Lesson 5: Tangent Lines and the Tangent Function

Student Outcomes

- Students construct a tangent line from a point outside a given circle to the circle (**G-C.A.4**).

Lesson Notes

This lesson is designed to address standard **G-C.A.4**, which involves constructing tangent lines to a given circle from a point outside the circle. The lesson begins by revisiting the geometric origins of the tangent function from Algebra II, Module 2, Lesson 6. Students then explore the standard by means of construction by compass and straightedge and by paper folding. Students are provided several opportunities to create mathematical arguments to explain their constructions.

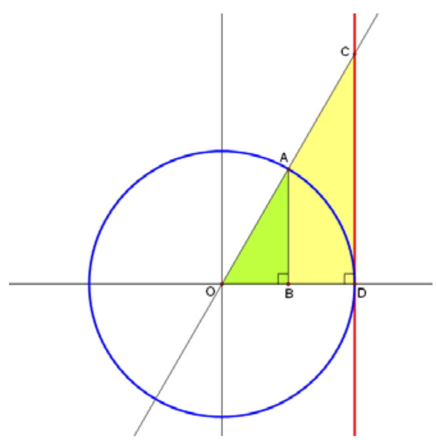
Classwork

Exercises 1–4 (4 minutes)

In this sequence of exercises, students recall the connections between the trigonometric functions and the geometry of the unit circle.

Exercises

The circle shown to the right is a unit circle, and the length of $\widehat{DA}$ is $\frac{\pi}{3}$ radians.



1. **Which segment in the diagram has length $\sin\left(\frac{\pi}{3}\right)$?**

   The sine of $\frac{\pi}{3}$ is the vertical component of point A, so $\overline{AB}$ has length $\sin\left(\frac{\pi}{3}\right)$.

2. **Which segment in the diagram has length $\cos\left(\frac{\pi}{3}\right)$?**

   The cosine of $\frac{\pi}{3}$ is the horizontal component of point A, so $\overline{OB}$ has length $\cos\left(\frac{\pi}{3}\right)$.

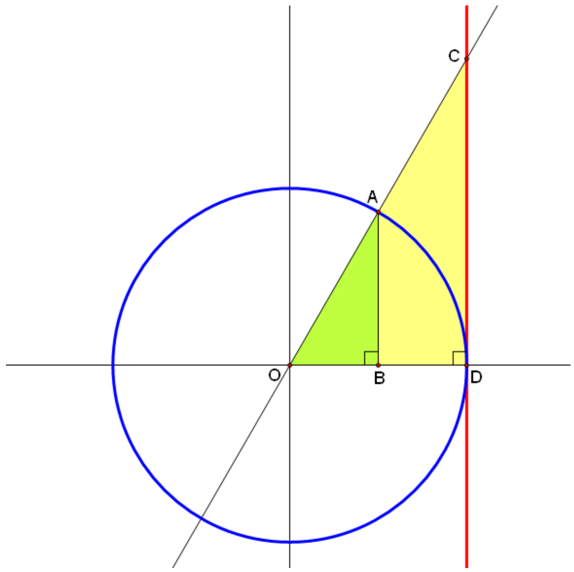
3. **Which segment in the diagram has length $\tan\left(\frac{\pi}{3}\right)$?**

   Since ΔOAB is similar to ΔOCD, $\frac{CD}{1} = \frac{AB}{OB} = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}$. Thus, $\overline{CD}$ has length $\tan\left(\frac{\pi}{3}\right)$.

4. **Which segment in the diagram has length $\sec\left(\frac{\pi}{3}\right)$?**

Since ΔOAB is similar to ΔOCD, $\frac{OD}{OB} = \frac{OC}{OA}$. Since $OA = OD = 1$ we have $OC = \frac{1}{OB} = \frac{1}{\cos\left(\frac{\pi}{3}\right)}$. Thus, $\overline{OC}$ has length $\sec\left(\frac{\pi}{3}\right)$.

