or incorrect answer and little evidence of reasoning or application of mathematics to solve the problem. | STEP 2 <br> Missing or incorrect answer but evidence of some reasoning or application of mathematics to solve the problem. | STEP 3 <br> A correct answer with some evidence of reasoning or application of mathematics to solve the problem, or an incorrect answer with substantial evidence of solid reasoning or application of mathematics to solve the problem. | STEP 4 <br> A correct answer supported by substantial evidence of solid reasoning or application of mathematics to solve the problem. |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | a $\text { A-REI.C. } 9$ | Student shows little or no understanding of matrix operations. | Student attempts to multiply matrices, making major mistakes, but equates terms to the identity matrix. | Student calculates the correct values for $a, b, c, d$, and $e$ but does not explain or does not clearly explain how values were obtained. | Student calculates the correct values for $a, b, c, d$, and $e$ and clearly explains how values were obtained.. |
|  | b <br> A-REI.C. 8 | Student shows little or no understanding of matrices. | Students writes one matrix ( $A, x$, or $b$ ) correctly. | Student writes two matrices ( $A, x$, or $b$ ) correctly. | Student writes all three matrices correctly. |
|  | $\begin{gathered} \text { c } \\ \text { A-REI.C. } 9 \end{gathered}$ | Student shows little or no understanding of matrices. | Student multiplies by the inverse matrix but makes mistakes in computations leading to only one correct value of $x, y$, or $z$. | Student multiplies by the inverse matrix but makes mistakes in computations leading to two correct values of $x, y$, or $z$. | Student multiplies by the inverse matrix arriving at the correct values of $x, y$, and $z$. |
| 2 | a <br> N-VM.A. 1 <br> N-VM.B.4a <br> N-VM.B.4c <br> N-VM.B. 5 | Student shows little or no evidence of vectors. | Student shows some knowledge of vectors drawing at least two correctly. | Student show reasonable knowledge of vectors drawing at least four correctly. | Student graphs and labels all vectors correctly. |


|  | b $\begin{aligned} & \text { N-VM.A. } 1 \\ & \text { N-VM.B.5b } \end{aligned}$ | Student show little or no understanding of the magnitude of a vector. | Student understands that the magnitude is 5 times the magnitude of the vector $\mathbf{v}$ but calculates the magnitude of $\mathbf{v}$ incorrectly. | Student calculates the magnitude of $\mathbf{v}$ correctly but does not multiply by 5 . | Student calculates the magnitude of $-5 \mathbf{v}$ correctly. |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | C <br> N-VM.A. 1 | Student shows little or no understanding of vectors and angles. | Student draws vectors with a common endpoint but does no calculations. | Student draws vectors with a common endpoint, but the angle identified is $45^{\circ}$. | Student draws vectors with a common endpoint and identifies the angle between them as $135^{\circ}$. |
| 3 | a $\begin{aligned} & \mathrm{N}-\mathrm{VM} . \mathrm{B} .4 \mathrm{a} \\ & \mathrm{~N}-\mathrm{VM} . \mathrm{B} .4 \mathrm{c} \end{aligned}$ | Student shows little or no understanding of vectors. | Student identifies vector components but does not add or subtract the vectors. | Student identifies the vector components and either adds or subtracts the vectors correctly. | Student identifies the vector components and adds and subtracts the vectors correctly. |
|  | b $\begin{aligned} & \text { N-VM.A. } 1 \\ & \text { N-VM.B.4a } \end{aligned}$ | Student shows little or no understanding of graphing vectors. | Student graphs either $\mathbf{V}$ or $\mathbf{w}$ correctly but does not graph $\mathbf{v}+\mathbf{w}$ or graphs $\mathbf{v}+\mathbf{w}$ incorrectly. | Student graphs $\mathbf{v}$ and $\mathbf{w}$ correctly but does not graph $\mathbf{v}+\mathbf{w}$ or graphs $\mathbf{v}+\mathbf{w}$ incorrectly. | Student graphs $\mathbf{v}, \mathbf{w}$, and $\mathbf{v}+\mathbf{w}$ correctly. |
|  | $\begin{gathered} \text { c } \\ \text { N-VM.B.4c } \end{gathered}$ | Student shows little or no understanding of vectors. | Student draws the vector correctly but does not write it in terms of $\mathbf{v}$ and $\mathbf{w}$ or identify the components. | Student draws the vector correctly and either writes the vector in terms of $\mathbf{v}$ and $\mathbf{w}$ correctly or identifies its components correctly. | Student draws the vector, writes it in terms of $\mathbf{v}$ and $\mathbf{w}$, and identifies its components correctly. |
|  | $\begin{gathered} d \\ \text { N-VM.B.4a } \end{gathered}$ | Student shows little or no understanding of vector magnitude. | Student finds one of the magnitudes of $\mathbf{v}, \mathbf{w}$, or $\mathbf{v}+\mathbf{w}$ correctly. | Student finds two of the magnitudes of $\mathbf{v}, \mathbf{w}$, and $\mathbf{v}+\mathbf{w}$ correctly. | Student finds the magnitudes of $\mathbf{v}, \mathbf{w}$, and $\mathbf{v}+\mathbf{w}$ correctly and shows that the magnitude of $\mathbf{v}$ plus the magnitude of $\mathbf{w}$ does not equal the magnitude of the sum of $\mathbf{v}$ and $\mathbf{w}$. |
| 3 | e N-VM.B.4a | Student shows little or no understanding of vectors. | Student identifies a vector $\mathbf{u}$, but it does not satisfy the conditions. | Student identifies a vector $\mathbf{u}=\mathbf{v}$ that satisfies the conditions but does not justify answer. | Student identifies a vector $\mathbf{u}=\mathbf{v}$ that satisfies the conditions and justifies the answer. |
|  | $\begin{gathered} f \\ \text { N-VM.B.4c } \end{gathered}$ | Student shows little or no understanding of vector magnitude. | Student finds two of the three magnitudes correctly. | Student finds the three magnitudes correctly but does not state that the magnitudes are all the same. | Student finds the three magnitudes correctly and states that the magnitudes are all the same. |


