Lesson 4: Addition and Subtraction Formulas

Classwork

Exercises

1. Derive formulas for the following:
   a. $\sin(2\theta)$

   b. $\cos(2\theta)$

2. Use the double-angle formulas for sine and cosine to verify these identities:
   a. $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$

   b. $\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$

c. $\sin(3\theta) = -4\sin^3(\theta) + 3\sin(\theta)$

3. Suppose that the position of a rider on the unit circle carousel is $(0.8, -0.6)$ for a rotation θ. What is the position of the rider after rotation by 2θ?

4. Use the double-angle formula for cosine to establish the identity $\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{\cos(\theta)+1}{2}}$.

5. Use the double-angle formulas to verify these identities:

a. $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$

b. $\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$

6. The position of a rider on the unit circle carousel is $(0.8, -0.6)$ after a rotation by θ where $0 \leq \theta < 2\pi$. What is the position of the rider after rotation by $\frac{\theta}{2}$?

7. Evaluate the following trigonometric expressions.

a. $\sin\left(\frac{3\pi}{8}\right)$

b. $\tan\left(\frac{\pi}{24}\right)$

Lesson Summary

The double-angle and half-angle formulas for sine, cosine, and tangent are summarized below.

For all real numbers θ for which the expressions are defined,

$$\begin{aligned}
\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\
&= 2\cos^2(\theta) - 1\\
&= 1 - 2\sin^2(\theta)\\
\tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)}\\
\sin\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos(\theta)}{2}}\\
\cos\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{\cos(\theta)+1}{2}}\\
\tan\left(\frac{\theta}{2}\right) &= \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}
\end{aligned}$$

Problem Set

1. Evaluate the following trigonometric expressions.
   a. $2\sin\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{8}\right)$
   b. $\frac{1}{2}\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$
   c. $4\sin\left(-\frac{5\pi}{12}\right)\cos\left(-\frac{5\pi}{12}\right)$
   d. $\cos^2\left(\frac{3\pi}{8}\right) - \sin^2\left(\frac{3\pi}{8}\right)$
   e. $2\cos^2\left(\frac{\pi}{12}\right) - 1$
   f. $1 - 2\sin^2\left(-\frac{\pi}{8}\right)$
   g. $\cos^2\left(-\frac{11\pi}{12}\right) - 2$
   h. $\frac{2\tan\left(\frac{\pi}{8}\right)}{1-\tan^2\left(\frac{\pi}{8}\right)}$
   i. $\frac{2\tan\left(-\frac{5\pi}{12}\right)}{1-\tan^2\left(-\frac{5\pi}{12}\right)}$
   j. $\cos^2\left(\frac{\pi}{8}\right)$
   k. $\cos\left(\frac{\pi}{8}\right)$

l. $\cos\left(-\frac{9\pi}{8}\right)$

m. $\sin^2\left(\frac{\pi}{12}\right)$

n. $\sin\left(\frac{\pi}{12}\right)$

o. $\sin\left(-\frac{5\pi}{12}\right)$

p. $\tan\left(\frac{\pi}{8}\right)$

q. $\tan\left(\frac{\pi}{12}\right)$

r. $\tan\left(-\frac{3\pi}{8}\right)$

2. Show that $\sin(3x) = 3\sin(x)\cos^2(x) - \sin^3(x)$. (Hint: Use $\sin(2x) = 2\sin(x)\cos(x)$ and the sine sum formula.)

3. Show that $\cos(3x) = \cos^3(x) - 3\sin^2(x)\cos(x)$. (Hint: Use $\cos(2x) = \cos^2(x) - \sin^2(x)$ and the cosine sum formula.)

4. Use $\cos(2x) = \cos^2(x) - \sin^2(x)$ to establish the following formulas.
   a. $\cos^2(x) = \frac{1+\cos(2x)}{2}$
   b. $\sin^2(x) = \frac{1-\cos(2x)}{2}$.

5. Jamia says that because sine is an odd function, $\sin\left(\frac{\theta}{2}\right)$ is always negative if θ is negative. That is, she says that for negative values of, $\sin\left(\frac{\theta}{2}\right) = -\sqrt{\frac{1-\cos(\theta)}{2}}$. Is she correct? Explain how you know.

6. Ginger says that the only way to calculate $\sin\left(\frac{\pi}{12}\right)$ is using the difference formula for sine since $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$. Fred says that there is another way to calculate $\sin\left(\frac{\pi}{12}\right)$. Who is correct, and why?

7. Henry says that by repeatedly applying the half-angle formula for sine we can create a formula for $\sin\left(\frac{\theta}{n}\right)$ for any positive integer n. Is he correct? Explain how you know.