

Lesson 10: Matrix Multiplication Is Not Commutative

Classwork

Opening Exercise

Consider the vector $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

If \mathbf{v} is rotated 45° counterclockwise and then reflected across the y -axis, what is the resulting vector?

If \mathbf{v} is reflected across the y -axis and then rotated 45° counterclockwise about the origin, what is the resulting vector?

Did these linear transformations commute? Explain.

Exercises 1–4

1. Let A equal the matrix that corresponds to a 45° rotation counterclockwise and B equal the matrix that corresponds to a reflection across the y -axis. Verify that matrix multiplication does not commute by finding the products AB and BA .

2. Let $A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. Verify that matrix multiplication does not commute by finding the products AB and BA .

3. Consider the vector $\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

- a. If \mathbf{v} is rotated 45° counterclockwise about the z -axis and then reflected across the xy -plane, what is the resulting vector?

- b. If \mathbf{v} is reflected across the xy -plane and then rotated 45° counterclockwise about the z -axis, what is the resulting vector?
- c. Verify algebraically that the product of the two corresponding matrices is the same regardless of the order in which they are multiplied.
4. Write two matrices in the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.
- a. Verify algebraically that the products of these two matrices are equal.

- b. Write each of your matrices as a complex number. Find the product of the two complex numbers.
- c. Why is it the case that any two matrices in the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ have products that are equal regardless of the order in which they are multiplied?

Problem Set

1. Let A be the matrix representing a dilation of 2, B the matrix representing a rotation of 30° , and $x = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$.
 - a. Evaluate AB .
 - b. Evaluate BA .
 - c. Find ABx .
 - d. Find BAx .

2. Let A be the matrix representing a reflection across the line $y = x$, B the matrix representing a rotation of 90° , and $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
 - a. Evaluate AB .
 - b. Evaluate BA .
 - c. Find Ax .
 - d. Find Bx .
 - e. Find ABx .
 - f. Find BAx .
 - g. Describe the linear transformation represented by AB .
 - h. Describe the linear transformation represented by BA .

3. Let matrices A, B represent scalars. Answer the following questions.
 - a. Would you expect $AB = BA$? Explain why or why not.
 - b. Let $A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and $B = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}$. Show $AB = BA$ through matrix multiplication and explain why.

4. Let matrices A, B represent complex numbers. Answer the following questions.
 - a. Let $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$. Show $AB = BA$ through matrix multiplication.
 - b. Would you expect $AC = CA$ for any matrix C ? Explain.
 - c. Let C be any 2×2 matrix, $\begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Show $AC \neq CA$ through matrix multiplication.
 - d. Summarize your results from Problems 4 and 5.

5. Quaternions are a number system that extends to complex numbers discovered by William Hamilton in 1843. Multiplication of quaternions is not commutative and is defined as the quotient of two vectors. They are useful in 3-dimensional rotation calculations for computer graphics. Quaternions are formed the following way:

$$i^2 = j^2 = k^2 = ijk = -1.$$

Multiplication by -1 and 1 works normally. We can represent all possible multiplications of quaternions through a table:

\times	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

- Is multiplication of quaternions commutative? Explain why or why not.
- Is multiplication of quaternions associative? Explain why or why not.