Lesson 11: Matrix Addition Is Commutative

Classwork

Opening Exercise

Kiamba thinks $A + B = B + A$ for all 2×2 matrices. Rachel thinks it is not always true. Who is correct? Explain.

Exercises 1–6

1. In two-dimensional space, let A be the matrix representing a rotation about the origin through an angle of $45°$, and let B be the matrix representing a reflection about the x-axis. Let x be the point $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   a. Write down the matrices A, B, and $A + B$.

b. Write down the image points of Ax, Bx, and $(A+B)x$, and plot them on graph paper.

c. What do you notice about $(A+B)x$ compared to Ax and Bx?

2. For three matrices of equal size, A, B, and C, does it follow that $A+(B+C)=(A+B)+C$?

   a. Determine if the statement is true geometrically. Let A be the matrix representing a reflection across the y-axis. Let B be the matrix representing a counterclockwise rotation of $30°$. Let C be the matrix representing a reflection about the x-axis. Let x be the point $\begin{pmatrix}1\\1\end{pmatrix}$.

   b. Confirm your results algebraically.

c. What do your results say about matrix addition?

3. If $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, what are the coordinates of a point y with the property $x + y$ is the origin $O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$?

4. Suppose $A = \begin{pmatrix} 11 & -5 & 2 \\ -34 & 6 & 19 \\ 8 & -542 & 0 \end{pmatrix}$, and matrix B has the property that $Ax + Bx$ is the origin. What is the matrix B?

5. For three matrices of equal size, A, B, and C, where A represents a reflection across the line $y = x$, B represents a counterclockwise rotation of $45°$, C represents a reflection across the y-axis, and $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$:

   a. Show that matrix addition is commutative: $Ax + Bx = Bx + Cx$.

b. Show that matrix addition is associative: $Ax + (Bx + Cx) = (Ax + Bx) + Cx$.

6. Let A, B, C, and D be matrices of the same dimensions. Use the commutative property of addition of two matrices to prove $A + B + C = C + B + A$.

Problem Set

1. Let A be matrix transformation representing a rotation of $45°$ about the origin and B be a reflection across the y-axis. Let $x = (3,4)$.
   a. Represent A and B as matrices, and find $A + B$.
   b. Represent Ax and Bx as matrices, and find $(A + B)x$.
   c. Graph your answer to part (b).
   d. Draw the parallelogram containing Ax, Bx, and $(A + B)x$.

2. Let A be matrix transformation representing a rotation of $300°$ about the origin and B be a reflection across the x-axis. Let $x = (2, -5)$.
   a. Represent A and B as matrices, and find $A + B$.
   b. Represent Ax and Bx as matrices, and find $(A + B)x$.
   c. Graph your answer to part (b).
   d. Draw the parallelogram containing Ax, Bx, and $(A + B)x$.

3. Let $\mathrm{A}, \mathrm{B}, \mathrm{C}$, and D be matrices of the same dimensions.
   a. Use the associative property of addition for three matrices to prove $(A + B) + (C + D) = A + (B + C) + D$.
   b. Make an argument for the associative and commutative properties of addition of matrices to be true for finitely many matrices being added.

4. Let A be an $m \times n$ matrix with element in the i^{th} row, j^{th} column a_{ij}, and B be an $m \times n$ matrix with element in the i^{th} row, j^{th} column b_{ij}. Present an argument that $A + B = B + A$.

5. For integers x, y, define $x \oplus y = x \cdot y + 1$, read "$x$ plus y" where $x \cdot y$ is defined normally.
   a. Is this form of addition commutative? Explain why or why not.
   b. Is this form of addition associative? Explain why or why not.

6. For integers x, y, define $x \oplus y = x$.
   a. Is this form of addition commutative? Explain why or why not.
   b. Is this form of addition associative? Explain why or why not.