|  | $\begin{gathered} \mathbf{g} \\ \text { N-VM.B.5a } \end{gathered}$ | Student shows little or no understanding of vectors. | Student either <br> multiplies $\mathbf{v}$ by $\frac{1}{4}$ or -1 <br> but not by $-\frac{1}{4}$. | Student multiplies $\mathbf{v}$ by $-\frac{1}{4}$ but does not identify the components of the resulting vector. | Student multiplies $\mathbf{v}$ by $-\frac{1}{4}$ and identifies the components of the resulting vector. |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 4 | N-VM.B.4b | Student shows little or no understanding of vectors. | Student draws the vectors correctly and attempts to determine magnitude and direction, but both are incorrect. | Student draws the vectors correctly and either the magnitude or direction are correct, but not both. | Student draws the vectors correctly and determines the correct magnitude and direction. |
| 5 | a $\begin{aligned} & \text { N-VM.A. } 2 \\ & \text { N-VM.B. } 4 \text { a } \end{aligned}$ | Student shows little or no understanding of vectors. | Student identifies the components of all three vectors correctly. | Student identifies the components of all three vectors correctly and shows their sum is 0 . | Student identifies the components of all three vectors correctly, shows their sum is 0 , and explains geometrically why this is expected. |
|  | b $\begin{gathered} \text { N-VM.A. } 2 \\ \text { N-VM.B.5b } \end{gathered}$ | Student shows little or no understanding of vectors. | Student determines the components of $\overrightarrow{M N}$ correctly. | Student determines the components of $\overrightarrow{M N}$ correctly and either shows that it is parallel to or half the magnitude of $\overrightarrow{B C}$. | Student determines the components of $\overrightarrow{M N}$ correctly and shows that it is parallel to and half the magnitude of $\overrightarrow{B C}$. |
|  | $\begin{gathered} \text { c } \\ \text { N-VM.A. } 2 \\ \text { N-VM.B.4a } \\ \text { N-VM.B.5b } \end{gathered}$ | Student shows little or no understanding of vector magnitude. | Student finds the components of $\overrightarrow{M G}$ and $\overrightarrow{M C}$. | Student finds the components of $\overrightarrow{M G}$ and $\overrightarrow{M C}$ and shows that $\overrightarrow{M C}=3 \overrightarrow{M G}$. | Student finds the components of $\overrightarrow{M G}$ and $\overrightarrow{M C}$ and shows the ratio is $\frac{1}{3}$. |
|  | $\begin{gathered} d \\ \text { N-VM.A. } 2 \\ \text { N-VM.B.4a } \\ \text { N-VM.B.5b } \end{gathered}$ | Student shows little or no understanding of vectors. | Student explains that $G$ must lie on $\overrightarrow{M C}$ since $\overrightarrow{M G}=\frac{1}{3} \overrightarrow{M C}$. | Student explains that $G$ must lie on $\overrightarrow{M C}$ and finds that $\overrightarrow{N G}=\frac{1}{3} \overrightarrow{N B}$ | Student explains that $G$ must lie on $\overrightarrow{M C}$ and also $\overrightarrow{N B}$. |
| 6 | $\begin{gathered} \text { a } \\ \text { N-VM.A. } 3 \end{gathered}$ | Student shows little or no understanding of the shape of the path or vectors. | Student states or draws the correct shape of the path but does not explain using vectors. | Students states or draws the correct shape of the path and uses vectors to support answer but makes errors in reasoning. | Student states or draws the correct shape of the path and clearly explains using vectors. |
|  | b $\text { N-VM.A. } 3$ | Student shows little or no understanding of vectors. | Student states that she will not land at point $B$ but shows little supporting work and does not state the point will be south of point $B$. | Student states that she will land at a point south of $B$, but supporting work is not clear or has simple mistakes. | Student states that she will land at a point south of $B$ and clearly explains answer. |


