Lesson 8: Composition of Linear Transformations

Classwork

Opening Exercise

Compute the product AB for the following pairs of matrices.

a. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

e. $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, B = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

Exploratory Challenge

1. For each pair of matrices A and B given below:
    i. Describe the geometric effect of the transformation $L_B\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = B \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.
    ii. Describe the geometric effect of the transformation $L_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.
    iii. Draw the image of the unit square under the transformation $L_B\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = B \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.
    iv. Draw the image of the transformed square under the transformation $L_A\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.
    v. Describe the geometric effect on the unit square of performing first L_B then L_A.
    vi. Compute the matrix product AB.
    vii. Describe the geometric effect of the transformation $L_{AB}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = AB \cdot \begin{bmatrix} x \\ y \end{bmatrix}$.

a. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

b. $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

c. $A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}, B = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

d. $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

e. $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, B = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

2. Make a conjecture about the geometric effect of the linear transformation produced by the matrix AB. Justify your answer.

Lesson Summary

The linear transformation produced by matrix AB has the same geometric effect as the sequence of the linear transformation produced by matrix B followed by the linear transformation produced by matrix A.

That is, if matrices A and B produce linear transformations L_A and L_B in the plane, respectively, then the linear transformation L_{AB} produced by the matrix AB satisfies

$$L_{AB}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = L_A\left(L_B\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right).$$

Problem Set

1. Let A be the matrix representing a dilation of $\frac{1}{2}$, and let B be the matrix representing a reflection across the y-axis.
   a. Write A and B.
   b. Evaluate AB. What does this matrix represent?
   c. Let $x = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$, $y = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, and $z = \begin{bmatrix} 8 \\ -2 \end{bmatrix}$. Find $(AB)x$, $(AB)y$, and (AB)z.

2. Let A be the matrix representing a rotation of $30°$, and let B be the matrix representing a dilation of 5.
   a. Write A and B.
   b. Evaluate AB. What does this matrix represent?
   c. Let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $(AB)x$.

3. Let A be the matrix representing a dilation of 3, and let B be the matrix representing a reflection across the line $y = x$.
   a. Write A and B.
   b. Evaluate AB. What does this matrix represent?
   c. Let $x = \begin{bmatrix} -2 \\ 7 \end{bmatrix}$. Find $(AB)x$.

4. Let $A = \begin{bmatrix} 3 & 0 \\ 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
   a. Evaluate AB.
   b. Let $x = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Find $(AB)x$.
   c. Graph x and $(AB)x$.

5. Let $A = \begin{bmatrix} \frac{1}{3} & 0 \\ 2 & \frac{1}{3} \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$.
   a. Evaluate AB.
   b. Let $x = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Find $(AB)x$.
   c. Graph x and $(AB)x$.

6. Let $A = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$.
   a. Evaluate AB.
   b. Let $x = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$. Find $(AB)x$.
   c. Graph x and (AB).

7. Let A, B, C be 2×2 matrices representing linear transformations.
   a. What does $A(BC)$ represent?
   b. Will the pattern established in part (a) be true no matter how many matrices are multiplied on the left?
   c. Does $(AB)C$ represent something different from $A(BC)$? Explain.

8. Let AB represent any composition of linear transformations in $\mathbb{R}^2$. What is the value of $(AB)x$ where $x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$?