Discussion (7 minutes): Connections Between Geometry and Trigonometry

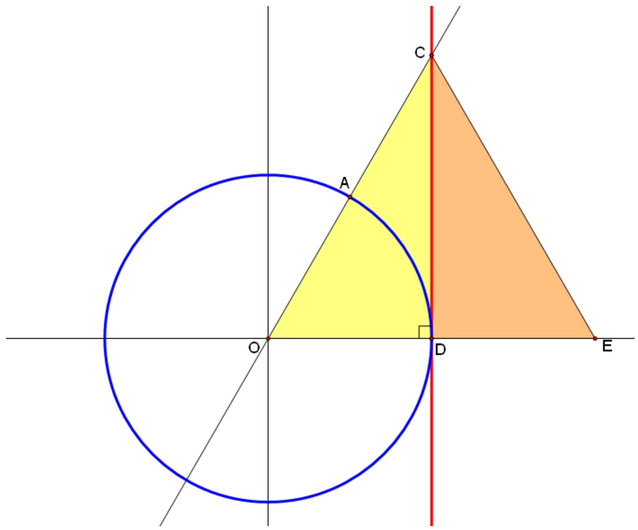


- Do you recall why the length of segment $\overline{CD}$ is called the tangent of $\frac{\pi}{3}$? And do you recall why the length of segment $\overline{OC}$ is called the secant of $\frac{\pi}{3}$? (Hint: Look at how lines containing $\overline{CD}$ and $\overline{OC}$ are related to the circle.)
  - *The line containing $\overline{CD}$ is tangent to the circle because it intersects the circle once, so it makes sense to refer to $\overline{CD}$ as a tangent segment and to refer to the length CD as the tangent of $\frac{\pi}{3}$.*
  - *The line containing $\overline{OC}$ is a secant line because it intersects the circle twice, so it makes sense to refer to $\overline{OC}$ as a secant segment and to refer to the length OC as the secant of $\frac{\pi}{3}$.*
- How can you be sure that the line containing $\overline{CD}$ is, in fact, a tangent line?
  - *We see that $\overleftrightarrow{CD}$ is perpendicular to segment $\overline{OD}$ at point D, and so $\overleftrightarrow{CD}$ must be tangent to the circle.*

MP.7

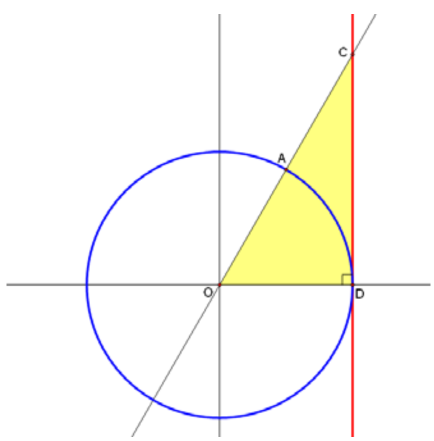
- Explain how this diagram can be used to show that $\tan\left(\frac{\pi}{3}\right) = \frac{\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}$.
  - *The triangles in the diagram have two pairs of congruent angles, so they are similar to each other. It follows that their sides are proportional, and so $\frac{\tan\left(\frac{\pi}{3}\right)}{\sin\left(\frac{\pi}{3}\right)} = \frac{1}{\cos\left(\frac{\pi}{3}\right)}$. If we multiply both sides by $\sin\left(\frac{\pi}{3}\right)$, we obtain the desired result.*
- Explain how this diagram can be used to show that $\sin^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right) = 1$.
  - *If we apply the Pythagorean theorem to ΔAOB, we find that $(AB)^2 + (OB)^2 = (OA)^2$, so $\sin^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right) = 1$.*

- Which trigonometric identity can be established by applying the Pythagorean theorem to ΔCOD?
  - $\tan^2\left(\frac{\pi}{3}\right) + 1^2 = \sec^2\left(\frac{\pi}{3}\right)$.
- What is the scale factor that relates the sides of ΔAOB to those of ΔCOD?
  - *Point B is the midpoint of $\overline{OD}$, so the sides of ΔCOD are twice the length of the corresponding sides of ΔAOB.*

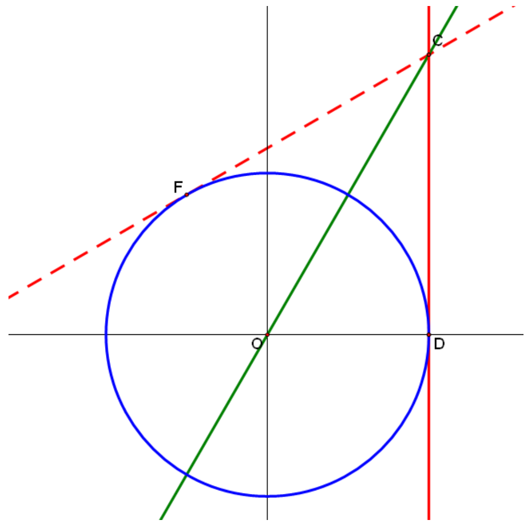


- Explain how the diagram above can be used to show that $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$.
  - *Since the length of $\widehat{DA}$ is $\frac{\pi}{3}$ radians, it follows that ΔCOE is equilateral, and so its sides must each be 2 units long. Applying the Pythagorean theorem to ΔCOD, we find $1^2 + CD^2 = 2^2$, which means that $CD^2 = 3$, and so $CD = \sqrt{3}$. Thus, $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$.*
- This diagram contains a host of information about trigonometry. But can we take this diagram even further? Let's press on to some new territory. Much of the remaining discussion will be devoted to the topic of constructions. Get ready to have some fun with paper folding and your compass and straightedge!