| 7 | N-VM.A. 3 | Student shows little or no understanding of vector components or magnitude. | Student identifies the components of the vector. | Student identifies the components of the vector and attempts to find the magnitude of the force component down the ramp but makes calculation mistakes. | Student identifies the components of the vector and calculates the magnitude of the force component down the ramp correctly. |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 8 | a $\begin{aligned} & \text { N-VM.A. } 3 \\ & \text { N-VM.C. } 11 \end{aligned}$ | Student shows little or no understanding of vectors. | Student calculates the components of $\overrightarrow{A B}$. | Student calculates the components of vector $\overrightarrow{A B}$ and understands that $P(t)=A+t A B$ but does not determine $P(t)$ in terms of $t$. | Student calculates the components of $\overrightarrow{A B}$ and finds the coordinates of $P(t)$ in terms of $t$. |
|  | $\begin{gathered} \text { b } \\ \text { N-VM.C. } 8 \\ \text { N-VM.C. } 11 \end{gathered}$ | Student shows little or no understanding of vectors. | Students states either $P(0)=A \text { or } P(1)=B .$ | Students states that $P(0)=A$ and $P(1)=$ $B$. | Students states that $P(0)=A, P(1)=B$, and that $P(0.5)$ is the midpoint of segment $A B$. |
|  | $\begin{gathered} \text { C } \\ \text { N-VM.C. } 8 \\ \text { N-VM.C. } 11 \end{gathered}$ | Student shows little or no knowledge of vectors and matrices. | Students sets up the matrices to determine $A^{\prime}$ and $B^{\prime}$ but makes errors in calculating both. | Student sets up the matrices to determine $A^{\prime}$ and $B^{\prime}$ and calculates one correctly. | Student sets up the matrices to determine $A^{\prime}$ and $B^{\prime}$ and calculates both correctly. |
|  | $\begin{gathered} \text { d } \\ \text { N-VM.C. } 8 \\ \text { N-VM.C. } 11 \end{gathered}$ | Student show little or no knowledge of vectors and matrices. | Students understands that $L(P(t))$ must be used to find the position of the second particle. | Student calculates either the position at $t=0$ as $L(P(0))$ or $t=1$ as $L(P(1))$ correctly. | Student calculates the position at $t=0$ as $L(P(0))$ and $t=1$ as $L(P(1))$ correctly. |
|  | e $\begin{gathered} \text { N-VM.C. } 8 \\ \text { N-VM.C. } 11 \end{gathered}$ | Student shows little or no understanding of vectors and matrices. | Student states the location of the second particle is $L(P(t))$. | Student states the location of the second particle is $L(P(t))$ and that $L(P(t))=A^{\prime}+$ $t \overrightarrow{A^{\prime} B^{\prime}}$. | Student states the location of the second particle is $L(P(t))$ and that $L(P(t))=A^{\prime}+$ $t \overrightarrow{A^{\prime} B^{\prime}}$ and explains the particle is moving on a straight line from $A^{\prime}$ to $B^{\prime}$ at uniform velocity given by $\overrightarrow{A^{\prime} B^{\prime}}$. |

Name $\qquad$ Date $\qquad$
1.
a. Find values for $a, b, c, d$, and $e$ so that the following matrix product equals the $3 \times 3$ identity matrix. Explain how you obtained these values.

$$
\left[\begin{array}{ccc}
a & -3 & 5 \\
c & c & 1 \\
5 & b & -4
\end{array}\right]\left[\begin{array}{lll}
1 & b & d \\
1 & c & e \\
2 & b & b
\end{array}\right]
$$

The row 1, column 1 entry of the product is $a-3+10$. This should equal 1, so $a=-6$.

