Lesson 14: Modeling with Inverse Trigonometric Functions

Classwork

Example 1

A designer wants to test the safety of a wheelchair ramp she has designed for a building before constructing it, so she creates a scale model. To meet the city's safety requirements, an object that starts at a standstill from the top of the ramp and rolls down it should not experience an acceleration exceeding $2.4\,\frac{m}{s^2}$.

a. A ball of mass 0.1 kg is used to represent an object that rolls down the ramp. As it is placed at the top of the ramp, the ball experiences a downward force due to gravity, which causes it to accelerate down the ramp. Knowing that the force applied to an object is the product of its mass and acceleration, create a sketch to model the ball as it accelerates down the ramp.

b. If the ball rolls at the maximum allowable acceleration of $2.4\ \frac{\text{m}}{\text{s}^2}$, what is the angle of elevation for the ramp?

c. If the designer wants to exceed the safety standards by ensuring the acceleration of the object does not exceed $2.0\frac{m}{s^2}$, by how much will the maximum angle of elevation decrease?

d. How does the mass of the object used in the scale model affect the value of θ? Explain your response.

Exercise 1

A vehicle with a mass of 1000 kg rolls down a slanted road with an acceleration of $0.07\frac{m}{s^2}$. The frictional force between the wheels of the vehicle and the wet concrete road is 2800 Newtons.

a. Sketch the situation.

b. What is the angle of elevation of the road?

c. What is the maximum angle of elevation the road could have so that the vehicle described would not slide down the road?

Example 2

The declination of the sun is the path the sun takes overhead the earth throughout the year. When the sun passes directly overhead, the declination is defined as $0°$, while a positive declination angle represents a northward deviation and a negative declination angle represents a southward deviation. Solar declination is periodic and can be roughly estimated using the equation$\delta = -23.44°(\cos\left(\frac{360}{365}\right)(N+10))$, where N represents a calendar date, e.g., $N = 1$ is January 1, and δ is the declination angle of the sun measured in degrees.

a. Describe the domain and range of the function.

b. Write an equation that represents N as a function of δ.

c. Determine the calendar date(s) for the given angles of declination:

i. $10°$

ii. $-5.2°$

iii. $25°$

d. When will the sun trace a direct path above the equator?

Exercises 2–3

2. The average monthly temperature in a coastal city in the United States is periodic and can be modeled with the equation $y = -8\cos\left((x-1)\left(\frac{\pi}{6}\right)\right) + 17.5$, where y represents the average temperature in degrees Celsius and x represents the month, with $x = 1$ representing January.

a. Write an equation that represents x as a function of y.

b. A tourist wants to visit the city when the average temperature is closest to $25°$ Celsius. What recommendations would you make regarding when the tourist should travel? Justify your response.

3. The estimated size for a population of rabbits and a population of coyotes in a desert habitat are shown in the table. The estimated population sizes were recorded as part of a long-term study related to the effect of commercial development on native animal species.

Years since initial count (n)	0	3	6	9	12	15	18	21	24
Estimated number of rabbits (r)	$14{,}989$	$10{,}055$	$5{,}002$	$10{,}033$	$15{,}002$	$10{,}204$	$4{,}999$	$10{,}002$	$14{,}985$
Estimated number of coyotes (c)	$1{,}995$	$2{,}201$	$2{,}003$	$1{,}795$	$1{,}999$	$2{,}208$	$2{,}010$	$1{,}804$	$2{,}001$

a. Describe the relationship between sizes of the rabbit and coyote populations throughout the study.

b. Plot the relationship between the number of years since the initial count and the number of rabbits. Fit a curve to the data.

c. Repeat the procedure described in part (b) for the estimated number of coyotes over the course of the study.

d. During the study, how many times was the rabbit population approximately $12{,}000$? When were these times?

e. During the study, when was the coyote population estimate below $2{,}100$?