Exploratory Challenge 1 (7 minutes): Constructing Tangents via Paper-Folding



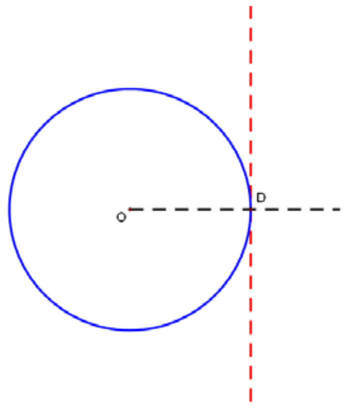
- Earlier we observed that $\overleftrightarrow{CD}$ is tangent to the circle. Can you visualize another line through point C that is also tangent to the circle? Try to draw a second tangent line through point C, and label its point of intersection with the circle as point F.



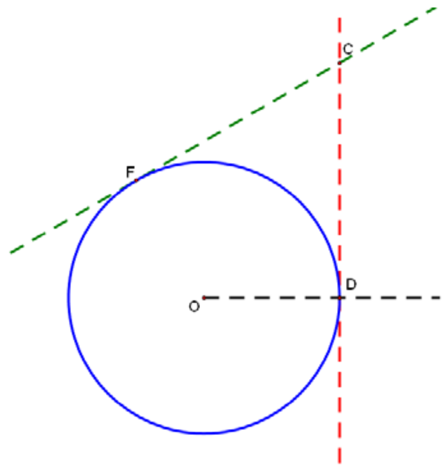
- First, let's examine this diagram through the lens of transformations. Can you see how to map $\overleftrightarrow{CD}$ onto $\overleftrightarrow{CF}$?
  - *It appears that you could map $\overleftrightarrow{CD}$ onto $\overleftrightarrow{CF}$ by reflecting it across $\overleftrightarrow{CO}$.*
- Let's get some hands-on experience with this by performing the reflection in the most natural way imaginable – by folding a piece of paper.

Give students an unlined piece of copy paper or patty paper, a ruler, and a compass.

- The goal of this next activity is essentially to recreate the diagram above. Students take one tangent line and use a reflection to produce a second tangent line.
- Choose a point in the middle of your paper, and label it O. Use your compass to draw a circle with center O. Choose a point on that circle, and label it D. Use your straightedge to draw the line through O and D, extending it beyond the circle. Can you see how to fold the paper in such a way that the crease is tangent to the circle at point D? Think about this for a moment.
  - *We want to create a line that is perpendicular to $\overleftrightarrow{OD}$ at point D, so we fold the paper in such a way that the crease is on point D and $\overleftrightarrow{OD}$ maps to itself.*



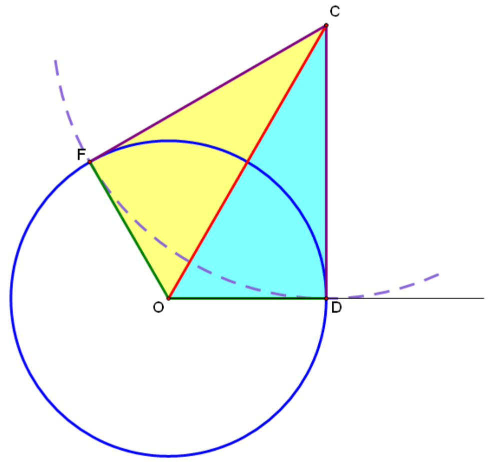
- Now choose a point on the crease, and label it C. Try to create a second tangent line through point C. Can you see how to do this?
  - *We just need to fold the paper so that the crease is on points C and O. We can see through the back of the paper where point D is, and this is where we mark a new point F, which is the other point of tangency. In other words, $\overleftrightarrow{CF}$ is also tangent to the circle.*