The row 2, column 1 entry of the product is $c+c+2$. This should equal zero, so $c=-$ 1.

The row 3, column 1 entry of the product is $5+b-8$. This should equal zero, so $b=3$.
The row 1, column 3 entry of the product is ad $-3 e+5 b=-6 d-3 e+15$, and this should equal zero. So, we should have $6 d+3 e=15$.

The row 2 , column 3 entry of the product is $c d+c d+b=-d-e+3$, and this should equal zero. So, we need $d+e=3$.

Solving the two linear equation is $d$ and $e$ gives $d=2$ and $e=1$.
So, in summary, we have $a=-6, b=3, c=-1, d=2$, and $e=1$.
b. Represent the following system of linear equations as a single matrix equation of the form $A x=b$, where $A$ is a $3 \times 3$ matrix and $x$ and $b$ are $3 \times 1$ column matrices.

$$
\begin{aligned}
& x+3 y+2 z=8 \\
& x-y+z=-2 \\
& 2 x+3 y+3 z=7
\end{aligned}
$$

We have $A x=b$ with $A=\left[\begin{array}{ccc}1 & 3 & 2 \\ 1 & -1 & 1 \\ 2 & 3 & 3\end{array}\right]$ and $x=\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$ and $b=\left[\begin{array}{c}8 \\ -2 \\ 7\end{array}\right]$.
c. Solve the system of three linear equations given in part (b).

The solution is given by $x=A^{-1} b$, if the matrix inverse exists. But part (a) shows that if we set

$$
B=\left[\begin{array}{ccc}
-6 & -3 & 5 \\
-1 & -1 & 1 \\
5 & 3 & -4
\end{array}\right]
$$

then we have $B A=1$. We could check that $A B=1$, as well (in which case $B$ is the matrix inverse of $A$ ), but even without knowing this, from

$$
A x=b \quad\left[\begin{array}{ccc}
1 & 3 & 2 \\
1 & -1 & 1 \\
2 & 3 & 3
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{c}
8 \\
-2 \\
7
\end{array}\right]
$$

we get

$$
\begin{array}{ll}
\begin{array}{ll}
B A x=B b
\end{array} & {\left[\begin{array}{ccc}
-6 & -3 & 5 \\
-1 & -1 & 1 \\
5 & 3 & -4
\end{array}\right]\left[\begin{array}{ccc}
1 & 3 & 2 \\
1 & -1 & 1 \\
2 & 3 & 3
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{ccc}
-6 & -3 & 5 \\
-1 & -1 & 1 \\
5 & 3 & -4
\end{array}\right]\left[\begin{array}{c}
8 \\
-2 \\
7
\end{array}\right]} \\
1 x=B b & {\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right]\left[\begin{array}{l}
x \\
y \\
z=B b
\end{array}\right]=\left[\begin{array}{ccc}
-6 & -3 & 5 \\
-1 & -1 & 1 \\
5 & 3 & -4
\end{array}\right]\left[\begin{array}{c}
8 \\
-2 \\
7
\end{array}\right]} \\
& {\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{ccc}
-6 & -3 & 5 \\
-1 & -1 & 1 \\
5 & 3 & -4
\end{array}\right]\left[\begin{array}{c}
8 \\
-2 \\
7
\end{array}\right]} \\
& {\left[\begin{array}{l}
x \\
y \\
z
\end{array}\right]=\left[\begin{array}{c}
-7 \\
1 \\
6
\end{array}\right] ;}
\end{array}
$$

that is, that $x=-7, y=1, z=6$ is a solution to the system of equations. (In fact, it can be the only solution to the system.)
2. The following diagram shows two two-dimensional vectors $\mathbf{v}$ and $\mathbf{w}$ in the place positioned to both have endpoint at point $P$.

a. On the diagram, make reasonably accurate sketches of the following vectors, again each with endpoint at $P$. Be sure to label your vectors on the diagram.
i. $2 \mathbf{v}$
ii. $-\mathbf{w}$
iii. $\mathbf{v}+3 \mathbf{w}$
iv. $\mathbf{w}-2 \mathbf{v}$
v. $\frac{1}{2} \mathbf{v}$

Vector $\mathbf{v}$ has magnitude 5 units, $\mathbf{w}$ has magnitude 3, and the acute angle between them is $45^{\circ}$.
b. What is the magnitude of the scalar multiple $-5 \mathbf{v}$ ?