Problem Set

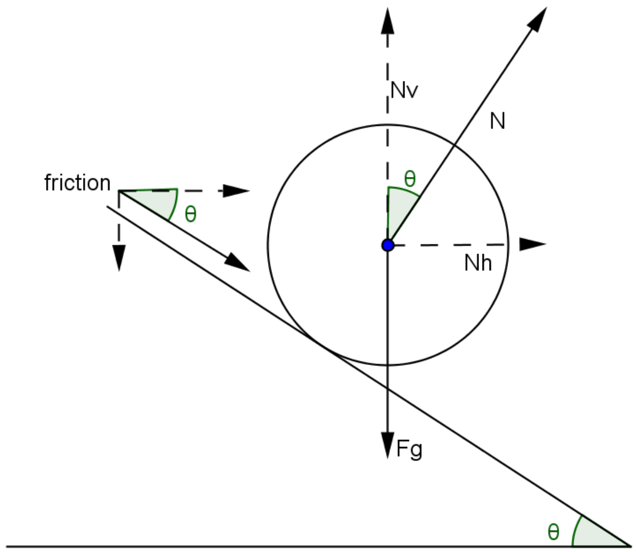
1. A particle is moving along a line at a velocity of $y = 3\sin\left(\frac{2\pi x}{5}\right) + 2\ \frac{\mathrm{m}}{\mathrm{s}}$ at location x meters from the starting point on the line for $0 \le x \le 20$.
   a. Find a formula that represents the location of the particle given its velocity.
   b. What is the domain and range of the function you found in part (a)?
   c. Use your answer to part (a) to find where the particle is when it is traveling $5\ \frac{\mathrm{m}}{\mathrm{s}}$ for the first time.
   d. How can you find the other locations the particle is traveling at this speed?

2. In general, since the cosine function is merely the sine function under a phase shift, mathematicians and scientists regularly choose to use the sine function to model periodic phenomena instead of a mixture of the two. What behavior in data would prompt a scientist to use a tangent function instead of a sine function?

3. A vehicle with a mass of $500\ \mathrm{kg}$ rolls down a slanted road with an acceleration of $0.04\ \frac{\mathrm{m}}{\mathrm{s}^2}$. The frictional force between the wheels of the vehicle and the road is 1800 Newtons.
   a. Sketch the situation.
   b. What is the angle of elevation of the road?
   c. The steepness of a road is frequently measured as grade, which expresses the slope of a hill as a percent that the change in height is of the change in horizontal distance. What is the grade of the hill described in this problem?

4. Canton Avenue in Pittsburgh, PA is considered to be one of the steepest roads in the world with a grade of 37%.
   a. Assuming no friction on a particularly icy day, what would be the acceleration of a $1000\ \mathrm{kg}$ car with only gravity acting on it?
   b. The force due to friction is equal to the product of the force perpendicular to the road and the coefficient of friction μ. For icy roads of a non-moving vehicle, assume the coefficient of friction is $\mu = 0.3$. Find the force due to friction for the car above. If the car is in park, will it begin sliding down Canton Avenue if the road is this icy?
   c. Assume the coefficient of friction for moving cars on icy roads is $\mu = 0.2$. What is the maximum angle of road that the car will be able to stop on?

5. Talladega Superspeedway has some of the steepest turns in all of NASCAR. The main turns have a radius of about 305 m and are pitched at $33°$. Let N be the perpendicular force on the car, and N_v and N_h be the vertical and horizontal components of this force, respectively. See the diagram below.



a. Let μ represent the coefficient of friction; recall that μN gives the force due to friction. To maintain the position of a vehicle traveling around the bank, the centripetal force must equal the horizontal force in the direction of the center of the track. Add the horizontal component of friction to the horizontal component of the perpendicular force on the car to find the centripetal force. Set your expression equal to $\frac{mv^2}{r}$, the centripetal force.

b. Add the vertical component of friction to the force due to gravity, and set this equal to the vertical component of the perpendicular force.

c. Solve one of your equations in part (a) or part (b) for m, and use this with the other equation to solve for v.

d. Assume $\mu = 0.75$, the standard coefficient of friction for rubber on asphalt. For the Talladega Superspeedway, what is the maximum velocity on the main turns? Is this about how fast you might expect NASCAR stock cars to travel? Explain why you think NASCAR takes steps to limit the maximum speeds of the stock cars.

e. Does the friction component allow the cars to travel faster on the curve or force them to drive slower? What is the maximum velocity if the friction coefficient is zero on the Talladega roadway?

f. Do cars need to travel slower on a flat roadway making a turn than on a banked roadway? What is the maximum velocity of a car traveling on a 305 m turn with no bank?

6. At a particular harbor over the course of 24 hours, the following data on peak water levels was collected (measurements are in feet above the MLLW):

Time	$1:30$	$7:30$	$14:30$	$20:30$
Water Level	-0.211	8.21	-0.619	7.518

   a. What appears to be the average period of the water level?
   b. What appears to be the average amplitude of the water level?
   c. What appears to be the average midline for the water level?
   d. Fit a curve of the form $y = A\sin\big(\omega(x-h)\big) + k$ or $y = A\cos\big(\omega(x-h)\big) + k$ modeling the water level in feet as a function of the time.
   e. According to your function, how many times per day will the water level reach its maximum?
   f. How can you find other time values for a particular water level after finding one value from your function?
   g. Find the inverse function associated with the function in part (d). What is the domain and range of this function? What type of values does this function output?