Lesson 6: Curves in the Complex Plane

Classwork

Opening Exercise

a. Consider the complex number $z = a + bi$.

   i. Write z in polar form. What do the variables represent?

   ii. If $r = 3$ and $\theta = 90°$, where would z be plotted in the complex plane?

   iii. Use the conditions in part (ii) to write z in rectangular form. Explain how this representation corresponds to the location of z that you found in part (ii).

b. Recall the set of points defined by $z = 3(\cos(\theta) + i\sin(\theta))$ for $0 \leq \theta < 360°$, where θ is measured in degrees.

   i. What does z represent graphically? Why?

   ii. What does z represent geometrically?

c. Consider the set of points defined by $z = 5\cos\theta + 3i\sin\theta$.

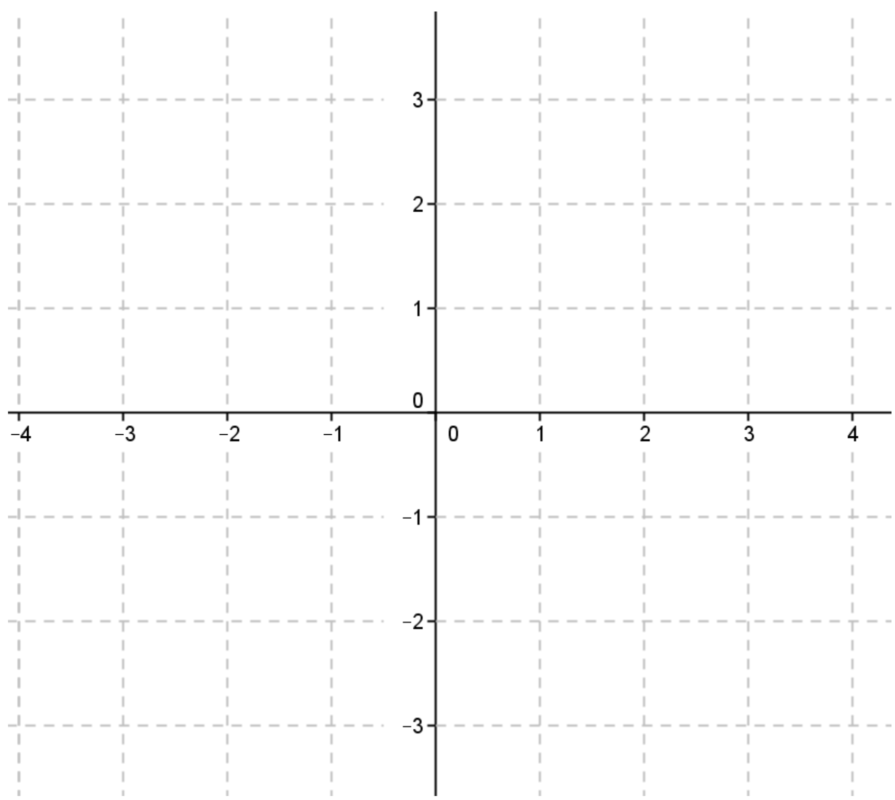
i. Plot z for $\theta = 0°, 90°, 180°, 270°, 360°$. Based on your plot, form a conjecture about the graph of the set of complex numbers.

ii. Compare this graph to the graph of $z = 3(\cos(\theta) + i\sin(\theta))$. Form a conjecture about what accounts for the differences between the graphs.

Example 1

Consider again the set of complex numbers represented by $z = 3(\cos(\theta) + i\sin(\theta))$ for $0 \le \theta < 360°$.