We have $\|-5 v\|=5\|v\|=5 \times 5=25$.
c. What is the measure of the smallest angle between $-5 \mathbf{v}$ and $3 \mathbf{w}$ if these two vectors are placed to have a common endpoint?


The two vectors in question have an angle of smallest measure $135^{\circ}$ between them.
3. Consider the two-dimensional vectors $\mathbf{v}=\langle 2,3\rangle$ and $\mathbf{w}=\langle-2,-1\rangle$.
a. What are the components of each of the vectors $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ ?
$v+w=\langle 2+(-2), 3+(-1)\rangle=\langle 0,2\rangle$ and $v-w=\langle 2-(-2), 3-(-1)\rangle=\langle 4,4\rangle$
b. On the following diagram, draw representatives of each of the vectors $\mathbf{v}, \mathbf{w}$, and $\mathbf{v}+\mathbf{w}$, each with endpoint at the origin.

c. The representatives for the vectors $\mathbf{v}$ and $\mathbf{w}$ you drew form two sides of a parallelogram, with the vector $\mathbf{v}+\mathbf{w}$ corresponding to one diagonal of the parallelogram. What vector, directed from the third quadrant to the first quadrant, is represented by the other diagonal of the parallelogram? Express your answer solely in terms of $\mathbf{v}$ and $\mathbf{w}$, and also give the coordinates of this vector.

Label the points $A$ and $B$ as shown. The vector we seek is $\overrightarrow{A B}$. To move from $A$ to $B$ we need to follow $-w$ and then $v$. Thus, the vector we seek is $-w+v$, which is the same as $v-w$.

Also, we see that to move from $A$ to $B$ we need to move 4 units to the right and 4 units upward. This is consistent with $v-w=\langle 2-(-2), 3-(-1)\rangle=\langle 4,4\rangle$.
d. Show that the magnitude of the vector $\mathbf{v}+\mathbf{w}$ does not equal the sum of the magnitudes of $\mathbf{v}$ and of w.

We have $\|v\|=\sqrt{2^{2}+3^{2}}=\sqrt{13}$ and $\|w\|=\sqrt{(-2)^{2}+(-1)^{2}}=\sqrt{5}$, so $\|v\|+\|w\|=\sqrt{13}+\sqrt{5}$.
Now, $\|v+w\|=\sqrt{0^{2}+2^{2}}=2$. This does not equal $\sqrt{13}+\sqrt{5}$.
e. Give an example of a non-zero vector $\mathbf{u}$ such that $\|\mathbf{v}+\mathbf{u}\|$ does equal $\|\mathbf{v}\|+\|\mathbf{u}\|$.

Choosing $u$ to be the vector $v$ works.

$$
\|v+u\|=\|v+v\|=\|2 v\|=2\|v\|=\|v\|+\|v\|
$$

(In fact, $u=k v$ for any positive real number $k$ works.)
f. Which of the following three vectors has the greatest magnitude: $\mathbf{v}+(-\mathbf{w}), \mathbf{w}-\mathbf{v}$, or $(-\mathbf{v})-(-\mathbf{w})$ ?
$(-v)-(-w)=-v+w=w-v$ and
$v+(-w)=v-w=-(w-v)$.
So, each of these vectors is either $w-v$ or the scalar multiple $(-1)(w-v)$, which is the same vector but with opposite direction. They all have the same magnitude.
g. Give the components of a vector one-quarter the magnitude of vector $\mathbf{v}$ and with direction opposite the direction of $\mathbf{v}$.

$$
-\frac{1}{4} v=-\frac{1}{4}\langle 2,3\rangle=\left\langle-\frac{1}{2},-\frac{3}{4}\right\rangle
$$

4. Vector $a$ points true north and has magnitude 7 units. Vector $b$ points $30^{\circ}$ east of true north. What should the magnitude of $b$ be so that $b-a$ points directly east?
a. State the magnitude and direction of $b-a$.

