Lesson 3: Counting Rules—Combinations

Classwork

Example 1

Seven speed skaters are competing in an Olympic race. The first-place skater earns the gold medal, the second-place skater earns the silver medal, and the third-place skater earns the bronze medal. In how many different ways could the gold, silver, and bronze medals be awarded? The letters A, B, C, D, E, F, and G will be used to represent these seven skaters.

How can we determine the number of different possible outcomes? How many are there?

Now consider a slightly different situation. Seven speed skaters are competing in an Olympic race. The top three skaters move on to the next round of races. How many different "top three" groups can be selected?

How is this situation different from the first situation? Would you expect more of fewer possibilities in this situation? Why?

Would you consider the outcome where skaters B, C, and A advance to the final to be a different outcome than A, B, and C advancing?

A permutation is an ordered arrangement (a sequence) of k items from a set of n distinct items.

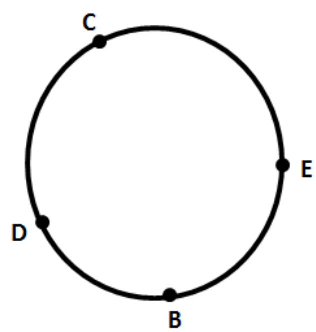
In contrast, a combination is an unordered collection (a set) of k items from a set of n distinct items.

When we wanted to know how many ways there are for seven skaters to finish first, second, and third, order was important. This is an example of a permutation of 3 selected from a set of 7. If we want to know how many possibilities there are for which three skaters will advance to the finals, order is not important. This is an example of a combination of 3 selected from a set of 7.

Exercises 1–4

1. Given four points on a circle, how many different line segments connecting these points do you think could be drawn? Explain your answer.

2. Draw a circle and place four points on it. Label the points as shown. Draw segments (chords) to connect all the pairs of points. How many segments did you draw? List each of the segments that you drew. How does the number of segments compare to your answer in Exercise 1?



You can think of each segment as being identified by a subset of two of the four points on the circle. Chord ED is the same as chord DE. The order of the segment labels is not important. When you count the number of segments (chords), you are counting combinations of two points chosen from a set of four points.

3. Find the number of permutations of two points from a set of four points. How does this answer compare to the number of segments you were able to draw?

4. If you add a fifth point to the circle, how many segments (chords) can you draw?

   If you add a sixth point, how many segments (chords) can you draw?

Example 2

Let's look closely at the four examples we have studied so far.

Choosing gold, silver, and bronze medal skaters	Choosing groups of the top three skaters
Finding the number of segments that can be drawn connecting two points out of four points on a circle	Finding the number of *unique* segments that can be drawn connecting two points out of four points on a circle

What do you notice about the way these are grouped?

The number of combinations of k items selected from a set of n distinct items is

$$ {}_nC_k = \frac{{}_nP_k}{k!} \text{ or } \quad {}_nC_k = \frac{n!}{k!\,(n-k)!}\,. $$

Exercises 5–11

5. Find the value of each of the following:
   a. $_{9}C_{2}$

   b. $_{7}C_{7}$

   c. $_{8}C_{0}$

   d. $_{15}C_{1}$

6. Find the number of segments (chords) that can be drawn for each of the following:
   a. 5 points on a circle

   b. 6 points on a circle

   c. 20 points on a circle

   d. n points on a circle

7. For each of the following questions, indicate whether the question posed involves permutations or combinations. Then provide an answer to the question with an explanation for your choice.

   a. A student club has 20 members. How many ways are there for the club to choose a president and a vice-president?

   b. A football team of 50 players will choose two co-captains. How many different ways are there to choose the two co-captains?

c. There are seven people who meet for the first time at a meeting. They shake hands with each other and introduce themselves. How many handshakes have been exchanged?

d. At a particular restaurant, you must choose two different side dishes to accompany your meal. If there are eight side dishes to choose from, how many different possibilities are there?

e. How many different four-letter sequences can be made using the letters A, B, C, D, E, and F if letters may not be repeated?

8. How many ways can a committee of 5 students be chosen from a student council of 30 students? Is the order in which the members of the committee are chosen important?