- Do you think you could construct an argument that $\overleftrightarrow{CF}$ is truly tangent to the circle? Think about this for a few minutes, and then share your thoughts with the students around you.
  - *A reflection is a rigid motion, preserving both distances and angles. Since $\overleftrightarrow{CD}$ is tangent to the circle, we know that $\angle CDO$ is a right angle. And since the distance from O to D is preserved by the reflection, it follows that point F, which is the image of D, is also located on the circle because $OF = OD$, both of which are radii of the circle. This means that F is on the circle and that $\angle CFO$ is a right angle, which proves that $\overleftrightarrow{CF}$ is indeed tangent to the circle.*

Exploratory Challenge 2 (5 minutes): Constructing Tangents using a Compass

- Let's try to produce this tangent line without using paper-folding. Can you see how to produce this diagram using only a compass and a straightedge? Take a few minutes to explore this problem.
  - *We use a straightedge to draw a line that is perpendicular to $\overleftrightarrow{OD}$ at point D and choose a point C on this perpendicular line as before. Next, we use a compass to draw a circle around point C that contains point D. This circle will intersect the original circle at a point F, which is the desired point of tangency.*

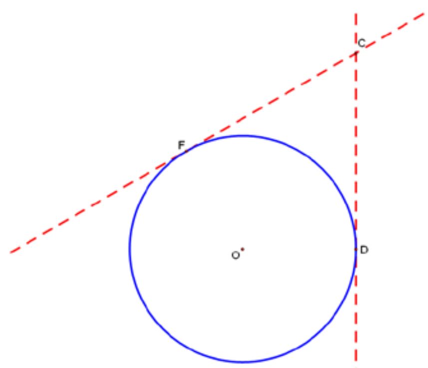


- Now make an argument that your construction produces a line that is truly tangent to the circle.
  - *Each point on the circle around point O is the same distance from O, so $OF = OD$. Also, each point on the circle around point C is the same distance from point C, so $CF = CD$. Since $CO = CO$, $\Delta OCD \cong \Delta OCF$ by the SSS criterion for triangle congruence. Since $\angle CDO$ is a right angle, it follows by $CPCTC$ that $\angle CFO$ is also a right angle. This proves that $\overleftrightarrow{CF}$ is indeed tangent to the circle.*

Exploratory Challenge 3 (7 minutes): Constructing the Tangents from an External Point

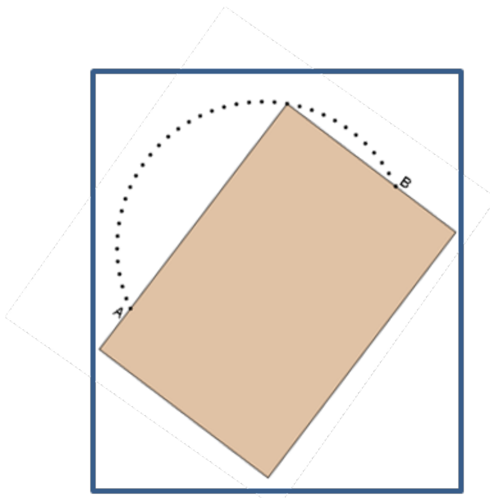
MP.1

- Now let's change the problem slightly. Suppose we are given a circle and an external point. How might we go about constructing the lines that are tangent to the given circle and pass through the given point? Take several minutes to wrestle with this problem, and then share your thoughts with a neighbor.



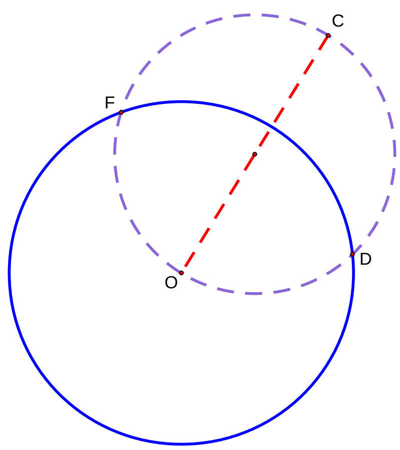
- This is a tough problem, isn't it? Let's see if we can solve it by using some logical reasoning. What do you know about the lines you want to create?
  - *We want the lines to be tangent to the circle, so they must be constructed in such a way that $\angle CFO$ and $\angle CDO$ are right angles.*
- Perhaps we can use this to our advantage. Let's consider the entire locus of points F such that $\angle CFO$ is a right angle. Can you describe this locus? Can you construct it? This is a challenge in its own right. Let's explore this challenge using a piece of paper, which comes with some built-in right angles.
- Take a blank piece of paper, and mark two points several inches apart; label them A and B. Now take a second

piece of paper, and line up one edge on A and the adjacent edge on B. Mark the point where the corner of the paper is, then shift the corner to a new spot, keeping the edges against A and B. Quickly repeat this 20 or so times until a clear pattern emerges. Then describe what you see.