We hope to have the following vector diagram incorporating a right triangle.

We have $\|a\|=7$, and for this $30-60-90$ triangle we need
$\|b\|=2\|b-a\|$ and $7=\|a\|=\sqrt{3}\|b-a\|$. This shows the magnitude of $b$ should be $\|b\|=\frac{14}{\sqrt{3}}$.

The vector $b$ - a points east and has magnitude $\frac{7}{\sqrt{3}}$.

b. Write $b-a$ in magnitude and direction form.
$\left\langle\frac{7}{\sqrt{3}}, 0^{\circ}\right\rangle ; b$ - a has a magnitude of $\frac{7}{\sqrt{3}}$ and a direction of $0^{\circ}$ measured from the horizontal.
5. Consider the three points $A=(10,-3,5), B=(0,2,4)$, and $C=(2,1,0)$ in three-dimensional space. Let $M$ be the midpoint of $\overline{A B}$ and $N$ be the midpoint of $\overline{A C}$.
a. Write down the components of the three vectors $\overrightarrow{A B}, \overrightarrow{B C}$, and $\overrightarrow{C A}$, and verify through arithmetic that their sum is zero. Also, explain why geometrically we expect this to be the case.

We have the following:

$$
\begin{aligned}
& \overrightarrow{A B}=\langle-10,5,-1\rangle \\
& \overrightarrow{B C}=\langle 2,-1,-4\rangle \\
& \overrightarrow{C A}=\langle 8,-4,5\rangle
\end{aligned}
$$

Their sum is $\langle-10+2+8,5-1-4,-1-4+5\rangle=\langle 0,0,0\rangle$.


This is to be expected as the three points $A, B$, and $C$ are vertices of a triangle (even in three-dimensional space), and the vectors $\overrightarrow{A B}, \overrightarrow{B C}$, and $\overrightarrow{C A}$, when added geometrically, traverse the sides of the triangle and have the sum effect of "returning to start." That is, the cumulative effect of the three vectors is no vectorial shift at all.
b. Write down the components of the vector $\overrightarrow{M N}$. Show that it is parallel to the vector $\overrightarrow{B C}$ and half its magnitude.

We have $M=\left(5,-\frac{1}{2}, \frac{9}{2}\right)$ and $N=\left(6,-1, \frac{5}{2}\right)$. Thus, $\overrightarrow{M N}=\left\langle 1,-\frac{1}{2},-2\right\rangle$.
We see that $\overrightarrow{M N}=\frac{1}{2}\langle 2,-1,4\rangle=\frac{1}{2} \overrightarrow{B C}$, which shows that $\overrightarrow{M N}$ has the same direction as $\overrightarrow{B C}$ (and hence is parallel to it) and half the magnitude.

Let $G=(4,0,3)$.
c. What is the value of the ratio $\frac{\|\overrightarrow{M G}\|}{\|\overrightarrow{M C}\|}$ ?
$\overrightarrow{M G}=\left\langle-1, \frac{1}{2},-\frac{3}{2}\right\rangle$ and $\overrightarrow{M C}=\left\langle-3, \frac{3}{2},-\frac{9}{2}\right\rangle=3 \overrightarrow{M G}$. Thus, $\frac{\|\overrightarrow{M G}\|}{\|\overrightarrow{M C}\|}=\frac{\|\overrightarrow{M G}\|}{3\|\overrightarrow{M C}\|}=\frac{1}{3}$.
d. Show that the point $G$ lies on the line connecting $M$ and $C$. Show that $G$ also lies on the line connecting $N$ and $B$.

That $\overrightarrow{M G}=\frac{1}{3} \overrightarrow{M C}$ means that the point $G$ lies a third of the way along the line segment $\overline{M C}$.

Check: $\overrightarrow{N G}=\left\langle-2,1, \frac{1}{2}\right\rangle=\frac{1}{3}\left\langle-6,3, \frac{3}{2}\right\rangle=\frac{1}{3} \overrightarrow{N B}$
So, $G$ also lies on the line segment $\overline{N B}$ (and one-third of the way along, too).
6. A section of a river, with parallel banks 95 ft . apart, runs true north with a current of $2 \mathrm{ft} / \mathrm{sec}$. Lashana, an expert swimmer, wishes to swim from point $A$ on the west bank to the point $B$ directly opposite it. In still water she swims at an average speed of $3 \mathrm{ft} / \mathrm{sec}$.

