Lesson 4: The Binomial Theorem

Classwork

Exercises

1. Show that $z = 1 + i$ is a solution to the fourth degree polynomial equation $z^4 - z^3 + 3z^2 - 4z + 6 = 0$.

2. Show that $z = 1 - i$ is a solution to the fourth degree polynomial equation $z^4 - z^3 + 3z^2 - 4z + 6 = 0$.

3. Based on the patterns seen in Pascal's triangle, what would be the coefficients of Rows 7 and 8 in the triangle? Write the coefficients of the triangle beneath the part of the triangle shown.

Row 0:							1							
Row 1:						1		1						
Row 2:					1		2		1					
Row 3:				1		3		3		1				
Row 4:			1		4		6		4		1			
Row 5:		1		5		10		10		5		1		
Row 6:	1		6		15		20		15		6		1	
Row 7:														
Row 8:														

4. Calculate the following factorials.
   a. $6!$

   b. $10!$

5. Calculate the value of the following factorial expressions.
   a. $\frac{7!}{6!}$

   b. $\frac{10!}{6!}$

   c. $\frac{8!}{5!}$

   d. $\frac{12!}{10!}$

6. Calculate the following quantities.
   a. $C(1,0)$ and $C(1,1)$

   b. $C(2,0)$, $C(2,1)$, and $C(2,2)$

   c. $C(3,0)$, $C(3,1)$, $C(3,2)$, and $C(3,3)$

   d. $C(4,0)$, $C(4,1)$, $C(4,2)$, $C(4,3)$, and $C(4,4)$

7. What patterns do you see in Exercise 5?

8. Expand the expression $(u+v)^3$.

9. Expand the expression $(u+v)^4$.

10.
   a. Multiply the expression you wrote in Exercise 4 by u.

   b. Multiply the expression you wrote in Exercise 4 by v.

   c. How can you use the results from parts (a) and (b) to find the expanded form of the expression $(u+v)^5$?

11. What do you notice about your expansions for $(u+v)^4$ and $(u+v)^5$? Does your observation hold for other powers of $(u+v)$?

12. Use the binomial theorem to expand the following binomial expressions.

a. $(x + y)^6$

b. $(x + 2y)^3$

c. $(ab + bc)^4$

d. $(3xy - 2z)^3$

e. $(4p^2qr - qr^2)^5$

Lesson Summary

Pascal's triangle is an arrangement of numbers generated recursively:

Row 0:						1					
Row 1:					1		1				
Row 2:				1		2		1			
Row 3:			1		3		3		1		
Row 4:		1		4		6		4		1	
Row 5:	1		5		10		10		5		1
	⋮		⋮		⋮		⋮		⋮		⋮

For an integer $n \geq 1$, the number $n!$ is the product of all positive integers less than or equal to n.
We define $0! = 1$.

The binomial coefficients $C(n, k)$ are given by $C(n, k) = \frac{n!}{k!(n-k)}!$ for integers $n \geq 0$ and $0 \leq k \leq n$.

THE BINOMIAL THEOREM: For any expressions u and v,

$$(u + v)^n = u^n + C(n, 1)u^{n-1}v + C(n, 2)u^{n-2}v^2 + \cdots + C(n, k)u^{n-k}v^k + \cdots + C(n, n - 1)u\, v^{n-1} + v^n.$$

That is, the coefficients of the expanded binomial $(u + v)^n$ are exactly the numbers in Row n of Pascal's triangle.

Problem Set

1. Evaluate the following expressions.
   a. $\frac{9!}{8!}$
   b. $\frac{7!}{5!}$
   c. $\frac{21!}{19!}$
   d. $\frac{8!}{4!}$

2. Use the binomial theorem to expand the following binomial expressions.
   a. $(x + y)^4$
   b. $(x + 2y)^4$
   c. $(x + 2xy)^4$
   d. $(x - y)^4$
   e. $(x - 2xy)^4$

3. Use the binomial theorem to expand the following binomial expressions.
   a. $\left(1+\sqrt{2}\right)^5$
   b. $(1+i)^9$
   c. $(1-\pi)^5$ (Hint: $1-\pi = 1+(-\pi)$.)
   d. $\left(\sqrt{2}+i\right)^6$
   e. $(2-i)^6$

4. Consider the expansion of $(a+b)^{12}$. Determine the coefficients for the terms with the powers of a and b shown.
   a. a^2b^{10}
   b. a^5b^7
   c. a^8b^4

5. Consider the expansion of $(x+2y)^{10}$. Determine the coefficients for the terms with the powers of x and y shown.
   a. x^2y^8
   b. x^4y^6
   c. x^5y^5

6. Consider the expansion of $(5p+2q)^6$. Determine the coefficients for the terms with the powers of p and q shown.
   a. p^2q^4
   b. p^5q
   c. p^3q^3

7. Explain why the coefficient of the term that contains u^n is 1 in the expansion of $(u+v)^n$.

8. Explain why the coefficient of the term that contains $u^{n-1}v$ is n in the expansion of $(u+v)^n$.

9. Explain why the rows of Pascal's triangle are symmetric. That is, explain why $C(n,k) = C\big(n,(n-k)\big)$.