

Lesson 5: The Binomial Theorem

Classwork

Opening Exercise

Write the first six rows of Pascal's triangle. Then, use the triangle to find the coefficients of the terms with the powers of u and v shown, assuming that all expansions are in the form $(u + v)^n$. Explain how Pascal's triangle allows you to determine the coefficient.

a. u^2v^4

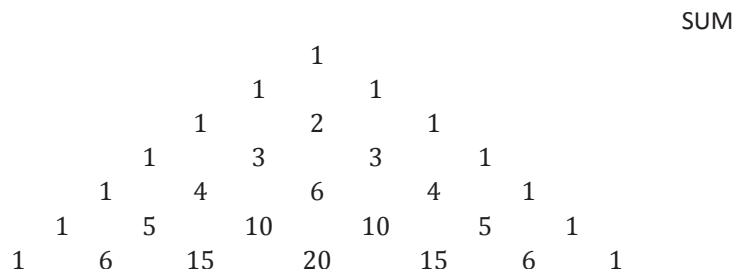
b. u^3v^2

c. u^2v^2

d. v^{10}

Exercises 1–2

1. Consider the Rows 0–6 of Pascal’s triangle.
 - a. Find the sum of each row.



- b. What pattern do you notice in the sums computed?

- c. Use the binomial theorem to explain this pattern.

2. Consider the expression 11^n .

- a. Calculate 11^n , where $n = 0, 1, 2, 3, 4$.

- b. What pattern do you notice in the successive powers?

- c. Use the binomial theorem to demonstrate why this pattern arises.
- d. Use a calculator to find the value of 11^5 . Explain whether this value represents what would be expected based on the pattern seen in lower powers of 11.

Example 2

We know that the volume $V(r)$ and surface area $S(r)$ of a sphere of radius r are given by these formulas:

$$V(r) = \frac{4}{3}\pi r^3$$

$$S(r) = 4\pi r^2$$

Suppose we increase the radius of a sphere by 0.01 units from r to $r + 0.01$.

- a. Use the binomial theorem to write an expression for the increase in volume.
- b. Write an expression for the average rate of change of the volume as the radius increases from r to $r + 0.01$.

- c. Simplify the expression in part (b) to compute the average rate of change of the volume of a sphere as the radius increases from r to $r + 0.01$.
- d. What does the expression from part (c) resemble?
- e. Why does it make sense that the average rate of change should approximate the surface area? Think about the geometric figure formed by $V(r + 0.01) - V(r)$. What does this represent?
- f. How could we approximate the volume of the shell using surface area? And the average rate of change for the volume?

Problem Set

1. Consider the binomial $(2u - 3v)^6$.
 - a. Find the term that contains v^4 .
 - b. Find the term that contains u^3 .
 - c. Find the third term.
2. Consider the binomial $(u^2 - v^3)^6$.
 - a. Find the term that contains v^6 .
 - b. Find the term that contains u^6 .
 - c. Find the fifth term.
3. Find the sum of all coefficients in the following binomial expansion.
 - a. $(2u + v)^{10}$
 - b. $(2u - v)^{10}$
 - c. $(2u - 3v)^{11}$
 - d. $(u - 3v)^{11}$
 - e. $(1 + i)^{10}$
 - f. $(1 - i)^{10}$
 - g. $(1 + i)^{200}$
 - h. $(1 + v)^{201}$
4. Expand the binomial $(1 + \sqrt{2}i)^6$.
5. Show that $(2 + \sqrt{2}i)^{20} + (2 - \sqrt{2}i)^{20}$ is an integer.
6. We know $(u + v)^2 = u^2 + 2uv + v^2 = u^2 + v^2 + 2uv$. Use this pattern to predict what the expanded form of each expression would be. Then, expand the expression, and compare your results.
 - a. $(u + v + w)^2$
 - b. $(a + b + c + d)^2$
7. Look at the powers of 101 up to the fourth power on a calculator. Explain what you see. Predict the value of 101^5 , and then find the answer on a calculator. Are they the same?
8. Can Pascal's triangle be applied to $\left(\frac{1}{u} + \frac{1}{v}\right)^n$ given $u, v \neq 0$?

9. The volume and surface area of a sphere are given by $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. Suppose we increase the radius of a sphere by 0.001 units from r to $r + 0.001$.
- Use the binomial theorem to write an expression for the increase in volume $V(r + 0.001) - V(r)$ as the sum of three terms.
 - Write an expression for the average rate of change of the volume as the radius increases from r to $r + 0.001$.
 - Simplify the expression in part (b) to compute the average rate of change of the volume of a sphere as the radius increases from r to $r + 0.001$.
 - What does the expression from part (c) resemble?
 - Why does it make sense that the average rate of changes should approximate the surface area? Think about the geometric figure formed by $V(r + 0.001) - V(r)$. What does this represent?
 - How could we approximate the volume of the shell using surface area? And the average rate of change for the volume?
 - Find the difference between the average rate of change of the volume and $S(r)$ when $r = 1$.
10. The area and circumference of a circle of radius r are given by $A(r) = \pi r^2$ and $C(r) = 2\pi r$. Suppose we increase the radius of a sphere by 0.001 units from r to $r + 0.001$.
- Use the binomial theorem to write an expression for the increase in area volume $A(r + 0.001) - A(r)$ as a sum of three terms.
 - Write an expression for the average rate of change of the area as the radius increases from r to $r + 0.001$.
 - Simplify the expression in part (b) to compute the average rate of change of the area of a circle as the radius increases from r to $r + 0.001$.
 - What does the expression from part (c) resemble?
 - Why does it make sense that the average rate of change should approximate the area of a circle? Think about the geometric figure formed by $A(r + 0.001) - A(r)$. What does this represent?
 - How could we approximate the area of the shell using circumference? And the average rate of change for the area?
 - Find the difference between the average rate of change of the area and $C(r)$ when $r = 1$.