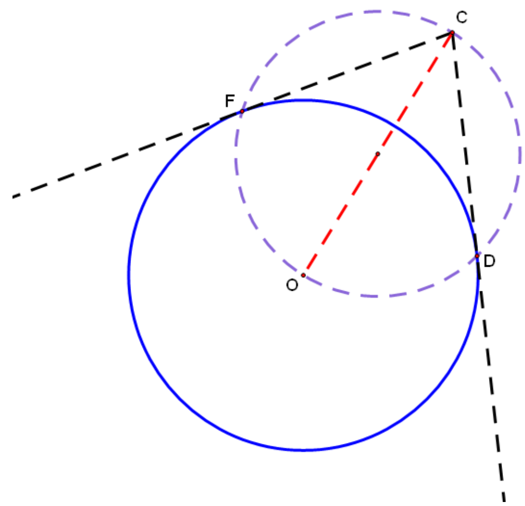


- *This locus of points is a circle with diameter $\overline{AB}$.*

- Now construct this circle using your compass.
  - *We just need to construct the midpoint of $\overline{AB}$, which is the center of the circle.*
- We are ready to apply this result to our problem involving tangents to a circle. Do you see how this relates to the problem of creating a tangent line?



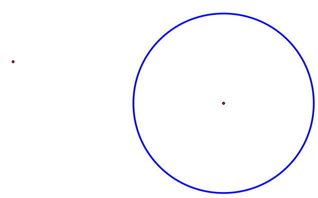
- *By drawing a circle with diameter $\overline{OC}$, we create two points F and D with two important properties. First, they are both on the circle around O. Second, they are on the circle with diameter $\overline{OC}$, which means that $\angle OFC$ and $\angle ODC$ are right angles, just as we require for tangent lines.*



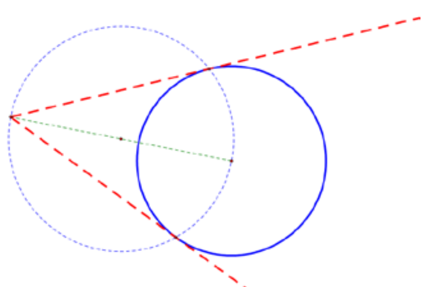
Exercises 5–7 (4 minutes)

Instruct students to perform the following exercises and to compare their results with a partner.

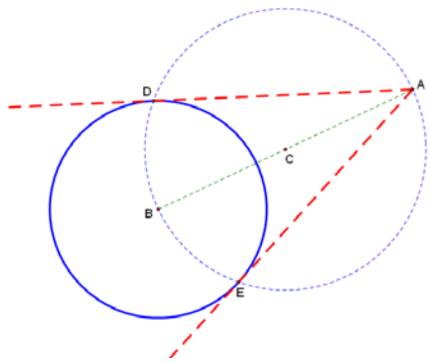
5. Use a compass to construct the tangent lines to the given circle that pass through the given point.



Sample solution:

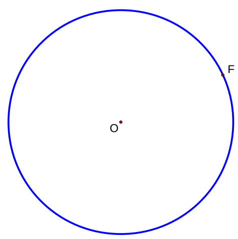


6. **Analyze the construction shown below. Argue that the lines shown are tangent to the circle with center B.**

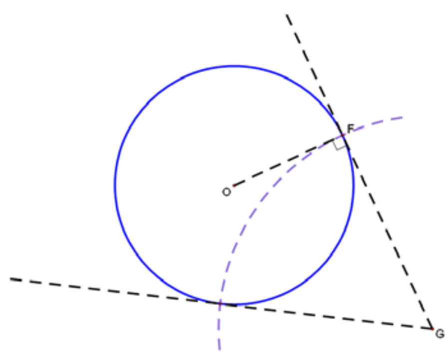


Since point D is on circle C, we know that $\angle BDA$ is a right angle. Since point D is on circle B and $\angle BDA$ is a right angle, it follows that $\overleftrightarrow{AD}$ is tangent to circle B. In the same way, we can show that $\overleftrightarrow{AE}$ is tangent to circle B.