The diagram to the right illustrates the situation.
To counteract the current, Lashana realizes that she is to swim at some angle $\theta$ to the east/west direction as shown.

With the simplifying assumptions that Lashana's swimming speed will be a constant $3 \mathrm{ft} / \mathrm{sec}$ and that the current of the water is a uniform $2 \mathrm{ft} / \mathrm{sec}$ flow northward throughout all regions of the river (we will ignore the effects of drag at the river banks, for example), at what angle $\theta$ to east/west direction should Lashana swim in order to reach the opposite bank precisely at point $B$ ? How long will her swim take?

a. What is the shape of Lashana's swimming path according to an observer standing on the bank watching her swim? Explain your answer in terms of vectors.

Lashana's velocity vector $v$ has magnitude 3 and resolves into two components as shown, a component in the east direction $v_{E}$ and a component in the south direction $v_{s}$.

We see


$$
\left\|v_{s}\right\|=\|v\| \sin \theta=3 \sin \theta
$$

Lashana needs this component of her velocity vector to counteract the northward current of the water. This will ensure that Lashana will swim directly toward point $B$ with no sideways deviation.

Since the current is $2 \mathrm{ft} / \mathrm{sec}$, we need $3 \sin \theta=2$, showing that $\theta=\sin ^{-1}\left(\frac{2}{3}\right) \approx 41.8^{\circ}$. Lashana will then swim at a speed of $\left\|v_{E}\right\|=3 \cos \theta \mathrm{ft} / \mathrm{sec}$ toward the opposite bank. Since $\sin \theta=\frac{2}{3}, \theta$ is part of a $2-\sqrt{5}-3$ right triangle, so $\cos \theta=\frac{\sqrt{5}}{3}$. Thus, $\left\|v_{E}\right\|=\sqrt{5} \mathrm{ft} / \mathrm{sec}$.

She needs to swim an east/west distance of 95 feet at this speed. It will take her $\frac{95}{\sqrt{5}}=19 \sqrt{5} \approx 42$ seconds to do this.
b. If the current near the banks of the river is significantly less than $2 \mathrm{ft} / \mathrm{sec}$, and Lashana swims at a constant speed of $3 \mathrm{ft} / \mathrm{sec}$ at the constant angle $\theta$ to the east/west direction as calculated in part (a), will Lashana reach a point different from $B$ on the opposite bank? If so, will she land just north or just south of $B$ ? Explain your answer.

As noted in the previous solution, Lashana will have no sideways motion in her swim. she will swim a straight-line path from $A$ to $B$.

If the current is slower than $2 \mathrm{ft} / \mathrm{sec}$ at any region of the river surface, Lashana's velocity vector component $v_{s}$, which has magnitude $2 \mathrm{ft} / \mathrm{sec}$, will be larger in magnitude than the magnitude of the current. Thus, she will swim slightly southward during these periods. Consequently, she will land at a point on the opposite bank south of $B$.
7. A 5 kg ball experiences a force due to gravity $\vec{F}$ of magnitude 49 Newtons directed vertically downwards. If this ball is placed on a ramped tilted at an angle of $45^{\circ}$, what is the magnitude of the component of this force, in Newtons, on the ball directed $45^{\circ}$ towards the bottom of the ramp? (Assume the ball is of sufficiently small radius that is reasonable to assume that all forces are acting at the point of contact of the ball with the ramp.)


The force vector can be resolved into two components as shown: $F_{\text {ramp }}$ and $F_{\text {perp. }}$.

We are interested in the component $F_{\text {ramp }}$.
We see a 45-90-45 triangle in this diagram, with hypotenuse of magnitude 49 N . This means that the magnitude of $F_{\text {ramp }}$ is $\frac{49}{\sqrt{2}} \approx 35 \mathrm{~N}$.

8. Let $A$ be the point $(1,1,-3)$ and $B$ be the point $(-2,1,-1)$ in three-dimensional space.

A particle moves along the straight line through $A$ and $B$ at uniform speed in such a way that at time $t=0$ seconds the particle is at $A$, and at $t=1$ second the particle is at $B$. Let $P(t)$ be the location of the particle at time $t$ (so, $P(0)=A$ and $P(1)=B$ ).
a. Find the coordinates of the point $P(t)$ each in terms of $t$.

