Lesson 9: Composition of Linear Transformations

Classwork

Opening Exercise

Recall from Problem 1, part (d) of the Problem Set of Lesson 7 that if you know what a linear transformation does to the three points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, you can find the matrix of the transformation. How do the images of these three points lead to the matrix of the transformation?

a. Suppose that a linear transformation L_1 rotates the unit cube by $90°$ counterclockwise about the z-axis. Find the matrix A_1 of the transformation L_1.

b. Suppose that a linear transformation L_2 rotates the unit cube by $90°$ counterclockwise about the y-axis. Find the matrix A_2 of the transformation L_2.

c. Suppose that a linear transformation L_3 scales by 2 in the x-direction, scales by 3 in the y-direction, and scales by 4 in the z-direction. Find the matrix A_3 of the transformation L_3.

d. Suppose that a linear transformation L_4 projects onto the xy-plane. Find the matrix A_4 of the transformation L_4.

© 2015 Common Core, Inc. Some rights reserved. commoncore.org
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

e. Suppose that a linear transformation L_5 projects onto the xz-plane. Find the matrix A_5 of the transformation L_5.

f. Suppose that a linear transformation L_6 reflects across the plane with equation $y = x$. Find the matrix A_6 of the transformation L_6.

g. Suppose that a linear transformation L_7 satisfies $L_7\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\\0\end{bmatrix}$, $L_7\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\\0\end{bmatrix}$, and $L_7\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\\frac{1}{2}\end{bmatrix}$. Find the matrix A_7 of the transformation L_7. What is the geometric effect of this transformation?

h. Suppose that a linear transformation L_8 satisfies $L_8\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\end{bmatrix}$, $L_8\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\1\\0\end{bmatrix}$, and $L_8\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\\1\end{bmatrix}$. Find the matrix of the transformation L_8. What is the geometric effect of this transformation?

© 2015 Common Core, Inc. Some rights reserved. commoncore.org
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Exploratory Challenge 1

Transformations L_1-L_8 refer to the linear transformations from the Opening Exercise. For each pair,

i. Make a conjecture to predict the geometric effect of performing the two transformations in the order specified.

ii. Find the product of the corresponding matrices, in the order that corresponds to the indicated order of composition. Remember that if we perform a transformation L_B with matrix B and then L_A with matrix A, the matrix that corresponds to the composition $L_A \circ L_B$ is AB. That is, L_B is applied first, but matrix B is written second.

iii. Use the GeoGebra applet TransformCubes.ggb to draw the image of the unit cube under the transformation induced by the matrix product in part (ii). Was your conjecture in part (i) correct?

a. Perform L_6 and then L_6.

b. Perform L_1 and then L_2.

© 2015 Common Core, Inc. Some rights reserved. commoncore.org
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

c. Perform L_4 and then L_5.

d. Perform L_4 and then L_3.

e. Perform L_3 and then L_7.

© 2015 Common Core, Inc. Some rights reserved. commoncore.org
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

f. Perform L_8 and then L_4.

g. Perform L_4 and then L_6.

h. Perform L_2 and then L_7.

© 2015 Common Core, Inc. Some rights reserved. commoncore.org
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

i. Perform L_8 and then L_8.

Exploratory Challenge 2 (Optional)

Transformations L_1-L_8 refer to the transformations from the Opening Exercise. For each of the following pairs of matrices A and B below, compare the transformations $L_A \circ L_B$ and $L_B \circ L_A$.

a. L_4 and L_5

b. L_2 and L_5

© 2015 Common Core, Inc. Some rights reserved. commoncore.org
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

c. L_3 and L_7

d. L_3 and L_6

e. L_7 and L_1

f. What can you conclude about the order in which you compose two linear transformations?

© 2015 Common Core, Inc. Some rights reserved. commoncore.org
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

Lesson Summary

- The linear transformation induced by a 3×3 matrix AB has the same geometric effect as the sequence of the linear transformation induced by the 3×3 matrix B followed by the linear transformation induced by the 3×3 matrix A.
- That is, if matrices A and B induce linear transformations L_A and L_B in $\mathbb{R}^3$, respectively, then the linear transformation L_{AB} induced by the matrix AB satisfies $L_{AB}\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = L_A\left(L_B\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)\right)$.

Problem Set

1. Let A be the matrix representing a dilation of $\frac{1}{2}$, and let B be the matrix representing a reflection across the yz-plane.
   a. Write A and B.
   b. Evaluate AB. What does this matrix represent?
   c. Let $x = \begin{bmatrix} 5 \\ 6 \\ 4 \end{bmatrix}, y = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$, and $z = \begin{bmatrix} 8 \\ -2 \\ -4 \end{bmatrix}$. Find $(AB)x$, $(AB)y$, and $(AB)\,z$.

2. Let A be the matrix representing a rotation of $30°$ about the x-axis, and let B be the matrix representing a dilation of 5.
   a. Write A and B.
   b. Evaluate AB. What does this matrix represent?
   c. Let $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Find $(AB)x, (AB)y,$ and $(AB)z$.

3. Let A be the matrix representing a dilation of 3, and let B be the matrix representing a reflection across the plane $y = x$.
   a. Write A and B.
   b. Evaluate AB. What does this matrix represent?
   c. Let $x = \begin{bmatrix} -2 \\ 7 \\ 3 \end{bmatrix}$. Find $(AB)x$.

© 2015 Common Core, Inc. Some rights reserved. commoncore.org

This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.

4. Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
   a. Evaluate AB.
   b. Let $x = \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix}$. Find $(AB)x$.
   c. Graph x and $(AB)x$.

5. Let $A = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & \frac{1}{3} \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
   a. Evaluate AB.
   b. Let $x = \begin{bmatrix} 0 \\ 3 \\ 2 \end{bmatrix}$. Find $(AB)x$.
   c. Graph x and $(AB)x$.

6. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.
   a. Evaluate AB.
   b. Let $x = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$. Find $(AB)x$.
   c. Graph x and $(AB)x$.
   d. What does AB represent geometrically?

7. Let A, B, C be 3×3 matrices representing linear transformations.
   a. What does $A(BC)$ represent?
   b. Will the pattern established in part (a) be true no matter how many matrices are multiplied on the left?
   c. Does $(AB)C$ represent something different from $A(BC)$? Explain.

8. Let AB represent any composition of linear transformations in $\mathbb{R}^3$. What is the value of $(AB)x$ where $x = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$?

© 2015 Common Core, Inc. Some rights reserved. commoncore.org
This work is licensed under a Creative Commons Attribution-NonCommercial-ShareAlike 3.0 Unported License.