7. **Use a compass to construct a line that is tangent to the circle below at point F. Then choose a point G on the tangent line, and construct another tangent to the circle through G.**



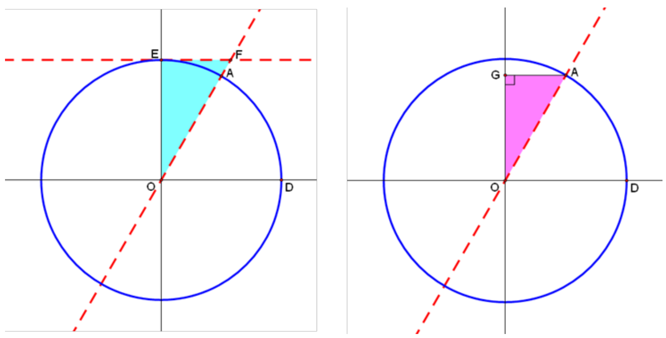
Sample solution:



Exercises 8–12 (5 minutes)

Instruct students to perform the following exercises and to compare their results with a partner.

8. **The circles shown below are unit circles, and the length of $\widehat{DA}$ is $\frac{\pi}{3}$ radians.**



Which trigonometric function corresponds to the length of $\overline{EF}$?

$\overline{EF}$ represents the cotangent of $\frac{\pi}{3}$.

9. **Which trigonometric function corresponds to the length of $\overline{OF}$?**

$\overline{OF}$ represents the cosecant of $\frac{\pi}{3}$.

10. **Which trigonometric identity gives the relationship between the lengths of the sides of ΔOEF?**

$\cot^2\left(\frac{\pi}{3}\right) + 1^2 = \csc^2\left(\frac{\pi}{3}\right)$

11. **Which trigonometric identities give the relationships between the corresponding sides of ΔOEF and ΔOGA?**

We have $\frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{\cot\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)}$ *and* $\frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{\csc\left(\frac{\pi}{3}\right)}{1}$.

12. **What is the value of $\csc\left(\frac{\pi}{3}\right)$? What is the value of $\cot\left(\frac{\pi}{3}\right)$? Use the Pythagorean theorem to support your answers.**

Let $x = EF$. Then we have $x^2 + 1^2 = (2x)^2 = 4x^2$, which means $3x^2 = 1$; therefore, $x = \sqrt{\frac{1}{3}}$. So $\cot\left(\frac{\pi}{3}\right) = \sqrt{\frac{1}{3}}$ and $\csc\left(\frac{\pi}{3}\right) = 2\sqrt{\frac{1}{3}}$.

Closing (2 minutes)

- If you are given a circle and an external point, how do you use a compass to construct the lines that pass through the given point and are tangent to the given circle? Use your notebook to write a summary of the main steps involved in this construction.
  - *Let's say the center of the circle is point A and the external point is B. First, we need to construct the midpoint of the segment joining A and B – call this point C. Next, we draw a circle around point C that goes through A and B. The points where this circle intersects the first circle are the points of tangency, so we just connect these points to point B to form the tangents.*

Exit Ticket (4 minutes)

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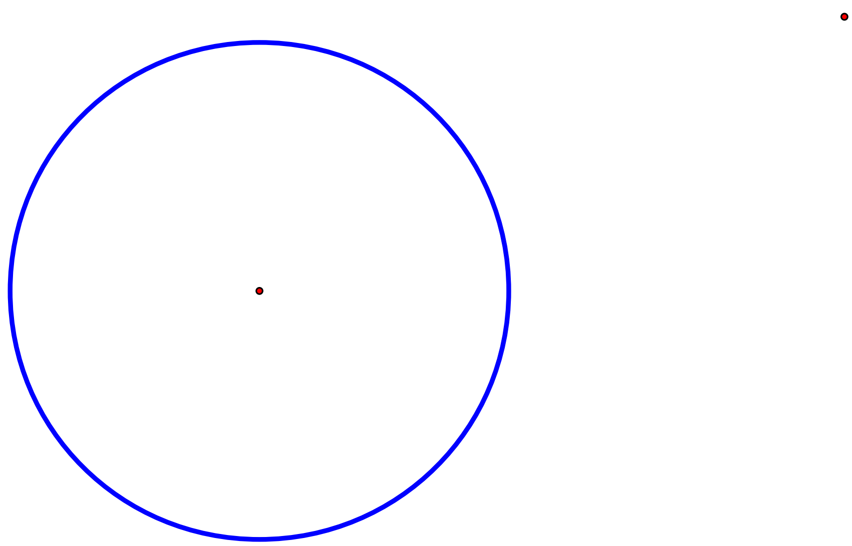
Lesson 5: Tangent Lines and the Tangent Function