We will write the coordinates of points as $3 \times 1$ column matrices, as is consistent for work with matrix notation.

The velocity vector of the particle is $\overrightarrow{A B}=\langle-3,0,2\rangle$. So, its position at time $t$ is
$P(t)=A+t \overrightarrow{A B}=\left[\begin{array}{c}1 \\ 1 \\ -3\end{array}\right]+t\left[\begin{array}{c}-3 \\ 0 \\ 2\end{array}\right]=\left[\begin{array}{c}1-3 t \\ 1 \\ -3+2 t\end{array}\right]$.
b. Give a geometric interpretation of the point $P(0.5)$.

Since $P(0)=A$ and $P(1)=B, P(0.5)$ is the midpoint of $\overline{A B}$.

Let $L$ be the linear transformation represented by the $3 \times 3$ matrix $\left[\begin{array}{lll}2 & 0 & 1 \\ 1 & 3 & 0 \\ 0 & 1 & 1\end{array}\right]$, and let $A^{\prime}=L A$ and $B^{\prime}=L B$ be the images of the points $A$ and $B$, respectively, under $L$.
c. Find the coordinates of $A^{\prime}$ and $B^{\prime}$.

$$
A^{\prime}=\left[\begin{array}{lll}
2 & 0 & 1 \\
1 & 3 & 0 \\
0 & 1 & 1
\end{array}\right]\left[\begin{array}{c}
1 \\
1 \\
-3
\end{array}\right]=\left[\begin{array}{c}
-1 \\
4 \\
-2
\end{array}\right] \text { and } B^{\prime}=\left[\begin{array}{lll}
2 & 0 & 1 \\
1 & 3 & 0 \\
0 & 1 & 1
\end{array}\right]\left[\begin{array}{c}
-2 \\
1 \\
-1
\end{array}\right]=\left[\begin{array}{c}
-5 \\
1 \\
0
\end{array}\right]
$$

A second particle moves through three-dimensional space. Its position at time $t$ is given by $L(P(t))$, the image of the location of the first particle under the transformation $L$.
d. Where is the second particle at times $t=0$ and $t=1$ ? Briefly explain your reasoning.

We see $L(P(O))=L(A)=A^{\prime}$ and $L(P(1))=L(B)=B^{\prime}$. Since the position of the particle at time $t$ is given by $L(P(t))$, to find the location at $t=0$ and $t=1$, evaluate $L(P(0))$ and $L(P(1))$.
e. Prove that the second particle is also moving along a straight line path at uniform speed.

At time $t$ the location of the second particle is

$$
L(P(t))=\left[\begin{array}{lll}
2 & 0 & 1 \\
1 & 3 & 0 \\
0 & 1 & 1
\end{array}\right]\left[\begin{array}{c}
1-3 t \\
1 \\
-3+2 t
\end{array}\right]=\left[\begin{array}{c}
2-4 t \\
4-3 t \\
-2+2 t
\end{array}\right]=\left[\begin{array}{c}
-1 \\
4 \\
-2
\end{array}\right]+t\left[\begin{array}{c}
-4 \\
-3 \\
2
\end{array}\right] .
$$

We recognize this as

$$
L(P(t))=A^{\prime}+t \overrightarrow{A^{\prime} B^{\prime}} .
$$

Thus, the second particle is moving along the straight line through $A^{\prime}$ and $B^{\prime}$ at a uniform velocity given by the vector $\overrightarrow{A^{\prime} B^{\prime}}=\langle-4,-3,2\rangle$.


[^0]:    ${ }^{1}$ Each lesson is ONE day, and ONE day is considered a 45-minute period.

[^1]:    ${ }^{2}$ These are terms and symbols students have seen previously.

[^2]:    ${ }^{1}$ Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson

[^3]:    ${ }^{1}$ Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson

[^4]:    ${ }^{1}$ Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson

[^5]:    ${ }^{1}$ Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson

[^6]:    The forces are all contained within the arch, so the arch will stand.

[^7]:    ${ }^{1}$ Lesson Structure Key: P-Problem Set Lesson, M-Modeling Cycle Lesson, E-Exploration Lesson, S-Socratic Lesson

[^8]:    Exploratory Challenge
    In this drawing task, the "eye" or the "camera" is the point, and the shaded figure is the "TV screen." The cube is in the 3-D universe of the computer game.

    By using lines drawn from each vertex of the cube to the point, draw the image of the 3-D cube on the screen.

