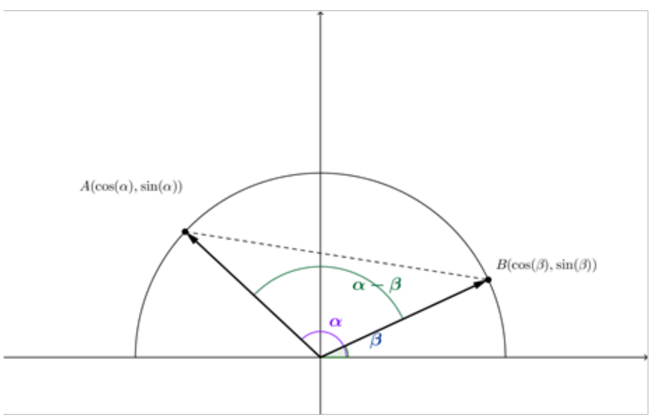
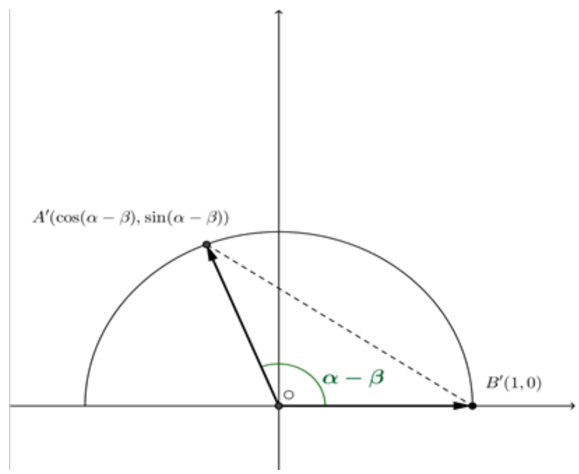
# Lesson 3: Addition and Subtraction Formulas

## Classwork

### Example 1

Consider the figures below. The figure on the right is obtained from the figure on the left by rotating by $-\beta$ about the origin.




a. Calculate the length of $\overline{AB}$ in the figure on the left.

b. Calculate the length of $\overline{A'B'}$ in the figure on the right.

c. Set $\overline{AB}$ and $\overline{A'B'}$ equal to each other, and solve the equation for $\cos(\alpha - \beta)$.

**Exercises 1–2**

1. Use the fact that $\cos(-\theta) = \cos(\theta)$ to determine a formula for $\cos(\alpha + \beta)$.

2. Use the fact that $\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$ to determine a formula for $\sin(\alpha - \beta)$.

## Example 2

Use the identity $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$ to show that $\tan(\alpha + \beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$.

## Exercises 3–5

2. Verify the identity $\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$ for all $(\alpha - \beta) \neq \frac{\pi}{2} + \pi n$.

3. Use the addition and subtraction formulas to evaluate the expressions shown.
   a. $\cos\left(-\frac{5\pi}{12}\right)$

   b. $\sin\left(\frac{23\pi}{12}\right)$

   c. $\tan\left(\frac{5\pi}{12}\right)$

4. Use the addition and subtraction formulas to verify these identities for all real-number values of $\theta$.
   a. $\sin(\pi - \theta) = \sin(\theta)$

   b. $\cos(\pi + \theta) = -\cos(\theta)$

### Lesson Summary

The sum and difference formulas for sine, cosine, and tangent are summarized below.

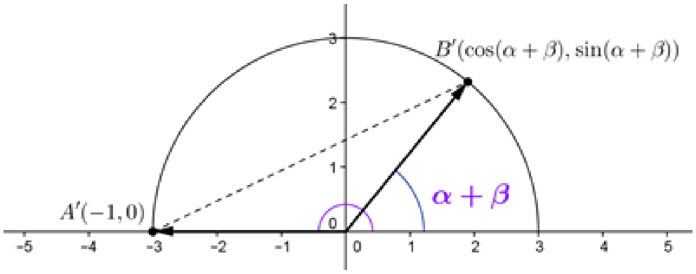
For all real numbers $\alpha$ and $\beta$ for which the expressions are defined,

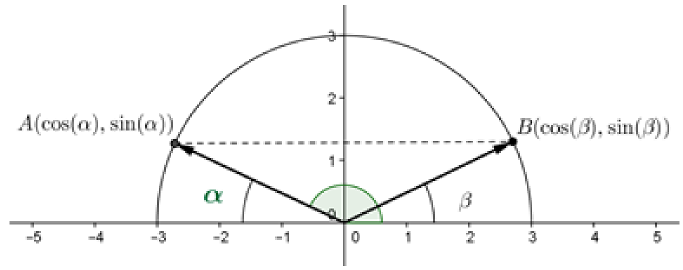
$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$
$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$
$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$
$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)}.$$

## Problem Set

1. Use the addition and subtraction formulas to evaluate the given trigonometric expressions.
   a. $\cos\left(\frac{\pi}{12}\right)$
   b. $\sin\left(\frac{\pi}{12}\right)$
   c. $\sin\left(\frac{5\pi}{12}\right)$
   d. $\cos\left(-\frac{\pi}{12}\right)$
   e. $\sin\left(\frac{7\pi}{12}\right)$
   f. $\cos\left(-\frac{7\pi}{12}\right)$
   g. $\sin\left(\frac{13\pi}{12}\right)$
   h. $\cos\left(-\frac{13\pi}{12}\right)$
   i. $\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)\sin\left(\frac{\pi}{12}\right)$
   j. $\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{6}\right)$
   k. $\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) + \sin\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}\right)$
   l. $\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right) - \sin\left(\frac{\pi}{8}\right)\sin\left(\frac{\pi}{8}\right)$
   m. $\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{12}\right) + \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{12}\right)$
   n. $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{12}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{12}\right)$

2. The figure below and to the right is obtained from the figure on the left by rotating the angle by $\alpha$ about the origin. Use the method shown in Example 1 to show that $\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$.




3. Use the sum formula for sine to show that $\sin(\alpha-\beta)=\sin(\alpha)\cos(\beta)-\cos(\alpha)\sin(\beta)$.

4. Evaluate $\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$ to show $\tan(\alpha+\beta)=\frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$. Use the resulting formula to show that $\tan(2\alpha)=\frac{2\tan(\alpha)}{1-\tan^2(\alpha)}$.

5. Show $\text{an}(x-y)=\frac{\tan(x)-\tan(y)}{1+\tan(x)\tan(y)}$.

6. Find the exact value of the following by using addition and subtraction formulas.
   a. $\tan\left(\frac{\pi}{12}\right)$
   b. $\tan\left(-\frac{\pi}{12}\right)$
   c. $\tan\left(\frac{7\pi}{12}\right)$
   d. $\tan\left(-\frac{13\pi}{12}\right)$
   e. $\frac{\tan\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{12}\right)}{1-\tan\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{12}\right)}$
   f. $\frac{\tan\left(\frac{\pi}{3}\right)-\tan\left(\frac{\pi}{12}\right)}{1+\tan\left(\frac{\pi}{3}\right)\tan\left(\frac{\pi}{12}\right)}$
   g. $\frac{\tan\left(\frac{\pi}{12}\right)+\tan\left(\frac{\pi}{12}\right)}{1-\tan\left(\frac{\pi}{12}\right)\tan\left(\frac{\pi}{12}\right)}$