Exit Ticket

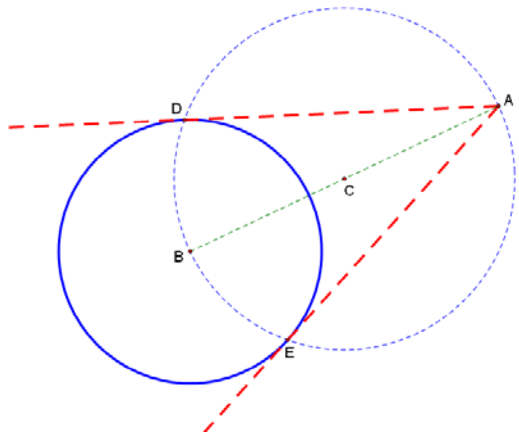
1. Use a compass and a straightedge to construct the tangent lines to the given circle that pass through the given point.



2. Explain why your construction produces lines that are indeed tangent to the given circle.

Exit Ticket Sample Solutions

1. Use a compass and a straightedge to construct the tangent lines to the given circle that pass through the given point.

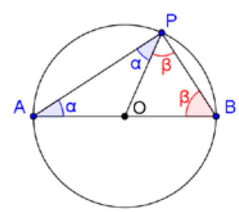


2. Explain why your construction produces lines that are indeed tangent to the given circle.

Since points D and E are on circle C, $\angle BDA$ and $\angle BEA$ are right angles. Thus, $\overleftrightarrow{AD}$ and $\overleftrightarrow{AE}$ are tangent to circle B.

Problem Set Sample Solutions

1. Prove Thales' theorem: If A, B, and P are points on a circle where $\overline{AB}$ is a diameter of the circle, then $\angle APB$ is a right angle.

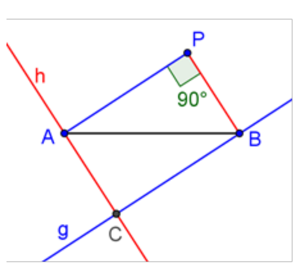


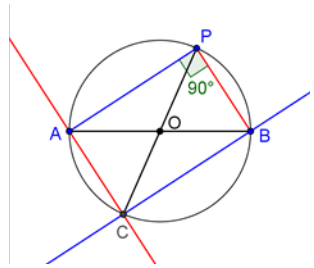
Since $\overline{OA} = \overline{OP} = \overline{OB}$, ΔOPA and ΔOPB are isosceles triangles. Therefore, $\angle OAP = \angle OPA$, and $\angle OPB = \angle OBP$.

Let $m\angle OPA = \alpha$ and $m\angle OPB = \beta$. The sum of three internal angles of ΔAPB equals 180°.

Therefore, $\alpha + (\alpha + \beta) + \beta = 180^\circ$, so $2\alpha + 2\beta = 180^\circ$, and $\alpha + \beta = 90^\circ$. Since $m\angle APB = \alpha + \beta$, we have $m\angle APB = 90°$, so $\angle APB$ is a right angle.

2. **Prove the converse of Thales' theorem: If $\overline{AB}$ is a diameter of a circle and P is a point so that $\angle APB$ is a right angle, then P lies on the circle for which $\overline{AB}$ is a diameter.**





Construct the right triangle $\triangle APB$.

Construct the line h that is parallel to $\overline{PB}$ through point A.

Construct the line g that is parallel to $\overline{AP}$ through point B.

Let C be the intersection of lines h and g.

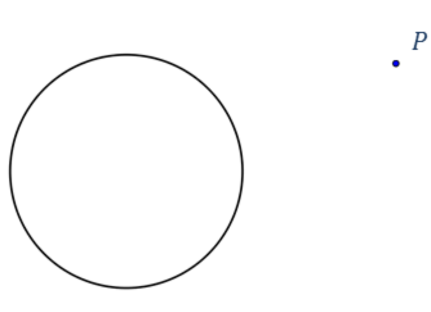
The quadrilateral $ACBP$ forms a parallelogram by construction.

By the properties of parallelograms, the adjacent angles are supplementary. Since $\angle APB$ is a right angle,

it follows that angles $\angle CAP$, $\angle BCA$, and $\angle PBC$ are also right angles. Therefore, the quadrilateral $ACBP$ is a rectangle.

