Lesson 7: Linear Transformations Applied to Cubes

Classwork

Opening Exercise

Consider the following matrices: $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$

a. Compute the following determinants.

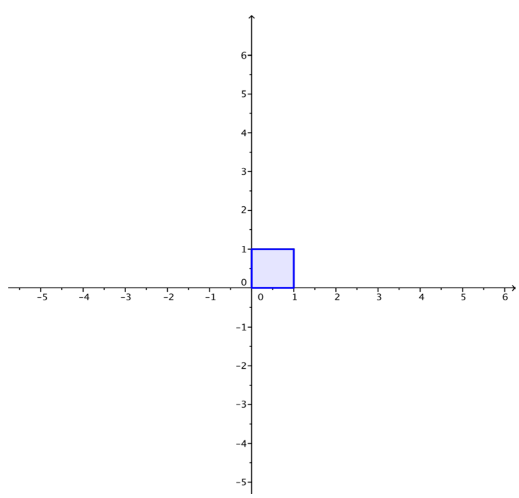
i. $\det(A)$

ii. $\det(B)$

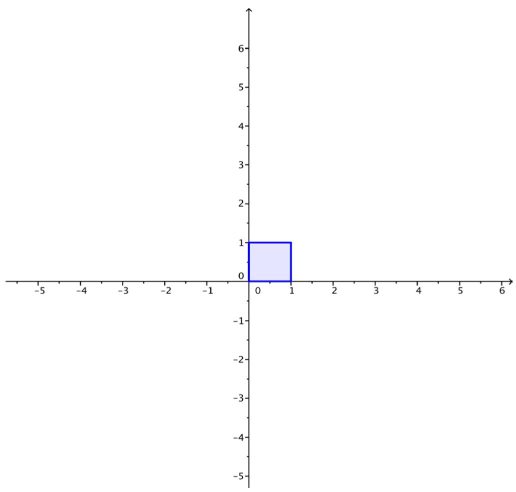
iii. $\det(C)$

b. Sketch the image of the unit square after being transformed by each transformation.

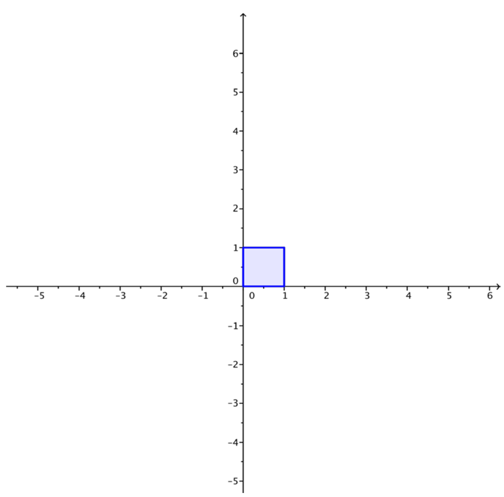
i. $L_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$



ii. $L_B\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$



iii. $L_C\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$



c. Find the area of each image of the unit square in Part 2.

d. Explain the connection between the responses to Parts 1 and 3.

Exploratory Challenge 1

For each matrix A given below:

i. Plot the image of the unit cube under the transformation.

ii. Find the volume of the image of the unit cube from part (i).

iii. Does the transformation have an inverse? If so, what is the matrix that induces the inverse transformation?

a. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

b. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

c. $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

d. Describe the geometric effect of a transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for numbers a, b, and c.

Describe when such a transformation is invertible.

Exploratory Challenge 2

a. Make a prediction: What would be the geometric effect of the transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(45°) & -\sin(45°) \\ 0 & \sin(45°) & \cos(45°) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ on the unit cube? Use the GeoGebra demo to test your conjecture.

b. For each geometric transformation below, find a matrix A so that the geometric effect of $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the specified transformation.

i. Rotation by $-45°$ about the x-axis.

ii. Rotation by $45°$ about the y-axis.

iii. Rotation by $45°$ about the z-axis.

iv. Rotation by $90°$ about the x-axis.

v. Rotation by $90°$ about the y-axis.

vi. Rotation by $90°$ about the z-axis.

vii. Rotation by θ about the x-axis.

viii. Rotation by θ about the y-axis.

ix. Rotation by θ about the z-axis.