$\boldsymbol{\theta}$	$\mathbf{3cos}(\boldsymbol{\theta})$	$\mathbf{3sin}(\boldsymbol{\theta})$	$(\mathbf{3cos}(\boldsymbol{\theta}), \mathbf{3}\boldsymbol{i}\,\mathbf{sin}(\boldsymbol{\theta}))$
0			
$\frac{\pi}{4}$			
$\frac{\pi}{2}$			
$\frac{3\pi}{4}$			
π			
$\frac{5\pi}{4}$			
$\frac{3\pi}{2}$			
$\frac{7\pi}{4}$			
2π			



a. Use an ordered pair to write a representation for the points defined by z as they would be represented in the coordinate plane.

b. Write an equation that is true for all the points represented by the ordered pair you wrote in part (a).

c. What does the graph of this equation look like in the coordinate plane?

Exercises 1–2

1. Recall the set of points defined by $z = 5\cos(\theta) + 3i\sin(\theta)$.
   a. Use an ordered pair to write a representation for the points defined by z as they would be represented in the coordinate plane.

   b. Write an equation in the coordinate plane that is true for all the points represented by the ordered pair you wrote in part (a).

2. Find an algebraic equation for all the points in the coordinate plane traced by the complex numbers $z = \sqrt{2}\cos(\theta) + i\sin(\theta)$.

Example 2

The equation of an ellipse is given by $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

a. Sketch the graph of the ellipse.

b. Rewrite the equation in complex form.

Exercise 3

3. The equation of an ellipse is given by $\frac{x^2}{9}+\frac{y^2}{26}=1$.
   a. Sketch the graph of the ellipse.

   b. Rewrite the equation of the ellipse in complex form.

Example 3

A set of points in the complex plane can be represented in the complex plane as $z = 2 + i + 7\cos(\theta) + i\sin(\theta)$ as θ varies.

a. Find an algebraic equation for the points described.

b. Sketch the graph of the ellipse.

Problem Set

1. Write the real form of each complex equation.
   a. $z = 4\cos(\theta) + 9i\sin(\theta)$
   b. $z = 6\cos(\theta) + i\sin(\theta)$
   c. $z = \sqrt{5}\cos(\theta) + \sqrt{10}i\sin(\theta)$
   d. $z = 5 - 2i + 4\cos(\theta) + 7i\sin(\theta)$

2. Sketch the graphs of each equation.
   a. $z = 3\cos(\theta) + i\sin(\theta)$
   b. $z = -2 + 3i + 4\cos(\theta) + i\sin(\theta)$
   c. $\frac{(x-1)^2}{9} + \frac{y^2}{25} = 1$
   d. $\frac{(x-2)^2}{3} + \frac{y^2}{15} = 1$

3. Write the complex form of each equation.
   a. $\frac{x^2}{16} + \frac{y^2}{36} = 1$
   b. $\frac{x^2}{400} + \frac{y^2}{169} = 1$
   c. $\frac{x^2}{19} + \frac{y^2}{2} = 1$
   d. $\frac{(x-3)^2}{100} + \frac{(y+5)^2}{16} = 1$

4. Carrie converted the equation $z = 7\cos(\theta) + 4i\sin(\theta)$ to the real form $\frac{x^2}{7} + \frac{y^2}{4} = 1$. Her partner Ginger said that the ellipse must pass through the point $(7\cos(0), 4\sin(0)) = (7,0)$ and this point does not satisfy Carrie's equation, so the equation must be wrong. Who made the mistake, and what was the error? Explain how you know.

5. Cody says that the center of the ellipse with complex equation $z = 4 - 5i + 2\cos(\theta) + 3i\sin(\theta)$ is $(4, -5)$, while his partner Jarrett says that the center of this ellipse is $(-4,5)$. Which student is correct? Explain how you know.

Extension:

6. Any equation of the form $ax^2 + bx + cy^2 + dy + e = 0$ with $a > 0$ and $c > 0$ might represent an ellipse. The equation $4x^2 + 8x + 3y^2 + 12y + 4 = 0$ is such an equation of an ellipse.
   a. Rewrite the equation $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ in standard form to locate the center of the ellipse (h, k).
   b. Describe the graph of the ellipse, and then sketch the graph.
   c. Write the complex form of the equation for this ellipse.