9. Brett has ten distinct t-shirts. He is planning on going on a short weekend trip to visit his brother in college. He has enough room in his bag to pack four t-shirts. How many different ways can he choose four t-shirts for his trip?

10. How many three-topping pizzas can be ordered from the list of toppings below? Did you calculate the number of permutations or the number of combinations to get your answer? Why did you make this choice?

Pizza Toppings

sausage	pepperoni	meatball	onions	olives	spinach
pineapple	ham	green peppers	mushrooms	bacon	hot peppers

11. Write a few sentences explaining how you can distinguish a question about permutations from a question about combinations.

Lesson Summary

A **combination** is a subset of k items selected from a set of n distinct items.

The number of combinations of k items selected from a set of n distinct items is

$$_{n}C_{k} = \frac{_{n}P_{k}}{k!} \quad \text{or} \quad _{n}C_{k} = \frac{n!}{k!\,(n-k)!}.$$

Problem Set

1. Find the value of each of the following:
   a. $_{9}C_{8}$
   b. $_{9}C_{1}$
   c. $_{9}C_{9}$

2. Explain why $_{6}C_{4}$ is the same value as $_{6}C_{2}$.

3. Pat has 12 books he plans to read during the school year. He decides to take 4 of these books with him while on winter break vacation. He decides to take *Harry Potter and the Sorcerer's Stone* as one of the books. In how many ways can he select the remaining 3 books?

4. In a basketball conference of 10 schools, how many conference basketball games are played during the season if the teams all play each other exactly once?

5. Which scenario(s) below is represented by $_{9}C_{3}$?
   a. Number of ways 3 of 9 people can sit in a row of 3 chairs.
   b. Number of ways to pick 3 students out of 9 students to attend an art workshop.
   c. Number of ways to pick 3 different entrees from a buffet line of 9 different entrees.

6. Explain why $_{10}C_{3}$ would not be used to solve the following problem:

   There are 10 runners in a race. How many different possibilities are there for the runners to finish first, second, and third?

7. In a lottery, players must match five numbers plus a bonus number. Five white balls are chosen from 59 white balls numbered from 1 to 59 and one red ball (the bonus number) is chosen from 35 red balls numbered 1 to 35. How many different results are possible?

8. In many courts, 12 jurors are chosen from a pool of 30 perspective jurors.
   a. In how many ways can 12 jurors be chosen from the pool of 30 perspective jurors?
   b. Once the 12 jurors are selected, 2 alternates are selected. The order of the alternates is specified. If a selected juror cannot complete the trial, the first alternate is called on to fill that jury spot. In how many ways can the 2 alternates be chosen after the 12 jury members have been chosen?

9. A band director wants to form a committee of 4 parents from a list of 45 band parents.
   a. How many different groups of 4 parents can the band director select?
   b. How many different ways can the band director select 4 parents to serve in the band parents' association as president, vice-president, treasurer, and secretary?
   c. Explain the difference between parts (a) and (b) in terms of how you decided to solve each part.

10. If you roll a cube with the numbers from 1 to 6 on the faces of the cube 4 times, how many different outcomes are possible?

11. Write a problem involving students that has an answer of ${}_6C_3$.

12. Suppose that a combination lock is opened by entering a three-digit code. Each digit can be any integer between 0 and 9, but digits may not be repeated in the code. How many different codes are possible? Is this question answered by considering permutations or combinations? Explain.

13. Six musicians will play in a recital. Three will perform before intermission, and three will perform after intermission. How many different ways are there to choose which three musicians will play before intermission? Is this question answered by considering permutations or combinations? Explain.

14. In a game show, contestants must guess the price of a product. A contestant is given nine cards with the numbers 1 to 9 written on them (each card has a different number). The contestant must then choose three cards and arrange them to produce a price in dollars. How many different prices can be formed using these cards? Is this question answered by considering permutations or combinations? Explain.

15.
   a. Using the formula for combinations, show that the number of ways of selecting 2 items from a group of 3 items is the same as the number of ways to select 1 item from a group of 3.
   b. Show that ${}_nC_k$ and ${}_nC_{n-k}$ are equal. Explain why this makes sense.