Exploratory Challenge 3 (Optional)

Describe the geometric effect of each transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for the given matrices A. Be as specific as you can.

a. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 0 \end{bmatrix}$

c. $A = \begin{bmatrix} 2 & 2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Lesson Summary

For a matrix A, the transformation $L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a function from points in space to points in space.

Different matrices induce transformations such as rotation, dilation, and reflection.

The transformation induced by a diagonal matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ will scale by a in the direction parallel to the x-axis, by b in the direction parallel to the y-axis, and by c in the direction parallel to the z-axis.

The matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$, $\begin{bmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{bmatrix}$, and $\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ induce rotation by θ about the x, y, and z axes, respectively.

Problem Set

1. Suppose that we have a linear transformation $\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, for some matrix $A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}$.

   a. Evaluate $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)$. How does the result relate to the matrix A?

   b. Evaluate $L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$. How does the result relate to the matrix A?

   c. Evaluate $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$. How does the result relate to the matrix A?

   d. James correctly said that if you know what a linear transformation does to the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, you can find the matrix of the transformation. Explain how you can find the matrix of the transformation given the image of these three points.

2. Use the result from Problem 1(d) to answer the following questions.

   a. Suppose a transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$, $L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$, and $L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$.

      i. What is the matrix A that represents the transformation L?

      ii. What is the geometric effect of the transformation L?

      iii. Sketch the image of the unit cube after the transformation by L.

b. Suppose a transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\-4\end{bmatrix}$.

i. What is the matrix A that represents the transformation L?

ii. What is the geometric effect of the transformation L?

iii. Sketch the image of the unit cube after the transformation by L.

c. Suppose a transformation $L: \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}-2\\0\\0\end{bmatrix}$, $L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\-2\end{bmatrix}$.

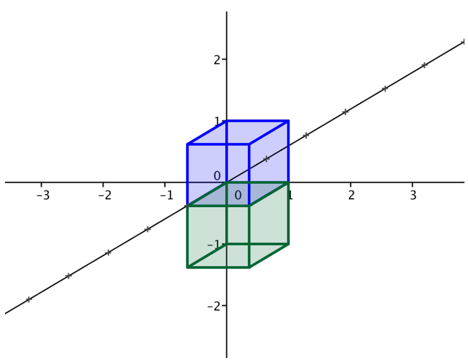
i. What is the matrix A that represents the transformation L?

ii. What is the geometric effect of the transformation L?

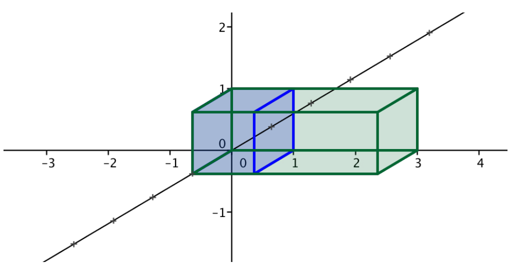
iii. Sketch the image of the unit cube after transformation by L.

3. Find the matrix of the transformation that will produce the following images of the unit cube. Describe the geometric effect of the transformation.

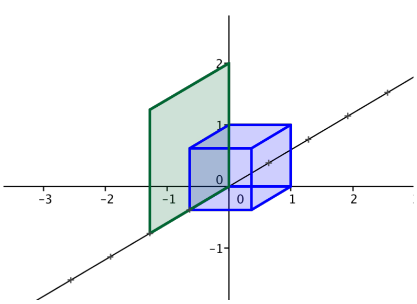
a.



b.



c.



d.