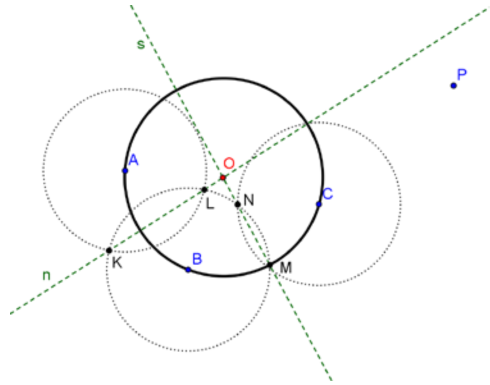
Let O be the intersection of the diagonals $\overline{AB}$ and $\overline{CP}$. Then, by the properties of parallelgrams, point O is the midpoint of $\overline{AB}$ and $\overline{CP}$, so $OA = OB = OC = OP$. Therefore, O is the center of the circumscribing circle, and the hypotenuse of the triangle $\overline{AB}$ is a diameter of the circle.

3. **Construct the tangent lines from point P to the circle given below.**

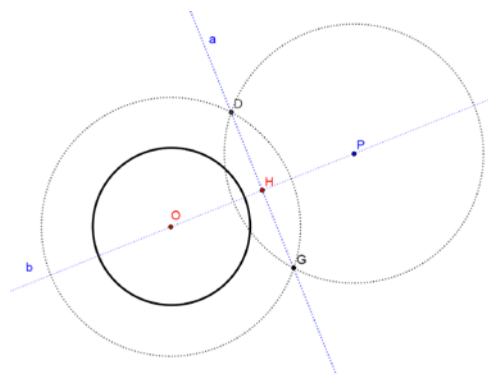


Mark any 3 points A, B, and C on the circle, and construct perpendicular bisectors of $\overline{AB}$ and $\overline{AC}$.

Let O be the intersection of the two perpendicular bisectors.



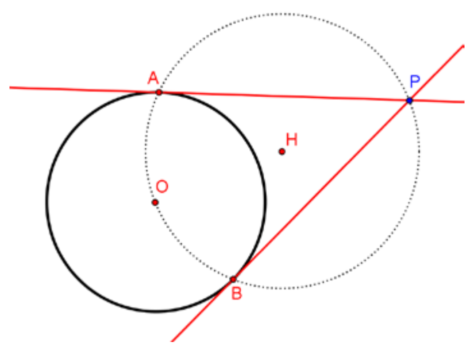
Construct the midpoint H of $\overline{OP}$.



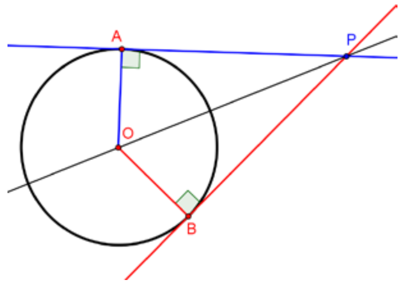
Construct a circle with center H and radius OH.

The circle centered at H will intersect the original circle O at points A and B.

Construct two tangent lines $\overleftrightarrow{PA}$ and $\overleftrightarrow{PB}$.

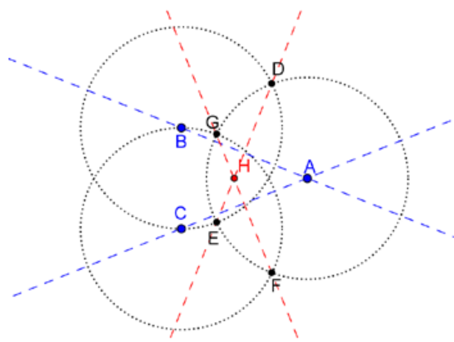


4. Prove that if segments from a point P are tangent to a circle at points A and B, then $\overline{PA} = \overline{PB}$.



Let P be a point outside of a circle with center O, and let A and B be points on the circle so that $\overline{PA}$ and $\overline{PB}$ are tangent to the circle. Then, $\overline{OA} = \overline{OB}$, $\overline{OP} = \overline{OP}$, and $m\angle OAP = m\angle OBP = 90^{\circ}$, so $\Delta PAO \cong \Delta PBO$ by the Hypotenuse Leg congruence criterion. Therefore, $\overline{PA} = \overline{PB}$ because corresponding parts of congruent triangles are congruent.

5. Given points A, B, and C so that $AB = AC$, construct a circle so that $\overline{AB}$ is tangent to the circle at B and $\overline{AC}$ is tangent to the circle at C.



Construct a perpendicular bisector of $\overline{AB}$.

Construct a perpendicular bisector of $\overline{AC}$.

The perpendicular bisectors will intersect at point H.

Construct a line through points A and H.

Construct a circle with center H and radius $\overline{HA}$.

The circle centered at H will intersect $\overleftrightarrow{HA}$ at I.

Construct a circle centered at I with radius $\overline{IA}$.